Recurrence Relation

Recursion → function is calling itself directly or indirectly

$3! = 3 \times 2 \times 1 = 6$

$0! = 1$

$1! = 1$

→ Base condition

T(n)

Psuedocode

```
fact(n):                    <- n >= 0
    if n==0 or n==1:   } -> O(1)
        return 1       }    c

    else:
                 n * T(n-1)
        return n * fact(n-1)
                     ↑
                 Recursion
```

Recurrence Relation
- Substitution method
- Recursive Tree
- Masters Theorem

$n \geq 0$

$$\boxed{T(n) = \begin{cases} c & n \leq 1 \\ n * T(n-1) & n > 1 \end{cases}}$$

← Recurrence Relation

Substitution Method

Example 1

$$T(n) = \begin{cases} 1 & n=1 \\ T(n-1)+n & n>1 \end{cases}$$

$T(1) = 1$

↓ Get Rid of recursive term

$T(n-1)$ → function is calling itself

substitute

$$\boxed{T(n) = T(n-1)+n}$$ — 1st time

$$T(n-1) = T(n-1-1) + n-1$$

$$T(n-1) = \underbrace{T(n-2)+n-1}$$

$$\boxed{T(n) = \underbrace{T(n-2)+n-1}_{T(n-1)}+n}$$ — 2nd time

$$\boxed{T(n) = \underbrace{T(n-3)+\overset{n-3+1}{n-2}}_{T(n-2)} + \underset{n-3+2}{n-1} + n}$$ 3rd time

$T(n) = T(n-4) + n-3+n-2+n-1+n$ ← 4th time

↓ k times

$$\boxed{n-k=1}$$

$n-1=k$

$T(1) = 1$

$$T(n) = T(n-k) + n-k+1+n-k+2+ - - - - + n-1+n$$

$$T(n) = T(n-(n-1)) + n-(n-1)+1 + n-(n-1)+2 + --- + n-1+n$$

$\cancel{n}-\cancel{n}+1$ $\quad \cancel{n}-\cancel{n}+1+1$

$= T(1) + 2 + 3 + \cdots + n-1 + n$

$\downarrow$

$= 1 + 2 + 3 + \cdots + (n-1) + n$

$\Downarrow$

Sum of n natural numbers

$$\Rightarrow \frac{n(n+1)}{2} = \frac{n^2+n}{2}$$

$$= O(n^2)$$

$1+2+3+4+5 =$

$$\frac{5*6}{2} = \frac{n(n+1)}{2}$$

Example 2

$$T(n) = \begin{cases} 1 & n = 1 \\ T(n-1) + \frac{1}{n} & n > 1 \end{cases}$$

$T(1) = 1$

$T(n-1)$ $\downarrow$ **Recursive Term**

$$\boxed{T(n) = T(n-1) + 1/n}$$ $\rightarrow$ 1st time

$$\boxed{T(n) = T(n-2) + \frac{1}{n-1} + \frac{1}{n}}$$ — 2nd time

$$\boxed{T(n) = T(n-3) + \frac{1}{\underset{n-3+1}{\underline{n-2}}} + \frac{1}{\underset{n-3+2}{\underline{n-1}}} + \frac{1}{n}}$$ — 3rd time

$\Big\downarrow$ k times

$$T(n) = \underline{T(n-k)} + \frac{1}{n-k+1} + \frac{1}{n-k+2} + \cdots + \frac{1}{n-1} + \frac{1}{n}$$

$n-k = 1$

$\boxed{n-1 = k}$

$$T(n) = T(\underbrace{n-(n-1)}_{\cancel{n}-\cancel{n}+1}) + \frac{1}{\underset{\cancel{n}-\cancel{n}+1+1}{n-(n-1)+1}} + \frac{1}{\underset{n-n+1+2}{n-(n-1)+2}} + \cdots + \frac{1}{n-1} + \frac{1}{n}$$

$$T(n) = \underline{T(1)} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{n}$$

$$= \underbrace{1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{n}}$$

$$= \underline{\underline{O(\log n)}}$$

Example 3

$$T(n) = \begin{cases} 1 & n = 1 \\ n * T(n-1) & n > 1 \end{cases}$$

$n = 1$ → Base case Condition

Recursion

$n = 5$ $T(5) = 5 * T(4)$ 24

factorial Recurrenu Relation $= 120$

$4 * T(3)$ 6

$3 * T(2)$ 2

$2 * T(1)$

1

$$T(n) = T(n-1) * n$$

$$T(n) = T(n-2) * n-1 * n$$

$$T(n) = T(n-3) * \underbrace{n-2}_{n-3+1} * \underbrace{n-1}_{n-3+2} * n$$

k times

$n - k = 1$

$\boxed{n-1 = k}$

$$T(n) = T(n-k) + n-k+1 * n-k+2 * \ldots * n-1 * n$$

$$= T(\underbrace{n-(n-1)}_{\cancel{n}-\cancel{n}+1}) * \underbrace{n-(n-1)+1}_{\cancel{n}-\cancel{n}+1+1} * \underbrace{n-(n-1)+2}_{\cancel{n}-\cancel{n}+1+2} * \ldots * n-1 * n$$

$$\Rightarrow \underbrace{T(1)}_{1} * 2 * 3 * \cdots * n-1 * n$$

$$\Rightarrow 1 * 2 * 3 * \cdots * n-1 * n = O(n!)$$

$5! = 5 \times 4 \times 3 \times 2 \times 1$

$= O(n^n)$

$n! < n^n$

$f(n) <= c \cdot g(n)$

$c = 1$

1) $T(n) = \begin{cases} 1 & n = 1 \\ 2T\left(\frac{n}{2}\right) + n & n > 1 \end{cases}$

Answer $= O(n\log n)$

Assignment Problem

3) $T(n) = \begin{cases} 1 & n = 1 \\ 8T\left(\frac{n}{2}\right) + n^2 & n > 1 \end{cases}$

Recursive Tree Approach

↳ More than one recursive term in the recurrence relation

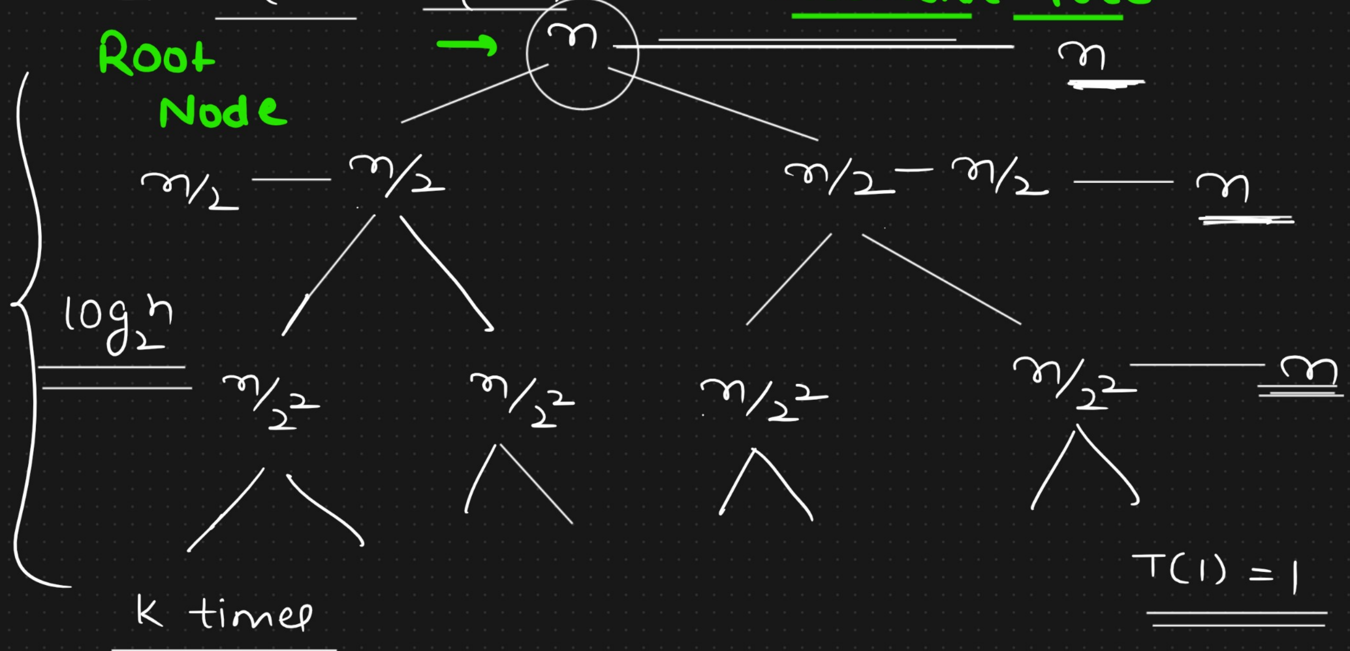
$$T(n) = \begin{cases} 1 & n = 1 \rightarrow \text{Base case condition} \\ T(n/2) + T(n/2) + n & n > 1 \end{cases}$$

Divide & conquer

$2T\left(\frac{n}{2}\right) + n \rightarrow$ substitution method

$T(n) = T(n/2) + T(n/2) + n$

Recursive Tree



Left side

$$\frac{n}{2^k} = 1$$

$$n = 2^k$$

$$\log_2 n = \log_2 2^k$$

$$\log_2 n = k \cancel{\log_2 2}^{\,1}$$

Right side

$$n/2^k = 1$$

$$n = 2^k$$

$$k = \log_2 n$$

$$n * k$$

$$\Rightarrow O(n \log_2 n)$$

find maximum

50 Data Points (70, 90)

$\rightarrow 90$

1 to 25 | 26 to 50

25 | 25

$\Downarrow$ | $\Downarrow$

70 | 90

Recursion

$$T(n) = \begin{cases} 1 & n = 1 \\ T\left(\frac{n}{5}\right) + T\left(\frac{4n}{5}\right) + n & n > 1 \end{cases}$$

n — n

$\frac{4n}{5} + \frac{n}{5} = \frac{\cancel{5}n}{\cancel{5}}$

Left side $\rightarrow * \frac{1}{5}$

Right side $\rightarrow * \frac{4}{5}$

$\frac{n}{5}$ — $\frac{n}{5}$ | $\frac{4n}{5}$ — $\frac{4n}{5}$ — n

n — $\frac{n}{5^2}$ — $\frac{4n}{5^2}$ — $\frac{4n}{5^2}$ — $\frac{4^2 n}{5^2}$

$$\frac{n}{\left(\frac{5}{4}\right)^k} = 1$$

$\frac{n}{5^3}$ | $\frac{4n}{5^3}$

Left side

$$\frac{n}{5^{k_L}} = 1$$

$$n = 5^{k_L}$$

$$\log_5 n = k_L \cancel{\log_5 5}^{\,1}$$

$$\boxed{k_L = \log_5 n}$$

Right side $\quad \dfrac{n}{(5/4)^{k_R}} = 1$

$$n = \left(\tfrac{5}{4}\right)^{k_R}$$

$$\log_{5/4} n = k_R \cancel{\log_{5/4} 5/4}^{\,1}$$

$$\boxed{k_R = \log_{5/4} n} \longrightarrow \text{Greater}$$

Time complexity $\rightarrow O(n * k_R)$

$$\Rightarrow O(n \log_{5/4} n) \Leftarrow \text{Precise}$$

$$= O(n \log n)$$

$$T(n) = T(n/5) + T\left(\frac{3n}{5}\right) + n$$

10 chocobars

1	2
2	2
3	2
4	2
5	2

$2+2+2+2+2 = 10$

$2 * 5 = 10$

1	1
2	2
3	3
4	4
5	6

$1+2+3+4+6$

$=$

n — $n\left(\frac{4}{5}\right)^0 n$

$n/5$ — $n/5$ $\qquad$ $3n/5$ — $3n/5 \Rightarrow \frac{4n}{5} = \left(\frac{4}{5}\right)^1 n$

$\frac{n}{5^2}$ — $\frac{n}{5^2}$ $\quad$ $\frac{3n}{5^2}$ — $\frac{3n}{5^2}$ $\quad$ $\frac{3n}{5^2}$ — $\frac{3n}{5^2}$ $\quad$ $\frac{3^2 n}{5^2}$ — $\frac{3^2 n}{5^2} \Rightarrow \frac{4^2 n}{5^2}$

$$\frac{n}{5^2} + \frac{3n}{5^2} + \frac{3n}{5^2} + \frac{9n}{5^2} = \frac{16n}{25}$$

Left side → $* \, 1/5$

Right side → $* \, 3/5$

$$\frac{n}{5^k} = 1$$

$$\boxed{k = \log_5 n}$$

$$\frac{n}{(5/3)^k} = 1$$

$$n = (5/3)^k$$

$$\boxed{k = \log_{5/3} n}$$

Greater

$$\left(\frac{4}{5}\right)^0 n + \left(\frac{4}{5}\right)^1 n + \left(\frac{4}{5}\right)^2 n + \cdots + \left(\frac{4}{5}\right)^{\log_{5/3} n} \cdot n$$

$$n\left(\left(\frac{4}{5}\right)^0 + \left(\frac{4}{5}\right)^1 + \left(\frac{4}{5}\right)^2 + \cdots + \left(4/5\right)^{\log_{5/3} n}\right)$$

GP series

$$r = 4/5 \qquad \underline{\underline{r < 1}}$$

$$a = \left(4/5\right)^0 = 1$$

$$\text{Sum of GP series} = \frac{a(1-r^n)}{1-r} = \frac{1\left(1-\left(4/5\right)^{\log_{5/3} n}\right)}{1-4/5}$$

$$= \underline{\text{constant}}$$

$$n\,(\text{constant}) \Rightarrow \underline{O(n)}$$

2) $T(n) = T(n/2) + T(n/3) + T(n/5) + c$ Assignment Problem