Bargaining With Unobservable Outside Options

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Abstract

I study a bargaining setting in which players are asymmetrically informed about the value of an outside option. A seller makes sequential offers to a buyer who has private information about the value of the good. The seller has an outside option whose value evolves over time. However, the buyer does not observe the outcome of this process. Further, if the seller exercises her outside option, the probability that the buyer trades depends on the value of the outside option. I construct a pooling equilibrium that exhibits Coasian dynamics and yields the seller a payoff equal to her payoff from an optimally timed take-it-or-leave-it offer. Trade dynamics follow a simple structure: the probability of trade initially increases at a continuous rate until some deterministic time at which point there is a discontinuous jump in trade probability.

1 Introduction

In this paper, I study a model of bargaining with asymmetric information about the outcome of exercising an outside option. In many bargaining settings, parties may have access to outside options. Further, the counter-party may be unsure of the possibility of trade if the option is exercised. For instance, consider an entrepreneur negotiating the sale of her firm with an acquirer. Two potential aspects of interest in this setting are: (1) the entrepreneur may be better informed than the acquirer about the interest among potential buyers in the market for acquiring her firm and (2) the entrepreneur always has the option of holding an auction by inviting bids. I study trade dynamics and potential causes of inefficiency in such settings when values are private and there are weak gains from trade. In particular, I consider how Coasian dynamics may operate in this setup.

I consider a model of two sided incomplete information where a seller(she) with an outside option that evolves stochastically over time, makes price offers to a buyer(he) who is privately informed of her valuation (which is either high or low). One way to interpret the increase in value is as an increase in demand for the seller's object. There are (weak) gains from trade in the sense that the value to the seller from exercising her outside option is equal to the low type buyer's valuation. The seller can choose to exercise her outside option in any period. If the seller exercises her outside option, the buyer receives the object if and only if the seller's outside option has not appreciated in value.

In contrast to standard Coasian settings with private values and weak gains from trade, there exists an equilibrium in stationary buyer strategies in this setting that involves inefficient delay. To better understand this phenomenon, I first consider an auxiliary benchmark setting in which the seller may only make a single offer to the buyer by way of negotiations. If the offer is rejected, the outside option is executed. Since the outside option evolves over time (or demand for the object accumulates over time), the seller has incentive to delay making her offer to the buyer. This game has a pooling equilibrium in which the seller delays trade until the buyer becomes sufficiently pessimistic about her chances of obtaining the object should the outside option be executed. The buyer's belief that the outside option has improved increases over time in equilibrium and the seller's choice of time for making her offer is uniquely pinned down in equilibrium as a function of her belief about the buyer's type.

In tandem with the observations in the benchmark setting, I construct a pooling equilibrium in which the seller delays trade in order to wear the buyer down. As opposed to classical incomplete information bargaining settings in which the buyer's trade probability jumps to one almost instantaneously, the trade probability in this equilibrium increases continuously over a period of time before jumping to one.

In this setting, the buyer's evolving uncertainty about the outside option is central to the existence of a stationary equilibrium with inefficient delay. The seller wishes to delay trade, not because it improves her outside option, but because it enhances her ability to extract a high price from the buyer by threatening him with the outside option. By virtue of construction, the seller is always guaranteed a payoff equal to her payoff from the benchmark setting in equilibrium. Since the gains made by the seller in the benchmark setting depend on the buyer's value as well as his belief about the outside option, the equilibrium in question inherits this dependence, resulting in inefficient delay.

Apart from the existence of a stationary equilibrium with delay, the probability of trade in equilibrium may also be of independent interest. The trade probability increases continuously over time and ends in an atom of trade. A point to note here is the absence of any 'gaps' (i.e., periods of time with zero probability of trade) in the probability distribution. This is distinct from equilibrium structures typically found in interdependent value settings (such as Deneckere and Liang (2006)), where in the 'gap' case (i.e., when the buyer's valuation is bounded away from zero), the equilibrium features gaps in trade sandwiched between atoms. In these settings, the incentive to delay trade remains constant over time at the gaps. In the equilibrium in our setting, however, the seller's incentive to delay trade changes over time, given her belief about the buyer's type. In particular, given the seller's belief, as the buyer's belief about the outside option increases, the seller's incentive to delay trade reduces over time. This results in an equilibrium that exhibits a trade dynamic the literature calls 'atomless trade'(see Fuchs and Skrzypacz (2010)).

Finally, I pin down the seller's payoff in this equilibrium. Consistent with findings in other settings that feature atomless delay (such as Fuchs and Skrzypacz (2010)), the seller's payoff is reduced to the minimum payoff guaranteed to her in equilibrium, which in this case, is the payoff from the benchmark case. The seller fails to do any better than the benchmark case for the same reason as in other settings that feature atomless delay: the presence of Coasian forces imply that if the seller delays trade in equilibrium, her incentive to delay trade must exactly match her incentive to speed up trade.

The rest of the paper is organized as follows: Section 2 discusses related literature, Section 3 describes the model, Section 4 solves for the equilibrium in the benchmark case, Section 6 discusses the main result and the final section concludes.

2 Related Literature

This paper is related to the dynamic bargaining with incomplete information literature as well as the literature on bargaining with outside options.

In the complete information bargaining literature, the question of outside options has been addressed both in static and dynamic settings. While these threat points are crucial in determining bargaining outcomes in static settings, introducing exogenous outside options in dynamic bargaining settings with complete information impacts bargaining dynamics and outcomes only marginally, as shown in Binmore et al. (1989). In this paper, bargaining dynamics are driven by incomplete information on both sides-the seller is unsure of the buyer's valuation and the buyer cannot observe the seller's search outcomes.

There is extensive work on outside options in the dynamic bargaining literature with one sided incomplete information. In the classic, 'Coasian' bilateral bargaining problem with one sided incomplete information (see Fudenberg et al. (1985), Gul et al. (1986)), the seller's temptation to revise price offers result in efficient trade with all buyer types as discount rate goes to one. As opposed to complete information models, introducing outside options into the standard one sided incomplete information model almost always breaks the efficiency result. For instance, Board and Pycia (2014) show that when the buyer has an exogenous outside option, the unique equilibrium outcome is that the seller obtains monopoly profits.

The dynamic bargaining literature with incomplete information has addressed the question of arriving outside options in a variety of settings. Among other papers, Ortner (2017) considers a setting where a monopolist's cost evolves stochastically, Daley and Green (2020) have exogenous arrival of information over time, Chaves (2019) studies a setting where entrants disrupt a negotiation endogenously. Similar to the equilibrium I construct, the equilibria in these settings exhibit Coasian dynamics in the sense that the seller's(buyer in the case of Daley and Green (2020)) payoff is reduced to her payoff from stalling negotiations (with the exception of Chaves (2019) who considers a private offers case where the payoff from stalling involves beliefs of the entrants). The main difference I would like to highlight between these papers and my setting is that the seller also possesses private information and the seller can credibly stall negotiations not necessarily because the outside option has some intrinsic value, but in order to influence the buyer's belief.

The work closest to this paper are Hwang and Li (2017) and Fuchs and Skrzypacz (2010) (henceforth, FS). Hwang and Li (2017) study a bargaining game between a seller and a privately informed buyer who has a stochastically arriving outside option. They compare bargaining outcomes when the arrival is public and when it is private. When the arrival is private, there are multiple equilibria including a deadlock equilibrium and a Coasian equilibrium. The forces that drive delay in their setting is distinct from those in my setting. In their setting, delay arises in the deadlock equilibrium because the buyer's opting out behavior neutralizes the change in belief from skimming, so that the seller can still extract rents from high valuation buyers. In my setting, delay is a result of growing buyer pessimism. Further, the equilibrium dynamics in their deadlock equilibrium and the equilibrium I consider are distinct. In the deadlock equilibrium, the seller's belief falls until it hits the point of deadlock where the seller randomizes between two offers. The equilibrium I characterize has no point of deadlock; the belief always falls and trade ends in finite time. Moreover, trade dynamics feature a smooth screening phase followed by an atom of trade.

Fuchs and Skrzypacz (2010) study a model in which an exogenous outside option arrives at a Poisson rate and ends the game. They characterize a stationary equilibrium in which the seller's payoff is reduced to her payoff from stalling trade until the outside option arrives and the equilibrium trade path is 'atomless',(i.e., screening is always smooth). The equilibrium I construct shares the feature that the seller's payoff is reduced to her payoff from stalling till the buyer becomes sufficiently pessimistic about the possibility of trade. Fuchs and Skrzypacz (2010) consider the equilibrium they construct in the 'no gap' case as a limit of 'DL equilibria'(Deneckere and Liang (2006)) in the gap case. In both papers, player valuations are correlated, which is not the case in my setting.

This paper is also closely related to Fudenberg et al. (1987). In their paper, the seller can end trade with the current buyer and start negotiations with a new buyer. They find that, under some conditions, in the unique stationary monotone equilibrium, the seller never haggles, i.e., the seller posts a price and switches buyers upon rejection. In contrast to their setting, in the setting I consider, the seller with an outside option has incentive to continue negotiations with the current buyer before exiting. Further, the seller has incentive to delay an agreement in order to extract some rents from the buyer. In the equilibrium I construct, the seller always haggles before exiting.

Finally, this paper is related to the literature on bargaining with two sided incomplete information. Earlier papers in this literature include Cho (1990), Cramton (1984), Cramton (1992) and Chatterjee and Samuelson (1988). More recently, Ortner (2023) considers a setting where the seller's cost evolves stochastically and is unobserved by the buyer. The paper characterizes the class of separating equilibria and pins down the most efficient separating equilibrium. While inefficient delay arises even in the most efficient separating equilibrium in Ortner (2023), the cause of delay is quite different from that in my setting. In Ortner (2023), delay emerges as a result of the information revelation constraint, i.e., delay must arise in a separating equilibrium so that a low cost seller is not tempted to mimic the high cost seller. On the other hand, delay emerges in my setting because the buyer grows pessimistic over time. This kind of consideration does not arise in Ortner (2023) because cost drops over time, so even in a pooling equilibrium the buyer grows more optimistic over time about lower future prices.

Cho (1990) considers a setting where the buyer and seller have private information about their valuation and cost respectively and an efficient separating equilibrium is characterized. The equilibrium I construct, on the other hand, highlights inefficiencies that emerge when the seller can delay trade to influence the buyer's belief about an arriving outside option. Cramton (1984),Cramton (1992) and Chatterjee and Samuelson (1988) also study models of dynamic bargaining with two sided asymmetric information. In these papers, reputation building is the source of delay as more obstinate types are more likely to hold out for better offers. In the equilibrium I construct, there is no signaling through offers and both types of the seller are equally likely to hold out for better offers. The buyer's belief changes only owing to the arriving outside option.

3 Model

Time is discrete $t = 0, \Delta, 2\Delta ..., N(\Delta)\Delta$, where $N(\Delta)\Delta = T$. We focus on outcomes when T is large. The seller of a good bargains with a buyer who has private information about her private value for the good. We call this buyer the primary buyer. The primary buyer's value for the good is either High (v_H) or Low(v_L). We denote the prior belief that the type is High by q_0. The seller makes a price offer to the buyer in each period, and the buyer decides whether to accept or reject the offer.

Outside Option: The seller has access to an outside option which yields her a payoff of v_L. If the seller exercises her outside option, the buyer's payoff depends on the status of the outside option, which is either 0 or 1. If the status of the outside option is zero, it increases to one at Poisson rate λ, where one is an absorbing state. The seller does not observe the outcome of the Poisson process and therefore, does not know the status of the outside option with certainty.

An interpretation: The above payoffs upon execution of the outside option are rationalized by the following reduced form game where the status of the outside option represents the arrival of a competitive buyer with valuation higher than v_H. In this scenario, the act of exercising outside option represents the seller's decision to hold a second price auction with reserve price v_L. If the status of the outside option is zero, the buyer has no competitors. If the status is one, as mentioned above, the seller has a competitor with a value higher than v_H. The equilibrium of this game in weakly dominant strategies yields the above payoffs to players.

The timeline of events in an arbitrary period t as follows:

1. The outside option arrives if $T^o = t$
2. The seller decides whether to exercise the outside option or make an offer
3. Buyer decides to accept or reject the offer. The game ends if the offer is accepted.

If the outside option is exercised, the seller can make a final offer before the outside option is executed.

We make the following assumption about the parameters

Assumption 1. $\dfrac{\lambda}{\lambda+r}[v_H q_0 + (1-q_0)v_L] > v_L$

We offer the following interpretation for Assumption 1– consider a game where the outcome of the Poisson random variable is observable to the primary buyer. If type H of the primary buyer agrees to trade at price v_H upon the change of status of the outside option, the seller prefers to wait for the status to change than trade with the buyer at price v_L.

4 TIOLI Benchmark

We first consider the seller's highest attainable payoff in the Take-It-Or-Leave-It benchmark where the seller makes a single offer, i.e., the seller makes a single take it or leave it offer to the buyer and immediately sells in the secondary market if the offer is rejected. In this setting, the seller chooses the optimal time to make the offer. Therefore, the seller's problem reduces to a stopping time problem. We consider the problem in continuous time for large T.

We first construct a pure strategy pooling equilibrium. For a pure stopping time strategy τ, the seller's payoff given μ, is given by

$$V_\tau(\mu) = \delta^\tau \left[(\mu_\tau v_H + (1-\mu_\tau)v_L) \frac{f_0}{1+f_0} + v_L \frac{1}{1+f_0} \right] \tag{1}$$

where $d\mu_t = (1-\mu_t)\lambda$. Therefore, the seller's problem reduces to choosing a threshold μ^* that maximizes

$$V_\tau(\mu) = \left(\frac{1-\mu^*}{1-\mu} \right)^{(r/\lambda)} \left[(\mu^* v_H + (1-\mu^*)v_L) \frac{f_0}{1+f_0} + v_L \frac{1}{1+f_0} \right] \tag{2}$$

Taking the first order condition and rearranging terms, we obtain a closed form solution for the threshold. The threshold is a function of f_0 and is given by

$$1 - \underline{\mu}(f_0) = \frac{r}{\lambda + r} \frac{v_H f_0 + v_L}{f_0(v_H - v_L)} \tag{3}$$

The seller therefore, follows the following strategy– the seller makes the offer iff $\mu_t \geq \underline{\mu}(f_0)$.

Note that $\underline{\mu}(.)$ is strictly increasing. In particular, this means that the more optimistic the seller is about the buyer's private value, the more she delays the offer. Intuitively, the more pessimistic type H buyer is about demand in the secondary market, the higher the offer she will accept. A higher likelihood of type H is associated with a higher marginal benefit from waiting, which in turn, rationalizes more delay. We denote the inverse of $\underline{\mu}(.)$ by $\underline{f}(.)$

Given (f, μ), we denote the seller's payoff from this TIOLI strategy by $W(f, \mu)$.

5 Strategies and Equilibrium

A public history at period t is a sequence of prices $\{p_s\}_{s=0}^{t-1}$. A private history at time t, denoted by $\hat{h}^t$ consists of a sequence $\{a_s\}_{s=0}^{t}$ where a_s denotes the status of the outside option at time s. An acceptance rule for the buyer is a mapping from a public history h^t

, the current offer and his type to a probability of acceptance. A (pure) offer strategy for the seller is a mapping from $h_{t-1} \sqcup \hat{h}^t$ to an offer. An (pure) exit strategy, denoted by e_t, for the seller with an outside option is a mapping from $h_{t-1} \sqcup \hat{h}_t$ to $\{1, 0\}$.

Let $W(N)$ denote the seller type with(without) an enhanced outside option. The respective equilibrium payoffs are denoted by $V_t^W(V_t^N)$. Note that the payoff to type L buyer is zero in any equilibrium. We denote type H buyer's payoff at time t by F_t. We let μ_t denote the buyer's belief that the seller has an outside option.

We focus on Perfect Bayesian Equilibria in pure seller strategies with the property that the buyer's strategy is stationary and depends only on the belief pair (f, μ). As in the standard incomplete information setting, the skimming property also holds in this setting.

6 Coasian Equilibrium

In this section, we establish the existence of an equilibrium akin to the unique equilibrium in the standard Coasian framework. We construct a pooling equilibrium in which the buyer's strategies only depend on the 'state' variables (f_t, μ_t) and the seller's strategy may additionally depend on the offer in the previous period. We call such an equilibrium a *stationary pooling equilibrium.* Importantly, in this equilibrium the seller's payoff depends only on the belief pair (f, μ).

Proposition 1. *There exists a stationary pooling equilibrium*

The proof of the statement can be found in the Appendix. The statement is proved using construction techniques standard in the literature.

We consider outcomes in the frequent offer limit of this equilibrium. The state space can be divided into three regions. We state the proposition formally. Let $1 - \overline{\mu}(f) = \dfrac{r}{\lambda + r}\dfrac{v_L}{f(v_H - v_L)}$. We say that trade is smooth at (f, μ) if the trade probability is continuous in the frequent offer limit.

Proposition 2. *In the frequent offer limit:*

- *There is no limit delay at (f, μ) if $\overline{\mu}(f) \geq \mu \geq \underline{\mu}(f)$*
- *There is limit delay at (f, μ) if $\mu < \underline{\mu}(f)$. Further, trade is always smooth at (f, μ)*

- *The seller's payoff is $W(f,\mu)$ at state (f,μ)*
- *On the equilibrium path, seller sells in the secondary market at (f,μ) iff $\mu > \overline{\mu}(f)$*

We provide a rough sketch of the proof. First, we note that in an equilibrium in which the seller's payoff weakly exceeds the TIOLI payoff from the benchmark case considered in Section 4, there is no delay at any (f,μ) if $\mu > \underline{\mu}(f)$. We prove this in the Appendix.

Next, we provide a heuristic argument for why the payoff is equal to $W(f,\mu)$ when $\mu < \underline{\mu}(f)$.

First suppose that trade is always smooth when $\mu < \mu'$ for some μ'. The seller's equilibrium payoff function is given by[1]

$$V(f,\mu) = p(f,\mu)(f - f_+) + \delta V(f_+,\mu_+)$$

Where (f_+,μ_+) are the states induced in the next period in equilibrium. Dividing by Δ and taking limits, we get

$$rV(f,\mu) = \frac{df}{dt}\left(p(f,\mu) - \frac{\partial V(f,\mu)}{\partial f}\right) + \frac{\partial V(f,\mu)}{\partial \mu}\frac{d\mu}{dt}$$

Due to the presence of Coasian forces, it has to be the case that $p(f,\mu) - \dfrac{\partial V(f,\mu)}{\partial f} = 0$. First notice that setting $p(f,\mu) - \dfrac{\partial V(f,\mu)}{\partial f} = 0$ yields an ODE. Substituting for $d\mu/dt$, we get

$$rV(f,\mu) = \lambda(1-\mu)\frac{\partial V(f,\mu)}{\partial \mu}\frac{d\mu}{dt}$$

Given the terminal condition at μ', the solution to the above ODE is the discounted payoff at (f,μ'). Since $df/dt < 0$, it cannot be the case that $p(f,\mu) - \dfrac{\partial V(f,\mu)}{\partial f} > 0$. Next, we claim that it cannot be the case that $p(f,\mu) - \dfrac{\partial V(f,\mu)}{\partial f} < 0$. If it is so, then the seller gains from lowering prices slightly and this would lead to a jump in the trade probability, contradicting the fact that trade is smooth. Therefore, if trade is smooth, then the payoff equals to the payoff in the TIOLI region. Moreover, by $p(f,\mu) = \partial V(f,\mu)/\partial f$, the price offer at (f,μ) is equal to the discounted price offer at (f,μ'). In words, this means that when trade is smooth, the seller's payoff is equal to her payoff from delaying trade for a

[1] We 'average' out the payoffs by multiplying by $(1+f)$.

fixed period of time and the price offer she makes is equal to the discounted payoff she makes at the end of the period of the delay.

The above argument relies on the assumption that trade is smooth at (f, μ) i.e., there are no atoms or gaps in the trade probability. It is not hard to see that when the trade probability is zero over a time period, the seller's payoff equals the discounted payoff at the end of the gap. Next, we show that there cannot be any gaps or atoms in the region of delay. We proceed in the following manner.

First, we note that if the seller strictly prefers not to delay trade at (f, μ), there must be an atom of trade. Let the size of the atom be $f_+ - f$. We show that the seller prefers not to delay trade iff the atom is followed by a gap.

1. If the seller strictly prefers not to delay trade $\iff$ atom of trade followed by a gap: If trade is smooth at (f_+, μ), the price offer is equal to the price offer made after a period of delay and the continuation payoff is equal to the payoff obtained at the end of the same period of delay. This would reduce the seller's payoff at (f, μ) to her payoff from delaying trade.

This implies that limit delay involves zero probability of trade at (f_+, μ), i.e., there is a gap in the trade probability.

Suppose an atom of trade is followed by a gap. Let μ_{+s} be the belief after a duration of length s. Then for $s > 0$ small enough

$$\begin{aligned}
\delta^s V(f, \mu_{+s}) &= \delta^s p(f, \mu_{+s})(f - f_+) + \delta^s V(f_+, \mu_{+s}) \\
&= \delta^s p(f, \mu_{+s})(f - f_+) + V(f_+, \mu) \\
&< p(f, \mu)(f - f_+) + V(f_+, \mu)
\end{aligned}$$

where the last line follows from the fact that $p(f, \mu) = p(f_+, \mu) = v_H(1 - \delta^s) + \delta^s p(f_+, \mu_{+s}) > p(f_+, \mu_{+s})$.

We next argue that if the seller is indifferent between trading and delaying trade at (f, μ) for f in some interval I, then $V(f, \mu) = W(f, \mu)$.

2. If the seller is indifferent between trading and delaying trade at (f, μ) for

f in some interval I, then the payoff over the interval is equal to the benchmark payoff: Suppose $V(f,\mu) > W(f,\mu)$ for $f \in I$. It must be the case that for some $m > \mu$ and $m < \underline{\mu}(f)$ there is an atom at (f,m).

Let T be the earliest time at which there is an atom of trade in I. Since an atom necessarily means that the probability of trade is non zero, we have that for beliefs in some interval I', the first atom occurs after a duration of length T. Moreover, if the length of the atom is k, for all beliefs $x \in (f, f+k)$, the size of the atom is $x - (f+k)$. This means that trade cannot be smooth over $(f, f+k)$. This is because $\partial V(.,\mu)/\partial f$ is constant over this interval. If there is an atom over this interval, then trade is smooth at $(f+k,\mu)$. This is because limit silent delay at (f_+,μ) implies that the seller strictly prefers not to delay trade at (f,μ). This implies the existence of an interval of beliefs at $f+k$ at which the seller's payoff is equal to her payoff from delaying trade. Since trade probability is necessarily continuous over this interval, we reach a contradiction.

Therefore if the seller is indifferent between trading and delaying trade at all (x,μ) for $x \in I$ for some interval I, then $V(x,\mu) = W(x,\mu)$.

This result has two implications. First, a region of smooth trade cannot be preceded by an atom of trade. This is because $W(f,\mu) < \partial W(g,\mu)/\partial g(f-g) + W(g,\mu)$ for all $g < f$. Second, if $V(f,\mu) > W(f,\mu)$, then there is necessarily at atom of trade at (f,μ). Conversely, if there is an atom of trade at (f,μ), then $V(f,\mu) > W(f,\mu)$.

Next, we argue that a gap cannot precede an atom.

3. A gap cannot precede an atom: Recall that an atom is followed by a gap. Suppose there is a gap before an atom. Suppose the initial gap occurs at f and the second gap at f_+ as illustrated in Figure 1. If for some duration of time $s > 0$ the length of the atom stays constant at $f - f_+$, we have for f' close to f, $V(f',\mu) - \delta^s V(f',\mu_{+s}) = v_H(1-\delta^s)(f' - f_+)$. By continuity of $V(.,\mu)$, this means that $V(f,\mu) > \delta^s V(f,\mu_{+s})$, which contradicts the fact that the size of the atom remains constant. Since there is limit delay at (f_+,μ), if there is a gap at (f,μ), the size of the atom cannot increase through a fall in f_+. Suppose the atom shrinks through an increase in f_+. Suppose for some $\mu' > \mu$ close to μ, the size of the atom at μ' shrinks to $f - f'_+$. Suppose the total time taken for the state to evolve from μ to μ' is t. For t small enough, t

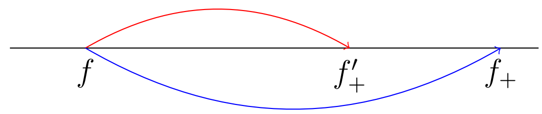

Figure 1: The red path denotes the atom at (f, μ_+) and the blue path denotes the atom at (f, μ)

is going to be less than the total delay at (f_+, μ). Let T denote the total delay at (f_+, μ). Since the atom shrinks between time 0 and time t, we have that $f - f_+ > f - f'_+$, i.e., $f'_+ > f_+$. However, because the seller can credibly delay trade at (f_+, μ'), the seller prefers to make an offer of $p(f_+, \mu')$ and immediately trade with probability $f'_+ - f_+$ when the state is (f'_+, μ'), contradicting the fact that there is a gap at (f'_+, μ). This is depicted in Figure 1.

Therefore an atom cannot occur between two gaps. Consequently, a gap in trade has to be followed by smooth trade. This in turn implies that if there is a gap in trade at (f, μ), the payoff at (f, μ) is equal to $W(f, \mu)$. Next, we argue that an atom cannot occur between a region of smooth trade and a gap.

4.An atom of trade cannot be preceded by smooth trade: Suppose we have a region of smooth trade followed by an atom of trade. We argue that the seller has a profitable deviation in this case. Since the region of smooth trade is followed by an atom of trade, by continuity of the payoff function, for every μ there exists a decreasing function $r(.)$ where $r(\mu)$ which marks the boundary between these two regions. By continuity of the payoff function, the seller is indifferent between delaying trade and trading with positive probability at $(r(), \mu)$. Suppose the size of the atom when the belief is μ is $r(\mu) - g$. Consequently there is a gap at g when the belief is μ. Recall that in a smooth trade region the seller must be indifferent between speeding up or slowing down trade, i.e., the seller must be indifferent between different rates of trade. Consequently, the seller obtains the same payoff from trading at different rates. In particular, the seller obtains her optimal payoff from trading at rate $\frac{dr(\mu)}{d\mu}\frac{d\mu}{dt}$. We argue that the seller has a profitable deviation. First, the seller's payoff at $(r(m), m)$ from trading at the above rate for small $t > 0$ is (approximately) equal to

$$
\begin{aligned}
&V(r(m),m) = p(r(m_t),m_t)(r(m)-r(m_t)) + e^{-rt}V(r(m_t),m_t) \\
\implies\ & e^{-rt}(V(r(m),m_t) - V(r(m_t),m_t)) = p(r(m_t),m_t)(r(m)-r(m_t)) + e^{-rt}V(r(m_t),m_t)
\end{aligned} \tag{4}
$$

where the second line follows from the fact that delaying trade is optimal at $(r(m), m)$. However, we also have

$$
\begin{aligned}
&V(r(m),m) = p(g,m)(r(m)-g) + W(g,m) \\
\implies\ & e^{-rt}(V(r(m),m_t) - V(r(m_t),m_t)) = (r(m)-r(m_t))p(g,m) + (r(m_t)-g)p(g,m_t)
\end{aligned} \tag{5}
$$

where the second line follows from the fact that $\delta^t V(r(m_t), m_t) = \delta^t[p(g, m_t)(r(m) - g) + W(g, m_t)]$, $W(g, m) = e^{-rt}W(g, m_t)$ and by the buyer's incentive compatibility, $p(g, m) = v_H(1 - \delta^t) + \delta^t p(g, m_t)$. Combining Equation (4) and Equation (5) and taking limits, we have

$$
-[p(r(m),m) - p(g,m)]\frac{dr(m)}{dm}\frac{dm}{dt} = (r(m)-g)rv_H
$$

This implies that $p(r(m), m) > p(g, m)$. But since the seller is indifferent between speeding up and slowing down trade, we also have $\dfrac{\partial V(r, m)}{\partial f} = p(g, m) = p(r(m), m)$, which yields a contradiction.

We conclude the argument by noting that a gap cannot be preceded by an atom of trade if there is no delay preceding the atom of trade. Once again, this is because as μ approaches $\underline{\mu}(.)$, the payoff at the atom falls below $W(., \mu)$. By right continuity of the reservation price function, a gap can also not be preceded by a region of smooth trade. We have argued that a gap cannot be preceded by an atom or by smooth trade. Therefore, there cannot be any gaps in the region of delay. Since gaps are necessary for the occurrence of atoms, we also rule out atoms in the region of delay. Therefore, trade is always smooth in the region of delay and the payoff is equal to the payoff in the benchmark case.

7 Discussion

In this paper I study how Coasian forces may operate in a two sided incomplete information setting where the seller of a good does not know the buyer's value for the object and the buyer is unsure of his possibility of being excluded from trade in the future. I construct an equilibrium in which the buyer's acceptance strategy depends only on her own type and the belief each player holds about the other's type. In contrast to stationary equilibria in typical private values settings, this equilibrium features limit delay. The presence of limit delay can be ascribed to two features of the equilibrium: (1) the threat of being excluded by the outside option introduces artificial interdependence in values and (2) the payoff from exercising the outside option evolves over time. The presence of these two features not only result in delay, but also yields distinctive, yet familiar, trade dynamics; the equilibrium trade probability increases continuously over time and terminates in an atom. The continuity of trade probabilities further pins down the seller's payoff. In equilibrium, the seller's payoff is reduced to her payoff from an optimally timed take-it-or-leave-it offer.

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8 Appendix

8.1 Proof of Proposition 1

We solve for the equilibrium by backward induction. Let $f_1^*(\mu;\Delta)$ be such that

$$f_1^*(\mu;\Delta)[v_H - \tilde{\delta}(1-\mu)(v_H - v_L)] + \delta v_L = f_1^*(\mu;\Delta)[v_H - (1-\mu)(v_H - v_L)] + v_L \qquad (6)$$

Suppose $t = N(\Delta) - 1$. Since the outside option is invoked in the next period, a type H buyer has a payoff from rejection equal to $v_H - \delta(v_H - (\mu_{N(\Delta)} v_H + (1 - \mu_{N(\Delta)}) v_L)) = v_H - \delta(1 - \mu_{N(\Delta)})(v_H - v_L)$. Then, for $0 < f < f_1^*(\mu_{N(\Delta)};\Delta)$, it is optimal for the seller to exercise the outside option while for $f > f_1^*(\mu_{N(\Delta)-1};\Delta)$ it is optimal for the seller to offer a price of $v_H - \delta(1-\mu_{N(\Delta)})(v_H - v_L)$, which is accepted by the type H buyer with probability one[2]. Note that $f_1^*(.)$ is strictly increasing.

We now show by induction that for any $t < N(\Delta) - 1$, (1) the seller exercises the outside option with probability one if $0 < f < f_1^*(\mu_t)$ and (2) type H buyer accepts the offer $v_H - \delta(1-\mu_{t+1})(v_H - v_L)$ with probability one. Suppose the statements above hold true for $t = N(\Delta) - k$. We show that it holds true for $t - 1 = N(\Delta) - (k+1)$. We first show that $v_H - \delta(1-\mu_t)(v_H - v_L)$ is accepted with probability one at $t-1$. Since $f_1^*(.)$ is strictly increasing, if $f < f_1^*(\mu_{t-1})$, then $f < f_1^*(\mu_t)$. Therefore, if type H buyer accepts an offer with probability less than one, the offer must be weakly greater than $v_H - \delta(1-\mu_t)(v_H - v_L)$. By the same logic as earlier, this implies that $v_H - \delta(1-\mu_t)(v_H - v_L)$ is accepted with probability one. By definition of $f_1^*(.)$ it follows that the seller prefers to exercise the outside option.

Next, let $f_2^*(\mu;\Delta)$ be such that

$$\begin{aligned}(f_2^*(\mu_+;\Delta) - f_1^*(\mu;\Delta))[v_H - \tilde{\delta}^2(1-\mu)(v_H - v_L)] + \delta V_1(f_1^*(\mu_+;\Delta), \mu_+;\Delta) \\ = f_2^*(\mu;\Delta)[v_H - \tilde{\delta}(1-\mu)(v_H - v_L)] + \delta v_L \qquad (7)\end{aligned}$$

where $V_1(f,\mu;\Delta) = f[v_H - \tilde{\delta}(1-\mu)(v_H - v_L)] + \delta v_L$.

[2]Since, at time $t = N(\Delta)$, the game is essentially an ultimatum game, the reservation price offer is accepted with probability one in equilibrium.

Next, let $f_k^*(\mu;\Delta)$ be such that

$$
\begin{aligned}
&(f_k^*(\mu;\Delta) - f_{k-1}^*(\mu_+;\Delta))[v_H - \tilde{\delta}^k(1-\mu)(v_H - v_L)] + \delta V_{k-1}(f_{k-1}^*(\mu_+;\Delta), \mu_+;\Delta) \\
&= (f_k^*(\mu;\Delta) - f_{k-2}^*(\mu_+;\Delta))[v_H - \tilde{\delta}^{k-1}(1-\mu)(v_H - v_L)] + \delta V_{k-2}(f_{k-2}^*(\mu_+;\Delta), \mu_+;\Delta)
\end{aligned} \tag{8}
$$

where $V_j(f,\mu;\Delta) = (f - f_{j-1}^*(\mu_+;\Delta))[v_H - \tilde{\delta}^j(1-\mu)(v_H - v_L)] + \delta V_{j-1}(f_{j-1}^*(\mu_+;\Delta), \mu_+;\Delta)$, for $j = 3, 4..k-1$.

We first show that $f_k^*(\mu;\Delta) < f_k^*(\mu_+;\Delta)$ for all μ. From Equation (6), it can be seen that $f_1^*(\mu;\Delta) < f_1^*(\mu_+;\Delta)$. Suppose, $f_k^*(\mu;\Delta) < f_k^*(\mu_+;\Delta)$ for all μ. We show that $f_{k+1}^*(\mu;\Delta) < f_{k+1}^*(\mu_+;\Delta)$ for all μ. Suppose not. Then, there exists μ for which $f_{k+1}^*(\mu;\Delta) \geq f_{k+1}^*(\mu_+;\Delta)$.

Note that $f_{k+1}^*(\mu;\Delta) \geq f_{k+1}^*(\mu_+;\Delta) > f_k^*(\mu_{++};\Delta) > f_k^*(\mu_+;\Delta) > f_k^*(\mu;\Delta) > f_{k-1}^*(\mu_+;\Delta)$ and $f_k^*(\mu_{++};\Delta) > f_k^*(\mu_+;\Delta) > f_{k-1}^*(\mu_{++};\Delta) > f_{k-1}^*(\mu_+;\Delta) > f_{k-2}^*(\mu_{++};\Delta)$.

Let $p_k(\mu) = v_H - \tilde{\delta}^k(1-\mu)(v_H - v_L)$. Since $f_{k+1}^*(\mu;\Delta) \geq f_{k+1}^*(\mu_+;\Delta) > f_k^*(\mu_+;\Delta) > f_k^*(\mu;\Delta)$, we have

$$
\begin{aligned}
&(f_{k+1}^*(\mu_+;\Delta) - f_k^*(\mu_+;\Delta))p_k(\mu) + V(f_k^*(\mu_+;\Delta), \mu;\Delta) \geq \\
&(f_{k+1}^*(\mu_+;\Delta) - f_k^*(\mu_+;\Delta))p_{k+1}(\mu) + \delta V(f_k^*(\mu_+;\Delta), \mu_+;\Delta)
\end{aligned} \tag{9}
$$

Further, we also have

$$
\begin{aligned}
&(f_{k+1}^*(\mu_+;\Delta) - f_k^*(\mu_+;\Delta))p_{k+1}(\mu_+) + \delta V(f_k^*(\mu_+;\Delta), \mu_{++};\Delta) > \\
&(f_{k+1}^*(\mu_+;\Delta) - f_k^*(\mu_+;\Delta))p_k(\mu_+) + V(f_k^*(\mu_+;\Delta), \mu_+;\Delta)
\end{aligned} \tag{10}
$$

Combining 9 and 10, we get,

$$(f^*_{k+1}(\mu;\Delta) - f^*_k(\mu_+;\Delta))(p_{k+1}(\mu_+) - p_{k+1}(\mu)) + V(q^*_k(\mu_+;\Delta),\mu;\Delta) - \delta V(q^*_k(\mu_+;\Delta),\mu_+;\Delta) >$$
$$(f^*_{k+1}(\mu;\Delta) - f^*_k(\mu_+;\Delta))(p_k(\mu_+) - p_k(\mu)) + V(f^*_k(\mu_+;\Delta),\mu_+;\Delta) - \delta V(f^*_k(\mu_+;\Delta),\mu_{++};\Delta) \tag{11}$$

Note that $p_{k+1}(\mu_+) - p_{k+1}(\mu) = \tilde{\delta}^{k+1}(1-\mu)(1-e^{-\lambda\Delta})(v_H - v_L) < \tilde{\delta}^k(1-\mu)(1-e^{-\lambda\Delta})(v_H - v_L) = p_k(\mu_+) - p_k(\mu)$. We show that $V(f^*_k(\mu_+;\Delta),\mu;\Delta) - \delta V(f^*_k(\mu_+;\Delta),\mu_+;\Delta) < V(f^*_k(\mu_+;\Delta),\mu_+;\Delta) - \delta V(f^*_k(\mu_+;\Delta),\mu_{++};\Delta)$.

Evaluating, $V(f^*_k(\mu_+;\Delta),\mu;\Delta) - \delta V(f^*_k(\mu_+;\Delta),\mu_+;\Delta)$,

$$\begin{aligned} V(f^*_k(\mu_+;\Delta),\mu;\Delta) - \delta V(f^*_k(\mu_+;\Delta),\mu_+;\Delta) &= (f^*_k(\mu;\Delta) - f^*_{k-1}(\mu_+;\Delta))[p_k(\mu) - \delta p_{k-1}(\mu_+)] \\ &= (f^*_k(\mu;\Delta) - f^*_{k-1}(\mu_+;\Delta))v_H(1-\delta) \end{aligned} \tag{12}$$

Evaluating $V(f^*_k(\mu_+;\Delta),\mu_+;\Delta) - \delta V(f^*_k(\mu_+;\Delta),\mu_{++};\Delta)$,

$$\begin{aligned} V(f^*_k(\mu_+;\Delta),\mu_+;\Delta) - \delta V(f^*_k(\mu_+;\Delta),\mu_{++};\Delta) &= (f^*_k(\mu;\Delta) - f^*_{k-1}(\mu_{++};\Delta))[p_k(\mu_+) - \delta p_{k-1}(\mu_{++})] + \\ &\quad (f^*_{k-1}(\mu_{++};\Delta) - f^*_{k-1}(\mu_+;\Delta))[p_k(\mu_+) - \delta p_{k-2}(\mu_{++})] \\ = (f^*_k(\mu;\Delta) - f^*_{k-1}(\mu_+;\Delta))v_H(1-\delta) &+ (f^*_{k-1}(\mu_{++};\Delta) - f^*_{k-1}(\mu_+;\Delta))\delta[p_{k-1}(\mu_{++}) - p_{k-2}(\mu_{++})] > \\ &(f^*_k(\mu;\Delta) - f^*_{k-1}(\mu_+;\Delta)-)v_H(1-\delta) \end{aligned} \tag{13}$$

So, we have a contradiction. Therefore, $f^*_{k+1}(\mu_+;\Delta) > f^*_{k+1}(\mu;\Delta)$.

We now describe players' strategies:

- Buyer:
  - Any buyer with value L accepts an offer iff the price offer is weakly less than v_L
  - If the outside option is exercised in period t, all buyers with value H accept an offer p iff $p \leq v_H - (1-\mu_t)(v_H - v_L)$
  - In period $t = N(\Delta) - k$ and $f > f_k(\mu_t;\Delta)$, type H buyer accepts an offer with probability $(f - f_k(\mu_t;\Delta))/(1+f)$ accepts an offer iff the offer is atmost

$v_H - \tilde{\delta}^k(1-\mu)(v_H - v_L)$. For $f \in (f_j(\mu_{t+1};\Delta), f_{j-1}(\mu_{t+1};\Delta)]$ for $j \leq k$, the buyer accepts an offer with probability $(f - f_{j-1}(\mu_{t+1};\Delta))/(1+f)$ if it is atmost $v_H - \tilde{\delta}^j(1-\mu_t)(v_H - v_L)$.

- Seller:
  - In period $t = N(\Delta) - k$, if $f_t > f_k(\mu_{N(\Delta)-k};\Delta)$, the seller offers $v_H - \tilde{\delta}^k(1 - \mu_{N(\Delta)-k})(v_H - v_L)$
  - In period $t = N(\Delta) - k$, if $f_j(\mu_{N(\Delta)-k};\Delta) > f_t > f_{j-1}(\mu_{N(\Delta)-k};\Delta)$ for $j < k$, the seller offers $v_H - \tilde{\delta}^{j-1}(1 - \mu_{N(\Delta)-k})(v_H - v_L)$. The seller takes the outside option iff $j = 1$
  - In period $t = N(\Delta) - k$, if $f_t = f_j(\mu_{N(\Delta)-k};\Delta)$ for $j \leq k$, the seller's offer depends on the offer made in the previous period p. The seller offers $p_j(\mu_{N(\Delta)-k}) = v_H - \tilde{\delta}^j(1 - \mu_{N(\Delta)-k})(v_H - v_L)$ with probability q and offers $p_{j-1}(\mu_{N(\Delta)-k}) = v_H - \tilde{\delta}^{j-1}(1 - \mu_{N(\Delta)-k})(v_H - v_L)$ with probability $1-q$ where q is such that

$$v_H - p = \delta(v_H - (qp_j(\mu_{N(\Delta)-k}) + (1-q)p_{j-1}(\mu_{N(\Delta)-k})))$$

    Additionally if $j = 1$, the seller exercises the outside option with probability $1-q$.

8.2 Proof of Proposition 2

8.2.1 Definitions

We define $\mu_{+t(\Delta)}(\mu;\Delta)$ as the belief t periods hence when current belief is μ,i.e., $(1 - \mu_{+t(\Delta)}(\mu;\Delta)) = e^{-t\lambda\Delta}(1-\mu)$ and $f_{+t}(f,\mu;\Delta)$ is the belief t period hence when the current state is (f,μ). Let $f_T(f,\mu) = \lim_{\Delta\to 0} f_{+t(\Delta)}(f,\mu;\Delta)$ and $(1-\mu_T) = \lim_{\Delta\to 0}(1-\mu_{+t(\Delta)})$ where $\lim_{\Delta\to 0} t(\Delta) = T$.

$$T(\mu,\mu') : \mu_{+T(\mu,\mu')}(\mu) = \mu'$$

$$T(f,\mu) = T(\mu, \underline{\mu}(f))$$

$$T(f,g,\mu;\Delta) = \min\{t \geq 0 | f_t(f,\mu;\Delta) \leq g\}$$

$$f_+(f,\mu) = \inf\{g \leq f | \lim_{\Delta \to 0} T(f,g,\mu;\Delta)\Delta = 0\}$$

$$\hat{T}(f,\mu) = \inf\{t \geq 0 | f - f_t(f,\mu;\Delta) > \varepsilon, \varepsilon > 0\}$$

$$\tilde{\mu}(f,\mu) = \inf\{\mu' \geq \mu | f_+(f,\mu') - f > \varepsilon, \varepsilon > 0\}$$

and $\tilde{T}(f,\mu)$ is such that

$$(1 - \tilde{\mu}(f,\mu)) = e^{-\lambda \tilde{T}(f,\mu)}(1-\mu)$$

$$\tilde{f}(f,\mu) = \lim_{\Delta \to 0} f_{+t(\Delta)}(f,\mu)$$

where $\lim_{\Delta \to 0} t(\Delta) = \tilde{T}(f,\mu)$.

We say that there is an **atom of trade** at (f,μ) if $f > f_+(\mu)$. We say that **limit trade is smooth** at (f,μ) if $f_+(\mu) = f$ and $\hat{T}(f,\mu) = 0$. We say that **limit trade is silent** at (f,μ) if $f_+(f,\mu) = f$ and $\hat{T}(f,\mu) > 0$. There is said to be **limit delay** at (f,μ) if limit trade is either smooth or silent at (f,μ).

8.2.2 Step 0: Ultimate Atom of Trade

Lemma 1. *There is no further limit delay at (f,μ) if $\mu > \underline{\mu}(f)$*

Proof. WLOG suppose $f_+(f,\mu) = 0$ and for all $\varepsilon > 0$, there is $g \in (f+\varepsilon, f]$ s.t. $g - f_+(g,\mu) = 0$. Suppose $f_n \to f$.

$$(v_H - \tilde{\delta}^{\Delta}(v_H - v_L)(1-\mu))(f_n - f_{+1}(f_n,\mu)) + \delta V(f_{+1}(f_n,\mu), \mu_{+1}(\mu)) \geq W(f_n,\mu)$$

Therefore,

$$
\begin{aligned}
&((1-\mu)(v_H - v_L) - F(f_n,\mu))(f_n - f_{+1}(f_n,\mu)) \\
&\geq W(f_n,\mu) - \delta V(f_{+1}(f_n,\mu),\mu_{+1}(\mu)) \\
\implies &\lim_{\Delta\to 0}\lim_{n\to\infty}((1-\mu)(v_H - v_L) - F(f_n,\mu))(f_n - f_{+1}(f_n,\mu)) \\
&\geq \lim_{\Delta\to 0}\lim_{n\to\infty} W(f_{+1}(f_n,\mu),\mu) - \delta V(f_{+1}(f_n,\mu),\mu_{+1}(\mu)) \\
\implies &0 \geq r[v_H f + v_L] - (1-\mu_{T_\sigma})(v_H - v_L)(\lambda + r)f
\end{aligned}
$$

But since $r[v_h f + v_L] = (1 - \underline{\mu}(f))(v_H - v_L)f > (1-\mu)(v_H - v_L)(\lambda + r)f > (1 - \mu_{T_\sigma})(v_H - v_L)(\lambda + r)f$, we have a contradiction. □

8.2.3 Step 1: Properties of a smooth trade region

We prove two main properties of a smooth trade region.

Lemma 2. *Suppose there is limit delay at (f,μ). Then, the seller's limit payoff at (f,μ) is $\delta^{\tilde{T}(f,\mu)}V(f,\tilde{\mu}(f,\mu))$.*

Proof. Let $\lim_{\Delta\to 0} t(T;\Delta) = T$, $p(f_{+1}(f_-(\mu;\Delta),\mu),\mu_{+1}) = p(f,\mu_{+1})$, let $T'(f,\mu) = \inf\{t \geq 0 | \lim_{\Delta\to 0} f_-(\mu_{t(t;\Delta)};\Delta) - f > 0\}$ and let $T''(f,\mu) = \min\{T'(f,\mu),\tilde{T}(f,\mu)\}$.

We first show that $V(f,\mu) = \delta^{T''(f,\mu)}V(f,\mu_{T''(f,\mu)})$.

Let $T < T''(f,\mu)$.[3]. Given Δ, consider the following function.

$$
f'_-(\mu;\Delta) = \max\{f_-(\mu_{+t(\Delta)}(\mu;\Delta);\Delta), f'_-(\mu_{+(t+1)(\Delta)}(\mu;\Delta);\Delta)\} \quad \mu \in [\mu_{+t(\Delta)}(\mu;\Delta),\mu_{+(t+1)(\Delta)}(\mu;\Delta)]
$$

Where $f'_-(\mu_{T-\Delta};\Delta) = f_-(\mu_{T-\Delta};\Delta)$. Note that $f'_-(\mu;\Delta)$ converges pointwise to f. Moreover, $f'_-(\mu;\Delta)$ is monotonic. Therefore, $f'_-(\mu;\Delta)$ converges uniformly.

For any $\Delta > 0$, it must be the case that

[3] We let $T = \tilde{T}(f,\mu)$ if $\lim_{\Delta\to 0} f_-(\mu_{t(\tilde{T}(f,\mu);\Delta)}(\mu;\Delta);\Delta) - f = 0$

$$p(f_-(\mu;\Delta),\mu)(f_-(\mu;\Delta)-f)+\delta V(f,\mu_{+1}) \geq p(f,\mu)(f_-(\mu;\Delta)-f)+V(f,\mu)$$
$$\implies (f_-(\mu;\Delta)-f)(p(f_-(\mu;\Delta),\mu)-p(f,\mu))+\delta V(f,\mu_+) \geq V(f,\mu) \tag{14}$$

Let $T(\Delta) \to T$ as $\Delta \to 0$. Iterating on Equation (14), we get

$$\sum_{j=0}^{T(\Delta)-1} \delta^j (f_-(\mu_{+j};\Delta)-f)(p(f_-(\mu_{+j};\Delta),\mu_{+j})-p(f,\mu_{+j}))+\delta^{T(\Delta)}V(f,\mu_{T(\Delta)}) \geq V(f,\mu)$$

Note that for any $\varepsilon > 0$, there exists $\Delta' > 0$ s.t. $\sup_{t \leq T} |f'_-(\mu_t;\Delta)-f| < \varepsilon$ for all $\Delta < \Delta'$. This in conjunction with the fact that $p(f_-(\mu_{+j};\Delta),\mu_{+j})-p(f,\mu_{+j}) \leq (1-\tilde{\delta}^2)v_H$ and $f'_-(\mu;\Delta) \geq f_-(\mu;\Delta)$, implies that for all $\Delta < \Delta'$

$$\begin{aligned}(1-\tilde{\delta}^2)v_H\varepsilon \sum_{j=0}^{T(\Delta)-1} \delta^j + \delta^{T(\Delta)}V(f,\mu_{T(\Delta)}) \\ \geq V(f,\mu)\end{aligned} \tag{15}$$

Therefore,

$$\delta^T V(f,\mu_T) \geq V(f,\mu)$$

This follows from the fact the first term goes to zero as $(1-\tilde{\delta}^2)\sum_{j=0}^{T(\Delta)-1}\delta^j$ is bounded and ε can be made arbitrarily small.

Therefore $V(f,\mu) \leq \delta^T V(f,\mu_T)$ for all $T < T''(f,\mu)$ and by continuity of $V(f,.)$(see Lemma[]), we have that $V(f,\mu) \leq \delta^{T''(f,\mu)}V(f,T''(f,\mu))$.

Next, we show that there exists a sequence $t_n \to 0$ s.t.

$$V(f,\mu) = \lim_{n\to 0} \delta^{\tilde{T}(f_{t_n}(f,\mu),\mu_{t_n}(\mu))} V(f_{t_n}(f,\mu),\tilde{\mu}(f_{t_n}(f,\mu),\mu_{t_n}(\mu)))$$

Suppose limit delay is smooth at (f,μ). Then there exists $\varepsilon > 0$ s.t. $\tilde{T}(f,\mu) \geq$

$\tilde{T}(f_t(f,\mu),\mu_t(\mu)) \geq T''(f_t(f,\mu),\mu_t(\mu)) > 0$ for $t \in (0,\varepsilon)$. If there exists a sequence $t_n \to 0$, such that $T''(f_{t_n}(f,\mu),\mu_{t_n}(\mu)) = \tilde{T}(f_{t_n}(f,\mu),\mu_{t_n}(\mu))$, then, $V(f_{t_n}(f,\mu),\mu_{t_n}(\mu)) = \delta^{\tilde{T}(f_{t_n}(f,\mu),\mu_{t_n}(\mu))}V(f_{t_n}(f,\mu),\tilde{\mu}(f_{t_n}(f,\mu),\mu_{t_n}(\mu)))$. Taking limits, by continuity, we have our result.

Suppose for any sequence $t_n \to 0$, there exists N s.t. for all $m > N$, $T''(f_{t_m}(f,\mu),\mu_{t_m}(f,\mu)) = T'(f_{t_m}(f,\mu),\mu_{t_m}(f,\mu)) < \tilde{T}(f_{t_m}(f,\mu),\mu_{t_m}(f,\mu))$. If for some $t \in (0,\varepsilon)$, $f_+(f',\mu_{T'(f_t(f,\mu),\mu_t(\mu))}(\mu)) = f_t$ for some $f' \geq f$, then $T'(f_t(f,\mu),\mu_t(\mu)) = \tilde{T}(f,\mu)$, because we also have $\tilde{T}(f,\mu) \leq T'(f_t(f,\mu),\mu_t(\mu))$. However, since $T'(f_t(f,\mu),\mu_t(\mu)) < \tilde{T}(f_{t_m}(f,\mu),\mu_{t_m}(f,\mu)) \leq \tilde{T}(f,\mu)$, yielding a contradiction. Therefore, for all $t \in (0,\varepsilon)$ with $f_+(f',\mu_{T'(f_t(f,\mu),\mu_t(\mu))}(\mu)) = f_t(f,\mu)$ $f' < f$.

Let $t_o \in (0,\varepsilon)$ and let $f_+(f_{t_{n+1}}(f,\mu),\mu_{T'(f_{t_n}(f,\mu),\mu_{t_n}(\mu))}) = f_{t_n}(f,\mu)$. Therefore $T'(f_{t_{n+1}}(f,\mu),\mu_{t_{n+1}}(\mu)) < \tilde{T}(f_{t_{n+1}}(f,\mu),\mu_{t_{n+1}}(\mu)) \leq T'(f_{t_n}(f,\mu),\mu_{t_n}(\mu))$. Therefore,

$$
\begin{aligned}
V(f_{t_n}(f,\mu),\mu_{t_n}(\mu)) &= \delta^{T'(f_{t_n}(f,\mu),\mu_{t_n}(\mu))}V(f_{t_n}(f,\mu),\mu_{T'(f_{t_n}(f,\mu),\mu_{t_n}(\mu))}) \\
&< \delta^{T'(f_{t_n}(f,\mu),\mu_{t_n}(\mu))}[(f_{t_{n+1}}(f,\mu) - f_{t_n}(f,\mu))p(f_{t_n}(f,\mu),\mu_{T'(f_{t_n}(f,\mu),\mu_{t_n}(\mu))})+ \\
&\qquad V(f_{t_n}(f,\mu),\mu_{T'(f_{t_n}(f,\mu),\mu_{t_n}(\mu))})] \\
&= \delta^{T'(f_{t_{n+1}}(f,\mu),\mu_{t_{n+1}}(\mu))}V(f_{t_{n+1}}(f,\mu),\mu_{T'(f_{t_{n+1}}(f,\mu),\mu_{t_{n+1}}(\mu))}) \\
&\leq \delta^{\tilde{T}(f_{t_{n+1}}(f,\mu),\mu_{t_{n+1}}(\mu))}V(f_{t_{n+1}}(f,\mu),\mu_{\tilde{T}(f_{t_{n+1}}(f,\mu),\mu_{t_{n+1}}(\mu))})
\end{aligned}
$$

Taking limits on both sides, we have $V(f,\mu) \leq \lim_{n\to\infty} \delta^{\tilde{T}(f_{t_n}(f,\mu),\mu_{t_n}(\mu))}V(f_{t_n}(f,\mu),\tilde{\mu}(f_{t_n}(f,\mu),\mu_{t_n}(\mu)))$. Further, note that $V(f,\mu) \geq \delta^{\tilde{T}(f_{t_n}(f,\mu),\mu_{t_n}(\mu))}V(f,\tilde{\mu}(f_{t_n}(f,\mu),\mu_{t_n}(\mu))) \geq \delta^{\tilde{T}(f_{t_n}(f,\mu),\mu_{t_n}(\mu))}V(f_{t_n}(f,\mu),\tilde{\mu}(f_{t_n}(f,\mu),\mu_{t_n}(\mu)))$. Therefore, we have our result.

We conclude by noting that $\limsup_{n\to\infty} \tilde{T}(f_{t_n}(f,\mu),\mu_{t_n}(\mu)) = \tilde{T}(f,\mu)$.

Since, $V(f,\mu) \geq \delta^{\tilde{T}(f,\mu)}V(f,\mu_{\tilde{T}(f,\mu)})$ (since the seller can always stall trade), we have our result. $\square$

Next, we show that for (f,μ) in a smooth trade region, $p(f,\mu) \leq \delta^{\tilde{T}(f,\mu)}p(f,\mu_{\tilde{T}(f,\mu)})$. The proof is similar to Lemma 5 in FS. However, for the sake of completeness, we provide the proof here.

Lemma 3. *Suppose trade is smooth at* (f,μ)*. Then* $p(f,\mu) \leq \delta^{\tilde{T}(f,\mu)} p(f,\tilde{\mu}(f,\mu))$*. Moreover if* $\lim_{g\uparrow f} \delta^{\tilde{T}(g,\mu)} p(g,\tilde{\mu}(g,\mu)) = \delta^{\tilde{T}(f,\mu)} p(f,\tilde{\mu}(f,\mu))$*, then* $p(f,\mu) = \delta^{\tilde{T}(f,\mu)} p(f,\tilde{\mu}(f,\mu))$

Proof. Suppose $p(f,\mu) > \delta^{\tilde{T}(f,\mu)} p(f,\tilde{\mu}(f,\mu))$. By continuity of the reservation price function, there must exist a $T > 0$ s.t. for all $t < T$ and a sequence $t(t,\Delta)$ with $\lim_{\Delta\to 0} t(t,\Delta) = t$, $\lim_{\Delta\to 0} p(f_{+t(\Delta)}(f,\mu), \mu_{+t(t,\Delta)}) > \delta^{\tilde{T}(f,\mu)-t} p(f,\tilde{\mu}(f,\mu))$ and $\lim_{\Delta\to 0} f_{+t(T,\Delta)}(f,\mu) < \tilde{f}(f,\mu)$. Because trade is smooth, we are guaranteed that $f_T < f$. Therefore,

$$\sum_{j=0}^{t(T,\Delta)-1} \delta^j p(f_{+j}(f,\mu),\mu_{+j})(f_{+j}(f,\mu) - f_{+(j+1)}(f,\mu)) + \delta^{t(T,\Delta)} V(f_{+t(T,\Delta)}(f,\mu), \mu_{+t(T,\Delta)})$$

We note that $\delta^j p(f_{+j}(f,\mu),\mu_{+j}) > \delta^{\tilde{T}(f,\mu)} p(f,\tilde{\mu}(f,\mu))$ for Δ small and $\lim_{\Delta\to 0} V(f_{+t(T,\Delta)}(f,\mu), \mu_{+t(T,\Delta)}) \geq \delta^{\tilde{T}(f,\mu)-T} \hat{V}(f_T, \tilde{\mu}(f,\mu))$ (where $\hat{V}(f_T,\tilde{\mu}(f,\mu))$ is the seller's limit payoff at $(f_T,\tilde{\mu}(f,\mu))$ from offering $p(f,\tilde{\mu}(f,\mu))$ at (f_T,μ_T)). By taking limits, we have

$$V(f,\mu) > \delta^{\tilde{T}(f,\mu)} p(f,\tilde{\mu}(f,\mu))(f - f_{t(T,\Delta)}(f,\mu)) + \delta^{\tilde{T}(f,\mu)} \hat{V}(f_T,\tilde{\mu}(f,\mu)) > \delta^{\tilde{T}(f,\mu)} V(f,\tilde{\mu}(f,\mu))$$

which contradicts Lemma 2. Next, suppose $p(f,\mu) < \delta^{\tilde{T}(f,\mu)} p(f,\tilde{\mu}(f,\mu))$ and $\lim_{g\uparrow f} \delta^{\tilde{T}(g,\mu)} p(g,\tilde{\mu}(g,\mu)) = \delta^{\tilde{T}(f,\mu)} p(f,\tilde{\mu}(f,\mu))$. Since $\lim_{g\uparrow f} \delta^{\tilde{T}(g,\mu)} p(g,\tilde{\mu}(g,\mu)) = \delta^{\tilde{T}(f,\mu)} p(f,\tilde{\mu}(f,\mu))$, there exists $T > 0$ and a sequence $t(t,\Delta)$ with $\lim_{\Delta\to 0} t(t,\Delta) = t$ s.t. for $p(f,\mu) < \delta^{\tilde{T}(f_T,\mu_T)} p(f_T,\tilde{\mu}(f_T,\mu_T))$. Therefore by taking limits, we have

$$V(f,\mu) < \delta^{\tilde{T}(f_T,\mu_T)} p(f_T,\tilde{\mu}(f,\mu_T))(f - f_T) + \delta^{\tilde{T}(f_T,\mu_T)} \hat{V}(f_T,\tilde{\mu}(f_T,\mu_T)) \leq \delta^{\tilde{T}(f_T,\mu_T)} \hat{V}(f,\tilde{\mu}(f_T,\mu_T))$$

where the second inequality follows from the upper bound on $p(f,\mu)$ and the fact that $V(f_T,\mu_T) = \delta^{\tilde{T}(f_T,\mu_T)-T} V(f_T,\tilde{\mu}(f_T,\mu_T))$. Therefore, we get a contradiction.

□

8.2.4 Step 3: Seller's payoff in region of delay

We first show that $V(f,\mu)$ is continuous and weakly increasing in both its arguments.

Lemma 4. $V(f,.)$ *is weakly increasing and continuous*

Proof. Suppose $V(f,.)$ is not increasing. Then for some μ and $\mu' < \mu$, there exists a $\Delta' > 0$ s.t. for all $\Delta < \Delta'$, $V(f,\mu';\Delta) > V(f,\mu(\Delta);\Delta)$, where $\lim_{\Delta\to 0}\mu(\Delta) = \mu$. We show that this inequality cannot hold for any $\Delta > 0$. We first show that $V(f^*_{T-1},\mu_{T-1}(\Delta);\Delta) \leq V(f^*_T,\mu_T(\Delta);\Delta)$ at the points of indifference. First note that at any $(f,\mu_{+1}(\mu;\Delta))$ where $v_L(1+f) = V(f,\mu) = f(v_H - \tilde{\delta}(1-\mu_{+1}(\mu;\Delta))(v_H - v_L)) + \delta v_L$, $V(f,\mu_{+1}(\mu;\Delta)) > V(f,\mu;\Delta)$. Next, we show that if $V(f,\mu_{+1}(\mu;\Delta);\Delta) \geq V(f,\mu;\Delta)$ when the seller is indifferent between ending trade in T and $T-1$ periods at $(f,\mu_{+1}(\mu;\Delta))$, then if the seller is indifferent between ending trade in T and $T+1$ periods at (f',μ), we have $V(f',\mu_{+1}(\mu;\Delta);\Delta) \geq V(f',\mu;\Delta)$. Suppose not. Then,

$$
\begin{gathered}
(f'-f)(v_H - \delta(v_H - p(f,\mu_{+1}(\mu;\Delta)))) + \delta V(f,\mu_{+1}) > (f'-f)p(f,\mu_{+1}(\mu;\Delta)) + V(f,\mu_{+1}(\mu;\Delta)) \\
\implies (f'-f)(v_H - p(f,\mu_{+1}(\mu;\Delta))) > V(f,\mu_{+1}(\mu;\Delta))
\end{gathered}
$$

By optimality, we also have

$$
\begin{gathered}
(f'-f)p(f,\mu) + V(f,\mu) = (f'-f)(v_H - \delta(v_H - p(f,\mu_{+1}(\mu;\Delta)))) + \delta V(f,\mu_{+1}(\mu;\Delta)) \\
\implies V(f,\mu) - \delta V(f,\mu_{+1}(\mu;\Delta)) = (f'-f)[(1-\delta)v_H + \delta p(f,\mu_{+1}(\mu;\Delta)) - p(f,\mu)]
\end{gathered}
\tag{16}
$$

Evaluating $(1-\delta)(v_H - p(f,\mu_{+1}(\mu;\Delta)))$, we get $(1-\delta)\tilde{\delta}\tilde{\delta}^{T-1}(v_H - v_L)(1-\mu_{+1}(\mu;\Delta))$. Evaluating the second term on the right hand side of Equation (16), we get $(f'-f)\delta(1-\tilde{\delta})\tilde{\delta}^{T-1}(v_H - v_L)(1-\mu_{+1}(\mu;\Delta))$. Therefore,

$$
\begin{gathered}
V(f,\mu) - \delta V(f,\mu_{+1}(\mu;\Delta)) = (f'-f)[(1-\delta)v_H + \delta p(f,\mu_{+1}(\mu;\Delta)) - p(f,\mu)] \\
\implies V(f,\mu_{+1}(\mu;\Delta))(1-\delta) \geq (f'-f)(1-\delta)\tilde{\delta}\tilde{\delta}^{T-1}(v_H - v_L)(1-\mu_{+1}(\mu;\Delta)) \\
\implies V(f,\mu_{+1}(\mu;\Delta))(1-\delta) \geq (f'-f)(1-\delta)(v_H - p(f,\mu_{+1}(\mu;\Delta)))
\end{gathered}
$$

which yields a contradiction. Therefore at any belief f that is a cutoff point at (f,μ), it must be the case that $V(f,\mu_{+1}(\mu;\Delta)) \geq V(f,\mu)$. We now show that the result can be extended to points that are not cutoff points. Suppose (f',μ) is not a cutoff point. Let $f > f'$ be such that $p(f,\mu) = p(f',\mu)$ and $p(f,\mu_{+1}(\mu;\Delta)) = p(f',\mu_{+1}(\mu;\Delta))$ and (f,μ) is a cutoff point (if such a cutoff point exists). If $p(f',\mu) > p(f',\mu_{+1}(\mu;\Delta))$, we have

$$
\begin{aligned}
&(f - f')p(f,\mu) > (f - f')p(f,\mu_{+1}(\mu;\Delta)) \\
\implies &V(f,\mu) - V(f'.\mu) > V(f,\mu_{+1}(\mu;\Delta)) - V(f',\mu_{+1}(\mu;\Delta)) \\
\implies &V(f,\mu) - V(f,\mu_{+1}(\mu;\Delta)) > V(f',\mu) - V(f',\mu_{+1}(\mu;\Delta))
\end{aligned}
$$

since $V(f,\mu) - V(f,\mu_{+1}(\mu;\Delta)) < 0$, we have that $V(f',\mu) < V(f',\mu_{+1}(\mu;\Delta))$. Suppose for (f',μ) the closest cutoff point that is greater than f', f is such that $p(f,\mu_{+1}(\mu;\Delta)) \neq p(f',\mu_{+1}(\mu;\Delta))$ and $p(f',\mu) > p(f',\mu_{+1}(\mu;\Delta))$. By our construction, for any $T > 0$, $f^*_T(\mu) < f^*_T(\mu_{+1}(\mu;\Delta))$ (where $f^*_T(.)$ are cutoffs as constructed in the Existence section). This implies that if $p(f,\mu_{+1}(\mu;\Delta)) \neq p(f',\mu_{+1}(\mu;\Delta))$, there exists at least one (albeit, a finite number) cutoff point $(f'',\mu_{+1}(\mu;\Delta))$ s.t. f'' is between f and f'. If f'' is the closest such belief to f, we have that $p(f'',\mu_{+1}(\mu;\Delta))$ is optimal at $(f,\mu_{+1}(\Delta)))$ and $p(f,\mu)$ is optimal at (f'',μ), and therefore $V(f'',\mu_{+1}(\mu;\Delta)) \geq V(f'',\mu)$. Iterating this argument finite number of times, we get our result.

Suppose $p(f',\mu) \leq p(f',\mu_{+1}(\mu;\Delta))$. Let $f'' < f'$ be such that $p(f',\mu)$ is optimal at (f'',μ) and $p(f',\mu_{+1}(\mu;\Delta))$ is optimal at $(f'',\mu_{+1}(\mu;\Delta))$ and either (f'',μ) or $(f'',\mu_{+1}(\mu;\Delta))$ is a cutoff point. Then, we have

$$
\begin{aligned}
V(f',\mu) &= p(f',\mu)(f' - f'') + V(f'',\mu) \\
&\leq p(f',\mu_{+1}(\mu;\Delta))(f' - f'') + V(f'',\mu_{+1}(\mu;\Delta)) \\
&= V(f',\mu_{+1})(\mu;\Delta))
\end{aligned}
$$

Therefore, $V(f,.)$ is weakly increasing. Continuity follows from the fact that $V(f,\mu) \geq \delta^T V(f,\mu_T)$ for any μ and T. Taking T to zero, we have $V(f,\mu) \geq \lim_{T\to 0} V(f,\mu_T)$. Since $\lim_{T\to 0} V(f,\mu_T) \geq V(f,\mu)$ by monotonicity, we have our result.

$\square$

Suppose $\tilde{T}(f,\mu) > 0$ and for some $f' < f$, $V(f',\mu) > \delta^T V(f',\mu_T(\mu))$ for any $T > 0$. Moreover, by Lemma 2, there must be an atom at (f',μ). Note that there is limit delay at $f_+(f',\mu)$.

Claim 1. *There is limit delay at $(f_+(f',\mu),\mu)$*

Proof. Suppose there is no limit delay at $(f_+(f',\mu),\mu)$. This implies that there exists $f'' <$

$f_+(f', \mu)$ s.t. $\lim_{\Delta\to 0} T(f_+(f',\mu), f'', \mu; \Delta)\Delta = 0$. Moreover we have, $\lim_{\Delta\to 0} T(f', f_+(f',\mu), \mu; \Delta)\Delta = 0$ by definition. Since $T(f', f'', \mu; \Delta) = T(f', f_+(f',\mu), \mu; \Delta) + T(f_+(f',\mu), f'', \mu; \Delta)$, we have that $\lim_{\Delta\to 0} T(f', f'', \mu; \Delta) = 0$, contradicting the definition of $f_+(f', \mu)$. $\square$

Next, we show that trade cannot be smooth at $f_+(f', \mu)$.

Claim 2. *Trade cannot be smooth at* $(f_+(f', \mu), \mu)$

Proof. Suppose it is smooth. Then the limit payoff at (f', μ) is

$$
\begin{aligned}
V(f', \mu) &= p(f_+(f',\mu), \mu)(f' - f_+(f'(\mu))) + V(f_+(f',\mu), \mu) \\
&\leq \delta^{\tilde{T}(f_+(f',\mu),\mu)} p(f_+(f',\mu), \mu(\tilde{T}(f_+(f',\mu),\mu))) + \delta^{\tilde{T}(f_+(f',\mu),\mu)} V(f_+(f',\mu), \mu(f_+(f',\mu),\mu)) \\
&\leq \delta^{\tilde{T}(f_+(f',\mu),\mu)} V(f', \tilde{\mu}(f_+(f',\mu),\mu))
\end{aligned}
$$

where the first inequality follows from Lemma 3 and Lemma 2. This inequality, however, contradicts the fact that $V(f', \mu) > \delta^T V(f, \mu_T(\mu))$ for all $T > 0$. $\square$

Therefore, limit delay is silent at $(f_+(f', \mu), \mu)$.

Suppose there is limit delay on some interval I. Let $T_L(x, y) = \sup\{t \geq 0 | V(x, y) = \delta^t V(x, \mu_t(y))\}$, i.e., $T_L(x, y)$ is the maximum length of delay at (x, y) and there is an atom of trade at time T_L. Let I be such that $\min_{x\in I} T_L(x, \mu) > 0$. Note that if there is an atom of size $x - f_+(x, m)$ at (x, m), then there is an atom at (y, m) for $y \in (x, f_+(x, m))$. Therefore there is an interval $I' \subset I$ s.t. for all $x, x' \in I'$, $T_L(x, \mu) = T_L(x', \mu)$. This in turn implies that $\dfrac{\partial V(x,\mu)}{\partial f} = \dfrac{\partial W(x',\mu)}{\partial f} = \delta^{T_L(x,\mu) - T(\mu,m)} p(x, \mu_{+T_L(x,m)}(\mu))$ for all $m \in [\mu, \mu_{+T_L(x,\mu)}(\mu)]$,i.e., delay is not smooth over the interval I' when the belief is μ. Since delay cannot be silent over the entire interval, we have that there is no delay over this interval. This in turn implies that the atom of trade at (x, μ) for $x \in I'$ terminates in smooth trade. Let $f_+ = f_+(x, \mu)$. It cannot be the case that $V(f_+, \mu) = W(f_+, \mu)$, since this would put the payoff at (x, μ) below $W(x, \mu)$. Then there is an interval I'' over which trade is smooth and $T_L(y, \mu) = T_L(y', \mu) > 0$ for $y, y' \in I''$, which yields a contradiction.

Therefore, it must be the case that if there is an interval I s.t. there is limit delay at (x, μ) for all $x \in I$, then $V(x, \mu) = W(x, \mu)$. We first note that smooth trade cannot be preceded by an atom of trade. So if there is an atom of trade, it must be followed by silent delay. Second, we argue that if there is silent delay at (x, μ), then for every $m > \mu$

there is an interval (x, x') s.t. for all $y \in (x, x')$, $V(y, m) = W(y, m)$. If not, then there is an $m > \mu$ s.t. $V(y, m) > W(y, m)$ for all y close to x. This in turn implies that there is an atom of trade at (y, m) followed by silent trade at $(f_+(y, m), m)$. We also have that $V(y, b) > \delta^t V(y, \mu_{+t}(b))$ for all $t > 0, b > \mu$. Therefore, there is an atom of trade at (y, b) for all y close to x. Therefore, at (x, y) where $z \in (\mu, m)$, the seller is indifferent between trading with probability $x - f_+(y, z))$ where $y < x$ is close to x. We first show that $f_+(y, .)$ cannot be constant in $[\mu, \min\{m, \hat{T}(f_+(y, \mu), \mu)\})$.

For $z \in (\mu, m)$, let

$$F_L(x, z) = (x - f_+(y, \mu))p(f_+(y, \mu), z) + V(f_+(y, \mu), z)$$

We first note that $F_L(x, \mu) > \delta^s F_L(x, \mu_{+s}(\mu))$ for all $s \in (0, T(m, \min\{m, \hat{T}(f_+(y, \mu), \mu)\}))$, $m \in [\mu, \min\{m, \hat{T}(f_+(y, \mu), \mu)\})$. Suppose not. Then,

$$\begin{aligned} F_L(x, \mu_{+s}(\mu)) &= (x - f_+(y, \mu)p(f_+(y, \mu), \mu_{+s}(\mu)) + V(f_+(y, \mu), \mu_{+s}(\mu)) \\ \implies \delta^s V(x, \mu_{+s}(\mu)) &= \delta^s(x - f_+(y, \mu)p(f_+(y, \mu), \mu_{+s}(\mu)) + \\ &\quad \delta^s V(f_+(y, \mu), \mu_{+s}(\mu)) \\ \implies V(x, \mu) &= \delta^s(x - f_+(y, \mu)p(f_+(y, \mu), \mu_{+s}(\mu)) + V(f_+(y, \mu), \mu) \end{aligned}$$

Since $V(x, \mu) = F_L(x, \mu) = p(f_+(y, \mu), \mu)(x - f_+(y, \mu)) + V(f_+(y, \mu), \mu)$, we have $p(f_+(y, \mu), \mu) = \delta^s p(f_+(y, \mu), \mu_{+s}(\mu))$, but this contradicts the fact that prices are decreasing in time.

Next, we show that $f_+(y, .)$ cannot be increasing in time.

Lemma 5. $f_+(y, .)$ *cannot be strictly increasing*

Proof. Let $z_n \uparrow z$ and WLOG let $f_+(y, z) = \lim_{n\to\infty} f_+(y, z_n)$. By continuity of the payoff function, $\lim_{n\to\infty} V(f', y_n) = V(f', y)$, i.e.,

$$\begin{aligned} \lim_{n\to\infty} p(y, z_n)(y - f_+(y, z_n)) + V(f_+(y, z_n), z_n) &= p(y, z)(y - f_+(y, z)) + V(f_+(y, z), z) \\ \implies \lim_{n\to\infty} p(y, z_n) &= p(y, z) \end{aligned}$$

This, in particular, implies that $\lim_{n\to\infty} \overline{T}(f_+(y, z_n), z_n) = \overline{T}(f_+(y, z), z)$. So for all $\varepsilon >$

0, there exists N s.t. for all $n > N$, $|\overline{T}(f_+(y,z),z) - \overline{T}(f_+(y,z_n),z)| < \varepsilon$. Suppose m is s.t. $|\delta^{\overline{T}(f_+(y,z),z)}(1-\mu_{\overline{T}(f_+(y,z),z)}) - \delta^{\overline{T}(f_+(y,z_m),z)}(1-\mu_{\overline{T}(f_+(y,z_m),z)})| < v_H(1-\delta^{\tilde{T}(f_+(y,z),z)})/2$. The reservation price at $(f_+(y,z_m),y)$ is $P(f_+(y,z_m),y) = v_H - \tilde{\delta}^{\overline{T}(f_+(y,z_m),z)}(v_H - v_L)(1-z)$. Therefore,

$$
\begin{aligned}
&V(f_+(y,z),z) - [P(f_+(y,z_m),z)(f_+(y,z) - f_+(y,z_m)) + V(f_+(y,z_m),z)] \\
&\leq \delta^{\tilde{T}(f_+(y,z),z)}[p(f_+(y,z),\tilde{\mu}(f_+(y,z),z))(f_+(y,z) - f_+(y,z_m)) + V(f_+(y,z_m),\tilde{\mu}(f_+(y,z),z))] \\
&\quad -[P(f_+(y,z_m),z)(f_+(y,z) - f_+(y,z_m)) + V(f_+(y,z_m),z)] \\
&\leq (f_+(y,z) - f_+(y,z_m))(\delta^{\tilde{T}(f_+(y,z),z)}p(f_+(y,z),\tilde{\mu}(f_+(y,z),z)) - P(f_+(y,z_m),z) \\
&= (f_+(y,z) - f_+(y,z_m))[\delta^{\overline{T}(f_+(y,z),z)}(v_H - v_L)(1-\mu_{\overline{T}(f_+(y,z),z)}) \\
&\quad -\delta^{\overline{T}(f_+(y,z_m),z)}(1-\mu_{\overline{T}(f_+(y,z_m),z)})(v_H - v_L) - v_H(1-\delta^{\tilde{T}(f_+(y,z),z)})] < 0
\end{aligned}
$$

which contradicts the optimality of the seller's strategy at $(f_+(y,z),z)$.

□

Therefore there exists $\varepsilon > 0$ s.t. for all $y \in (x, x+\varepsilon)$, $V(y,\mu) = W(y,\mu)$. This in turn means that $V(x,\mu) = W(x,\mu)$. We next show that if there is a gap at (g,μ) then there cannot be limit delay at (f,μ) for $f > g$. Suppose there is silent delay at (g,μ) and limit delay at (f,μ) for $f > g$. Let $T'(f,g;\mu)$ denote the minimum time taken for the state to transition from f to g. For every $t < \hat{T}(g,\mu)$, there is an atom of trade that precedes the gap at $(g,\mu_{+t}(\mu))$. Let $r(m) = \inf\{x > g | \tilde{T}(x,m) > 0\}$. Note that there is necessarily limit delay at (x,m) for $x > r(m)$ close to $r(m)$. Moreover, $r(.)$ is decreasing.

Claim 3. *Let $m \in (\mu, \hat{\mu}(g,\mu))$. If $x < r(m)$, then $f_{+t}(x,m) \geq r(\mu_{+t}(m))$ for all $t \in [0,\hat{T}(g,\mu)]$.*

Proof. Note that by definition of $r(.)$, $T'(x,g) > 0$. Suppose $f_{+t}(x,m) < r(\mu_{+t}(m))$ for some $t \in (0,\hat{T}(g,\mu))$, then the price at time t is $p(g,\mu_{+t}(m))$. Therefore the all prices following the initial price at (x,m) are strictly less than $v_H(1-\delta^t) + \delta^t p(g,\mu_{+t}(m)) = p(g,m)$. Since there is limit delay between g and x, the payoff would be lower than the payoff from charging $p(g,m)$, yielding a contradiction. □

Corollary 1. $r(.)$ *is strictly decreasing*

Proof. That $r(.)$ is weakly decreasing follows from the previous claim. Suppose for some $m > m'$, $r(m) > r(m')$. Then $f_{+T(m',m)}(r(m),m') \leq r(m')$. Note that equality cannot hold because the seller would be better off offering $p(g, m')$. Therefore, we get a contradiction. Suppose $r(.) = l$ over some interval $[k_1, k_2]$. For $x > l$ close to l and $\varepsilon > 0$ small, the payoff at (x, k_1) is atmost

$$\begin{aligned}(x - l + \varepsilon)v_H + \delta^{T(k_1,k_2)}V(l - \varepsilon, k_2) \\ < V(l - \varepsilon, k_1)\end{aligned}$$

Since, $V(x, k_1) \geq V(g, k_1)$, we get a contradiction. □

First we note that $V(r(m), m) = W(r(m), m)$ and let $m' = \mu_{+\hat{T}(g,\mu)}(\mu)$. Next, we have

$$W(r(m), m) = (r(m) - g)[v_H(1 - \delta^{\hat{T}(g,m)} + \delta^{\hat{T}(g,m)}p(g, m')] + W(g, m)$$

Let $m_+ = \mu_{+t}(m)$. Noting that $v_H(1 - \delta^{\hat{T}(g,m)}) + \delta^{\hat{T}(g,m)}p(g, m') = v_H(1 - \delta^t) + \delta^t p(g, m_+)$, we have,

$$\delta^t W(r(m), m_+) - \delta^t W(r(m_+), m_+) = (r(m) - r(m_+))p(g, m) + (r(m_+) - g)v_H(1 - \delta^t)$$

Dividing by t and taking $t \to 0$

$$\begin{aligned}-\frac{\partial W(r(m), m)}{\partial f}\frac{dr(m)}{d\mu}\frac{d\mu}{dt} &= -\frac{dr(m)}{d\mu}\frac{d\mu}{dt}p(g, m) + (r(m) - g)rv_H \\ \implies -\frac{dr(m)}{d\mu}\frac{d\mu}{dt}\left[\frac{\partial W(r(m), m)}{\partial f} - p(g, m)\right] &= (r(m) - g)rv_H \qquad (17)\end{aligned}$$

Since $r(.)$ denote the indifference points, the rate of trade is given by $\dfrac{dr(.)}{dt}$. Since $r(m_t) \to g$ as $t \to \hat{T}(g, m_t)$, we have that $p(r(m), m) = p(g, m)$, which contradicts Equation (17).

Therefore, if there is a at (g, μ), there cannot be any limit delay at (f, μ) for $f > g$. But this once again, yields a contradiction as $m \to \hat{\mu}(g, \mu)$, $V(f, m) < W(f, m)$.