Why Better Outside Options May Erode Bargaining Power

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Abstract

I study a bargaining game with outside options in an interdependent values setting. A seller makes sequential offers to a buyer who has private information about the value of the object. The seller has an exercisable outside option that is valued at $\alpha < 1$ proportion of the buyer's value. In the frequent offer limit of any equilibrium, the seller is able to extract full rents from a subset of buyer types. However, Coasian forces are present on two levels; the seller is not only tempted to lower prices to conclude trade, but is less inclined to exercise her outside option as she becomes more pessimistic about the value of the object (and therefore, the value of her outside option). In equilibrium, an improvement in the outside option (through an increase in α) could make the seller less willing to lower prices and helps her extract more rents from the middle type. As she is able to extract more rents, she is less willing to exit, which undermines her ability to extract rents from the high type. When the middle type occurs with sufficiently high probability, a marginal increase in α may make the seller worse off for some values of α.

1 Introduction

The role of outside options in bargaining is a topic that has received a lot of attention in the literature. This question has been studied in a variety of settings, including the standard dynamic bargaining setting with one sided private information. In the standard one-sided incomplete information model sans outside options, the seller's temptation to lower prices in the future prevents her from securing a high price in the present. In essence, the seller's present self, who wishes to service high valuations of the buyer, is engaged in a price competition with her future self, who services lower valuations that rejected high offers in the past. This dynamic, however, unravels with the introduction of outside options. If the buyer has an outside option (see Board and Pycia (2014)), lower valuations of the buyer opt out in equilibrium early in the game, eliminating the seller's temptation to lower prices in the future. If the seller has an outside option in a private values setting (see Fudenberg et al. (1987)), the seller's future self would rather exercise her outside option than service buyer types with low valuations. The seller's problem (which is reminiscent of standard commitment problems) is resolved either because the seller's future self has no control over terminating negotiations or when they do have control, the future and present selves agree on the value of terminating negotiations, thus aligning future and present incentives in that regard. In light of this argument, if the value of the outside option depends on the buyer's value, the seller's present and future incentives to exercise the outside option may possibly be misaligned. In this paper, I study the robustness of Coasian dynamics to the introduction of outside options in an interdependent values setting.

In particular, I study the following model: A seller makes sequential offers to a buyer who has private information about the value of the object. The seller has an outside option that she may exercise at any point during the game. The seller receives $\alpha < 1$ proportion (therefore, there are always gains from trade) of the value of the object if she exercises her outside option, while the buyer receives nothing. Following the literature, we study Weak Markov Equilibria of this game, where the seller's belief acts like a state variable.

There are many bargaining settings in which the buyer of the object knows more about the (common) value of the object that the seller. For instance, art dealers are likely to be more informed about the value of the artwork than the seller of the artwork. When the seller has an outside option in such settings, she learns about the value of the object *and* the value of her outside option through negotiations; an obstinate buyer makes the seller more pessimistic about the value of the object and consequently, the value of her outside option. In this paper, I study bargaining dynamics and outcomes that emerge in such settings in the frequent offer limit[1]. I further study how changes in the cost of exercising the outside option affects the seller's payoff.

I first show existence for large discount factors(Proposition 1). I then (uniquely) pin down equilibrium dynamics and the seller's payoff in the frequent offer limit as a function of α (Proposition 2). Finally, I characterize necessary and sufficient conditions on the prior that ensure non-monotonicity of the seller's payoff function when the buyer's valuation can take three values(Theorem 1).

The limit structure of equilibria pinned down in Proposition 2 have the same pattern for all values of α that put the ex-ante value of the outside option above the lowest valuation in the support of the value distribution. In equilibrium, an initial burst of trade is followed by immediate exit. The price in the initial burst of trade is (approximately) equal to the valuation of the marginal type that is excluded from trade when the seller subsequently exits. This equilibrium structure resembles the equilibrium dynamic in Fudenberg et al. (1987). However, in Fudenberg et al. (1987), the seller exercises her outside option when there are no gains from trade. When there are gains from trade in a private values setting, the seller never exercises her outside option. In an interdependent values setting, on the other hand, the seller may exercise her outside option even though there are gains from trade.

The seller exits in equilibrium with positive probability. Moreover, the seller's (limit) payoff from exercising her outside option is always weakly lower than her equilibrium payoff from negotiating with the buyer. Therefore, the seller exercises her outside option when she is exactly indifferent between exercising her outside option and continuing negotiations. We show, in Proposition 2 that such indifference points exist.

As a consequence of Coasian forces, the seller's payoff may not be monotonically increasing in α. Theorem 1 lays down a sufficient and necessary condition for three value distributions that obtains a non monotonic payoff function for the seller. We provide a brief intuition for why this may happen. The intuition is discussed in detail in Section 5.

The seller may get a higher payoff for lower values of α because she may be able to extract a higher price from the buyer when the value of α is low. For the purpose of providing a coarse intuition, consider a seller who is optimistic on day one, agnostic on day two and pessimistic on day three. The seller's ability to extract a high price from the buyer today depends on her willingness to exit tomorrow. Conversely, the seller's willingness to exit today depends on the price she is able to extract today, should she

[1]Essentially, as players become arbitrarily patient.

choose to trade. The key point of difference between trade dynamics for low and high values of α is that the seller may prefer to exercise her outside option on day three for high values of α, while she prefers to trade at a low price for low values of α. As a seller with a better outside option will exercise her outside option on day three when she is pessimistic, the buyer is willing to pay a high price on day two. Consequently, the seller is unwilling to exercise her outside option on day two. On the other hand, when the value of α is low, the buyer will not accept a high price on day two, anticipating the price to fall on day three. This could make it worthwhile for the seller to opt out of negotiations on day two. The same argument implies that in period one, the buyer may accept a high price on day one when the value of α is low but may not accept a high price when α is high.

At the heart of the seller's dilemma are familiar Coasian forces, that now act on two fronts. The seller simultaneously faces a commitment problem on two fronts. When the buyer rejects a price offer targeted at a particular type, the seller becomes more pessimistic about the value of the object. This tempts her to not only lower her price offers, but also weakens her incentive to exercise her outside option. An increase in α affects the seller's commitment problem in two ways. On the one hand, an increase in α alleviates the seller's commitment problem with respect to her pricing decision. On the other hand, if the seller has more control over her pricing decision, the commitment problem with respect to her exit decision is exacerbated, i.e., she is less willing to opt out.

Finally, I consider a variation of the original game where the seller has to make an unobservable investment in the outside option in order to maintain it. I construct an equilibrium in which the seller stops investing on the equilibrium path. Under some conditions, this equilibrium gives the seller a higher payoff than any equilibrium in which she maintains investment in the outside option. In equilibrium, if the seller does not exit after the first offer is rejected, the buyer believes it to be unlikely that the seller has access to the outside option, and is therefore unwilling to accept a high price in the second period. This rationalizes the seller's decision to exit at the beginning of the second period.

The rest of the paper is organized as follows: Section 2 provides a literature review, Section 3 describes the model, Section 4 describes strategies and lays down the equilibrium concept,Section 6 explains the main results, Section 8 discusses the unobserved investment case and the final section concludes.

2 Related Literature

This paper is related to the literature on outside options in bargaining with one sided asymmetric information. The classic Coasian force, examined in Gul et al. (1986) (henceforth,GSW) and Stokey (1981) also manifests in our setting, albeit in two ways. In context of bargaining with one sided asymmetric information, the question of outside options has been studied extensively. Fudenberg et al. (1987) study a setting where the seller has an outside option that she may exercise at any time. In their analysis, they discuss how the self fulfilling nature of equilibria means that the seller could end up playing either soft or tough in equilibrium. Board and Pycia (2014) study a game where the buyer has the outside option as well as private information about his type. The Coase conjecture breaks down in this setting and the seller gets her commitment payoff. In contrast, our setting examines the commitment problem the seller faces with respect to exercising her outside option. Nava and Schiraldi (2019) study a setting where the seller sells two products.

The role of interdependent values with and without outside options has been studied in a variety of contexts. Deneckere and Liang (2006) (henceforth, DL) study a dynamic bargaining model in a lemons market and find that trade happens in 'bursts' punctuated by periods of delay. In our setting, the seller quits negotiations instead of delaying negotiations. Fuchs and Skrzypacz (2010) study a bilateral trading game where the seller has an outside option that arrives at some exogenous rate and ends the game. In their setting, the value of the outside option is correlated with the buyer's value for the object. They find that in an atomless stationary equilibrium, the seller's payoff is reduced to the payoff she would get from simply waiting for the outside option to arrive. As a consequence, the seller is indifferent between different rates of trade in equilibrium. A similar logic is applied to our setting. The option value in our setting lies in quitting negotiations altogether. While she is able to extract some rents in the first few periods, the seller's payoff is eventually reduced to the payoff from her outside option. However, the fact that the outside option can be exercised at will by the seller plays a crucial role in the non monotonicity of the seller's payoff in α. In Fuchs and Skrzypacz (2010), an increase in the exogenous arrival rate is unambiguously better for the seller as her payoff is exactly equal to the payoff she would get from waiting for the outside option to arrive. In Chaves (2019), negotiations between the seller and the buyer is observed by a third party that can endogenously disrupt negotiations. The paper examines the role of offer transparency in this setting. Daley and Green (2020) study trade dynamics in a lemons market when news arrives over time about the object's value. Similar to Fuchs and Skrzypacz (2010), they find that the buyer's payoff is reduced to the payoff from waiting to become suf-

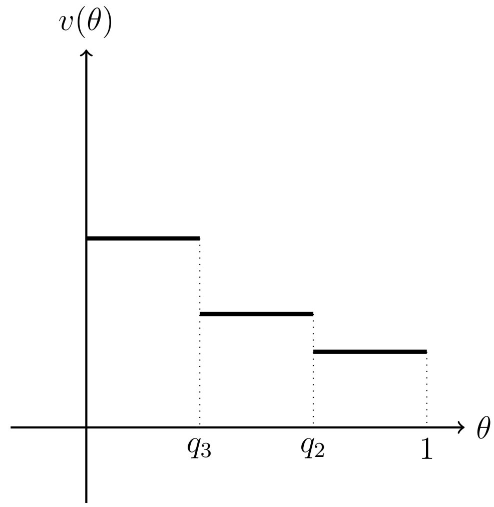

Figure 1: Example of value distribution

ficiently optimistic about the object's value. An endogenous interdependence arises in Ortner (2017) where the seller has stochastic cost. In this setting, the seller is tempted to lower her price and cater to low valued buyers as costs fall.

3 Model

Time is discrete and each period is of length Δ. We are interested in outcomes as Δ goes to zero. A seller negotiates with a buyer who is privately informed of her valuation. The buyer's valuation takes value in the set $\{v_1, v_2, v_3, ..v_N\}$ where $v_1 < v_2 < .. < v_N$. Following DL, we let the value of the buyer depend on the realization of a random variable $q \sim U[0, 1]$, i.e.,

$$v(q) = v_i \quad q \in (q_{i+1}, q_i] \tag{1}$$

for $i = 1, 2, ..N$, where $q_{N+1} = 0$ and $q_1 = 1$. We refer to the realization of q as the buyer's 'type'. Figure 1 illustrates the value distribution.

The seller has an outside option whose value depends on the buyer's valuation. In particular, if the buyer's type is q, the seller's payoff from opting out is $\alpha v(q)$. If the seller opts out, the buyer's payoff is zero.

The timeline of the game is as follows

- Seller decides whether or not to exit
- If seller doesn't exit, she makes an offer from the set of available offers, or makes no offer
- If an offer is made, buyer decides whether to reject or accept the offer

4 Strategies and Equilibrium

A public history consists of all past offers, i.e., a time t public history, denoted by h^t is the sequence of all offers made till period $t-1$. The set of all time t public histories is denoted by $\mathcal{H}^t$. Let p^t denote the price offer made at time t. The buyer's acceptance strategy at time t is a mapping from their type (i.e., the realization of q), time t public history and the current offer to an accept or reject decision. We denote the buyer's strategy by the function $\sigma^t : [0,1] \times \mathcal{H}^t \times \mathbf{R}_+ \to \{0,1\}$.

The seller makes two decisions-one with respect to her exit decision and the other with respect to the price offers. The seller's (pure) exit strategy is a mapping from a public history to $\{0,1\}$ where 1 indicates exit. A pure offer strategy is a function $\sigma : \mathcal{H}^t \to \mathcal{R}_{++}$.

The skimming property is a standard result in the classic Coasian framework which shows that if lower valuation buyers accept a price, it must be acceptable to high valuation buyers. An implication of the skimming property is that in any PBE, the belief over the set of types following rejection of an offer is a right truncation of the prior belief. Thus, beliefs over types in each period can be summarized by a cutoff type at which truncation occurs.We show that the skimming property holds in any PBE.

Lemma 1. *Suppose $v_{\theta'} > v_\theta$ and suppose a positive mass of both types remain. If a type with valuation θ accepts an offer p with positive probability, then all types with valuation θ' accept p with probability 1*

Proof. Let σ_b' be any other arbitrary strategy for the buyer and let σ_s be the seller's strategy in equilibrium. By optimality, we have

$$v_\theta - p \geq E_{(\sigma_b', \sigma_s)}[\delta^t (v_\theta - p)] \tag{2}$$

Next,consider

$$
\begin{aligned}
&E_{(\sigma_b',\sigma_s)}[\delta^t(v_{\theta'}-p)] - E_{(\sigma_b',\sigma_s)}[\delta^t(v_\theta-p)] \\
&\quad = E_{(\sigma_b',\sigma_s)}[\delta^t(v_{\theta'}-v_\theta)] < v_{\theta'} - v_\theta \\
\implies\ & v_{\theta'} - E_{(\sigma_b',\sigma_s)}[\delta^t(v_{\theta'}-p)] > v_\theta - E_{(\sigma_b',\sigma_s)}[\delta^t(v_\theta-p)] \qquad (3)
\end{aligned}
$$

Subtracting p on both sides, we get

$$
v_{\theta'} - p - E_{(\sigma_b',\sigma_s)}[\delta^t(v_{\theta'}-p)] > v_\theta - p - E_{(\sigma_b',\sigma_s)}[\delta^t(v_\theta-p)] \geq 0 \qquad (4)
$$

where the last inequality follows from Equation (2). This implies that θ' accepts p with probability 1.

□

We consider a subclass of PBE called Weak Markov Equilibria (see Ausubel and Deneckere (1989),Fudenberg et al. (1985)). A PBE is a Weak Markov Equilibrium if the buyer's acceptance strategy depends only on the current price and the buyer's type (i.e., q) and the seller's offer and exit strategy depend only on the current belief and the seller's offer in the previous period. This class of equilibria has been studied extensively in the literature, is tractable and provide a natural point of comparison to standard Coasian results.

5 Illustration

In this section, we provide a heuristic argument for the non monotonicity of the seller's ex ante payoff. Suppose the buyer's valuation takes values in the set $\{v_1, v_2, v_3\}$ and suppose $\alpha < v_2/v_3$. We first examine when the seller offers v_1 and when she can get a higher price accepted with positive probability.

Intuitively, the seller's ability to extract a high price depends on the credibility of her threat to exit if her offer is rejected. Figures 3 and 2 denote the belief simplex. The black dot denotes the prior belief. In both figures, v_1 is greater than the payoff from the outside option in the white region. Coasian forces operate in this region, which means that the seller offers v_1 almost immediately in the frequent offer limit. In the blue and

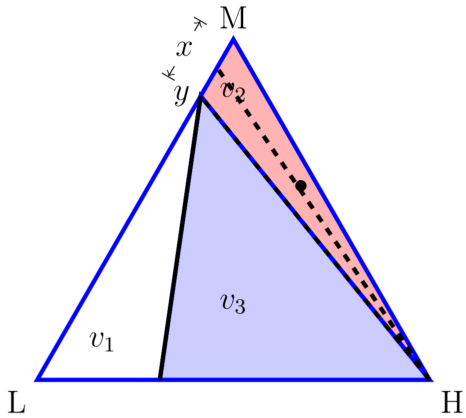

Figure 2: The black dot represents the prior

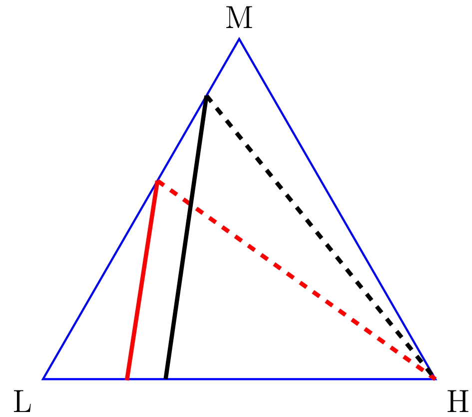

Figure 3: Black lines correspnd to α_1 and red lines to α_2 where $\alpha_1 < \alpha_2$

pink regions, the seller would rather opt out than trade at v_1 with probability one. Along the solid black line, the seller is indifferent between trading at v_1 and exiting. We argue that the seller is able to extract a higher price with positive probability in the blue and pink regions. Further, the seller is able to trade at v_3 with positive probability in the blue region, but not the red region.

The seller can extract v_3 from type H with positive probability subject to two restrictions– (1) the posterior upon rejection of v_3[2] satisfies Bayes' plausibility and (2) Type H has incentive to accept the offer v_3, i.e., the seller does not strictly prefer to wait for lower future prices. The first point implies that in Figure 2, the seller's posterior belief upon rejection must lie along the dashed black line. The second point implies that the seller must not lower prices in the future, which in turn implies that the seller exits with probability one in the future. In Figure 2, the seller's posterior upon rejection must lie on the solid black line.[3]

When the seller's prior lies in the blue region, there is exactly one posterior belief that satisfies the above conditions, denoted by the point of intersection between the solid black line and the dashed black line. It is, therefore, possible for the seller to extract price v_3

[2]Note that v_3 is rejected with positive probability by the buyer. If the buyer is of types M or L, she rejects the offer.

[3]There is one further step involved here, which is to show that the seller's posterior cannot lie in the interior of the blue or pink region upon rejection. Suppose the seller exits at some point in the interior of the blue region, where the seller strictly prefers to opt out. If, at this point the seller makes an offer that concedes some small rents to type H, it has to be accepted with positive probability, which in turn implies that the seller does not strictly prefer to opt out.

from type H with positive probability in equilibrium.

We now turn our attention to the pink region. Since $\alpha < v_2/v_3$, the seller never opts out if she can trade at a price of v_2 with type H. Indeed, if the belief lies on the line segment labeled x, the seller can get M to accept v_2 in equilibrium with positive probability since the point y satisfies Bayes' plausibility (which implies that the posterior must lie along the blue line joining the M and L vertices) and M's IC constraint (which implies that the posterior must lie on the black solid line). By the skimming property, it must be the case that type H accepts v_2 with probability one.

Since the seller never allows the price to fall to v_1 in the pink region, there are two possibilities: either the seller can get H to accept v_3 with positive probability or the seller trades with type H at price v_2.

Towards a contradiction, suppose there is a way to get type H to accept v_3 with positive probability. Once again, the seller's posterior must lie at the intersection of the dashed black line and the solid black line. However, note that the dashed black line has no intersection with the solid black line in the pink region. Since, the seller can always charge v_2 in the pink region, there is no incentive to opt out at any point in this region. The seller cannot credibly follow through on her threat to opt out of negotiations after an offer of v_3. Therefore, the seller's initial price offer is v_2 in equilibrium.

Why is the seller unable to extract v_3 from type H in the pink region? The reason is that the presence of type M acts as a sort of buffer against lowering prices to v_1. When the belief assigned to type M isn't sufficiently high, the seller knows that if she stays in negotiations for too long, she is likely to lower prices quickly. Given her (future) temptation to lower prices, she preemptively opts out while she is still optimistic. This enables her to extract v_3 (with some probability) from type H. In the pink region, the probability assigned to type M is sufficiently high that the seller is not tempted to lower prices to v_1, regardless of type H's acceptance probability. In fact, even if H accepts v_3 with probability one, the posterior upon rejection assigns a very high probability to type M, which prevents the seller from lowering prices. Consequently, the seller is able to extract price v_2 from type M before exiting. The seller's temptation to trade with type M adversely affects her ability to trade with H at price v_3.

We shall now see what may happen when α increases. Suppose the initial value of α is α_1 and we increase α to α_2. This makes the price v_1 unacceptable to the seller for a larger set of priors. The contraction of the white region is illustrated in Figure 3. The solid black line is the indifference set associated with α_1 and the red line is the indifference set associated with type α_2. Another consequence of an increase in α is the expansion of the pink region at the expense of the blue region. Therefore some priors that lay in the blue region under α_1 lies in the pink region under α_2. Consequently, there is a fall in the initial price offer. However, since the initial price offer is lower, there is also a corresponding increase in the probability of trade. Since the dashed line shifts continuously with α, for small changes in α, the fall in price dominates the increase in trade probability, causing a fall in the seller's payoff. An increase in α improves the marginal value of the outside option given the expected value of the object. This means that for a larger region of priors that assign a sufficiently high probability to type M, v_1 is an unacceptable trade price. This consequently guarantees that the seller can get the buyer to accept v_2 with positive probability later in the game, which in turn adversely affects her ability to charge v_3 earlier in the game. The seller is able to extract v_3 from type H because she can credibly exit in the future before lowering prices any further and she can credibly exit in the future because she lacks confidence in her future conduct, i.e., she may hastily lower prices. When α increases, the seller does not run the risk of lowering prices as long as the belief assigns a sufficiently high probability to type M. This means that the seller can *always* get the buyer to accept v_2 with positive probability. The seller, thus, settles for the lower, albeit, guaranteed price of v_2.

6 Existence and Uniqueness of Limit Outcomes

We first present the existence results for the general model for small values of Δ. Aside from some details, the proof proceeds as the existence proof in DL.

Proposition 1. *[**Existence for small** Δ] There exists $\Delta' > 0$ s.t. for $\Delta < \Delta'$, there exists an equilibrium.*

The proof is constructive and as in DL, we 'backward induct' the seller's payoff function, the buyer's reservation price strategy and the induced state in the next period. The induction process stops at state zero or when the seller's payoff is equal to her payoff from the outside option, whichever happens earlier. Our assumption that Δ is small is key at this point. When Δ is small, the seller never makes a price offer that induces a state above the point of indifference. This allows us to repeat the induction process for states below the indifference point in the same manner. For large values of Δ, there may exist states below the indifference point at which the seller prefers to induce a state above the indifference point, making the proof for existence much more challenging.

Proposition 2. ***[Uniqueness of Limit Outcomes]*** *As Δ goes to zero, the seller's payoff is uniquely pinned down.*

We outline the proof for uniqueness of limit outcomes below. Suppose $W(q) = E[v(x)|x \geq q]$. Let $\overline{q}_1$ be such that

$$v_1 = \alpha W(\overline{q}_1) \tag{5}$$

and $\overline{q}_i$ is s.t.

$$\alpha W(\overline{q}_i) = v_i \frac{\overline{q}_{i-1} - \overline{q}_i}{1 - \overline{q}_i} + \frac{1 - \overline{q}_{i-1}}{1 - \overline{q}_i} \alpha W(\overline{q}_{i-1}) \tag{6}$$

for $i \leq N$, where $\overline{q}_i \in [0, 1]$. Intuitively, $\overline{q}_i$ represent points at which the seller's incentive to lower prices is matched by her incentive to opt out. If the seller initially prefers to opt out than lower prices, she will continue to stay in only if she anticipates that her future selves will not lower prices too quickly. However, upon rejection of her offers, the seller becomes pessimistic over time, which in turn makes her more willing to lower offers and less inclined to opt out. In equilibrium, the seller's future incentive to lower prices must match her willingness to opt out. In order to rationalize trade in the current period, the seller must opt out in the future with sufficiently high probability.

The proof involves three steps.

Step 1: Whenever $\overline{q}_i$ is between zero and one, for Δ small enough, there exists $\overline{q}_i(\Delta)$ where the seller is indifferent between continuing trade and opting out. As Δ goes to zero, $\overline{q}_i(\Delta)$ converges to $\overline{q}_i$.

Step 2: When Δ is small, if $\overline{q}_{i+1}(\Delta) < q < \overline{q}_i(\Delta)$, then the induced belief in the next period is atmost $\overline{q}_i(\Delta)$.

Step 3: When $\overline{q}_{i+1}(\Delta) < q < \overline{q}_i(\Delta)$, the price charged converges to $v(\overline{q}_i(\Delta))$ as Δ goes to zero.

Let $V(\alpha)$ denote the unique limit ex-ante payoff of the seller as a function of α. We first note that $V(\alpha)$ is continuous almost everywhere. If $V(.)$ is decreasing at α, there are two possibilities-(1) $V(.)$ is discontinuous at α or (2) $V(.)$ is continuous at α. Regardless of whether $V(.)$ is continuous or discontinuous at the point at which it decreases, we have that for an interval $[\underline{\alpha}, \overline{\alpha}]$, there exists $a < \underline{\alpha}$ s.t. $V(a) > V(x)$, for every $x \in [\underline{\alpha}, \overline{\alpha}]$, i.e., a lower α yields a higher payoff. In the next section, we examine sufficient and necessary conditions under which the seller's payoff may be non-monotonic in the three values case.

7 Necessary and Sufficient Condition: Three Values Case

Let $\beta_{ij} = v_i/v_j$. We assume that $v_1/W(q_3) \neq \beta_{23}$. We first describe a condition on the prior.

Definition 1. *The prior is said to exhibit* ***median prominence*** *if*

$$\frac{v_1}{W(q_3)} < \frac{1-\beta_{13}}{2-\beta_{23}-\beta_{12}} \tag{7}$$

We now state the main result:

Theorem 1. *[**Decreasing Payoffs**] For any prior distribution there exists $\alpha \in (0, 1)$ s.t. $V(.)$ is decreasing at α iff the prior exhibits median prominence*

The fundamental tradeoff that the seller faces is encapsulated in the relation between $\overline{q}_1$ and $\overline{q}_2$. Ideally, the seller would prefer higher both values to be as high as possible, as an increase in these values increase the probability of trade. However, changes in α affect $\overline{q}_1$ and $\overline{q}_2$ differently: an increase in α increases the former quantity and but may decrease the latter in equilibrium.

The explanation for the increase in $\overline{q}_1$ is straightforward. An increase in α increases the value of the outside option at the original value of $\overline{q}_1$: the value of the outside option is now strictly greater than v_1 when the state is $\overline{q}_1$, i.e., the left hand side of Equation (5) is strictly lower than the right hand side. In order to offset the increase in α in Equation (5), the *expected* value of the object must fall and so the new $\overline{q}_1$ must be larger. Therefore, an increase in α increases the probability of trade with the middle type.

Why may an increase in α cause $\overline{q}_2$ to fall? The value $\overline{q}_2$ represents the state at which the seller is indifferent between opting out and offering v_2, which in equilibrium is accepted by all types between $\overline{q}_2$ and $\overline{q}_1$. If $\overline{q}_1$ increases, the seller gets a higher payoff from trade and so is less willing to exit at $\overline{q}_2$. This inverse relation between $\overline{q}_1$ and $\overline{q}_2$ can be obtained from Equation (6). Upon manipulating Equation (6), we get

$$(\alpha v_3 - v_2)(q_3 - \overline{q}_2) = v_2(1-\alpha)(\overline{q}_1 - q_3) \tag{8}$$

Fixing α, we depict the inverse relationship between $\overline{q}_1$ and $\overline{q}_2$ in Figure 5 by the black solid line. We will refer to this line as the $\overline{q}_2, \overline{q}_1$ feasibility line, as it pins down feasible (i.e., the seller can credibly take the outside option at $\overline{q}_2$) $\overline{q}_2$ as a function of $\overline{q}_1$. An increase in α shifts the line up, depicted by the dashed black line. The exact value of $\overline{q}_1$ is determined independently of $\overline{q}_2$, depicted in Figure 5 by the vertical solid red line. An increase in α shifts the red line to the right (depicted by the dashed red line). The point of intersection of the solid(dashed) red line and the solid (dashed) black line pins down the value of $\overline{q}_2$ in equilibrium. If the upward shift in the black line is smaller than the shift in the red line, $\overline{q}_2$ drops with an increase in α[4].

We now trace the seller's indifference curves in the $\overline{q}_1$,$\overline{q}_2$ plane. The seller's ex-ante payoff is given by

$$v_3\overline{q}_2 + v_2(\overline{q}_1 - \overline{q}_2) + v_1(1 - \overline{q}_1) \tag{9}$$

Therefore, the slope of the indifference curve is given by $-\dfrac{v_2 - v_1}{v_3 - v_2}$ while the slope of the $\overline{q}_2, \overline{q}_1$ feasibility line is $-\dfrac{v_2(1-\alpha)}{\alpha v_3 - v_2}$.

Now, we examine the median prominence condition. Consider an α between $v_1/W(q_3)$ and $\dfrac{1-\beta_{13}}{2-\beta_{23}-\beta_{12}}$[5]. Rearranging $\alpha < \dfrac{1-\beta_{13}}{2-\beta_{23}-\beta_{12}}$, we get

$$-\frac{v_2 - v_1}{v_3 - v_2} > -\frac{v_2(1-\alpha)}{\alpha v_3 - v_2} \tag{10}$$

i.e., the condition implies the existence of an α that makes the slope of the $\overline{q}_2, \overline{q}_1$ feasibility line steeper than the slope of the indifference curve. This is depicted in Figure 4.

[4]This may happen when α is close to $v_1/W(q_3)$

[5]Provided $v_1/W(q_3) > v_2/v_3$

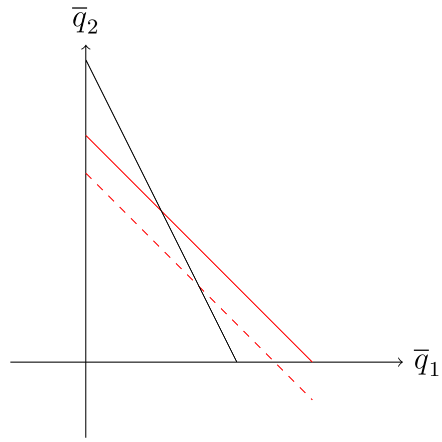

Figure 4: The red lines are indifference curves and the black line is the $\overline{q}_2, \overline{q}_1$ feasibility line. The dashed line corresponds to a higher value of α

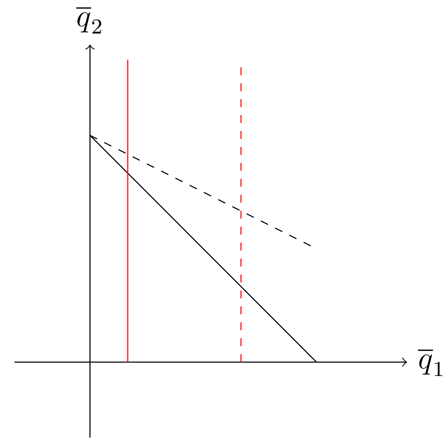

Figure 5: The black lines are $\overline{q}_2, \overline{q}_1$ feasibility lines. The red lines mark the value of $\overline{q}_1$ associated with each α. The dashed lines correspond to a higher value of α

When α is close to $v_1/W(q_3)$, both $\overline{q}_1$ and $\overline{q}_2$ are close to q_3, and so benefit to be had from an increase in α is a relatively small quantity compared to the loss the seller bears from a decrease in $\overline{q}_2$ due to an increase in $\overline{q}_1$. The slope of the $\overline{q}_2, \overline{q}_1$ feasibility line therefore is a good approximation of the effect on $\overline{q}_2$ of a marginal change in α when α is close to $v_1/W(q_3)$. When the $\overline{q}_2, \overline{q}_1$ feasibility line is steeper than the indifference curve, the increase in payoff from an increase in $\overline{q}_1$ does not compensate for the loss in payoff from a fall in $\overline{q}_2$ and so, the seller's payoff falls.

Before we proceed to the next section, we state a sufficiency condition for non-monotonicity of the seller's payoff in the general N value case:

Proposition 3. *$V(.)$ is non-monotonic in α if*

$$\frac{v_1}{W(q_N)} < \frac{v_{N-1}}{v_N}$$

8 Unobservable Investment in Outside Options

So far, we have considered the seller's optimal exit and offer strategies when she is exogenously given access to an outside option. In this section, we allow the seller to make unobservable investments (to be defined shortly) in her outside option. We show that there exists a partial investment equilibrium that does better than any equilibrium in which the seller always invests. We describe the model and strategies below. In this section we focus on the three values case.

8.1 Model Revisited

The seller makes an unobservable investment in each period to maintain her outside option. The outside option is available to the seller in any given period, if she has invested in all previous periods. If, in any period, the seller ceases to invest in the outside option, the outside option becomes unavailable to her in all future periods. The seller's investment decision is unobservable to the buyer. In the worker-firm setting, for example, this would mean that the worker chooses whether or when to (irreversibly) stop searching for other employment opportunities. Further, the firm does not observe the worker's search process. We describe the timeline below.

- Seller decides whether to continue investing in outside option, if she has invested in all previous periods.
- Seller decides whether or not to opt out.
- If she doesn't opt out, she makes an offer to the buyer.
- The buyer chooses whether or not to accept the offer

8.2 Strategies Revisited

Let o^{t-1} denote the availability of the outside option at the beginning of period t, where $o^{t-1} = 1$ if the outside option is available at the beginning of period t and zero otherwise. We denote the seller's time t private history by $\{o^s\}_{s=1}^t$. A (pure) investment strategy is a function $\sigma_i^t : \mathcal{H}^t \times \{0,1\} \to \{0,1\}$ with the constraint that $\sigma_i^t(h^t, 0) = 0$ for any time t history where the seller has ceased to invest in the past. If the seller ceases to invest in period t, then $o^t = 0$. The seller's exit strategy is given by $\sigma_e^t : \mathcal{H}^t \times \{0,1\} \to [0,1]$ where $\sigma_e^t(h^t, 0) = 0$ for any time t history at any period t. For any $h^t \in \mathcal{H}^t$, $\sigma_e^t(h^t, 1)$ denotes the probability with which the seller opts out. A pure offer strategy for the seller is a function $\sigma_p^t : \mathbf{H}^t \times \{0,1\} \to \mathbf{R}_+$.

We consider two classes of equilibria-the full investment equilibrium and the partial investment equilibrium. An equilibrium is a *full investment equilibrium* if $\sigma_i^t(h^t, 1) = 1$ for any history h^t at any time t. An equilibrium is a *partial investment equilibrium* if it is not a full investment equilibrium.

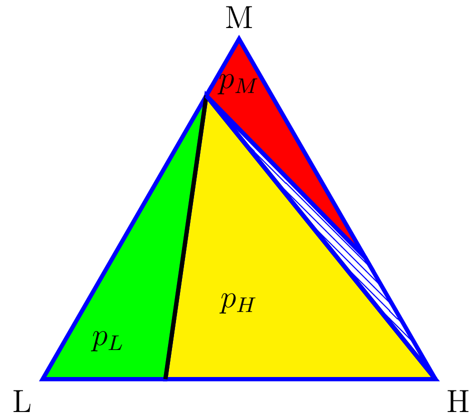

Figure 6: Partial Investment Equilibrium

8.3 A Partial Investment Equilibrium: Example

We first illustrate the partial investment equilibrium with a three price example. Suppose there are three feasible prices: $\{p_L, p_M, p_H\}$.

We now look at a partial investment equilibrium which yields a higher payoff than any retain equilibrium. Consider the following equilibrium play

- The proposer destroys the outside option with a small probability ε
- Negotiation Stage: The proposer's first offer is p_H which is accepted by all types with value H
- Exit Stage: At the beginning of the second period, the proposer exits with some positive probability. The belief that the proposer has disarmed if he doesn't exit is $\dfrac{v_2 - p_M}{\delta(v_2 - p_L)}$
- Deadlock Stage: If the seller does not exit, she makes no offer for T^* periods[6]before making an offer p_M.
- Negotiation Stage II: At the end of T^* periods p_M is offered which is accepted by all types between $\hat{q}$ and q_3
- Exit Stage II: After p_M is offered, the proposer who is able to exit does so, while the type that has destroyed its outside option offers p_L

Where T^* is the extent of delay that makes the seller indifferent between opting out and offering p_M. The above equilibrium play is supported by the following off path play

[6]We take a sequence of Δ such that T^* is an integer

- If the proposer offers p_M when some types with valuation H are yet to accept an offer, all types till $\hat{q}$ accept the offer, following which there is suitable randomization

- If the proposer deviates by offering p_M during the deadlock stage, the respondent rejects the offer. In the next period, the proposer exits if she has an outside option and offers p_L otherwise

- If at any stage p_L is offered, it is accepted by all types

The blue checked region in Figure 6 represents the region of beliefs at which a partial investment equilibrium does better.

8.4 A Partial Investment Equilibrium

We now analyze the general model. Much of the intuition from the example in the previous section carries over to the general case. However, in the general model, the equilibrium does not feature delay.

Proposition 4. *Suppose $v_1/v_2 < v_2/v_3$. Then for any α in $(v_1/v_2, v_2/v_3)$ there exist functions $q_2(\alpha)$ and $q_3(q_2, \alpha)$ s.t. for $q_2 \in (q_2(\alpha), 1)$ and $q_3 \in (q_3(q_2, \alpha), \overline{q}_1(q_2, \alpha))$, there is a partial investment equilibrium for Δ small enough, that yields a limit payoff higher than the limit payoff from any full investment equilibrium.*

The equilibrium construction can be found in the Appendix. The partial investment equilibrium does better than any full investment equilibrium for the same reason that the seller's payoff is non monotone in α.

9 Conclusion

In this paper, I study a bilateral trade setting where the buyer has private information about the value of the object and the seller has an outside option whose value depends on the value of the object. In particular, the value of the object is either high, medium or low and the value of the outside option is $\alpha < 1$ times the value of the object. I find that under some conditions, an increase in the value of α may hurt the seller. The seller's limit payoff is non monotone in the scale parameter α if and only if the prior assigns a sufficiently high probability to the middle type. I call this condition median prominence. Intuitively, when the middle type occurs with a sufficiently high probability, the seller is able to extract surplus from the middle type as α increases. This makes her unwilling to exclude the middle type from trade, compromising her ability to extract surplus from the high type.

This paper highlights a potential risk involved in improving the value of outside options in interdependent value settings. As long as the seller is unable to resume negotiations with the buyer once she has walked away, the main result suggests the existence of a force that adversely affects the seller's ability to exercise her outside option when the value of her outside option improves.

This result also has potential policy implications in labor market settings. A recent report on the state of labor market competition (Department of Treasury (2022)) cites workers' "informational disadvantage relative to firms" as a source of market power for firms that hire them. They point out that workers often do "not [know] what other, similarly placed workers earn, the competitive wages for their labor, or the existence of workplace problems like discriminatory conduct or unsafe working conditions". Further, "workers also may have a limited or no ability to switch locations and occupations quickly and may lack the financial resources to support themselves while they search for jobs that pay more and better match their skills and abilities". To summarize the concerns raised by the report, not only do firms have more information than workers, but workers may also be constrained in their ability to switch jobs. Policy makers motivated to alleviate this problem may be inclined to implement policies that improve workers' outside option (for eg., policies that effectively reduce search costs)[7]. In this paper, I argue that certain interventions targeted at improving outside options may make the party with the exit option (workers, in this case) worse off.

The non monotonicty of the seller's payoff in α is a manifestation of familiar Coasian forces in a setting with outside options: the seller's temptation to extract surplus from the buyer prevents her from breaking off negotiations. This problem is, in fact, so severe that the seller's payoff falls in spite of a decrease in the cost of exercising the outside option.

In the final section of a paper, I study a separate, but related question: what if the seller can make an unobservable investment in her outside option? Analogous to the main result of the paper, I find that the seller may be better off in an equilibrium where she under invests in her outside option. In a partial investment equilibrium, the seller leverages the buyer's skepticism about her outside option to quit early in the game. This enables her to extract rents from the high type buyer.

[7] Indeed, the report proposes measures that promote competition among firms and improve job mobility

References

Simon Board and Marek Pycia. Outside options and the failure of the coase conjecture. *American Economic Review*, 104(2):656–671, 2014.

Drew Fudenberg, David K Levine, and Jean Tirole. Incomplete information bargaining with outside opportunities. *The Quarterly Journal of Economics*, 102(1):37–50, 1987.

Faruk Gul, Hugo Sonnenschein, and Robert Wilson. Foundations of dynamic monopoly and the coase conjecture. *Journal of economic Theory*, 39(1):155–190, 1986.

Nancy L Stokey. Rational expectations and durable goods pricing. *The Bell Journal of Economics*, pages 112–128, 1981.

Francesco Nava and Pasquale Schiraldi. Differentiated durable goods monopoly: a robust coase conjecture. *American Economic Review*, 109(5):1930–1968, 2019.

Raymond Deneckere and Meng-Yu Liang. Bargaining with interdependent values. *Econometrica*, 74(5):1309–1364, 2006.

William Fuchs and Andrzej Skrzypacz. Bargaining with arrival of new traders. *American Economic Review*, 100(3):802–836, 2010.

Isaías N Chaves. Privacy in bargaining: The case of endogenous entry. *Available at SSRN 3420766*, 2019.

Brendan Daley and Brett Green. Bargaining and news. *American Economic Review*, 110 (2):428–474, 2020.

Juan Ortner. Durable goods monopoly with stochastic costs. *Theoretical Economics*, 12 (2):817–861, 2017.

Lawrence M Ausubel and Raymond J Deneckere. Reputation in bargaining and durable goods monopoly. *Econometrica: Journal of the Econometric Society*, pages 511–531, 1989.

Drew Fudenberg, David Levine, and Jean Tiróle. *Infinite Horizon models of bargaining with one-sided incomplete information*, chapter 5. Cambridge University Press, 1985.

Department of Treasury. The state of labor market competition. report, 2022. URL `https://home.treasury.gov/system/files/136/State-of-Labor-Market-Competition-2022.pdf`.

Lawrence M Ausubel and Raymond J Deneckere. A generalized theorem of the maximum. *Economic Theory*, 3(1):99–107, 1993.

Nancy L Stokey, Robert Lucas, and Edward Prescott. *Recursive methods in economic dynamics*. Harvard University Press, 1989.

10 Appendix

10.1 Existence

10.1.1 Stationary Triplet Construction

Lemma 2. *If $v_1 > \alpha W(q)$, then the seller never opts out in any PBE*

Proof. Consider any history h at which $q(h) \geq q_2$, where $q(h)$ is the cutoff at history h. First we show that in any equilibrium, type L's payoff is 0. Suppose not. Let Π denote the set of equilibrium payoffs for L when the belief cutoff is q where $q \geq q_2$ and let π denote the supremum of Π(which exists because prices are bounded below by 0 and so payoff is bounded above by v_1). For any $x > 0$ s.t. $x \in \Pi$, it must be the case that the seller trades with positive probability in some equilibrium (σ, μ) (otherwise the buyer's payoff is 0). The seller's payoff associated with this equilibrium is bounded above by $v_1 - x$ (since the total surplus is v_1 and the buyer gets x). It must then be the case that $v_1 - x \geq \alpha v_1$ (since the seller trades with positive probability). Since this must hold for all $x \in \Pi$, $v_1 - \pi \geq \alpha v_1$. Suppose the seller offers $v_1 - \delta\pi - \varepsilon$ where $\varepsilon < (1-\delta)\pi$. Since $v_1 - (v_1 - \delta\pi - \varepsilon) > \delta\pi$, the buyer accepts the offer with probability 1 in any equilibrium. Also, $v_1 - \delta\pi - \varepsilon > v_1 - \pi \geq \alpha v_1$, so having this offer accepted is better than opting out. Since $v_1 - \delta\pi - \varepsilon > v_1 - x$, the seller has a profitable deviation as the seller would rather offer $v_1 - \delta\pi - \varepsilon$ when $x \in (\delta\pi + \varepsilon, \pi]$. But then this means that for all $x \in (\delta\pi + \varepsilon, \pi]$, $x \notin \Pi$. This implies that there exists $0 < \varepsilon' < (1-\delta)\pi - \varepsilon$ s.t. for all $x \in \Pi$, $x < \pi - \varepsilon'$. This in turn implies that π cannot be the supremum of Π, a contradiction. So, type L's payoff is always zero.

Next, exercising the outside option cannot be optimal since an offer of $v_1 - \varepsilon > 0$ is always accepted with probability 1 by the buyer. For ε small enough, the seller would prefer to make this offer than exercise her outside option.

Therefore, there exists an equilibrium in which v_1 is offered and accepted by the buyer with probability 1 and this is the unique equilibrium when $q \geq q_2$.

Now suppose $q < q_2$ and $v_1(1-q) > \alpha W(q)$. If an offer is accepted by L with positive probability (not equal to 1) in equilibrium, this offer must be v_1. Owing to the skimming property, if type L accepts an offer with positive probability, type M accepts it with probability 1. This means that the next period's cutoff is greater than q_2 and the offer is v_1. This means that the offer that is accepted with positive probability is also v_1. Therefore, if an offer less than v_1 is made in equilibrium, it must be accepted by all types with probability 1. Suppose there is an equilibrium in which the seller offers $p < v_1$ and L accepts this offer with probability 1. Let p^* be the infimum of such offers. Now suppose the seller offers $\delta p^* + (1-\delta)v_1 - \varepsilon$ where $\varepsilon < (1-\delta)(v_1 - p^*)$. The buyer would accept this price with probability 1 since $v_1 - (\delta p^* + (1-\delta)v_1 - \varepsilon) = \delta(v_1 - p^*) + \varepsilon > \delta(v_1 - p^*)$[8] and the seller gets a payoff strictly higher than p^*. So any price offer between $[p^*, \delta p^* + (1-\delta)v_1 - \varepsilon)$ is not optimal. This contradicts the fact that $p^* < v_1$ is the infimum of all price offers that are accepted with probability one. Therefore, type L's payoff in equilibrium is always 0 and an offer of v_1 is always accepted in any equilibrium(the same argument above implies that the seller does not opt out). And since v_1 is always accepted by all types in equilibrium, opting out is suboptimal in any equilibrium. □

Lemma 3. *There exists $q_1 < q_2$ s.t. for all $q > q_1$, it is optimal to offer v_1 in any PBE*

Proof. First we show that the optimal commitment payoff in a setting without outside options forms an upper bound to equilibrium payoffs for the seller in this setting when q is close enough to q_2. For q close enough to q_2, $v_1 > \alpha W(q)$ and so the seller never opts out (by the previous Lemma, the seller could get a higher payoff from offering v_1). Consider an arbitrary equilibrium (σ, μ). Since the seller never opts out following any history at which the state is q, the equilibrium outcome is feasible in the commitment setting without outside options. Therefore, for q close enough to q_2, the commitment payoff is an upper bound(cite). Since the optimal mechanism is a posted price mechanism and since the optimal price is v_1, the commitment payoff can be obtained by offering v_1 (which, as we noted in the previous Lemma, is accepted with probability 1 by the buyer). In particular, the optimal offer with commitment is v_1 iff $q > \hat{q}_1$ for some $\hat{q}_1 > 0$. Let $q_1' := \max\{\hat{q}_1, \hat{q}_2\}$ where $v_1(1 - \hat{q}_2) = \alpha W(\hat{q}_2)$. For $q > q_1'$, the outside option is never invoked and the optimal offer price with commitment is p_L and so v_1 is offered in any equilibrium. □

Lemma 4. *[ADAPTED FROM DL] $P(q) = v(q)(1-\delta) + \delta v_1$ is a reservation price strategy*[9] *on $(q_1', q_2]$*

Proof. Suppose $p > P(q)$ is offered and accepted by all types $q' > q$. In the next period, the price is v_1 (since the state is atleast $q > q_1'$). So, $v(q') - p = v(q') - P(q) + [P(q) - p] =$

[8] Although p^* is the infimum of offers at any history where the belief cutoff is q, the buyer cannot (a) accept the proposed alternative with probability 1(only the offer v_1 is accepted with positive probability) or (b) reject it with probability one (the payoff in the next period is atmost $v - p^*$)

[9] see Gul et al. (1986)

$\delta(v(q')-v_1)-(1-\delta)(v(q)-v(q'))+(P(q)-p) < \delta(v(q')-v_1)$, which gives us a contradiction.

Suppose $p < P(q)$ is offered and rejected by some $r < q$. Next period price offer is atleast v_1 and so $v(q) - p > v(q) - P(q) = \delta[v(q) - v_1]$, a contradiction. $\square$

Following DL, given a left continuous, weakly decreasing function $P(.)$, we define

$$V(q) = \max_{z \geq q} P(z)\frac{z-q}{1-q} + \delta V(z)\frac{1-z}{1-q} \tag{11}$$

$$t(q) = \min \arg\max_{z \geq q} P(z)\frac{z-q}{1-q} + \delta V(z)\frac{1-z}{1-q} \tag{12}$$

We say that $(V(.), P(.), t(.))$ is a *consistent triplet* if $P(.)$ is non increasing and left continuous and given $P(.)$, $V(.)$ and $t(.)$ satisfy Equation (11) and Equation (12) respectively. We say that a stationary triplet $(V(.), P(.), t(.))$ is consistent with $P(.)$ if $(V(.), P(.), t(.))$ is a consistent stationary triplet.

We say that a consistent triplet is *generated* by a stationary equilibrium if there exists a stationary equilibrium s.t. (1) the buyer follows a reservation price strategy $P(.)$ (2) the seller's equilibrium payoff is given by $V(.)$ and (3) the induced state in the next period given today's state q is $t(q)$.

We show that given the reservation price strategy $P(.)$ on $[q_1', 1]$ there exists a stationary triplet on $[q_1', 1]$.

Lemma 5. *Given the reservation price strategy $P(q) = (1-\delta)v(q) + \delta v_1$ on $(q_1', 1]$, there exists a stationary triplet on $(q_1', 1]$ that is consistent with $P(.)$ and generated by a stationary equilibrium.*

Proof. Define

$$V_1(q) = \max_{z \in [q, q_2]} P(z)\frac{z-q}{1-q} + \delta\frac{1-z}{1-q}v_1 \tag{13}$$

and

$$t_1(q) = \arg\max_{z \in [q, q_2]} P(z)\frac{z-q}{1-q} + \delta\frac{1-z}{1-q}v_1 \tag{14}$$

and let $V(q) = v_1$ and $t(q) = 1$. We show that $(V(.), P(.), t(.))$ constitute a consistent triplet.

We first note that $V_1(q_1') < v_1$. This follows directly from the definition of q_1'. By Lemma 2, v_1 is accepted with probability one and consequently $t(q) = 1$. This means that the triplet is consistent with $P(.)$.

Next, we show that $(V(.)P(.), t(.))$ is generated by a stationary equilibrium.

Consider the following strategies for players:

- Seller:The seller offers v_1 when $q \in (q_1, 1]$
- Buyer: Buyer follows the reservation price strategy $P(.)$

We note that the above strategies constitute an equilibrium. Suppose $q > q_1$. From the fact that $V_1(q) < v_1$, it is optimal for the seller to offer v_1, if v_1 is accepted with probability one. Since $P(q) > v_1$ for $q \in [q_1, q_2]$ and $P(q) = v_1$ for $q \in [q_2, 1]$, it follows that v_1 is indeed accepted with probability one. From Lemma 4, it follows that the buyer's strategy is also optimal. Furthermore, the seller's payoff in this equilibrium is v_1 and $t(q) = 1$. Thus, the triplet is generated by a stationary equilibrium.

□

Suppose there exists a consistent stationary triplet $(V(.), P(.), t(.))$ on $(q, 1]$. Define[10]

$$G(x; q) \equiv \max_{y \geq q} P(y)\frac{y - x}{1 - x} + \delta V(y)\frac{1 - y}{1 - x} - \alpha W(x) \tag{15}$$

and

$$x(q) := \max\{0, \max\{x \leq q | G(x; q) = 0\}\} \tag{16}$$

We next show that the stationary triplet on $(q, 1]$ can be extended either (1) over the entire unit interval or (2) until some point q at which $x(q) = q$. But first, we prove a result which shall help us establish the continuity of the value function. We first state a preliminary result from Ausubel and Deneckere (1993) that shall help us prove this result.

Lemma 6. *[Theorem 2,Ausubel and Deneckere(1993)] Let X be a regular topological space and let Λ be a topological space, and $\gamma : \Lambda \to X$ be a u.h.c correspondence that is non-empty and compact valued. Suppose $f : X \times \Lambda \to \mathbf{R}$ is a u.s.c function and $\Pi : \Lambda \to \mathbf{R}$ a*

[10] If the set $\{x \leq q | G(x; q) = 0\}$ is empty,

l.h.c correspondence. Then (a) $M(\lambda) = \max_{x\in\gamma(\lambda)} f(x;\lambda)$ is a continuous function and (b) $m(\lambda) = \arg\max_{x\in\gamma(\lambda)} f(x;\lambda)$ is a non-empty and compact valued, u.h.c correspondence.

Where $\Pi(\lambda) := \{y|y \le f(x;\lambda), x \in \gamma(\lambda)\}$.

Define

$$V_1(q) = \max_{x\geq q} P(x)\frac{x-q}{1-q} + \delta V(x)\frac{1-x}{1-q} \tag{17}$$

$$\mathcal{T}_1(q) = \arg\max_{x\geq q} P(x)\frac{x-q}{1-q} + \delta V(x)\frac{1-x}{1-q} \tag{18}$$

Lemma 7. *Suppose $V(.)$ is a continuous function and $P(.)$ is left continuous and weakly decreasing. Then, $V_1(.)$ is continuous and $\mathcal{T}_1(.)$ is non empty, compact valued and a u.h.c correspondence*

Proof. We show that the conditions of Lemma 6 are satisfied. Let $f(x;q) := P(x)\frac{x-q}{1-q} + \delta V(x)\frac{1-x}{1-q}$ Define $\gamma(q) \equiv [q,1]$.

That $\gamma(q)$ is u.h.c is a standard result (see for eg., pg 59, Stokey et al. (1989)). We note that $P(.)$ is u.s.c. Let $x_n \to x$ be an increasing sequence, then $\lim_{n\to\infty} P(x_n)$ is a decreasing sequence and by left continuity of $P(.)$, we have that $\lim_{n\to\infty} P(x_n) = P(x)$. Suppose $x_n \to x$ is a decreasing sequence, then since $P(x_n)$ is weakly increasing, we have that $\lim_{n\to\infty} P(x_n) \le P(x)$. Thus $P(.)$ is u.s.c. Since $V(.)$ is continuous and $P(.)$ is u.s.c, and since $f(.,.)$ is the sum of two upper continuous functions, it is u.s.c.

Next, we show that $\Pi(.)$ is l.h.c. Suppose $y \in \Pi(x)$ and $x_n \to x$. By definition, there exists $z \in [x,1]$ s.t. $y \le f(z;x)$. Suppose $z > x$. Then there exists N s.t. $z \in [x_n,1]$ for all $n > N$. Since $f(z;.)$ is continuous, we have that $y_n \equiv f(z;x_n) \in \Pi(x_n)$[11] converges to $f(z;x) \ge y$. Similarly, if $z = x$, $f(z;x) = \delta V(x)$. Since $V(.)$ is continuous, taking the sequence $x_n \in [x_n,1]$ gives us our result.

Applying Lemma 6 gives us our result. □

We show that $\mathcal{T}_1(.)$ is an increasing correspondence, i.e., if $x > y$ and $a \in \mathcal{T}_1(y)$, then $a \le b$ for all $b \in \mathcal{T}_1(x)$.

[11] For n large enough

Lemma 8. *If $x > y$ and $a \in \mathcal{T}_1(y)$, then $a \geq b$ for all $b \in \mathcal{T}_1(x)$.*

Proof. Let $x > y$. First note that for any $a > x$,

$$f(a;y) = P(a)\frac{x-y}{1-y} + \frac{1-x}{1-y}f(a;x)$$
$$\implies f(a;y)(1-y) - f(a;x)(1-x) = P(a)(x-y)$$

And so if $x < a < b$,

$$f(a;y)(1-y) - f(a;x)(1-x) = P(a)(x-y) \geq P(b)(x-y) \tag{19}$$
$$= f(b;y)(1-y) - f(b;x)(1-x) \tag{20}$$

So, if $(f(a;x) - f(b;x))(1-x) \geq 0$, then $(f(a;y) - f(b;y))(1-y) \geq 0$, which gives us our result.

□

Given some stationary triplet $(V(.), P(.), \mathcal{T})$, if $V(.)$ is continuous and $\mathcal{T}$ is uhc, compact valued and increasing, we say that the stationary triplet is *continuous*. In the next lemma, we show that a continuous stationary triplet defined on an interval can be extended to a larger interval.

Lemma 9. *Suppose there exists a continuous stationary triplet on $[q_n, 1]$ and suppose $V(q) > \alpha W(q)$ for $q \in [q_n, 1]$. Then there exists $q_{n+1} < q_n$ s.t. a continuous stationary triplet exists on $[q_{n+1}, 1]$ with the property that $V(q) \geq \alpha W(q)$ for all $q \in [q_{n+1}, 1]$ and $V(q) > \alpha W(q)$ for $q \in (q_{n+1}, 1]$.*

Proof. Note that $G(x; q_n)$ is continuous in x and if $G(q_n; q_n) > 0$, and if for some $x < q_n$, $G(x; q_n) < 0$, then there exists $x' \in (x, q_1)$ s.t. $G(x'; q_1) = 0$. Let $\{P(.), V(.), t(.)\}$ denote a stationary triplet on $[q_n, 1]$, where $q_n > x(q_n)$. Let

$$V_1(q) = \max_{x \geq q_n} P(x)\frac{x-q}{1-q} + \delta V(x)\frac{1-x}{1-q} \tag{21}$$

$$V_2(q) = \max_{x \in (q, q_n]} P_1(x)\frac{x-q}{1-q} + \delta V_1(q)\frac{1-x}{1-q} \tag{22}$$

Where $P_1(x) = \delta P(t_1(x)) + (1-\delta)v(x)$ and $t_1(x) = \min \mathcal{T}_1(x) \equiv \arg\max_{x \geq q_n} P(x)\dfrac{x-q}{1-q} + \delta V(x)\dfrac{1-x}{1-q}$.

Next, let $q_{n+1} := \max\{q \in [x(q_n), q_n] | V_1(q) \leq V_2(q)\}$[12]. So for all $q > q_{n+1}$, $V_1(q) > V_2(q)$. It is easy to see that $q_{n+1} < q_n$. Let $(V_1(.), P_1(.), t_1(.))$ be the candidate stationary triplet on $[q_{n+1}, q_n]$. We consider two cases:

***Case I** (*$q_{n+1} > x(q_n)$*):* Let $V(.) = V_1(.)$,$P(.) = P_1(.)$ and $t(.) = t_1(.)$ on $[q_{n+1}, q_n]$. We show that $(V(.), P(.), t(.))$ is a continuous and consistent triplet.

That $(V(.), P(.), t(.))$ is continuous follows from Lemma 7. Given $P(.)$, for all $q \in (q_{n+1}, q_n]$, $V_1(q) > V_2(q)$. This follows from the definition of q_{n+1}. Consequently, $t_1(q)$ as defined above is an optimizer.

Further, note that for $q < q_{n+1}$, we must have that $V_2(q) \geq V_1(q)$. We omit the proof of this fact as it is identical to the proof in DL.

Note that since $q_{n+1} > x(q_n)$, we have that $G(q_{n+1}; q_{n+1}) > 0$, i.e., $V(q_{n+1}) > \alpha W(q_{n+1})$. By definition of $x(q)$, $G(q_{n+1}; q_n) > 0$ which implies that $G(q_{n+1}; q_{n+1}) > 0$. Consequently, $q_{n+1} > x(q_{n+1})$.

***Case II**(*$q_{n+1} = x(q_n)$*):* Note that when the buyer follows the reservation price strategy given by $P(.)$, we once again have that $V_1(q) > V_2(q)$ for $q \geq q_{n+1}$. Consequently, $V(q) = V_1(q)$ for $q > q_{n+1}$ and so, if $G(q_{n+1}; q_n) = 0$, this implies that $G(q_{n+1}; q_{n+1}) = 0$.

We now show that either (1) for some finite k, $q_k = x(q_k)$ or (2) there exists a finite k s.t. $q_k \leq 0$. Suppose (1) doesn't hold and suppose there exists a sequence $\{q_k\}_{k=1}^{\infty}$ with $q_k > x(q_k)$ and $q_k \to q^*$.

Note that

[12] If the set is empty, then set $q_{n+1} = 0$

$$
\begin{aligned}
V(q_{k+2}) &= P(t(q_{k+2}))\frac{t(q_{k+2}) - q_{k+2}}{1 - q_{k+2}} + \delta V(t(q_{k+2}))\frac{1 - t(q_{k+2})}{1 - q_{k+2}} \\
&< v_3 \frac{q_k - q_{k+2}}{1 - q_{k+2}} + \delta V(q_{k+2}) \\
&\implies V(q_{k+2})(1 - \delta) < v_3 \frac{q_k - q_{k+2}}{1 - q_{k+2}}
\end{aligned}
$$

Where the first inequality follows from the fact that $V()$ is decreasing, $P(t(q_{k+2})) < v_3$(since it is accepted with positive probability) and $q_k \geq t(q_{k+2})$. Since $V(q) > 0$ for any $q \in [0, 1]$ (since the seller could simply offer v_1), the left hand side of the inequality is strictly bounded away from zero, even as $k \to \infty$. But this contradicts the fact that q_k is convergent. Thus, we have that iterations end after $k < \infty$ rounds. Therefore, after $k < \infty$ rounds, $q_k = 0$ or $q_k = x(q_k)$.

□

Let q_Δ be such that $x(q_\Delta) = q_\Delta$. Note that the construction above is valid for any $\Delta > 0$. We now show that for Δ small enough, (1) $q_k = x(q_k)$ for $k < \infty$ and (2) it is possible to construct a stationary equilibrium when the state is less than q_Δ.

First we note that if there exists $q_k(\Delta) = 0$ and $x(q_k(\Delta)) < 0$, even as Δ goes to zero, the Coase Conjecture(see Lemma 10, GSW or DL), implies that the initial price offer (and the seller's payoff) converges to v_1[13]. However, this contradicts the fact that $\overline{q}_1 \in (0, q_2)$. And so, as Δ goes to zero, there exists $k(\Delta) < \infty$ s.t. $q_{k(\Delta)} = x(q_{k(\Delta)})$.

We have constructed a stationary equilibrium for states $x \geq q_{k(\Delta)}$(for Δ small). We now proceed to construct an equilibrium when the belief is less than $q_{k(\Delta)}$ for Δ small.

For $q \geq q_{k(\Delta)}$ and $x < q$, let

$$
F_1(x; q) := P(q)\frac{q - x}{1 - x} + \delta V(q)\frac{1 - q}{1 - x} - \alpha W(x)
$$

and $\overline{F}_1(x) = \max_{q \geq q_{k(\Delta)}} F_1(x; q)$. Let $S_1^F(\Delta) := \{x < q_{k(\Delta)} | \overline{F}_1(x) \geq 0\}$. Claim 1 shows that for Δ small $S_1^F(\Delta)$ is empty. We now proceed to construct the equilibrium.

[13] Note that by definition, under our hypothesis, $q_l(\Delta) > x(q_l(\Delta))$ and so for any $x \geq q_k(\Delta)$, $G(x; x) > 0$

If $v(q_k(\Delta)) = v_3$, then we construct an equilibrium in the following way. Let $P(q) = v_3$ for $q \leq q_{k(\Delta)}$. For $q > q_{k(\Delta)}$, the equilibrium is as constructed in the previous section.

If $v(q_{k(\Delta)}) = v_2$, then, let $P(q) = v_2$ for $q_3 < q \leq q_{k(\Delta)}$. We inductively construct an equilibrium as in the previous section.

10.1.2 Equilibrium Strategies

- Buyer's Strategies
  - Type q buyer accepts a price iff $p \leq P(q)$
- Seller's Strategies
  - If the state is anything but $\overline{q}_i(\Delta) \in (0,1)$, the seller never exits
  - If the state is $\overline{q}_i(\Delta) \in (0,1)$ and the previous period offer is p the seller exits with probability x, where[14]

$$v(\overline{q}_i(\Delta)) - p = \delta(1-x)(v(\overline{q}_i(\Delta)) - P(t'(\overline{q}_i(\Delta)))$$

    where $t'(\overline{q}_i(\Delta))$ is a member of $\arg\max \overline{F}_i(\overline{q}_i(\Delta))$.
  - If the state is q and the previous period's offer is p, the seller makes randomizes suitably between offers that induce states in $\mathcal{T}(q)$.

10.2 Limit Outcomes

We first prove that trade probability in each period is uniformly bounded below even as Δ goes to zero. The proof is similar to DL Lemma C-1.

Lemma 10. *There exists $\eta > 0$ s.t. if trade occurs with positive probability at any $q \in [0, 1-\eta)$, the probability of trade is atleast η in any Weak Markov Equilibrium*

Proof. Let $\overline{V}(q) \equiv (1-q)V(q)$. Let q be such that $t(q) < q_2$ as $\Delta \to 0$. At any such on path q, we have

$$\overline{V}(q) = P(t(q))(t(q) - q) + \delta\overline{V}(t(q))$$
$$\implies P(t(q))(t(q) - q) = \overline{V}(q) - \delta\overline{V}(t(q))$$

[14] In the generic case that $\overline{q}_i(\Delta)$ is not exactly equal to q_3

By optimality,

$$\begin{aligned} P(t(q))(t(q)-q) &= \overline{V}(q) - \delta\overline{V}(t(q)) \\ &\geq P(t^2(q))(t(q)-q) + \overline{V}(t(q))(1-\delta) \end{aligned}$$

where $t^2(q) = t(t(q))$. And so,

$$\begin{aligned} [P(t(q)) - P(t^2(q))](t(q)-q) &\geq \overline{V}(t(q))(1-\delta) \\ (1-\delta)[v(t(q)) - P(t^2(q))](t(q)-q) &\geq \overline{V}(t(q))(1-\delta) \\ (t(q)-q) &\geq \frac{\overline{V}(t(q))}{v(t(q)) - P(t^2(q))} \\ &\geq \frac{v_1(1-t(q))}{v_3 - v_1} \\ &\geq \frac{v_1(1-q_2)}{v_3 - v_1} \equiv \eta \end{aligned}$$

where the second inequality comes from the fact that $P(t(q)) = \delta P(t^2(q)) + (1-\delta)v(t(q))$, the third inequality comes from the fact that $\overline{V}(q) \geq v_1(1-q)$ for any $q \in [0,1]$ and that $v(q) \leq v_3$ and $P(q) \geq v_1$ for any $q \in [0,1]$. The final inequality comes from the fact that $t(q) \leq q_2$.

$\square$

Note that this implies that there exists $N < \infty$ s.t. trade ends in N periods even as Δ goes to zero.

As earlier, we define $G(x;q)$ and $x(q)$ for each $x \leq q$.

$$G(x;q) = \max_{y \geq q} P(y)\frac{y-x}{1-x} + \delta V(y)\frac{1-y}{1-x} - \alpha W(x)$$

$$x(q) := \max\{0, \max\{x \leq q | G(x,q) = 0\}\}$$

If $G(x,q) > 0$ for some $x \leq q$, but $G(x,q) \neq 0$ for all $x \leq q$, then set $x(q) = 0$.[15] Let $\hat{S}_\Delta(q) := [x(q), q]$ for all $q > 0$ and $\hat{S}_\Delta(0) = \{0\}$. (If $G(x,q) < 0$ for all $x \leq q$, then set

[15] Note that by continuity of $G(x;q)$ in x, if $G(x,q) > 0$ for some $x \leq q$ and $G(x';q) < 0$ for some $x' \leq q$, then there must exist some x'' between x and x' s.t. $G(x'';q) = 0$

$\hat{S}_\Delta(q) = \phi$).

Let $q_{k+1} = x(q_k)$ and $q_1 = \overline{q}_1$. First, note that for $k < \infty$, $\hat{S}_\Delta(q_k)$ is non empty. We know that $\hat{S}_\Delta(q_1) \neq \phi$(since $G(q_1, q_1) > 0$). Suppose $\hat{S}_{k-1}(\Delta)$ is non empty and $q_k = 0$. Then $q_{k'} = 0$ for all $k' \geq k$ and $\hat{S}_{k'} = \{0\}$ and so is non empty. Suppose $q_k > 0$. Then since $G(q_k, q_{k-1}) = 0$, it must be that $G(q_k, q_k) \geq 0$ and so $\hat{S}_\Delta(q_k)$ is non empty. Next, we show that if $x \in \hat{S}_\Delta(q_k)$, then $G(x, q_k) \geq 0$.

Lemma 11. *For any $k \in N$, $G(x, q_k) \geq 0$ for all $x \in \hat{S}_\Delta(q_k)$. Moreover, if $x > x(q_k)$, then $G(x, q_k) > 0$.*

Proof. If $G(q_k, q_k) = 0$ then $q_{k'} = q_k$ for all $k' \geq k$, and so we have our result. Suppose for some $k < \infty$ and $x \in \hat{S}_\Delta(q_k)$, $x < q_k$, it is the case that $G(x, q_k) < 0$. Since $x < q_k$ belongs in $\hat{S}_\Delta(q_k)$, $G(q_k, q_k) > 0$(Note that $G(q_k, q_{k-1}) = 0$, so $G(q_k, q_k) \geq 0$). Further $x \in [q_{k+1}, q_k]$, so $x > q_{k+1}$. By continuity of $G(x, q)$ in x, there must be $x' \in (x, q_k)$ s.t. $G(x', q_k) = 0$, which contradicts the definition of $x(q)$. Further, if $x > x(q_k)$, then it must be the case that, by definition of $x(.)$, $G(x, q_k) > 0$. □

Note that $q_{k+1} \leq q_k$ and the sequence is bounded below by 0. Suppose $q_k \to q$. We show that $x(q) = q$. First, note that for $x \in (q_1, q)$ that is not part of the sequence, $G(x, x) > 0$. Since $q_1 > x$ and by convergence of q_k, there exists n s.t. $q_n < x$. Let q_n be the smallest member of the sequence below x. Then $q_{n-1} > x > q_n$, i.e., $x \in \hat{S}_\Delta(q_{n-1})$. By the previous lemma, we have that $G(x, q_{n-1}) > 0$ and so $G(x, x) > 0$. We also have that $G(q, q) \geq 0$. Suppose $G(q, q) < 0$. Then for $x > q$ close enough to q, $G(x, x) < 0$, which is a contradiction. This implies that $\hat{S}_\Delta(q) \neq \phi$ as q belongs in the set. We now prove a preliminary lemma that helps us solve the fixed point result. Let $t_q(y) \equiv \min \arg\max G(y, q)$.

Lemma 12. *Suppose $q_n \downarrow q$, where $x(q_n) < q_n$ and suppose there exists $y \in \hat{S}_\Delta(q)$, $y < q$. Then there exists a subsequence q_m s.t. $y' \in \hat{S}_\Delta(q_m)$, where $y' < q$ and $y' \in \hat{S}_\Delta(q)$.*

Proof. Let $q_n \downarrow q$. Suppose there exists $y < q$ s.t. $t_q(y) > q$ and $y \in \hat{S}_\Delta(q)$. Since $y < q$ belongs in $\hat{S}_\Delta(q)$ and since the set is convex, there exists $y' < q$ close enough to q s.t. $y' \in \hat{S}_\Delta(q)$ and $t(y') > q$. Note that $G(q, q) > 0$ and $x(q) < y$. Consider $z \in (y', q)$. Since $z > y'$, it must be that $t_q(z) \geq t_q(y')$. So, for n large enough s.t. $q_n < t(y')$, $G(z, q_n) = G(z, q) > 0$ for all $z \in [y', q]$. Let M be s.t. $q_m < t(q)$ for all $m \geq M$. Let $\varepsilon > 0$ be s.t. $G(x, x) > 0$ for all $x \in (q, q + \varepsilon)$ (It is possible to find such a ε since $G(x, x)(1-x) \geq (t(q)-x)P(t(q)) + \delta(1-t(q))V(t(q)) - \alpha W(x)(1-x)$ and the term on the right hand side of the inequality is continuous in x. So, for $\varepsilon > 0$ small enough, the term on the right hand side is strictly positive as $G(q,q) > 0$). Let M' be s.t. $q_m < q + \varepsilon$ for all $m > M'$. So, for $m > M'$ and any $z \in [q, q_2]$, we have that $G(z,z) = G(z,q_2) > 0$ (since $t(q) \le t(z)$, we have that $G(z,z) = G(z,q_2)$ when $q_m < t(q)$). Let $K = \max\{M, M', N\}$. For all $m > K$, we have that $G(z, q_m) > 0$ for all $z \in [y', q_m]$. This in turn means that $y' \in \hat{S}_\Delta(q_m)$. □

Since $q_k \to q$ and since $x(q_k)$ is a decreasing sequence, for any k, $y < q$ does not belong to $\hat{S}_\Delta(q_k)$ and so cannot belong to $\hat{S}_\Delta(q)$[If it did then by the previous Lemma, it is possible to construct a subsequence of q'_k s.t. $y' \in \hat{S}_\Delta(q'_k)$ for some $y' < q$, which contradicts the fact that $x(q'_k) = q'_{k+1} > q$]. This implies that $x(q) \ge q$ which implies (by definition) that $x(q) = q$.

We show that q_Δ where $x_\Delta(q_\Delta) = q_\Delta$ (constructed in the preceding paragraphs) converges to $\overline{q}_1$ as Δ goes to zero. Let $\lim_{\Delta \to 0} q_\Delta = q^*$[16]. First note that $G_\Delta(x,x) > 0$ for any $x > q_\Delta$[17]. And so the the seller never opts out when the state is greater than q_Δ. So for any $x > q^*$, there exists Δ small enough such that the seller never opts out if the state is atleast x and continues to trade with all types. By Lemma 10 , the number of periods until trade ends is bounded above even as Δ goes to zero. Hence, q^* cannot be strictly less than $\overline{q}_1$. By construction, $q_\Delta \le \overline{q}_1$ and so q^* cannot exceed $\overline{q}_1$. Hence $q^* = \overline{q}_1$.

As in the previous section, for $q \ge q_\Delta$ and $x < q$, let

$$F_1(x;q) := P(q)\frac{q-x}{1-x} + \delta V(q)\frac{1-q}{1-x} - \alpha W(x)$$

and $\overline{F}_1(x) = \max_{q \ge q_{k(\Delta)}} F_1(x;q)$. Let $S_1^F(\Delta) := \{x < q_{k(\Delta)} | \overline{F}_1(x) \ge 0\}$.

We show that $S_1^F(\Delta)$ is empty for Δ small enough.

Claim 1. *For Δ small enough, $S_1^F(\Delta)$ is empty.*

Proof. Since q_Δ converges to $\overline{q}_1$, for Δ small enough, $|\overline{q}_1 - q_\Delta| < \eta/2$, where η is as defined in Lemma 10. Since $t_{q_\Delta}(q_\Delta) > q_\Delta + \eta$, it must be that for Δ small enough $t_{q_\Delta}(q_\Delta) > \overline{q}_1 + \eta/2 \equiv q'$. Note that by Lemma 10, as Δ goes to zero, $P(q')$ approaches v_1 and so does

[16]Take any convergent subsequence if the limit does not exist

[17]We noted earlier that $G(x,x) > 0$ if x does not belong to the sequence. If x is a member of the sequence, either $G(x,x) = 0$ in which case, $x = q_\Delta$, or $x(x) < x$, in which case $G(x,x) > 0$

$\delta P(q')+(1-\delta)v(\overline{q}_1)$. Note that given $P(.)$ in any WME, $v(x)-P(x) \geq \delta[v(x)-P(t(x)))]$[18] which implies that $P(x) \leq \delta P(t(x))+(1-\delta)v(x)$. Therefore, we have for Δ small enough s.t. $v(q_\Delta)=v(q_1)$ that[19]

$$\begin{aligned} P_{q_\Delta}(q_\Delta) &\leq \delta P(t_{q(\Delta}(q_\Delta)) + (1-\delta)v(q_\Delta) \\ &< \delta P(q') + (1-\delta)v(\overline{q}_1) \end{aligned}$$

where the inequality follows from the fact that $t_{q_\Delta}(q_\Delta) > q'$ and $P(.)$ is weakly decreasing. Therefore, for Δ small enough, $P_{q_\Delta}(q_\Delta)$ is close to v_1 and so, $\alpha v(q_1) > P_{q_\Delta}(q_\Delta)$. So, for $x < q_\Delta$[20]

$$\begin{aligned} \overline{F}_1(x) &= P(t_{q_\Delta}(x))\frac{t_{q_\Delta}(x)-x}{1-x} + \delta\frac{1-t_{q_\Delta}(x)}{1-x}V(t_{q_\Delta}(x)) \\ &= P(t_{q_\Delta}(x))\frac{q_\Delta - x}{1-x} + F_1(q_\Delta, t_{q_\Delta}(x))\frac{1-q_\Delta}{1-x} \\ &\leq P_{q_\Delta}(q_\Delta)(\frac{q_\Delta - x}{1-x}) + \alpha W(q_\Delta)\frac{1-q_\Delta}{1-x} \\ &< \alpha v(q_1)\frac{q_\Delta - x}{1-x} + \alpha W(q_\Delta)\frac{1-q_\Delta}{1-x} \\ &\leq \alpha W(x) \end{aligned}$$

where the first equality comes from rearranging terms, the second inequality comes from the fact that $F_1(q_\Delta, t_{q_\Delta}(x)) \leq \overline{F}(q_\Delta) = \alpha W(q_\Delta)$ and that $t_{q_\Delta}(x) \geq q_\Delta$ and $P(.)$ is weakly decreasing. The final inequality follows from the fact that $P_{q_\Delta}(q_\Delta) > \alpha v(q_1)$ for Δ small enough.

□

Next, we show that for Δ small enough, in any equilibrium, $P(q_\Delta) = v(q_\Delta)$.

Lemma 13. *For Δ small enough, $P(q_\Delta) = v(q_\Delta)$ in any equilibrium.*

Proof. Suppose $P(q_\Delta) < v(q_\Delta)$. Suppose the offer $p = P(q_\Delta)\delta + (1-\delta)v(q_\Delta) - \varepsilon$ is made for $\varepsilon > 0$ small s.t. $p > P(q_\Delta)$. We show that the induced state in the next period is q_Δ,

[18]Suppose x prefers to strictly reject the offer $\delta P(t(x)) + (1-\delta)v(x)$. Then, the next period state is strictly less than x, which means that the offer in the next period is atleast $P(t(x))$ and therefore type x's payoff from rejection is atmost $\delta P(t(x)) + (1-\delta)v(x)$, a contradiction.

[19]Where $P_{q_\Delta}(q_\Delta) = \lim_{q \downarrow q_\Delta} P(q)$

[20]If $t_{q_\Delta}(x) = q_\Delta$, then $P(t_{q_\Delta}(x)) = P_{q_\Delta}(q_\Delta)$

which contradicts the fact that $P(q_\Delta)$ is type q_Δ's reservation price. Since $P(q)$ is non increasing, the induced belief q' cannot be greater than q_Δ. So $q' \leq q_\Delta$.

Suppose $q_\Delta > q'$ and suppose $v(q') = v(q_\Delta)$. Rearranging $p = P(q_\Delta)\delta + (1-\delta)v(q_\Delta) - \varepsilon$, we get $v(q_\Delta) - p = \delta(v(q_\Delta) - P(q_\Delta)) + \varepsilon$. Since $v(q_\Delta) = v(q')$, we have that $v(q') - p = \delta(v(q') - P(q_\Delta)) + \varepsilon > \delta(v(q') - P(q_\Delta))$. This implies that in the next period, an offer $p'(\varepsilon)$ which is strictly less than $P(q_\Delta)$ is made with positive probability. Combined with the fact that $P(q)$ is non increasing, it must be that the induced belief is weakly greater than q_Δ. If the induced belief upon offering $p'(\varepsilon)$ is q_Δ, then it is not optimal for the seller to offer $p'(\varepsilon)$ when the state is q since offering $P(q_\Delta) > p'(\varepsilon)$ gives a higher payoff since the probability of trade is the same, but the price is higher. So, the induced belief upon offering $p'(\varepsilon)$ for any $\varepsilon > 0$ is strictly greater than q_Δ. But since $S_1^F(\Delta)$ is empty, the seller strictly prefers to opt out than offer $P(x)$ for some $x > q_\Delta$, we get a contradiction.

Suppose $q' < q_\Delta$ and $v(q') = v_3$ and $v(q_\Delta) = v_2$. Then, $v_3 - p > \delta(v_3 - P(q_\Delta))$[21]. Suppose the offer made in the following period is $p'(\varepsilon)$. Then, $\delta(v_3 - p'(\varepsilon)) = v_3 - p > \delta(v_3 - P(q_\Delta))$, which in turn implies that $p'(\varepsilon) < P(q_\Delta)$. We apply the same arguments as in the previous paragraph to obtain a contradiction. $\square$

Finite iterations of the above arguments gives us our result.

We summarize the limit ex-ante payoffs below.

- If $v_1 \geq \alpha W(0)$, then the limit payoff is v_1.
- If $\overline{q}_1 \in (0, q_3]$, then the limit payoff is $\overline{q}_1 v_3 + \alpha(1 - \overline{q}_1)W(\overline{q}_1)$.
- If $\overline{q}_1 \in (q_3, q_2)$ and $v_2\overline{q}_1 + \alpha(1 - \overline{q}_1)W(\overline{q}_1) \geq \alpha W(0)$, then the payoff is $v_2\overline{q}_1 + \alpha W(\overline{q}_1)(1 - \overline{q}_1)$.
- Finally if $\overline{q}_1 \in (q_3, q_2)$ and $\overline{q}_2 \in (0, q_3)$, then the payoff is $v_3\overline{q}_2 + \alpha W(\overline{q}_2)(1 - \overline{q}_2)$.

10.3 Theorem 1

We consider two cases. We first show that if $v_1/W(q_3) < v_2/v_3$ there exists α s.t. the seller's payoff is decreasing at α.

[21]Since $p < v_2(1-\delta) + \delta P(q_\Delta) < v_3(1-\delta) + \delta P(q_\Delta)$

Let $H((q,\alpha);(q',\alpha')) := [v_2 q+(1-q)\alpha W(q)]-[q' v_3+(1-q')\alpha' W(q')]$ where $q > q_3 > q'$. Simplifying $H((q,\alpha);(q',\alpha'))$, we get

$$H((q,\alpha);(q',\alpha')) = [v_2(q-q'+q'-q_3+q_3)+\alpha v_2(q_2-q)+(1-q_2)\alpha v_1]-$$
$$[v_3 q'+(q_3-q')\alpha' v_3+\alpha' v_2(q_2-q_3)+(1-q_2)\alpha' v_1]$$
$$= (v_2-v_3)q'+(v_2-\alpha v_3)(q_3-q')+(1-\alpha)v_2(q-q_3)+(\alpha-\alpha')[v_2(q_2-q)+v_1(1-q_2)]$$

Let $\underline{\varepsilon} = (v_3-v_2)/[(v_2-v_3\alpha+(v_3 v_1/W(q_3)))]N$ and $\overline{\varepsilon} = (v_3-v_2)/[(v_2-v_2 v_1 \alpha+(v_2/W(q_3)))]N$, where $\alpha = v_1/W(q_3)$ and N is large enough s.t. $q_3-\underline{\varepsilon} > 0$ and $q_3+\overline{\varepsilon} < q_2$. Let $\underline{\alpha} = v_1/W(\underline{q})$, where $\underline{q} = q_3-\underline{\varepsilon}$. Let $\overline{\alpha} = v_1/W(\overline{q})$ where $\overline{q} = q_3+\overline{\varepsilon}$.

Let $A = [\underline{\alpha},\overline{\alpha}]$. Note that $\overline{q}_1(a) \in [\underline{q},\overline{q}]$ for $a \in A$. Moreover, $\overline{q}_1(a)$ is increasing in a. If $a = v_1/W(x)$, then, $\overline{q}_1 = x$. Let $q_3 > q' > \max\{2/N,\underline{q}\}$. Let $\overline{q} > q > q_3$. Let $a = v_1/W(q)$ and $a' = v_1/W(q')$. Thus,

$$H((q,a);(q',a'))$$
$$= (v_2-v_3)q'+(v_2-av_3)(q_3-q')+(1-a)v_2(q-q_3)+$$
$$(a-a')[v_2(q_2-q)+v_1(1-q_2)]$$

Note that

$$(a-a')[v_2(q_2-q)+v_1(1-q_2)]$$
$$= (a-a')W(q)(1-q)$$
$$= \frac{W(q')(1-q)-W(q)(1-q')}{W(q)W(q')}W(q)v_1$$
$$< \frac{W(q')(1-q')-W(q)(1-q)}{W(q')}v_1$$
$$= \frac{v_1 v_3(q_3-q')+v_1 v_2(q-q_3)}{W(q')}$$

and so

$$
\begin{aligned}
&H((q,a);(q',a')) \\
&< (v_2 - v_3)q' + (v_2 - av_3)(q_3 - q') + (1-a)v_2(q - q_3) + \\
&\frac{v_1 v_3 (q_3 - q') + v_1 v_2 (q - q_3)}{W(q')} \\
&= (v_2 - v_3)q' + (v_2 - av_3 + (v_1 v_3)/W(q'))(q_3 - q') + ((1-a)v_2 + +(v_1 v_2)/W(q'))(q - q_3) \\
&< (v_2 - v_3)q' + (v_2 - av_3 + (v_1 v_3)/W(q'))\underline{\varepsilon} + ((1-a)v_2 + (v_1 v_2)/W(q'))\overline{\varepsilon} \\
&< (v_2 - v_3)[q' - 2/N] < 0
\end{aligned}
$$

Where the final inequality comes from the fact that $\underline{\varepsilon} < \dfrac{v_3 - v_2}{N(v_2 - av_3 + (v_3 v_1/W(q'))}$ and $\overline{\varepsilon} < \dfrac{v_3 - v_2}{N(v_2 - av_2 + (v_2 v_1/W(q'))}$ as $a > \alpha$ and $W(q') > W(q_3)$. Since $H((q,a);(q',a')) < 0$, we have our result.

Next we consider the $v_2/v_3 < v_1/W(q_3)$ case. We show that $V(.)$ is decreasing at $v_1/W(q_3)$ iff the condition holds.

First note that, where both quantities are well defined, the following relation holds between $\overline{q}_2$ and $\overline{q_1}$

$$
(q_3 - \overline{q}_2)(\alpha v_3 - v_2) = v_2(1-\alpha)(\overline{q}_1 - q_3) \tag{23}
$$

By the implicit function theorem,

$$
(q_3 - \overline{q}_2)v_3 + v_2(\overline{q}_1 - q_3) - \frac{d\overline{q}_2}{d\alpha}(\alpha v_3 - v_2) - \frac{d\overline{q}_1}{d\alpha}(v_2(1-\alpha)) = 0 \tag{24}
$$

which implies that

$$
\frac{d\overline{q}_2}{d\alpha} \geq -\frac{d\overline{q}_1}{d\alpha}\frac{v_2(1-\alpha)}{(\alpha v_3 - v_2)} \tag{25}
$$

and so

$$
(\frac{d\overline{q}_2}{d\alpha})/(\frac{d\overline{q}_1}{d\alpha}) \geq -\frac{v_2(1-\alpha)}{(\alpha v_3 - v_2)} \tag{26}
$$

Next, consider the ex-ante payoff of the seller

$$v_3\overline{q}_2 + (\overline{q}_1 - \overline{q}_2)v_2 + v_1(1 - \overline{q}_1) \tag{27}$$

Differentiating Equation (27), we get

$$(v_3 - v_2)\frac{d\overline{q}_2}{d\alpha} + (v_2 - v_1)\frac{d\overline{q}_1}{d\alpha} \tag{28}$$

Equation (28) is less than zero if

$$(\frac{d\overline{q}_2}{d\alpha})/(\frac{d\overline{q}_1}{d\alpha}) < -\frac{v_2 - v_1}{v_3 - v_2} \tag{29}$$

A necessary condition for this to hold is

$$-\frac{v_2 - v_1}{v_3 - v_2} > -\frac{v_2(1-\alpha)}{(\alpha v_3 - v_2)} \tag{30}$$

which implies that

$$\alpha < \frac{1 - \beta_{13}}{2 - \beta_{12} - \beta_{23}}$$

Since $\alpha > v_1/W(q_3)$, we have our result.

Suppose $v_1/W(q_3) < \frac{1 - \beta_{13}}{2 - \beta_{12} - \beta_{23}}$. This implies that Equation (30) is satisfied. We show that it is possible to find an α s.t. $v_1/W(q_3) < \alpha < \frac{1 - \beta_{13}}{2 - \beta_{12} - \beta_{23}}$ at which the seller's payoff is decreasing. From Equation (23) and Equation (24), we have

$$v_2[1 + \frac{v_3(1-\alpha)}{\alpha v_3 - v_2}](\overline{q}_1 - q_3) - \frac{d\overline{q}_2}{d\alpha}(\alpha v_3 - v_2) - \frac{d\overline{q}_1}{d\alpha}(v_2(1-\alpha)) = 0 \tag{31}$$

Let $\varepsilon' = 1/N(v_2[1 + \frac{v_3(1-\alpha)}{\alpha v_3 - v_2}])$ where $\alpha = v_1/W(q_3)$. When α is $v_1/W(q')$ where $q' = q_3 + \varepsilon'$, and for N large enough, we have that Equation (29) holds.

10.4 Partial Investment Equilibrium

Let $q_2(\alpha) \equiv (1-\alpha)v_1/\alpha(v_2-v_1)$, $\overline{q}_1(q_2,\alpha) = \overline{q}_1$, $q_3(q_2,\alpha) = \max\{\overline{q}_1 - \frac{v_1(1-q_2)}{v_3-v_1}, \frac{\overline{q}_1(q_2,\alpha)(v_2(1-\alpha))}{v_3-\alpha v_2}\}$, $\underline{\alpha} = v_1/v_2$ and $\overline{\alpha} = v_2/v_3$.

Let $v_1/v_2 < \alpha < v_2/v_3$. We verify that $\overline{q}_1 \in (q_3, q_2)$. First, $q_2 > (1-\alpha)v_1/\alpha(v_2-v_1)$ implies that $\overline{q}_1$ is between zero and q_2. Next, note that $\overline{q}_1 > q_3(q_2,\alpha)$ since $v_2(1-\alpha) < v_3 - \alpha v_2$ and $\overline{q}_1 > 0$. So it is possible to choose a q_3 in the relevant range, i.e., $\overline{q}_1(q_2,\alpha) > q_3$. So, $\overline{q}_1 \in (q_3, q_2)$. Next, since $\alpha v_3 < v_2$, $\overline{q}_1 v_2 + \alpha W(\overline{q}_1) > q_3 \alpha v_3 + (\overline{q}_1 - q_3)\alpha v_2 + \alpha W(\overline{q}_1) = \alpha W(0)$. So the limit payoff in any retain equilibrium is $v_2 \overline{q}_1 + \alpha W(\overline{q}_1)$.

We construct a partial investment equilibrium (for Δ small enough) in which the limit payoff is $v_3 q_3 + \alpha W(q_3)$. We first verify that this payoff is higher than any full investment equilibrium payoff. We see that the partial investment payoff is higher iff $\overline{q}_1 < \frac{(v_3-\alpha v_2)q_3}{v_2(1-\alpha)}$, i.e., $q_3 > \frac{\overline{q}_1(q_2,\alpha)(v_2(1-\alpha))}{v_3-\alpha v_2}$.

Let Δ be small enough that $S_1^F(q_\Delta) = \phi$ (where $S_1^F(.)$ is as defined in the existence section), $v_2(\hat{q}_\Delta - q_3) + \delta W(\hat{q}_\Delta) - \alpha W(q_3) > 0$ and $q_\Delta - q_3 < \frac{v_1(1-q_2)}{v_3-v_1}$ (this is possible since $\overline{q}_1 - q_3 < \frac{v_1(1-q_2)}{v_3-v_1}$). Since the reservation price schedule is flat between cutoffs and since $t(q) - q$ is atleast $\frac{v_1(1-q_2)}{v_3-v_1}$, this ensures that the price offer is the same at q_Δ and q_3 in the absence of outside options. Let μ_Δ be s.t. $(v_2 - \delta(1-\mu_\Delta)(v_2 - P(t(\hat{q}_\Delta))))(\hat{q}_\Delta - q_3) + \delta W(\hat{q}_\Delta) - \alpha W(q_3) = 0$. Existence of such a μ_Δ is guaranteed by the fact that the lhs is strictly negative for $\mu = 0$, strictly positive for $\mu = 1$ and the lhs is continuous in μ. Consider the following on-path play

- In the first period, the seller discards with a small probability ε less than one
- She then makes the offer $p(0) = v_3 - \delta(1-\varepsilon)(v_3 - p(q_3))$, where $p(q_3)$ is the offer made at q_3 (to be described shortly). This offer is accepted by all types till q_3
- In the second period, she exits with probability $1 - \mu_\Delta \varepsilon/(1-\mu_\Delta)(1-\varepsilon)$. If she doesn't exit, she makes an offer $p(q_3) = v_2 - \delta(1-\mu_\Delta)(v_2 - P(t(\hat{q}_\Delta)))$. This offer is accepted by all types till $\hat{q}_\Delta$
- In the third period, the seller exits with probability 1 if she has the outside option and continues to bargain otherwise.

The above on path play is supported by the following off path behavior. Let q_1 be such that

$$p(0)(q_3 - q_1) + \delta W(q_3) = [v_2 - \delta^2(1-\mu_\Delta)(v_2 - P(t(\hat{q}_\Delta)))](\hat{q}_\Delta - q_1) + \delta W(\hat{q}_\Delta) \qquad (32)$$

Note that as $\delta \to 0$, $v_2 - \delta^2(1-\mu_\Delta)(v_2 - P(t(\hat{q}_\Delta)))$ goes to $\alpha v_2 < \lim_{\Delta \to 0} p(0)$, and so q_1 goes to q_3.

- At any stage if an offer of $P(q')$ is made for $q' > \overline{q}_\Delta$ all types till q' accept the offer and play proceeds as prescribed by the Coasian equilibrium.
- If the belief state is q_3, $\mu = \mu_\Delta$ and the offer is between $p(q_3)$ and $\lim_{q \downarrow \hat{q}_\Delta} P(q)$, all types till $\hat{q}_\Delta$ accept the offer, following which there is suitable randomization (both in opting out behavior and next period offers).
- If $q < q_3$, and the offer is greater than $v_3(1-\delta) + p(0)\delta$, it is rejected and the offer $p(0)$ is made in the next period
- If $q < q_3$ and an offer between $v_3(1-\delta) + p(0)\delta$ and $p(0)$ is made, all types till q_1 accept the offer, following which there is suitable randomization.
- If the offer is between $p(0)$ and $v_2 - \delta^2(1-\varepsilon)(v_2 - P(t(\hat{q}_\Delta)))$ all types till q_3 accept the offer, following which play proceeds as on path
- For offers between $v_2 - \delta^2(1-\varepsilon)(v_2 - P(t(q_\Delta)))$ and $v_2 - \delta^2(1-\mu_\Delta)(v_2 - P(t(q_\Delta)))$ all types till q_3 accept the offer following which the seller discards with positive probability and then exits in a way that makes value M buyers indifferent between accepting and rejecting. The belief upon staying in the game at q_3 is μ_Δ and play proceeds as on path
- For any offer between $v_2 - \delta^2(1-\varepsilon)(v_2 - P(t(\hat{q}_\Delta)))$ and $\lim_{q\downarrow} \hat{q}_\Delta P(q)$, all types till q_Δ accept the offer, following which there is suitable randomization (with discarding if necessary).