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Hands-On
Session-4

Python Based Statistics and its Application in Astrophysics & Cosmology

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Key Points from Previous Lecture

Revise:

- Observational dataset: x_i, $y_{\rm obs}(x_i)$, σ_{y_i}
- Theoretical model: $y_{\rm th}(x_i; a, b) = f(x_i; a, b)$
- Define Chi-square: $\chi^2 = \sum_{i=1}^{N} \left(\frac{y_{\rm obs}(x_i) - y_{\rm th}(x_i; a, b)}{\sigma_{y_i}} \right)^2$
- $\chi^2_{\rm min} \Rightarrow$ *Best fit value of parameters*
- Draw Confidence Level: $\chi^2_{n\sigma} = \chi^2_{\rm min} + \Delta\chi^2_{n\sigma}$
- Estimate error in parameters: $a = a_{\rm best} \pm \sigma_a$ and $b = b_{\rm best} \pm \sigma_b$

Bayesian Statistics

Bayes Theorem:

$$\mathbb{P}(A|B,C) = \frac{\mathbb{P}(B|A,C)\mathbb{P}(A|C)}{\mathbb{P}(B|C)}$$

We set:

- $A \leftarrow \Theta$: the parameters of a physical model,
- $B \leftarrow \mathcal{D}$: the experimental data,
- $C \leftarrow \mathcal{M}$: the physical model that includes all assumptions made in an analysis,

$$\therefore \quad \mathbb{P}(\Theta|\mathcal{D},\mathcal{M}) = \frac{\mathbb{P}(\mathcal{D}|\Theta,\mathcal{M})\mathbb{P}(\Theta|\mathcal{M})}{\mathbb{P}(\mathcal{D}|\mathcal{M})}$$

Bayesian Statistics

Bayes Theorem:
$$\mathbb{P}(\Theta|\mathcal{D},\mathcal{M}) = \frac{\mathbb{P}(\mathcal{D}|\Theta,\mathcal{M})\mathbb{P}(\Theta|\mathcal{M})}{\mathbb{P}(\mathcal{D}|\mathcal{M})}$$

- Posterior probability distribution: $\mathbb{P}(\Theta|\mathcal{D},\mathcal{M})$
- Prior probability distribution: $\mathbb{P}(\Theta|\mathcal{M})$
- Likelihood function: $\mathbb{P}(\mathcal{D}|\Theta,\mathcal{M})$
- Model evidence: $\mathbb{P}(\mathcal{D}|\mathcal{M}) = \int \mathbb{P}(\mathcal{D}|\Theta,\mathcal{M})\mathbb{P}(\Theta|\mathcal{M})d\Theta$

Schematically,

$$\boxed{\text{posterior} = \frac{\text{likelihood} \times \text{prior}}{\text{evidence}}}.$$

... updating our degree of belief when new information, in the form of data, becomes available

Markov Chain Monte Carlo:

Bayesian Statistics

$$\mathbb{P}(\Theta|\mathcal{D},\mathcal{M}) = \frac{\mathbb{P}(\mathcal{D}|\Theta,\mathcal{M})\mathbb{P}(\Theta|\mathcal{M})}{\mathbb{P}(\mathcal{D}|\mathcal{M})}$$

Model evidence: Integral of the likelihood and the prior over all possible parameters

$$\mathbb{P}(\mathcal{D}|\mathcal{M}) = \int \mathbb{P}(\mathcal{D}|\Theta,\mathcal{M})\mathbb{P}(\Theta|\mathcal{M})d\Theta$$

$\hookrightarrow$ multidimensional integral: extremely difficult to solve

Metropolis-Hastings Algorithm

Steps of the Metropolis-Hastings algorithm:

- Initial Seeds: Choose an initial value for the Markov chain state, x_i.
- Prior Range: Set prior range for each parameters.
- Likelihood Function: Likelihood type depends open the observational datasets that we have collected before the experiments $\rightarrow$ collected under Gaussian distribution.
- Posterior distribution: proportional to prior$\times$likelihood
- Proposal Function: Propose a new position $x_i \rightarrow x^*$ for the Markov chain from the proposal distribution $\mathbb{Q}(x^*|x_i) \sim \mathcal{N}[x_i, \sigma^2]$

Metropolis-Hastings Algorithm

Steps of the Metropolis-Hastings algorithm:

- Acceptance Probability: $\alpha = \dfrac{\mathbb{F}(x^*)}{\mathbb{F}(x_i)}$
  - If $\alpha > 1$; accept proposed state and set $x_{i+1} \leftarrow x^*$
  - If $\alpha < 1$; draw uniform random number $u = \mathcal{U}[0,1]$
    - If $\alpha > u$; accept proposed state and set $x_{i+1} \leftarrow x^*$
    - If $\alpha < u$; reject proposed state and set $x_{i+1} \leftarrow x_i$
  - Increment $i = i + 1$ and repeat this process.

Metropolis-Hastings Algorithm

Steps of the Metropolis-Hastings algorithm:

- Burn-in Phase Period: *discard an initial portion of a Markov chain sample.→which is not relevant and not close to the converging phase.*