Controls Final project

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0.1 Introduction

The project simulates the behaviour of a set of pendulums with mass m_1 and m_2 attached at the end of two cables of lengths l_1 and l_2. This two pendulums are attached with a move-able cart of mass M which can translate on a one dimensional track. The surface on which the cart moves is assumed to be friction less. The following sections will describe the dynamical equations of cart with a double pendulum system. The next section will than use these derived equations and linearize the system around a stable point. After which in the next section the system is checked for controllability.

0.2 Part A: The equations of motion for the system and non-linear state space representation

For finding the equations of motion the Euler Lagrange approach will be followed where the total energy stored in the system will be taken out after which the set of second order equations is derivated with respect to the system parameters like cart movement x, angular displacement θ_1 and θ_2.

We will first find the Potential energy of the system: To find the potential energy of the pendulums the height of the pendulums is required. Let height h_1 be represented for pendulum m_1:

Then h_1 is given by:

$$h_1 = -l_1 cos(\theta_1) \tag{1}$$

Similarly for mass m_2, the height h_2 is given by:

$$h_2 = -l_2 cos(\theta_2) \tag{2}$$

The general form of Potential energy **E** of the system is written as:

$$\mathbf{E} = m_1 g h_1 + m_2 g h_2 \tag{3}$$

Substituting h_1 and h_2 from (1) and (2)

$$\mathbf{E} = -m_1 g l_1 cos(\theta_1) - m_2 g l_2 cos(\theta_2) \tag{4}$$

For finding the Kinetic energy **K** of the system, we need to have the instantaneous velocities of pendulums. Note that the instantaneous velocity of cart is given by $\dot{x}$.

The displacement vector $r_1(t)$ for mass m_1 is given by:

$$r_1(t) = [x(t) - l_1 sin(\theta_1(t))]\hat{i} + l_1 cos(\theta_1)\hat{j} \tag{5}$$

The instantaneous velocity for mass m_1 is found by differentiating (5) with respect to time:

$$\dot{r_1}(t) = [\dot{x}(t) - \dot{\theta}_1 l_1 cos(\theta_1)]\hat{i} - \dot{\theta}_1 l_1 sin(\theta_1)\hat{j} \tag{6}$$

Similarly the displacement vector and velocity vector for mass m_2 is given by r_2 as shown below:

$$r_2(t) = [x(t) - l_2 sin(\theta_2(t))]\hat{i} + l_2 cos(\theta_2)\hat{j} \tag{7}$$

The instantaneous velocity for mass m_2 is found by differentiating equation (5) with respect to time:

$$\dot{r_2}(t) = [\dot{x}(t) - \dot{\theta}_2 l_1 cos(\theta_2)]\hat{i} - \dot{\theta}_2 l_2 sin(\theta_2)\hat{j} \tag{8}$$

Now the Kinetic Energy **K** can be written as:

$$\mathbf{K} = \frac{1}{2} m_1 \dot{r}_1^2 + \frac{1}{2} m_2 \dot{r}_2^2 \tag{9}$$

Where $\dot{r}_1^2$ and $\dot{r}_2^2$ are dot product of the velocity vectors derived earlier and those are given as below:

$$\dot{r}_1^2 = (\dot{x}(t) - \dot{\theta}_1 l_1 cos(\theta_1))^2 + l_1^2 \dot{\theta}_1^2 sin^2(\theta_1) \tag{10}$$

$$\dot{r}_2^2 = (\dot{x}(t) - \dot{\theta}_2 l_2 cos(\theta_2))^2 + l_2^2 \dot{\theta}_2^2 sin^2(\theta_2) \tag{11}$$

Simplifying the above equation (10):

$$\dot{r}_1^2 = \dot{x}^2 - 2\dot{x}l_1\dot{\theta}_1 cos\theta_1 + l_1^2\dot{\theta}_1^2 \tag{12}$$

Simplifying the above equation (11):

$$\dot{r}_2^2 = \dot{x}^2 - 2\dot{x}l_2\dot{\theta}_2 cos\theta_2 + l_2^2\dot{\theta}_2^2 \tag{13}$$

The Kinetic energy after substituting equations (12) and (13) in equation (9) can be written as:

$$\mathbf{K} = \frac{1}{2}(M + m_1 + m_2)\dot{x}^2 + \frac{1}{2}(m_1 l_1\dot{\theta}_1^2 + m_2 l_2\dot{\theta}_2^2) - \dot{x}(m_1 l_1\dot{\theta}_1 cos\theta_1 + m_2 l_2\dot{\theta}_2 cos\theta_2) \tag{14}$$

The Lagrangian $\mathbf{L}$ is given as:

$$\mathbf{L} = \mathbf{K} - \mathbf{E} \tag{15}$$

Substituting terms of $\mathbf{K}$ and $\mathbf{E}$ from equation (4) and (14) in equation (15) of the Lagrangian gives:

$$\mathbf{L} = \frac{1}{2}(M + m_1 + m_2)\dot{x}^2 + \frac{1}{2}(m_1 l_1\dot{\theta}_1^2 + m_2 l_2\dot{\theta}_2^2) + m_1 l_1 cos\theta_1(g - \dot{\theta}_1\dot{x}) + m_2 l_2 cos\theta_2(g - \dot{\theta}_2\dot{x}) \tag{16}$$

Next step is to find the Euler Lagrange equation of motion for parameters x, θ_1 and θ_2, This will give the force $\mathbf{F}$ on the system, and torques τ_1 and τ_2 for pendulums m_1 and m_2

The Euler Lagrange equation is given by:

$$\frac{d}{dt}\frac{\partial \mathbf{L}}{\partial \dot{q}} - \frac{\partial \mathbf{L}}{\partial q} = \mathbf{Q} \tag{17}$$

Where,

$$q = x, \theta_1, \theta_2$$

and

$$\mathbf{Q} = \mathbf{F}, \tau_1, \tau_2$$

To find the Force $\mathbf{F}$, the Lagrangian equation is written as:

$$\frac{d}{dt}\frac{\partial \mathbf{L}}{\partial \dot{x}} - \frac{\partial \mathbf{L}}{\partial x} = \mathbf{F} \tag{18}$$

Solving for the term $\frac{\partial \mathbf{L}}{\partial x}$, we find that:

$$\frac{\partial \mathbf{L}}{\partial x} = 0 \tag{19}$$

Since there are no terms of x in equation 16 the differentiation would result in a zero.

Solving for the term $\frac{\partial \mathbf{L}}{\partial \dot{x}}$,

$$\frac{\partial \mathbf{L}}{\partial \dot{x}} = \dot{x}(M + m_1 + m_2) - m_1 l_1 \dot{\theta}_1 cos\theta_1 - m_2 l_2 \dot{\theta}_2 cos\theta_2 \tag{20}$$

Differentiating equation (20) with respect to time,

$$\frac{d}{dt}\frac{\partial \mathbf{L}}{\partial \dot{x}} = (M + m_1 + m_2)\ddot{x} + (\dot{\theta}_1^2 m_1 l_1 sin\theta_1 - \ddot{\theta}_1 m_1 l_1 cos\theta_1) + (\dot{\theta}_2^2 m_2 l_2 sin\theta_2 - \ddot{\theta}_2 m_2 l_2 cos\theta_2) \tag{21}$$

The force in equation (18) is given by substituting equations (19) and (21) as:

$$\mathbf{F} = (M + m_1 + m_2)\ddot{x} + (\dot{\theta}_1^2 m_1 l_1 sin\theta_1 - \ddot{\theta}_1 m_1 l_1 cos\theta_1) + (\dot{\theta}_2^2 m_2 l_2 sin\theta_2 - \ddot{\theta}_2 m_2 l_2 cos\theta_2) \tag{22}$$

To find the Torque τ_1, the Lagrangian equation is written as:

$$\frac{d}{dt}\frac{\partial \mathbf{L}}{\partial \dot{\theta}_1} - \frac{\partial \mathbf{L}}{\partial \theta_1} = \tau_1 \tag{23}$$

Solving for the term $\frac{\partial \mathbf{L}}{\partial \theta_1}$,

$$\frac{\partial \mathbf{L}}{\partial \theta_1} = -m_1 l_1 g sin\theta_1 + m_1 l_1 \dot{\theta}_1 \dot{x} sin\theta_1 \tag{24}$$

Solving for the term $\frac{\partial \mathbf{L}}{\partial \dot{\theta}_1}$,

$$\frac{\partial \mathbf{L}}{\partial \dot{\theta}_1} = m_1 l_1^2 \dot{\theta}_1 - m_1 l_1 \dot{x} cos\theta_1 \tag{25}$$

Taking time derivative of equation (25),

$$\frac{d}{dt}\frac{\partial \mathbf{L}}{\partial \dot{\theta}_1} = m_1 l_1^2 \ddot{\theta}_1 - m_1 l_1 \ddot{x} cos\theta_1 + m_1 l_1 \dot{x} \dot{\theta}_1 sin\theta_1 \tag{26}$$

Substituting equations (24) and (26) in equation (23) to find the torque τ_1,

$$m_1 l_1^2 \ddot{\theta}_1 - m_1 l_1 \ddot{x} cos\theta_1 + m_1 l_1 \dot{x} \dot{\theta}_1 sin\theta_1 - (-m_1 l_1 g sin\theta_1 + m_1 l_1 \dot{\theta}_1 \dot{x} sin\theta_1) = \tau_1 \tag{27}$$

Simplifying the above equation we get,

$$\tau_1 = m_1 l_1^2 \ddot{\theta}_1 + m_1 l_1 g sin\theta_1 - m_1 l_1 \ddot{x} cos\theta_1 \tag{28}$$

To find the Torque τ_2, the Lagrangian equation is written as:

$$\frac{d}{dt}\frac{\partial \mathbf{L}}{\partial \dot{\theta}_2} - \frac{\partial \mathbf{L}}{\partial \theta_2} = \tau_2 \tag{29}$$

Solving for the term $\frac{\partial \mathbf{L}}{\partial \theta_2}$,

$$\frac{\partial \mathbf{L}}{\partial \theta_2} = -m_2 l_2 g sin\theta_2 + m_2 l_2 \dot{\theta}_2 \dot{x} sin\theta_2 \tag{30}$$

Solving for the term $\frac{\partial \mathbf{L}}{\partial \dot{\theta}_2}$,

$$\frac{\partial \mathbf{L}}{\partial \dot{\theta}_2} = m_2 l_2^2 \dot{\theta}_2 - m_2 l_2 \dot{x} cos\theta_2 \tag{31}$$

Taking time derivative of equation (31),

$$\frac{d}{dt}\frac{\partial \mathbf{L}}{\partial \dot{\theta}_2} = m_2 l_2^2 \ddot{\theta}_2 - m_2 l_2 \ddot{x} cos\theta_2 + m_2 l_2 \dot{x} \dot{\theta}_2 sin\theta_2 \tag{32}$$

Substituting equations (30) and (32) in equation (29) to find the torque τ_2,

$$m_2 l_2^2 \ddot{\theta}_2 - m_2 l_2 \ddot{x} cos\theta_2 + m_2 l_2 \dot{x} \dot{\theta}_2 sin\theta_2 - (-m_2 l_2 g sin\theta_2 + m_2 l_2 \dot{\theta}_2 \dot{x} sin\theta_2) = \tau_2 \tag{33}$$

Simplifying the above equation we get,

$$\tau_2 = m_2 l_2^2 \ddot{\theta}_2 + m_2 l_2 g sin\theta_2 - m_2 l_2 \ddot{x} cos\theta_2 \tag{34}$$

Since the torques τ_1 and τ_2 are acting around the point of suspension of the cables and since the tension of the cables l_1 and l_2 passes through the point of suspension, the torques will be equal to zero,

$$\tau_1 = \tau_2 = 0$$

Applying the above stated conditions in equations (28) and (34), and rewriting those equations below:

$$m_1 l_1^2 \ddot{\theta}_1 + m_1 l_1 g sin\theta_1 - m_1 l_1 \ddot{x} cos\theta_1 = 0 \tag{35}$$

$$m_2 l_2^2 \ddot{\theta}_2 + m_2 l_2 g sin\theta_2 - m_2 l_2 \ddot{x} cos\theta_2 = 0 \tag{36}$$

Rewriting the above equations,

$$m_1 l_1 \ddot{\theta}_1 = -m_1 (g sin\theta_1 - \ddot{x} cos\theta_1) \tag{37}$$

$$m_2 l_2 \ddot{\theta}_2 = -m_2 (g sin\theta_2 - \ddot{x} cos\theta_2) \tag{38}$$

Substituting equation (37) & (38) in equation (22) to find the Force and subsequently $\ddot{x}$ in terms of Force $\mathbf{F}$,

$$\mathbf{F} = (M + m_1 + m_2)\ddot{x} - cos\theta_1 (m_1 \ddot{x} cos\theta_1 - m_1 g sin\theta_1) + \dot{\theta}_1^2 m_1 l_1 sin\theta_1 - cos\theta_2 (m_2 \ddot{x} cos\theta_2 - m_2 g sin\theta_2) + \dot{\theta}_2^2 m_2 l_2 sin\theta_2$$

Rearranging the terms in the above equation,

$$\mathbf{F} = (M + m_1 + m_2 - m_1 cos^2\theta_1 - m_2 cos^2\theta_2)\ddot{x} + m_1 g sin\theta_1 cos\theta_1 + \dot{\theta}_1^2 m_1 l_1 sin\theta_1 + m_2 g sin\theta_2 cos\theta_2 + \dot{\theta}_2^2 m_2 l_2 sin\theta_2$$

Simplifying the above equation,

$$\mathbf{F} = (M + m_1 sin^2\theta_1 + m_2 sin^2\theta_2)\ddot{x} + m_1 g sin\theta_1 cos\theta_1 + \dot{\theta}_1^2 m_1 l_1 sin\theta_1 + m_2 g sin\theta_2 cos\theta_2 + \dot{\theta}_2^2 m_2 l_2 sin\theta_2 \tag{39}$$

Rearranging the terms, Rewriting equation 39 to find $\ddot{x}$,

$$\ddot{x} = \frac{\mathbf{F} - m_1 g sin\theta_1 cos\theta_1 - \dot{\theta}_1^2 m_1 l_1 sin\theta_1 - m_2 g sin\theta_2 cos\theta_2 - \dot{\theta}_2^2 m_2 l_2 sin\theta_2}{M + m_1 sin^2\theta_1 + m_2 sin^2\theta_2} \tag{40}$$

Now rewriting the equations (37) and (38), Since m_1 and m_2 can be canceled from each side and representing the whole equation in terms of $\ddot{\theta}_1$ and $\ddot{\theta}_2$,

$$\ddot{\theta}_1 = -(\frac{g}{l_1} sin\theta_1 - \frac{\ddot{x}}{l_1} cos\theta_1) \tag{41}$$

$$\ddot{\theta}_2 = -(\frac{g}{l_2} sin\theta_2 - \frac{\ddot{x}}{l_2} cos\theta_2) \tag{42}$$

Substituting result of $\ddot{x}$ equation (40) in equation (41) and (42) and simplifying,

$$\ddot{\theta}_1 = (\frac{\mathbf{F} - m_1 g sin\theta_1 cos\theta_1 - \dot{\theta}_1^2 m_1 l_1 sin\theta_1 - m_2 g sin\theta_2 cos\theta_2 - \dot{\theta}_2^2 m_2 l_2 sin\theta_2}{M + m_1 sin^2\theta_1 + m_2 sin^2\theta_2})\frac{cos\theta_1}{l_1} - \frac{g}{l_1} sin\theta_1 \tag{43}$$

$$\ddot{\theta}_2 = (\frac{\mathbf{F} - m_1 g sin\theta_1 cos\theta_1 - \dot{\theta}_1^2 m_1 l_1 sin\theta_1 - m_2 g sin\theta_2 cos\theta_2 - \dot{\theta}_2^2 m_2 l_2 sin\theta_2}{M + m_1 sin^2\theta_1 + m_2 sin^2\theta_2})\frac{cos\theta_2}{l_2} - \frac{g}{l_2} sin\theta_2 \tag{44}$$

The standard state space representation of the system is written as below:

$$\dot{\vec{\mathbf{X}}} = A\vec{\mathbf{X}} + B\vec{\mathbf{U}} \tag{45}$$

In our case the state vector of the system $\vec{\mathbf{X}}$ can be written as following:

$$\vec{\mathbf{X}} = \begin{bmatrix} x \\ \dot{x} \\ \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix} \tag{46}$$

The time derivative of state vector $\vec{\mathbf{X}}$ gives the complete state of the system, Upon substituting the expressions from Equations (40), (43) and (44) it is observed that the system is non-linear in nature and is given by the following equations,

$$\dot{\vec{\mathbf{X}}} = \begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \frac{\mathbf{F} - m_1 g sin\theta_1 cos\theta_1 - \dot{\theta}_1^2 m_1 l_1 sin\theta_1 - m_2 g sin\theta_2 cos\theta_2 - \dot{\theta}_2^2 m_2 l_2 sin\theta_2}{M + m_1 sin^2\theta_1 + m_2 sin^2\theta_2} \\ \dot{\theta}_1 \\ (\frac{\mathbf{F} - m_1 g sin\theta_1 cos\theta_1 - \dot{\theta}_1^2 m_1 l_1 sin\theta_1 - m_2 g sin\theta_2 cos\theta_2 - \dot{\theta}_2^2 m_2 l_2 sin\theta_2}{M + m_1 sin^2\theta_1 + m_2 sin^2\theta_2}) \frac{cos\theta_1}{l_1} - \frac{g}{l_1} sin\theta_1 \\ \dot{\theta}_2 \\ (\frac{\mathbf{F} - m_1 g sin\theta_1 cos\theta_1 - \dot{\theta}_1^2 m_1 l_1 sin\theta_1 - m_2 g sin\theta_2 cos\theta_2 - \dot{\theta}_2^2 m_2 l_2 sin\theta_2}{M + m_1 sin^2\theta_1 + m_2 sin^2\theta_2}) \frac{cos\theta_2}{l_2} - \frac{g}{l_2} sin\theta_2 \end{bmatrix} \tag{47}$$

This concludes the Part A of the project.

0.3 Part B: Linearizing around a equilibrium point

For this project, Jacobian linearization will be used, where the given equilibrium points are $(x, \theta_1, \theta_2) = (0, 0, 0)$ The Jacobian linearization is done using the following approach:

$$A = \frac{\partial f}{\partial x}; B = \frac{\partial f}{\partial u} \tag{48}$$

where $f = \dot{\vec{\mathbf{X}}}$, $x = \vec{\mathbf{X}}$ and $u = \mathbf{F}$ this will give a 6x6 Matrix for A and a 6x1 Matrix for B as shown below:

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} & \frac{\partial f_1}{\partial x_4} & \frac{\partial f_1}{\partial x_5} & \frac{\partial f_1}{\partial x_6} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} & \frac{\partial f_2}{\partial x_4} & \frac{\partial f_2}{\partial x_5} & \frac{\partial f_2}{\partial x_6} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} & \frac{\partial f_3}{\partial x_4} & \frac{\partial f_3}{\partial x_5} & \frac{\partial f_3}{\partial x_6} \\ \frac{\partial f_4}{\partial x_1} & \frac{\partial f_4}{\partial x_2} & \frac{\partial f_4}{\partial x_3} & \frac{\partial f_4}{\partial x_4} & \frac{\partial f_4}{\partial x_5} & \frac{\partial f_4}{\partial x_6} \\ \frac{\partial f_5}{\partial x_1} & \frac{\partial f_5}{\partial x_2} & \frac{\partial f_5}{\partial x_3} & \frac{\partial f_5}{\partial x_4} & \frac{\partial f_5}{\partial x_5} & \frac{\partial f_5}{\partial x_6} \\ \frac{\partial f_6}{\partial x_1} & \frac{\partial f_6}{\partial x_2} & \frac{\partial f_6}{\partial x_3} & \frac{\partial f_6}{\partial x_4} & \frac{\partial f_6}{\partial x_5} & \frac{\partial f_6}{\partial x_6} \end{bmatrix}_{(0,0,0)} ; B = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \\ \frac{\partial f_3}{\partial u} \\ \frac{\partial f_4}{\partial u} \\ \frac{\partial f_5}{\partial u} \\ \frac{\partial f_6}{\partial u} \end{bmatrix}_{(0,0,0)} \tag{49}$$

where $f = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix}$ and $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} x \\ \dot{x} \\ \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix}$

Evaluating the individual elements of the A matrix below, Note that for differentiation of Rows 2,4 and 6 of A Matrix here Sympy library has been used. (The code is attached)

$$\frac{\partial f_1}{\partial x_1} = 0 \tag{50}$$

Since no element of the state space is dependant on linear displacement of cart x, the differentiation of f_1 with respect to x_1 (or x) is zero.

$$\frac{\partial f_1}{\partial x_2} = \frac{\partial \dot{x}}{\partial \dot{x}} = 1 \tag{51}$$

$$\frac{\partial f_1}{\partial x_3} = \frac{\partial \dot{x}}{\partial \theta_1} = 0 \tag{52}$$

$$\frac{\partial f_1}{\partial x_4} = \frac{\partial \dot{x}}{\partial \dot{\theta}_1} = 0 \tag{53}$$

$$\frac{\partial f_1}{\partial x_5} = \frac{\partial \dot{x}}{\partial \theta_2} = 0 \tag{54}$$

$$\frac{\partial f_1}{\partial x_6} = \frac{\partial \dot{x}}{\partial \dot{\theta}_2} = 0 \tag{55}$$

Next calculating Jacobian for f_3,

$$\frac{\partial f_3}{\partial x_1} = \frac{\partial \dot{\theta}_1}{\partial x} = 0 \tag{56}$$

$$\frac{\partial f_3}{\partial x_2} = \frac{\partial \dot{\theta}_1}{\partial \dot{x}} = 0 \tag{57}$$

$$\frac{\partial f_3}{\partial x_3} = \frac{\partial \dot{\theta}_1}{\partial \theta_1} = 0 \tag{58}$$

$$\frac{\partial f_3}{\partial x_4} = \frac{\partial \dot{\theta}_1}{\partial \dot{\theta}_1} = 1 \tag{59}$$

$$\frac{\partial f_3}{\partial x_5} = \frac{\partial \dot{\theta}_1}{\partial \theta_2} = 0 \tag{60}$$

$$\frac{\partial f_3}{\partial x_6} = \frac{\partial \dot{\theta}_1}{\partial \dot{\theta}_2} = 0 \tag{61}$$

Next calculating Jacobian for f_5,

$$\frac{\partial f_5}{\partial x_1} = \frac{\partial \dot{\theta}_2}{\partial x} = 0 \tag{62}$$

$$\frac{\partial f_5}{\partial x_2} = \frac{\partial \dot{\theta}_2}{\partial \dot{x}} = 0 \tag{63}$$

$$\frac{\partial f_5}{\partial x_3} = \frac{\partial \dot{\theta}_2}{\partial \theta_1} = 0 \tag{64}$$

$$\frac{\partial f_5}{\partial x_4} = \frac{\partial \dot{\theta}_2}{\partial \dot{\theta}_1} = 0 \tag{65}$$

$$\frac{\partial f_5}{\partial x_5} = \frac{\partial \dot{\theta}_2}{\partial \theta_2} = 0 \tag{66}$$

$$\frac{\partial f_5}{\partial x_6} = \frac{\partial \dot{\theta}_2}{\partial \dot{\theta}_2} = 1 \tag{67}$$

Next are the matrix elements of Row 2 calculated from Jacobian of f_2

$$\frac{\partial f_2}{\partial x_1} = \frac{\partial \ddot{x}}{\partial x} = 0 \tag{68}$$

$$\frac{\partial f_2}{\partial x_2} = \frac{\partial \ddot{x}}{\partial \dot{x}} = 0 \tag{69}$$

$$\frac{\partial f_2}{\partial x_3} = \frac{\partial \ddot{x}}{\partial \theta_1} = -\frac{2m_1\left(F - gm_1\sin(\theta_1)\cos(\theta_1) - gm_2\sin(\theta_2)\cos(\theta_2) - l_1m_1\dot{\theta}_1^2\sin(\theta_1) - l_2m_2\dot{\theta}_2^2\sin(\theta_2)\right)\sin(\theta_1)\cos(\theta_1)}{\left(M + m_1\sin^2(\theta_1) + m_2\sin^2(\theta_2)\right)^2} + \frac{gm_1\sin^2(\theta_1) - gm_1\cos^2(\theta_1) - l_1m_1\dot{\theta}_1^2\cos(\theta_1)}{M + m_1\sin^2(\theta_1) + m_2\sin^2(\theta_2)} \tag{70}$$

Equation (70) was calculated using Sympy, Substituting $\theta_1 = 0$, $\theta_2 = 0$ (By extension $\dot{\theta}_1$ and $\dot{\theta}_2$ are also zero) in the above equation, we get,

$$\frac{\partial f_2}{\partial x_3} = \frac{\partial \ddot{x}}{\partial \theta_1} = -\frac{gm_1}{M} \tag{71}$$

$$\frac{\partial f_2}{\partial x_4} = \frac{\partial \ddot{x}}{\partial \dot{\theta}_1} = -\frac{2 l_1 m_1 \dot{\theta}_1 \sin(\theta_1)}{M + m_1 \sin^2(\theta_1) + m_2 \sin^2(\theta_2)} \tag{72}$$

Equation (72) was calculated using Sympy, Substituting $\theta_1 = 0$, $\theta_2 = 0$ (By extension $\dot{\theta}_1$ and $\dot{\theta}_2$ are also zero) in the above equation, we get,

$$\frac{\partial f_2}{\partial x_4} = \frac{\partial \ddot{x}}{\partial \dot{\theta}_1} = 0 \tag{73}$$

$$\frac{\partial f_2}{\partial x_5} = \frac{\partial \ddot{x}}{\partial \theta_2} = -\frac{2 m_2 \left(F - g m_1 \sin(\theta_1)\cos(\theta_1) - g m_2 \sin(\theta_2)\cos(\theta_2) - l_1 m_1 \dot{\theta}_1^2 \sin(\theta_1) - l_2 m_2 \dot{\theta}_2^2 \sin(\theta_2)\right)\sin(\theta_2)\cos(\theta_2)}{\left(M + m_1 \sin^2(\theta_1) + m_2 \sin^2(\theta_2)\right)^2} + \frac{g m_2 \sin^2(\theta_2) - g m_2 \cos^2(\theta_2) - l_2 m_2 \dot{\theta}_2^2 \cos(\theta_2)}{M + m_1 \sin^2(\theta_1) + m_2 \sin^2(\theta_2)} \tag{74}$$

Equation (74) was calculated using Sympy, Substituting $\theta_1 = 0$, $\theta_2 = 0$ (By extension $\dot{\theta}_1$ and $\dot{\theta}_2$ are also zero) in the above equation, we get,

$$\frac{\partial f_2}{\partial x_5} = \frac{\partial \ddot{x}}{\partial \theta_2} = -\frac{g m_2}{M} \tag{75}$$

$$\frac{\partial f_2}{\partial x_6} = \frac{\partial \ddot{x}}{\partial \dot{\theta}_2} = -\frac{2 l_2 m_2 \dot{\theta}_2 \sin(\theta_2)}{M + m_1 \sin^2(\theta_1) + m_2 \sin^2(\theta_2)} \tag{76}$$

Equation (76) was calculated using Sympy, Substituting $\theta_1 = 0$, $\theta_2 = 0$ (By extension $\dot{\theta}_1$ and $\dot{\theta}_2$ are also zero) in the above equation, we get,

$$\frac{\partial f_2}{\partial x_6} = \frac{\partial \ddot{x}}{\partial \dot{\theta}_2} = 0 \tag{77}$$

Next calculating Jacobian for f_4,

$$\frac{\partial f_4}{\partial x_1} = \frac{\partial \ddot{\theta}_1}{\partial x} = 0 \tag{78}$$

$$\frac{\partial f_4}{\partial x_2} = \frac{\partial \ddot{\theta}_1}{\partial \dot{x}} = 0 \tag{79}$$

$$\frac{\partial f_4}{\partial x_3} = \frac{\partial \ddot{\theta}_1}{\partial \theta_1} = (contd..below) \tag{80}$$

$$= -\frac{g\cos(\theta_1)}{l_1} - \frac{2 m_1 \left(F - g m_1 \sin(\theta_1)\cos(\theta_1) - g m_2 \sin(\theta_2)\cos(\theta_2) - l_1 m_1 \dot{\theta}_1^2 \sin(\theta_1) - l_2 m_2 \dot{\theta}_2^2 \sin(\theta_2)\right)\sin(\theta_1)\cos^2(\theta_1)}{l_1\left(M + m_1 \sin^2(\theta_1) + m_2 \sin^2(\theta_2)\right)^2} + \frac{\left(g m_1 \sin^2(\theta_1) - g m_1 \cos^2(\theta_1) - l_1 m_1 \dot{\theta}_1^2 \cos(\theta_1)\right)\cos(\theta_1)}{l_1\left(M + m_1 \sin^2(\theta_1) + m_2 \sin^2(\theta_2)\right)} - \frac{\left(F - g m_1 \sin(\theta_1)\cos(\theta_1) - g m_2 \sin(\theta_2)\cos(\theta_2) - l_1 m_1 \dot{\theta}_1^2 \sin(\theta_1) - l_2 m_2 \dot{\theta}_2^2 \sin(\theta_2)\right)\sin(\theta_1)}{l_1\left(M + m_1 \sin^2(\theta_1) + m_2 \sin^2(\theta_2)\right)} \tag{81}$$

Equation (81) was calculated using Sympy, Substituting $\theta_1 = 0$, $\theta_2 = 0$ (By extension $\dot{\theta}_1$ and $\dot{\theta}_2$ are also zero) in the above equation, we get,

$$\frac{\partial f_4}{\partial x_3} = \frac{\partial \ddot{\theta}_1}{\partial \theta_1} = \frac{g(-M - m_1)}{M l_1} \tag{82}$$

$$\frac{\partial f_4}{\partial x_4} = \frac{\partial \ddot{\theta}_1}{\partial \dot{\theta}_1} = -\frac{2m_1\dot{\theta}_1 \sin(\theta_1)\cos(\theta_1)}{M + m_1 \sin^2(\theta_1) + m_2 \sin^2(\theta_2)} \tag{83}$$

Equation (83) was calculated using Sympy, Substituting $\theta_1 = 0$, $\theta_2 = 0$ (By extension $\dot{\theta}_1$ and $\dot{\theta}_2$ are also zero) in the above equation, we get,

$$\frac{\partial f_4}{\partial x_4} = \frac{\partial \ddot{\theta}_1}{\partial \dot{\theta}_1} = 0 \tag{84}$$

$$\frac{\partial f_4}{\partial x_5} = \frac{\partial \ddot{\theta}_1}{\partial \theta_2} = (contd..below) \tag{85}$$

$$= -\frac{2m_2\left(F - gm_1 \sin(\theta_1)\cos(\theta_1) - gm_2 \sin(\theta_2)\cos(\theta_2) - l_1 m_1 \dot{\theta}_1^2 \sin(\theta_1) - l_2 m_2 \dot{\theta}_2^2 \sin(\theta_2)\right)\sin(\theta_2)\cos(\theta_1)\cos(\theta_2)}{l_1\left(M + m_1 \sin^2(\theta_1) + m_2 \sin^2(\theta_2)\right)^2} + \frac{\left(gm_2 \sin^2(\theta_2) - gm_2 \cos^2(\theta_2) - l_2 m_2 \dot{\theta}_2^2 \cos(\theta_2)\right)\cos(\theta_1)}{l_1\left(M + m_1 \sin^2(\theta_1) + m_2 \sin^2(\theta_2)\right)} \tag{86}$$

Equation (86) was calculated using Sympy, Substituting $\theta_1 = 0$, $\theta_2 = 0$ (By extension $\dot{\theta}_1$ and $\dot{\theta}_2$ are also zero) in the above equation, we get,

$$\frac{\partial f_4}{\partial x_5} = \frac{\partial \ddot{\theta}_1}{\partial \theta_2} = -\frac{gm_2}{Ml_1} \tag{87}$$

$$\frac{\partial f_4}{\partial x_6} = \frac{\partial \ddot{\theta}_1}{\partial \dot{\theta}_2} = -\frac{2l_2 m_2 \dot{\theta}_2 \sin(\theta_2)\cos(\theta_1)}{l_1\left(M + m_1 \sin^2(\theta_1) + m_2 \sin^2(\theta_2)\right)} \tag{88}$$

Equation (88) was calculated using Sympy, Substituting $\theta_1 = 0$, $\theta_2 = 0$ (By extension $\dot{\theta}_1$ and $\dot{\theta}_2$ are also zero) in the above equation, we get,

$$\frac{\partial f_4}{\partial x_6} = \frac{\partial \ddot{\theta}_1}{\partial \dot{\theta}_2} = 0 \tag{89}$$

Next calculating Jacobian for f_6,

$$\frac{\partial f_6}{\partial x_1} = \frac{\partial \ddot{\theta}_2}{\partial x} = 0 \tag{90}$$

$$\frac{\partial f_6}{\partial x_2} = \frac{\partial \ddot{\theta}_2}{\partial \dot{x}} = 0 \tag{91}$$

$$\frac{\partial f_6}{\partial x_3} = \frac{\partial \ddot{\theta}_2}{\partial \theta_1} = (contd..below) \tag{92}$$

$$= -\frac{2m_1\left(F - gm_1 \sin(\theta_1)\cos(\theta_1) - gm_2 \sin(\theta_2)\cos(\theta_2) - l_1 m_1 \dot{\theta}_1^2 \sin(\theta_1) - l_2 m_2 \dot{\theta}_2^2 \sin(\theta_2)\right)\sin(\theta_1)\cos(\theta_1)\cos(\theta_2)}{l_2\left(M + m_1 \sin^2(\theta_1) + m_2 \sin^2(\theta_2)\right)^2} + \frac{\left(gm_1 \sin^2(\theta_1) - gm_1 \cos^2(\theta_1) - l_1 m_1 \dot{\theta}_1^2 \cos(\theta_1)\right)\cos(\theta_2)}{l_2\left(M + m_1 \sin^2(\theta_1) + m_2 \sin^2(\theta_2)\right)} \tag{93}$$

Equation (93) was calculated using Sympy, Substituting $\theta_1 = 0$, $\theta_2 = 0$ (By extension $\dot{\theta}_1$ and $\dot{\theta}_2$ are also zero) in the above equation, we get,

$$\frac{\partial f_6}{\partial x_3} = \frac{\partial \ddot{\theta}_2}{\partial \theta_1} = -\frac{gm_1}{Ml_2} \tag{94}$$

$$\frac{\partial f_6}{\partial x_4} = \frac{\partial \ddot{\theta}_2}{\partial \dot{\theta}_1} = -\frac{2 l_1 m_1 \dot{\theta}_1 \sin(\theta_1)\cos(\theta_2)}{l_2\left(M + m_1 \sin^2(\theta_1) + m_2 \sin^2(\theta_2)\right)} \tag{95}$$

Equation (95) was calculated using Sympy, Substituting $\theta_1 = 0$, $\theta_2 = 0$ (By extension $\dot{\theta}_1$ and $\dot{\theta}_2$ are also zero) in the above equation, we get,

$$\frac{\partial f_6}{\partial x_4} = \frac{\partial \ddot{\theta}_2}{\partial \dot{\theta}_1} = 0 \tag{96}$$

$$\frac{\partial f_6}{\partial x_5} = \frac{\partial \ddot{\theta}_2}{\partial \theta_2} = (contd..below) \tag{97}$$

$$\begin{aligned}= -\frac{g\cos(\theta_2)}{l_2} - \frac{2m_2\left(F - gm_1\sin(\theta_1)\cos(\theta_1) - gm_2\sin(\theta_2)\cos(\theta_2) - l_1 m_1 \dot{\theta}_1^2 \sin(\theta_1) - l_2 m_2 \dot{\theta}_2^2 \sin(\theta_2)\right)\sin(\theta_2)\cos^2(\theta_2)}{l_2\left(M + m_1\sin^2(\theta_1) + m_2\sin^2(\theta_2)\right)^2} \\ + \frac{\left(gm_2\sin^2(\theta_2) - gm_2\cos^2(\theta_2) - l_2 m_2 \dot{\theta}_2^2\cos(\theta_2)\right)\cos(\theta_2)}{l_2\left(M + m_1\sin^2(\theta_1) + m_2\sin^2(\theta_2)\right)} \\ - \frac{\left(F - gm_1\sin(\theta_1)\cos(\theta_1) - gm_2\sin(\theta_2)\cos(\theta_2) - l_1 m_1 \dot{\theta}_1^2\sin(\theta_1) - l_2 m_2 \dot{\theta}_2^2\sin(\theta_2)\right)\sin(\theta_2)}{l_2\left(M + m_1\sin^2(\theta_1) + m_2\sin^2(\theta_2)\right)}\end{aligned} \tag{98}$$

Equation (98) was calculated using Sympy, Substituting $\theta_1 = 0$, $\theta_2 = 0$ (By extension $\dot{\theta}_1$ and $\dot{\theta}_2$ are also zero) in the above equation, we get,

$$\frac{\partial f_6}{\partial x_5} = \frac{\partial \ddot{\theta}_2}{\partial \theta_2} = \frac{g(-M - m_2)}{M l_2} \tag{99}$$

$$\frac{\partial f_6}{\partial x_6} = \frac{\partial \ddot{\theta}_2}{\partial \dot{\theta}_2} = (contd..below) \tag{100}$$

$$\begin{aligned}= -\frac{g\cos(\theta_2)}{l_2} - \frac{2m_2\left(F - gm_1\sin(\theta_1)\cos(\theta_1) - gm_2\sin(\theta_2)\cos(\theta_2) - l_1 m_1 \dot{\theta}_1^2 \sin(\theta_1) - l_2 m_2 \dot{\theta}_2^2 \sin(\theta_2)\right)\sin(\theta_2)\cos^2(\theta_2)}{l_2\left(M + m_1\sin^2(\theta_1) + m_2\sin^2(\theta_2)\right)^2} \\ + \frac{\left(gm_2\sin^2(\theta_2) - gm_2\cos^2(\theta_2) - l_2 m_2 \dot{\theta}_2^2\cos(\theta_2)\right)\cos(\theta_2)}{l_2\left(M + m_1\sin^2(\theta_1) + m_2\sin^2(\theta_2)\right)} \\ - \frac{\left(F - gm_1\sin(\theta_1)\cos(\theta_1) - gm_2\sin(\theta_2)\cos(\theta_2) - l_1 m_1 \dot{\theta}_1^2\sin(\theta_1) - l_2 m_2 \dot{\theta}_2^2\sin(\theta_2)\right)\sin(\theta_2)}{l_2\left(M + m_1\sin^2(\theta_1) + m_2\sin^2(\theta_2)\right)}\end{aligned} \tag{101}$$

Equation (101) was calculated using Sympy, Substituting $\theta_1 = 0$, $\theta_2 = 0$ (By extension $\dot{\theta}_1$ and $\dot{\theta}_2$ are also zero) in the above equation, we get,

$$\frac{\partial f_6}{\partial x_6} = \frac{\partial \ddot{\theta}_2}{\partial \dot{\theta}_2} = 0 \tag{102}$$

The final A matrix can be written from substituting above calculated values in equation (49),

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{gm_1}{M} & 0 & -\frac{gm_2}{M} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{g(-M-m_1)}{Ml_1} & 0 & -\frac{gm_2}{Ml_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{gm_1}{Ml_2} & 0 & \frac{g(-M-m_2)}{Ml_2} & 0 \end{bmatrix} \tag{103}$$

Now calculating the B matrix for the system, The element wise Jacobian of the column vector as given in Equation (49) is given by:

$$\frac{\partial f_1}{\partial u} = \frac{\partial \dot{x}}{\partial \mathbf{F}} = 0 \tag{104}$$

$$\frac{\partial f_3}{\partial u} = \frac{\partial \dot{\theta}_1}{\partial \mathbf{F}} = 0 \tag{105}$$

$$\frac{\partial f_5}{\partial u} = \frac{\partial \dot{\theta}_2}{\partial \mathbf{F}} = 0 \tag{106}$$

Now for rows 2, 4 and 6 Sympy has been used to evaluate differentiation of the elements,

$$\frac{\partial f_2}{\partial u} = \frac{\partial \ddot{x}}{\partial \mathbf{F}} = \frac{1}{M + m_1 \sin^2(\theta_1) + m_2 \sin^2(\theta_2)} \tag{107}$$

Equation (107) was calculated using Sympy, Substituting $\theta_1 = 0$, $\theta_2 = 0$ (By extension $\dot{\theta}_1$ and $\dot{\theta}_2$ are also zero) in the above equation, we get,

$$\frac{\partial f_2}{\partial u} = \frac{\partial \ddot{x}}{\partial \mathbf{F}} = \frac{1}{M} \tag{108}$$

$$\frac{\partial f_4}{\partial u} = \frac{\partial \ddot{\theta}_1}{\partial \mathbf{F}} = \frac{\cos(\theta_1)}{l_1 \left(M + m_1 \sin^2(\theta_1) + m_2 \sin^2(\theta_2)\right)} \tag{109}$$

Equation (109) was calculated using Sympy, Substituting $\theta_1 = 0$, $\theta_2 = 0$ (By extension $\dot{\theta}_1$ and $\dot{\theta}_2$ are also zero) in the above equation, we get,

$$\frac{\partial f_4}{\partial u} = \frac{\partial \ddot{\theta}_1}{\partial \mathbf{F}} = \frac{1}{Ml_1} \tag{110}$$

$$\frac{\partial f_6}{\partial u} = \frac{\partial \ddot{\theta}_2}{\partial \mathbf{F}} = \frac{\cos(\theta_2)}{l_2 \left(M + m_1 \sin^2(\theta_1) + m_2 \sin^2(\theta_2)\right)} \tag{111}$$

Equation (111) was calculated using Sympy, Substituting $\theta_1 = 0$, $\theta_2 = 0$ (By extension $\dot{\theta}_1$ and $\dot{\theta}_2$ are also zero) in the above equation, we get,

$$\frac{\partial f_6}{\partial u} = \frac{\partial \ddot{\theta}_2}{\partial \mathbf{F}} = \frac{1}{Ml_2} \tag{112}$$

The final B matrix can be written from substituting above calculated values in equation (49),

$$B = \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{1}{Ml_1} \\ 0 \\ \frac{1}{Ml_2} \end{bmatrix} \tag{113}$$

The final state space equation as in (45) can be rewritten as:

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{gm_1}{M} & 0 & -\frac{gm_2}{M} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{g(-M-m_1)}{Ml_1} & 0 & -\frac{gm_2}{Ml_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{gm_1}{Ml_2} & 0 & \frac{g(-M-m_2)}{Ml_2} & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{1}{Ml_1} \\ 0 \\ \frac{1}{Ml_2} \end{bmatrix} F \tag{114}$$

0.4 Part C: Controllability conditions

The controllability of a Linear Time Invariant system is given by the rank of controllability matrix which can be written as below:

$$rank(E) = rank\begin{bmatrix} B & AB & A^2B & A^3B & A^4B & A^5B \end{bmatrix} \tag{115}$$

For the system to be controllable the rank should be equal to the dimension of matrix n for an n x n controllability matrix E.

In part C we are asked to find conditions of controllability, For that first we will find the conditions for which the system is uncontrollable. The system is uncontrollable if $rank(E) < 6$ or in simple terms $det(E) = 0$, Calculating the determinant of equation (115) and equating it to zero we get,

$$det(E) = \frac{g^6\left(-l_1^2 + 2l_1l_2 - l_2^2\right)}{M^6 l_1^6 l_2^6} = 0 \tag{116}$$

The above equation can also be rewritten as,

$$\left(l_1^2 - 2l_1l_2 + l_2^2\right) = 0 \tag{117}$$

Simplifying we get the condition for $rank(E) < 6$,

$$(l_1 - l_2)^2 = 0 \Rightarrow l_1 = l_2 \tag{118}$$

Hence, when the cable lengths of the pendulums of mass m_1 and m_2 are equal than the system becomes uncontrollable, except that the system is controllable (Barring impossible conditions like $g = 0$ or $M \to \infty$ in equation (116)).

The above equations for rank and determinants are calculated in MATLAB and in Python (Using Sympy Package), the code of which is attached in the attachments.

0.5 Part D: Design, simulation of LQR Controller for Linear and Non-Linear System, Use of Lyapunov's Indirect Method to Verify Closed Loop Stability

For the given values of pendulum bob's mass and lengths m_1, m_2 and l_1, l_2 respectively as well as Cart's Mass M, A Matlab script is written in which the state space matrices are defined. Once those state space matrices are defined, a MATLAB function called $ctrb(A, B)$ is used.

This function gives us the controllability matrix as mentioned in Equation (115). Our point of interest is the rank of that matrix, since the system is only controllable iff the matrix rank matches the dimensions nxn of the A matrix.

The following shows the controllability matrix after substituting the values as states above from Matlab's $ctrb$ function:

$$\begin{pmatrix} 0 & \frac{1}{1000} & 0 & -\frac{338958922354413}{2305843009213693952} & 0 & \frac{1306573659368811}{9223372036854775808} \\ \frac{1}{1000} & 0 & -\frac{338958922354413}{2305843009213693952} & 0 & \frac{1306573659368811}{9223372036854775808} & 0 \\ 0 & \frac{1}{20000} & 0 & -\frac{2350115194990597}{73786976294838206464} & 0 & \frac{1674185909292917}{73786976294838206464} \\ \frac{1}{20000} & 0 & -\frac{2350115194990597}{73786976294838206464} & 0 & \frac{1674185909292917}{73786976294838206464} & 0 \\ 0 & \frac{1}{10000} & 0 & -\frac{8315792228428267}{73786976294838206464} & 0 & \frac{4597367655677375}{36893488147419103232} \\ \frac{1}{10000} & 0 & -\frac{8315792228428267}{73786976294838206464} & 0 & \frac{4597367655677375}{36893488147419103232} & 0 \end{pmatrix} \tag{119}$$

The rank of the above matrix is 6, and hence the system is controllable.

Next step is to design the LQR controller for the above system, For a pair (A, B) a cost function J can be defined such that the system can be stabilizable with minimum costs,

$$J(K, X(0)) = \int_0^\infty \vec{X}^T(t)Q\vec{X}(t) + \vec{U}_k^T(t)R\vec{U}_k(t)\,dt \tag{120}$$

where Q and R are symmetric positive definite matrices.

The methodology used to tune K using the appropriate Q and R values is mentioned below:

- It is assumed that the control is cheap and hence R = 0.0001 is considered, essentially stating that we have over specification motors/actuators which have a very good response or the electricity or energy required to run the system is cheap.
- Secondly the penalties on individual state variables is defined in Q, In the simulation the penalty for deviation on θ_1, $\dot{\theta}_1$ and similarly θ_2, $\dot{\theta}_2$ as compared to cart's linear displacement x and $\dot{x}$.

According to that the following simulation results are obtained for Linear and Non linear system of equations as shown in Figure 1 and Figure 2:

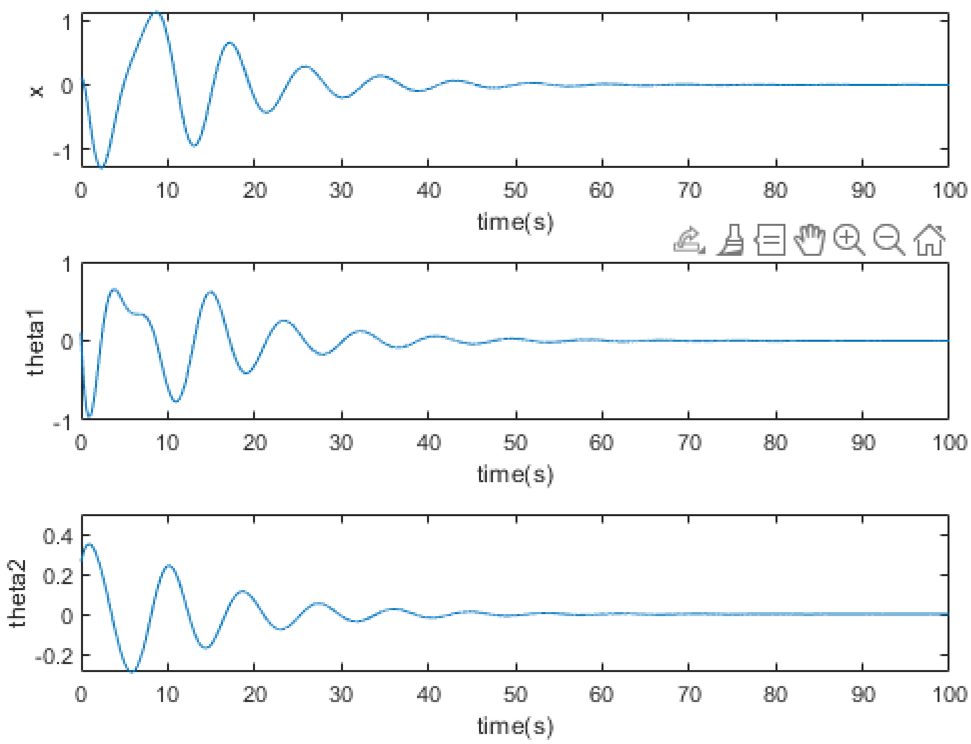

Figure 1: Linearized system state response of x,θ_1,θ_2

0.5.1 Lyapunov's Indirect method to certify closed loop stability:

The revised $(A - B_k K)$ matrix is written below with mass, lengths and gravity values substituted, The eigen values of this matrix are found in MATLAB,

$$A - B_k K = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{\sqrt{10}}{10} & -\frac{5015247022514233}{4503599627370496} & -\frac{7596763303165983}{2251799813685248} & -\frac{1702000984552337}{562949953421312} & -\frac{1264600535532603}{281474976710656} & -\frac{6902238806119779}{9007199254740992} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\frac{\sqrt{10}}{200} & -\frac{4012197618011387}{72057594037927936} & -\frac{5932880295456283}{9007199254740992} & -\frac{2723201575283739}{18014398509481984} & -\frac{4046721713704329}{18014398509481984} & -\frac{345111940305989}{9007199254740992} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -\frac{\sqrt{10}}{100} & -\frac{4012197618011387}{36028797018963968} & -\frac{3038705321266393}{9007199254740992} & -\frac{2723201575283739}{9007199254740992} & -\frac{6436888491675251}{4503599627370496} & -\frac{345111940305989}{4503599627370496} \end{pmatrix}$$

The eigen values as displayed in MATLAB are, Note that all $Re(\lambda) < 0$ hence the system is stable as per Lyapunov's criterion:

$$\lambda of(A - B_k K) = \begin{pmatrix} -\frac{1964405266495813}{9007199254740992} + \frac{1150229891073851}{1125899906842624}\mathrm{i} \\ -\frac{1964405266495813}{9007199254740992} - \frac{1150229891073851}{1125899906842624}\mathrm{i} \\ -\frac{3308468847515699}{9007199254740992} + \frac{6430611243299141}{18014398509481984}\mathrm{i} \\ -\frac{3308468847515699}{9007199254740992} - \frac{6430611243299141}{18014398509481984}\mathrm{i} \\ -\frac{96035655328705}{1125899906842624} + \frac{3257400182114611}{4503599627370496}\mathrm{i} \\ -\frac{96035655328705}{1125899906842624} - \frac{3257400182114611}{4503599627370496}\mathrm{i} \end{pmatrix}$$

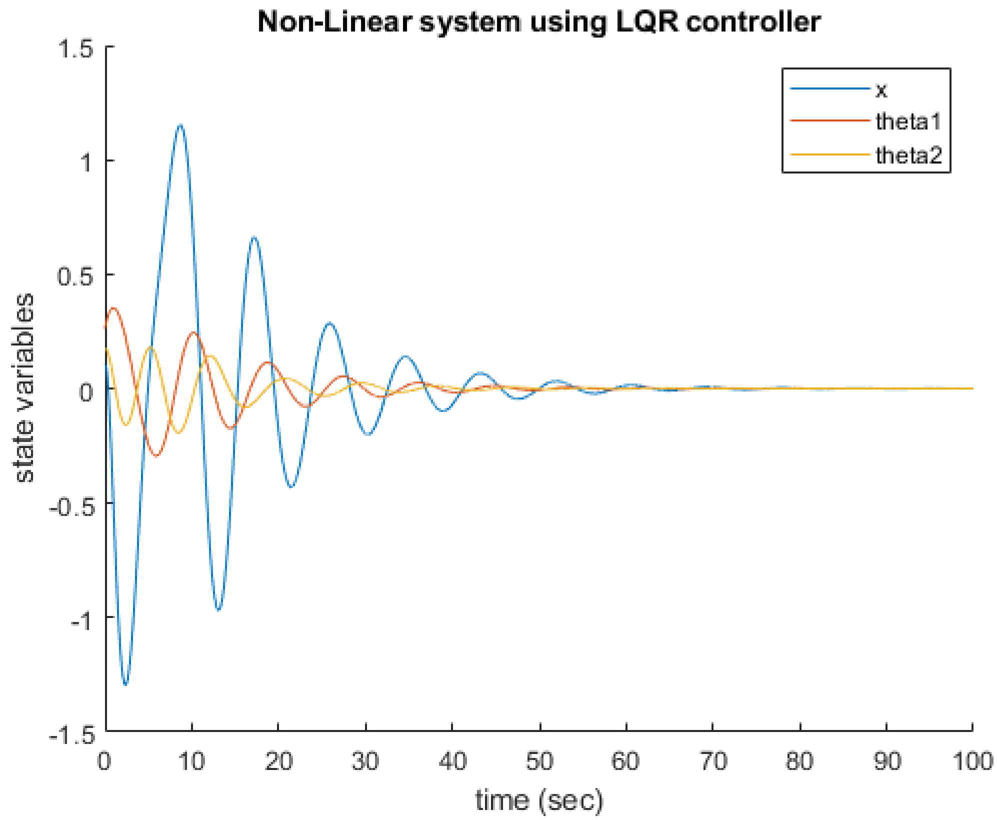

Figure 2: Non-Linear system state response of x,θ_1,θ_2

0.6 Part E: Determine the output vectors for which the system is Observable

To determine whether the state is observable using the given measurements the rank of the observability matrix is measured for the LTI system.

$$rank \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \\ CA^4 \\ CA^5 \end{bmatrix} = 6 \tag{121}$$

The rank is 6 in our case since the number of state variables are 6.

The C matrices for the given cases can be empirically derived as:

- When $x(t)$ is measured for calculating the whole state

$$C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{122}$$

- When θ_1 and θ_2 is measured for calculating the whole state

$$C_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{123}$$

- When $x(t)$ and θ_2 is measured for calculating the whole state

$$C_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{124}$$

- When $x(t)$, θ_1 and θ_2 is measured for calculating the whole state

$$C_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{125}$$

The following steps evaluate the observability of the above system as per the measurements provided in the question, Note MATLAB package was used to get the ranks of this matrix,

0.6.1 When Output vector $x(t)$ is only available:

We calculate the observability of the system in symbolic as well as when the values are substituted, The following matrix is calculated for observability in symbolic form Ob_1,

$$Ob_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{g\,m_1}{M} & 0 & -\frac{g\,m_2}{M} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{g\,m_1}{M} & 0 & -\frac{g\,m_2}{M} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{g^2\,m_1\,(M+m_1)}{M^2\,l_1} + \frac{g^2\,m_1\,m_2}{M^2\,l_2} & 0 & \frac{g^2\,m_2\,(M+m_2)}{M^2\,l_2} + \frac{g^2\,m_1\,m_2}{M^2\,l_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{g^2\,m_1\,(M+m_1)}{M^2\,l_1} + \frac{g^2\,m_1\,m_2}{M^2\,l_2} & 0 & \frac{g^2\,m_2\,(M+m_2)}{M^2\,l_2} + \frac{g^2\,m_1\,m_2}{M^2\,l_1} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The observability matrix when the values of M, m_1, m_1, m_2, l_1 and l_2 are substituted,

$$Ob_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{49}{50} & 0 & -\frac{49}{50} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{49}{50} & 0 & -\frac{49}{50} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{31213}{50000} & 0 & \frac{55223}{50000} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{31213}{50000} & 0 & \frac{55223}{50000} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The rank of both the matrices are 6 and hence the matrix is observable when $x(t)$ is the only available measurable output. The rank is calculated using MATLAB code.

0.6.2 When Output vector θ_1, θ_2 are only available:

We calculate the observability of the system in symbolic as well as when the values are substituted, The following matrix is calculated for observability in symbolic form Ob_2,

$$Ob_2 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{g\,(M+m_1)}{M\,l_1} & 0 & -\frac{g\,m_2}{M\,l_1} & 0 \\
0 & 0 & -\frac{g\,m_1}{M\,l_2} & 0 & -\frac{g\,(M+m_2)}{M\,l_2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{g\,(M+m_1)}{M\,l_1} & 0 & -\frac{g\,m_2}{M\,l_1} \\
0 & 0 & 0 & -\frac{g\,m_1}{M\,l_2} & 0 & -\frac{g\,(M+m_2)}{M\,l_2} \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{g^2\,(M+m_1)^2}{M^2\,{l_1}^2}+\frac{g^2\,m_1\,m_2}{M^2\,l_1\,l_2} & 0 & \frac{g^2\,m_2\,(M+m_1)}{M^2\,{l_1}^2}+\frac{g^2\,m_2\,(M+m_2)}{M^2\,l_1\,l_2} & 0 \\
0 & 0 & \frac{g^2\,m_1\,(M+m_2)}{M^2\,{l_2}^2}+\frac{g^2\,m_1\,(M+m_1)}{M^2\,l_1\,l_2} & 0 & \frac{g^2\,(M+m_2)^2}{M^2\,{l_2}^2}+\frac{g^2\,m_1\,m_2}{M^2\,l_1\,l_2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{g^2\,(M+m_1)^2}{M^2\,{l_1}^2}+\frac{g^2\,m_1\,m_2}{M^2\,l_1\,l_2} & 0 & \frac{g^2\,m_2\,(M+m_1)}{M^2\,{l_1}^2}+\frac{g^2\,m_2\,(M+m_2)}{M^2\,l_1\,l_2} \\
0 & 0 & 0 & \frac{g^2\,m_1\,(M+m_2)}{M^2\,{l_2}^2}+\frac{g^2\,m_1\,(M+m_1)}{M^2\,l_1\,l_2} & 0 & \frac{g^2\,(M+m_2)^2}{M^2\,{l_2}^2}+\frac{g^2\,m_1\,m_2}{M^2\,l_1\,l_2}
\end{pmatrix}$$

The observability matrix when the values of M, m_1, m_1, m_2, l_1 and l_2 are substituted,

$$Ob_2 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{539}{1000} & 0 & -\frac{49}{1000} & 0 \\
0 & 0 & -\frac{49}{500} & 0 & -\frac{539}{500} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{539}{1000} & 0 & -\frac{49}{1000} \\
0 & 0 & 0 & -\frac{49}{500} & 0 & -\frac{539}{500} \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1330016552753937}{4503599627370496} & 0 & \frac{5709339348407145}{72057594037927936} & 0 \\
0 & 0 & \frac{5709339348407145}{36028797018963968} & 0 & \frac{5255187354783849}{4503599627370496} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1330016552753937}{4503599627370496} & 0 & \frac{5709339348407145}{72057594037927936} \\
0 & 0 & 0 & \frac{5709339348407145}{36028797018963968} & 0 & \frac{5255187354783849}{4503599627370496}
\end{pmatrix}$$

The rank of both the matrices are 4 and hence the matrix is not observable when θ_1 and $\theta_2$is the only available measurable output. The rank is calculated using MATLAB code.

0.6.3 When $x(t)$ and θ_2 are only available:

We calculate the observability of the system in symbolic as well as when the values are substituted, The following matrix is calculated for observability in symbolic form Ob_3,

$$Ob_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{g\,m_1}{M} & 0 & -\frac{g\,m_2}{M} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{g\,m_1}{M\,l_2} & 0 & -\frac{g\,(M+m_2)}{M\,l_2} & 0 \\ 0 & 0 & 0 & -\frac{g\,m_1}{M} & 0 & -\frac{g\,m_2}{M} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{g\,m_1}{M\,l_2} & 0 & -\frac{g\,(M+m_2)}{M\,l_2} \\ 0 & 0 & \frac{g^2\,m_1\,(M+m_1)}{M^2\,l_1}+\frac{g^2\,m_1\,m_2}{M^2\,l_2} & 0 & \frac{g^2\,m_2\,(M+m_2)}{M^2\,l_2}+\frac{g^2\,m_1\,m_2}{M^2\,l_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{g^2\,m_1\,(M+m_2)}{M^2\,{l_2}^2}+\frac{g^2\,m_1\,(M+m_1)}{M^2\,l_1\,l_2} & 0 & \frac{g^2\,(M+m_2)^2}{M^2\,{l_2}^2}+\frac{g^2\,m_1\,m_2}{M^2\,l_1\,l_2} & 0 \\ 0 & 0 & 0 & \frac{g^2\,m_1\,(M+m_1)}{M^2\,l_1}+\frac{g^2\,m_1\,m_2}{M^2\,l_2} & 0 & \frac{g^2\,m_2\,(M+m_2)}{M^2\,l_2}+\frac{g^2\,m_1\,m_2}{M^2\,l_1} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{g^2\,m_1\,(M+m_2)}{M^2\,{l_2}^2}+\frac{g^2\,m_1\,(M+m_1)}{M^2\,l_1\,l_2} & 0 & \frac{g^2\,(M+m_2)^2}{M^2\,{l_2}^2}+\frac{g^2\,m_1\,m_2}{M^2\,l_1\,l_2} \end{pmatrix}$$

The observability matrix when the values of M, m_1, m_1, m_2, l_1 and l_2 are substituted,

$$Ob_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{49}{50} & 0 & -\frac{49}{50} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{49}{500} & 0 & -\frac{539}{500} & 0 \\ 0 & 0 & 0 & -\frac{49}{50} & 0 & -\frac{49}{50} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{49}{500} & 0 & -\frac{539}{500} \\ 0 & 0 & \frac{31213}{50000} & 0 & \frac{55223}{50000} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{5709339348407145}{36028797018963968} & 0 & \frac{5255187354783849}{4503599627370496} & 0 \\ 0 & 0 & 0 & \frac{31213}{50000} & 0 & \frac{55223}{50000} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{5709339348407145}{36028797018963968} & 0 & \frac{5255187354783849}{4503599627370496} \end{pmatrix}$$

The rank of both the matrices are 6 and hence the matrix is not observable when $x(t)$ and θ_2 is the only available measurable output. The rank is calculated using MATLAB code.

0.6.4 When Output Vector $x(t)$, θ_1 and θ_2 are available:

We calculate the observability of the system in symbolic as well as when the values are substituted, The following matrix is calculated for observability in symbolic form Ob_4,

$$Ob_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{g\,m_1}{M} & 0 & -\frac{g\,m_2}{M} & 0 \\ 0 & 0 & -\frac{g\,(M+m_1)}{M\,l_1} & 0 & -\frac{g\,m_2}{M\,l_1} & 0 \\ 0 & 0 & -\frac{g\,m_1}{M\,l_2} & 0 & -\frac{g\,(M+m_2)}{M\,l_2} & 0 \\ 0 & 0 & 0 & -\frac{g\,m_1}{M} & 0 & -\frac{g\,m_2}{M} \\ 0 & 0 & 0 & -\frac{g\,(M+m_1)}{M\,l_1} & 0 & -\frac{g\,m_2}{M\,l_1} \\ 0 & 0 & 0 & -\frac{g\,m_1}{M\,l_2} & 0 & -\frac{g\,(M+m_2)}{M\,l_2} \\ 0 & 0 & \frac{g^2\,m_1\,(M+m_1)}{M^2\,l_1} + \frac{g^2\,m_1\,m_2}{M^2\,l_2} & 0 & \frac{g^2\,m_2\,(M+m_2)}{M^2\,l_2} + \frac{g^2\,m_1\,m_2}{M^2\,l_1} & 0 \\ 0 & 0 & \frac{g^2\,(M+m_1)^2}{M^2\,{l_1}^2} + \frac{g^2\,m_1\,m_2}{M^2\,l_1\,l_2} & 0 & \frac{g^2\,m_2\,(M+m_1)}{M^2\,{l_1}^2} + \frac{g^2\,m_2\,(M+m_2)}{M^2\,l_1\,l_2} & 0 \\ 0 & 0 & \frac{g^2\,m_1\,(M+m_2)}{M^2\,{l_2}^2} + \frac{g^2\,m_1\,(M+m_1)}{M^2\,l_1\,l_2} & 0 & \frac{g^2\,(M+m_2)^2}{M^2\,{l_2}^2} + \frac{g^2\,m_1\,m_2}{M^2\,l_1\,l_2} & 0 \\ 0 & 0 & 0 & \frac{g^2\,m_1\,(M+m_1)}{M^2\,l_1} + \frac{g^2\,m_1\,m_2}{M^2\,l_2} & 0 & \frac{g^2\,m_2\,(M+m_2)}{M^2\,l_2} + \frac{g^2\,m_1\,m_2}{M^2\,l_1} \\ 0 & 0 & 0 & \frac{g^2\,(M+m_1)^2}{M^2\,{l_1}^2} + \frac{g^2\,m_1\,m_2}{M^2\,l_1\,l_2} & 0 & \frac{g^2\,m_2\,(M+m_1)}{M^2\,{l_1}^2} + \frac{g^2\,m_2\,(M+m_2)}{M^2\,l_1\,l_2} \\ 0 & 0 & 0 & \frac{g^2\,m_1\,(M+m_2)}{M^2\,{l_2}^2} + \frac{g^2\,m_1\,(M+m_1)}{M^2\,l_1\,l_2} & 0 & \frac{g^2\,(M+m_2)^2}{M^2\,{l_2}^2} + \frac{g^2\,m_1\,m_2}{M^2\,l_1\,l_2} \end{pmatrix}$$

The observability matrix when the values of M, m_1, m_1, m_2, l_1 and l_2 are substituted,

$$Ob_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{49}{50} & 0 & -\frac{49}{50} & 0 \\ 0 & 0 & -\frac{539}{1000} & 0 & -\frac{49}{1000} & 0 \\ 0 & 0 & -\frac{49}{500} & 0 & -\frac{539}{500} & 0 \\ 0 & 0 & 0 & -\frac{49}{50} & 0 & -\frac{49}{50} \\ 0 & 0 & 0 & -\frac{539}{1000} & 0 & -\frac{49}{1000} \\ 0 & 0 & 0 & -\frac{49}{500} & 0 & -\frac{539}{500} \\ 0 & 0 & \frac{31213}{50000} & 0 & \frac{55223}{50000} & 0 \\ 0 & 0 & \frac{1330016552753937}{4503599627370496} & 0 & \frac{5709339348407145}{72057594037927936} & 0 \\ 0 & 0 & \frac{5709339348407145}{36028797018963968} & 0 & \frac{5255187354783849}{4503599627370496} & 0 \\ 0 & 0 & 0 & \frac{31213}{50000} & 0 & \frac{55223}{50000} \\ 0 & 0 & 0 & \frac{1330016552753937}{4503599627370496} & 0 & \frac{5709339348407145}{72057594037927936} \\ 0 & 0 & 0 & \frac{5709339348407145}{36028797018963968} & 0 & \frac{5255187354783849}{4503599627370496} \end{pmatrix}$$

The rank of both the matrices are 6 and hence the matrix is not observable when $x(t)$, θ_1 θ_2 is the only available measurable output. The rank is calculated using MATLAB code.

0.7 Part F: Obtain Best Luenberger Observer for output vectors which are observable:

The best Luenberger observer was obtained from the Matlab simulation, The following plots show the simulated outputs for the developed L Observer for unit step response, Linear and Non-Linear systems,

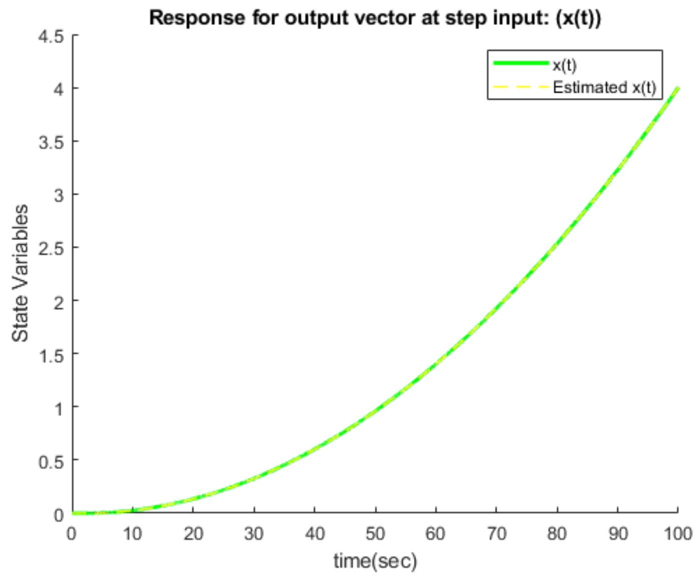

Figure 3: Unit step Response of Luenberger Observer when $x(t)$ is measured

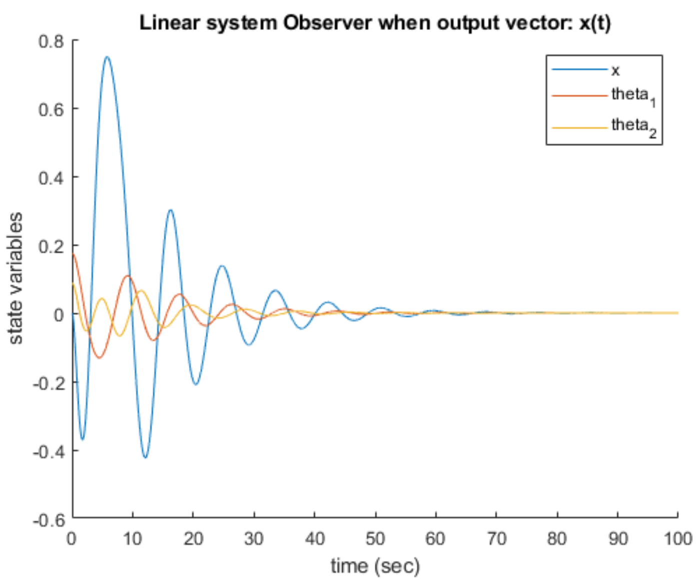

Figure 4: Linear Response of Luenberger Observer when $x(t)$ is measured

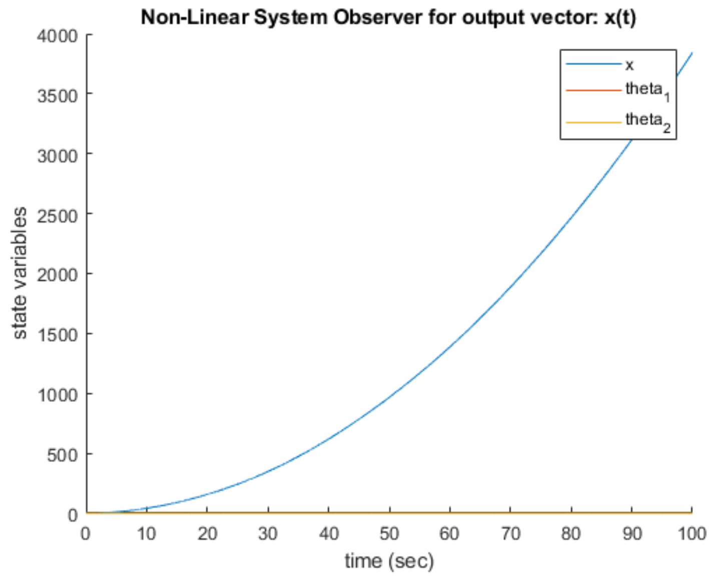

Figure 5: Non Linear Response of Luenberger Observer when $x(t)$ is measured

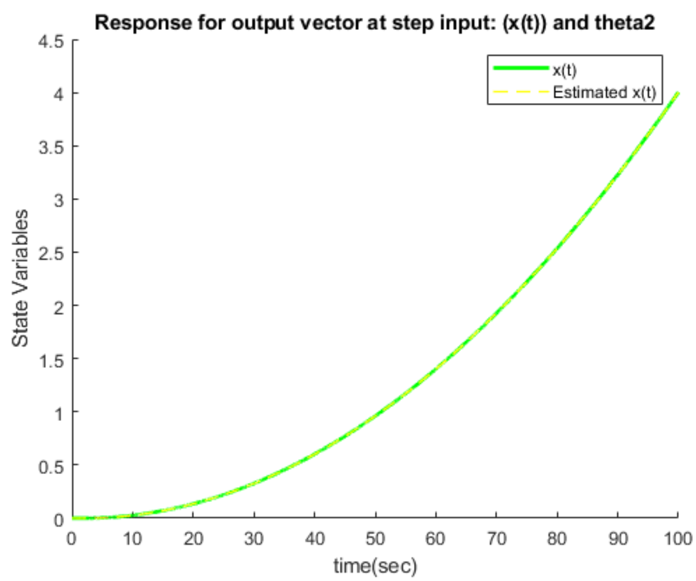

Figure 6: Non Linear Response of Luenberger Observer when $x(t)$ and θ_2 is measured

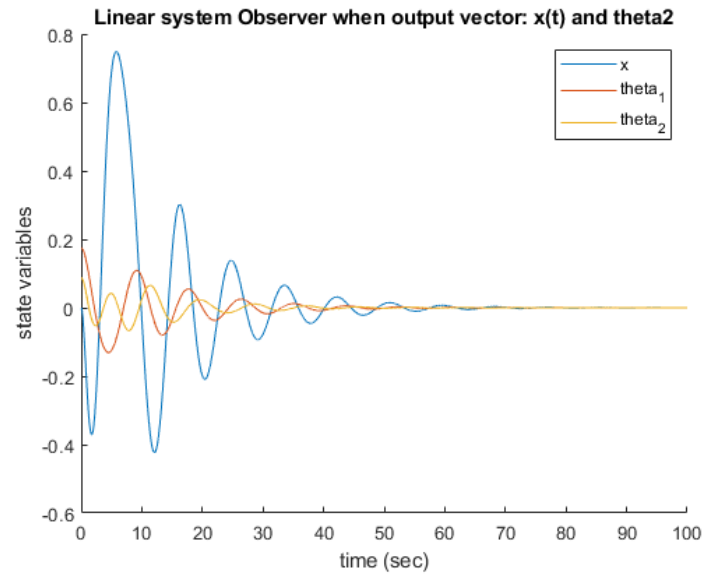

Figure 7: Non Linear Response of Luenberger Observer when $x(t)$ and θ_2 is measured

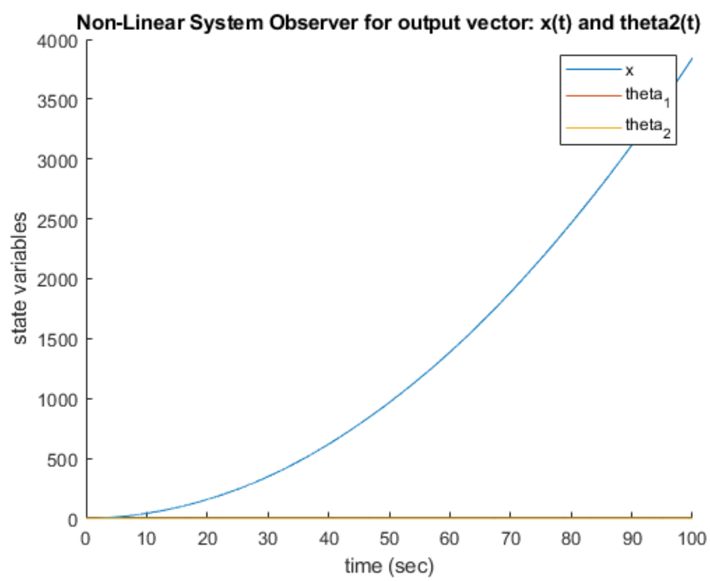

Figure 8: Non Linear Response of Luenberger Observer when $x(t)$ and θ_2 is measured

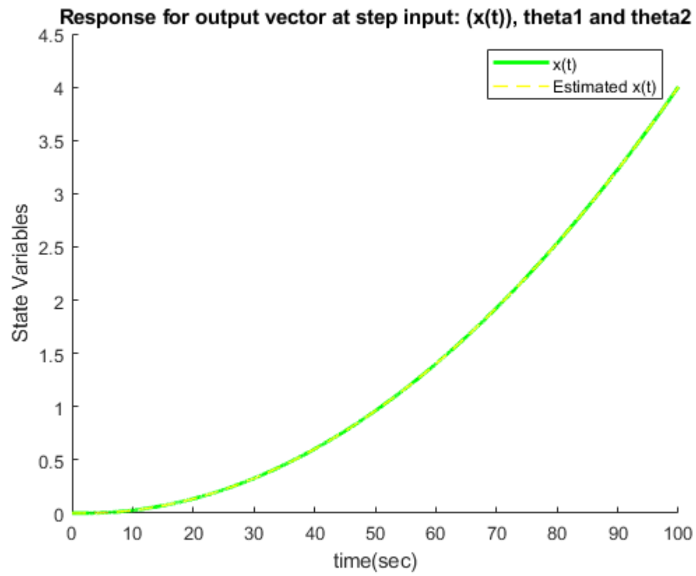

Figure 9: Unit Step Response of Luenberger Observer when $x(t)$, θ_1 and θ_2 is measured

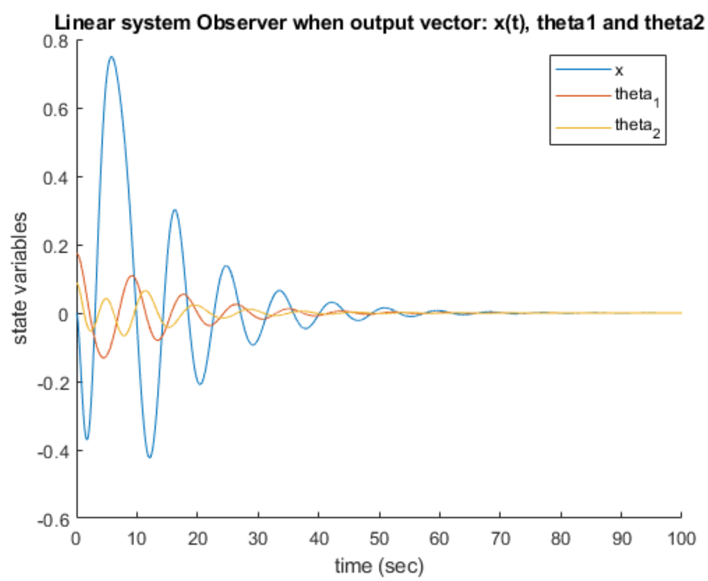

Figure 10: Linear Response of Luenberger Observer when $x(t)$, θ_1 and θ_2 is measured

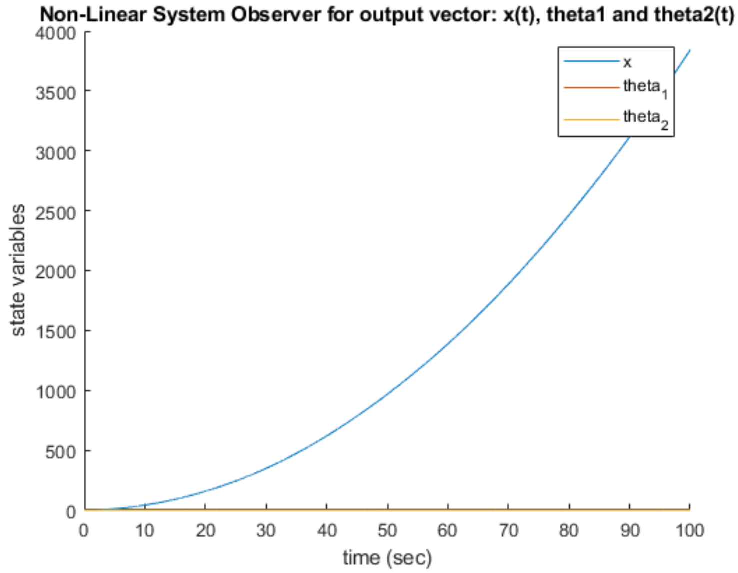

Figure 11: Non Linear Response of Luenberger Observer when $x(t)$, θ_1 and θ_2 is measured

0.8 LQG Controller Design

The "smallest" Output Vector is the one which was considered in Case 1 in Section 6.1.

How would you reconfigure your controller to asymptotically track a constant reference on x? I will set the Q penalty matrix such that the penalty on $x(t)$ is more than other states or set the penalties of other state to zero and keep the penalty of $x(t)$ as 1.

Will your design reject constant force disturbances applied to the cart? Yes it is strong enough to reject most of the disturbances applied to the cart's position x. As it is evident from the figure in the next page the noise in the state is minimal and hence it rejects disturbances as those are modeled in the controller design itself

The graph of the given condition when White Noise $V_n = 0.001$, System Noise $B_d = 0.01$Magnitude 6x6 matrix. The Q matrix $(1, 1)$ element is set to 1000 and rest of the diagonal elements are zero, The final state of $x = 20$ and initial state is $x_0 = 1$. Refer the photo on the next page. Figure 12

0.9 References

- MATLAB Documentation of LQR, LQG, and observability and controllability functions
- Sympy Documentation in python
- ENPM667 Lecture notes for derivation of Euler Lagrange Equations, Linearization parts of the projects

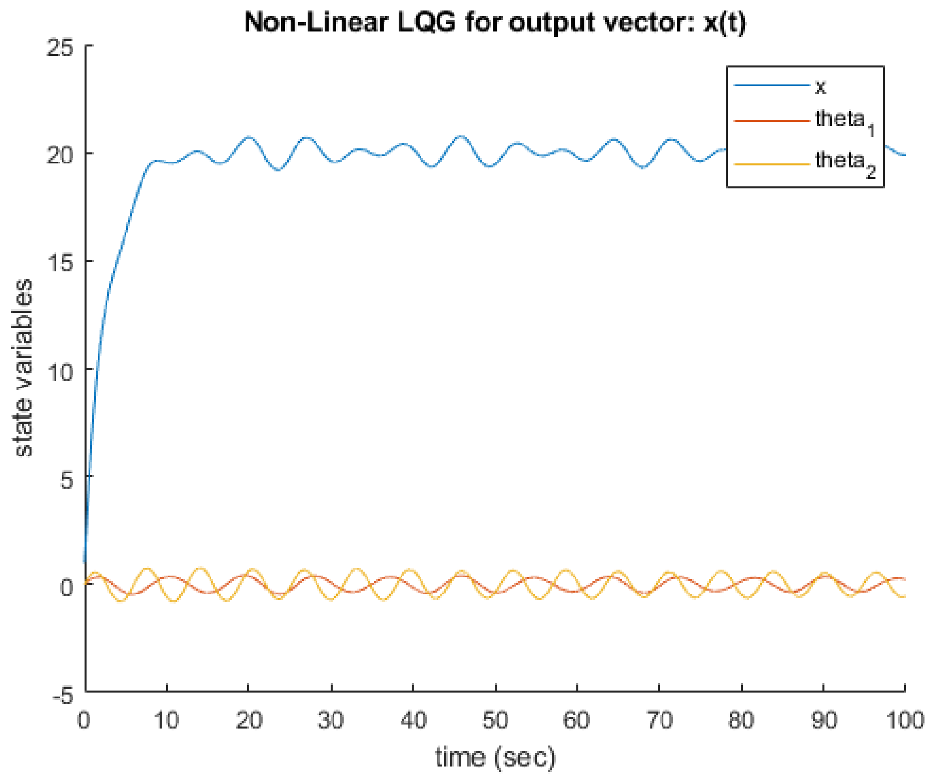

Figure 12: LQG Controller for Non Linear system

0.10 Appendix(CODES OF MATLAB)

```
######PART C MATLAB CODE FOR LQR DESIGN AND SIMULATION######
clc
clear all
syms  m_1 m_2 l_2 l_1 g M
A=[0 1 0 0 0 0;
   0 0 (-g*m_1)/M 0 (-g*m_2)/M 0;
   0 0 0 1 0 0;
   0 0 (-g*(M+m_1))/(l_1*M) 0 (-g*m_2)/(M*l_1) 0;
   0 0 0 0 0 1;
   0 0 (-g*m_1)/(M*l_2) 0 (-g*(M+m_2))/(M*l_2) 0];
B=[0;1/M;0;1/(M*l_1);0;1/(M*l_2)];
Controllability_Mat = [B, A*B, A^2*B, A^3*B, A^4*B, A^5*B];
r_c_mat=rank(Controllability_Mat);
disp("The rank of the matrix of controllability is = ");
r_c_mat
disp("The determinant of controllability matrix is=");
det(Controllability_Mat)
#############################################################

####PART D MATLAB CODE FOR CHECKING OBSERVABILITY OF DIFFERENT STATES####
clc
clear all
syms  m_1 m_2 l_2 l_1 g M
A=[0 1 0 0 0 0;
   0 0 (-g*m_1)/M 0 (-g*m_2)/M 0;
   0 0 0 1 0 0;
```

```
   0 0 (-g*(M+m_1))/(l_1*M) 0 (-g*m_2)/(M*l_1) 0;
   0 0 0 0 0 1;
   0 0 (-g*m_1)/(M*l_2) 0 (-g*(M+m_2))/(M*l_2) 0];
B=[0;
    1/M;
    0;
    1/(M*l_1);
    0;
    1/(M*l_2)];

%%Checking Controllability
A_0 = double(subs(A,[M m_1 m_2 l_1 l_2 g], [1000 100 100 20 10 9.8]));
B_0 = double(subs(B,[M m_1 m_2 l_1 l_2 g], [1000 100 100 20 10 9.8]));
disp("Controllability of the system is given by getting rank of output of MATLAB's ctrb function: ");
disp(rank(ctrb(A_0,B_0)));

%% Section for Calculating the LQR controller gain
syms  m_1 m_2 l_2 l_1 g M
A=[0 1 0 0 0 0;
   0 0 (-g*m_1)/M 0 (-g*m_2)/M 0;
   0 0 0 1 0 0;
   0 0 (-g*(M+m_1))/(l_1*M) 0 (-g*m_2)/(M*l_1) 0;
   0 0 0 0 0 1;
   0 0 (-g*m_1)/(M*l_2) 0 (-g*(M+m_2))/(M*l_2) 0];
B=[0;
    1/M;
    0;
    1/(M*l_1);
    0;
    1/(M*l_2)];

A_1 = double(subs(A,[M m_1 m_2 l_1 l_2 g], [1000 100 100 20 10 9.8]));
B_1 = double(subs(B,[M m_1 m_2 l_1 l_2 g], [1000 100 100 20 10 9.8]));
Q = [10 0 0 0 0 0;
    0 10 0 0 0 0;
    0 0 1000 0 0 0;
    0 0 0 1000 0 0;
    0 0 0 0 1000 0;
    0 0 0 0 0 1000];
R= 0.0001;

K_2 = lqr(A_1, B_1, Q, R);

X_initial = [0.1 ; 0.1; deg2rad(15); 0.2; deg2rad(10); 0.01];
X_final = [0;0;0;0;0;0];
tinterval = 0:0.01:100;
U = @(X) -K_2*(X - X_final);     %% Control input
[t,X] = ode45(@(t,X)statespace(A_1, B_1, X, U(X)), tinterval, X_initial);

figure(2)
title('LQR on the linear controller');
subplot(3,1,1);
plot(t, X(:,1));
xlabel('time(s)');
ylabel('x');
subplot(3,1,2);
```

```
plot(t, X(:,2));
xlabel('time(s)');
ylabel('theta1');
subplot(3,1,3);
plot(t, X(:,3));
xlabel('time(s)');
ylabel('theta2');
%% Checking stability using Lyapunov's criterion
A_new = A_1 - B_1*K_2;
lat11=latex(simplify(sym(A_new)));
disp ("Eigen values of the closed loop matrix is given by: ");
lat12=latex(sym(eig(A_new)));
disp (eig(A_new));




%% Simulating for a Non-Linear System
syms  m_1 m_2 l_2 l_1 g M
A=[0 1 0 0 0 0;
   0 0 (-g*m_1)/M 0 (-g*m_2)/M 0;
   0 0 0 1 0 0;
   0 0 (-g*(M+m_1))/(l_1*M) 0 (-g*m_2)/(M*l_1) 0;
   0 0 0 0 0 1;
   0 0 (-g*m_1)/(M*l_2) 0 (-g*(M+m_2))/(M*l_2) 0];
B=[0;
    1/M;
    0;
    1/(M*l_1);
    0;
    1/(M*l_2)];

A_2 = double(subs(A,[M m_1 m_2 l_1 l_2 g], [1000 100 100 20 10 9.8]));
B_2 = double(subs(B,[M m_1 m_2 l_1 l_2 g], [1000 100 100 20 10 9.8]));

X_initial = [0.1 ; 0.1; deg2rad(15); 0.2; deg2rad(10); 0.01];
X_final = [0;0;0;0;0;0];
tinterval = 0:0.01:100;
[t,X2] = ode45(@(t,X2)NonLinear_for_LQR(t, X2, -K_2*X2), tinterval, X_initial);
figure(3);
hold on
plot(t,X2(:,1))
plot(t,X2(:,3))
plot(t,X2(:,5))
ylabel('state variables')
xlabel('time (sec)')
title('Non-Linear system using LQR controller')
legend('x','theta1','theta2')
disp(K_2);
%% Functions definitions used in this code
%Constructing a function to find state space matrix to feed into the ode45
%function
function Xdot = statespace(A, B, X, U)
    Xdot = A*X + B*U
end
function dX = NonLinear_for_LQR(t,x,f)
```

```
    m1 = 100; m2 = 100; M=1000; L1 = 20; L2 = 10; g = 9.81;
    X_ = x(1);
    X_dot = x(2);
    theta_1 = x(3);
    theta_dot1 = x(4);
    theta_2 = x(5);
    theta_dot2 = x(6);
    dX = zeros(6,1);
    dX(1) = X_dot;
    dX(2) = (f - m1*L1*sin(theta_1)*theta_dot1^2 - m2*L2*sin(theta_2)*theta_dot2^2 - m1*g*sin(theta_1)*cos
    dX(3) = theta_dot1;
    dX(4) = cos(theta_1)*dX(2)/L1 - g*sin(theta_1)/L1;
    dX(5) = theta_dot2;
    dX(6) = cos(theta_2)*dX(2)/L2 - g*sin(theta_2)/L2;
end
##################################################################
OUTPUT OF THE CODE:


#####PART E MATLAB CODE FOR CHECKING OBSERVABILITY#########
clc
clear all
syms  m_1 m_2 l_2 l_1 g M
%Case 1: When x of the system is only measurable or observable
A=[0 1 0 0 0 0;
   0 0 (-g*m_1)/M 0 (-g*m_2)/M 0;
   0 0 0 1 0 0;
   0 0 (-g*(M+m_1))/(l_1*M) 0 (-g*m_2)/(M*l_1) 0;
   0 0 0 0 0 1;
   0 0 (-g*m_1)/(M*l_2) 0 (-g*(M+m_2))/(M*l_2) 0];
C_1=[1 0 0 0 0 0;
    0 0 0 0 0 0;
    0 0 0 0 0 0];
O_1 = [C_1 ; C_1*A ; C_1*A^2 ; C_1*A^3 ; C_1*A^4 ; C_1*A^5];
lat_1 = O_1;
lat__1 = latex(lat_1);
disp("Observability of system when only x is measureable is given by Rank of the observability matrix");
disp("Rank using manual derivation of observability matrix is given by:")
disp(rank(O_1));
if(rank(O_1)<6)
    disp("System is not observable using the given observable parameters");
else
    disp("System is observable since matrix is of full rank");
end
disp("______________________________________________________________________");

%% Substituting values of M, m_1, m_2, l_1, l_2
%Case 1: When x of the system is only measurable or observable with values
%substituted
A_0 = double(subs(A,[M m_1 m_2 l_1 l_2 g], [1000 100 100 20 10 9.8]));
disp("Rank using Matlab function of observability is given as(after substituting variables: ");
Observ_using_matlab = rank(obsv(A_0,C_1));
disp(Observ_using_matlab);
O1_ = obsv(A_0,C_1);
lat_ = latex(sym(O1_));
if(Observ_using_matlab<6)
```

```
    disp("System is not observable using the given observable parameters");
else
    disp("System is observable since matrix is of full rank");
end
disp("______________________________________________________________________");
%% Case 2: When theta_1 and theta_2 of the system is only measurable or observable
C_2=[0 0 0 0 0 0;
    0 0 1 0 0 0;
    0 0 0 0 1 0];
O_2 = [C_2 ; C_2*A ; C_2*A^2 ; C_2*A^3 ; C_2*A^4 ; C_2*A^5];
disp("Observability of system when only theta_1 and theta_2 are observed is given by Rank of the observabi
disp("Rank using manual derivation of observability matrix is given by:")
disp(rank(O_2));
lat_2 = O_2;
lat__2 = latex(simplify(lat_2));
if(rank(O_2)<6)
    disp("System is not observable using the given observable parameters");
else
    disp("System is observable since matrix is of full rank");
end
disp("______________________________________________________________________");
%% Case2 : When theta-1 and theta-2 are observed with values of Mass and lengths substituted
A_1 = double(subs(A,[M m_1 m_2 l_1 l_2 g], [1000 100 100 20 10 9.8]));
disp("Rank using Matlab function of observability is given as(after substituting variables: ");
Observ_using_matlab_1 = rank(obsv(A_1,C_2));
disp(Observ_using_matlab_1);
O2_ = obsv(A_1,C_2);
lat_1_ = latex(simplify(sym(O2_)));
if(Observ_using_matlab_1<6)
    disp("System is not observable using the given observable parameters");
else
    disp("System is observable since matrix is of full rank");
end
disp("______________________________________________________________________");
%% Case3: When x and theta_2 are taken as feedback for the system
C_3=[1 0 0 0 0 0;
    0 0 0 0 0 0;
    0 0 0 0 1 0];
O_3 = [C_3 ; C_3*A ; C_3*A^2 ; C_3*A^3 ; C_3*A^4 ; C_3*A^5];
disp("Observability of system when only x and theta_2 are observed is given by Rank of the observability m
disp("Rank using manual derivation of observability matrix is given by:")
disp(rank(O_3));
lat_3 = latex(O_3);
if(rank(O_3)<6)
    disp("System is not observable using the given observable parameters");
else
    disp("System is observable since matrix is of full rank");
end
disp("______________________________________________________________________");
%%  Case3: When x and theta_2 are taken as feedback for the system with values of mass and lengths substit
A_2 = double(subs(A,[M m_1 m_2 l_1 l_2 g], [1000 100 100 20 10 9.8]));
disp("Rank using Matlab function of observability is given as(after substituting variables: ");
Observ_using_matlab_2 = rank(obsv(A_2,C_3));
disp(Observ_using_matlab_2);
lat__3 = latex(sym(obsv(A_2,C_3)));
if(Observ_using_matlab_2<6)
```

```
    disp("System is not observable using the given observable parameters");
else
    disp("System is observable since matrix is of full rank");
end
disp("______________________________________________________________________");
%% Case4 : when x, theta_1, theta_2 are observable
C_4=[1 0 0 0 0 0;
    0 0 1 0 0 0;
    0 0 0 0 1 0];
O_4 = [C_4 ; C_4*A ; C_4*A^2 ; C_4*A^3 ; C_4*A^4 ; C_4*A^5];
disp("Observability of system when only x, theta_1 and theta_2 are observed is given by Rank of the obser
disp("Rank using manual derivation of observability matrix is given by:")
disp(rank(O_4));
lat_4 = latex(O_4);
if(rank(O_4)<6)
    disp("System is not observable using the given observable parameters");
else
    disp("System is observable since matrix is of full rank");
end
disp("______________________________________________________________________");
%% Case4 : when x, theta_1, theta_2 are observable
A_3 = double(subs(A,[M m_1 m_2 l_1 l_2 g], [1000 100 100 20 10 9.8]));
disp("Rank using Matlab function of observability is given as(after substituting variables: ");
Observ_using_matlab_2 = rank(obsv(A_3,C_4));
disp(Observ_using_matlab_2);
lat__4 = latex(sym(obsv(A_3,C_4)));
if(Observ_using_matlab_2<6)
    disp("System is not observable using the given observable parameters");
else
    disp("System is observable since matrix is of full rank");
end
disp("______________________________________________________________________");

###########OUTPUT OF THE CODE##############
Observability of system when only x is measureable is given by Rank of the observability matrix
Rank using manual derivation of observability matrix is given by:
     6

System is observable since matrix is of full rank
______________________________________________________________________
Rank using Matlab function of observability is given as(after substituting variables:
     6

System is observable since matrix is of full rank
______________________________________________________________________
Observability of system when only theta_1 and theta_2 are observed is given by Rank of the observability m
Rank using manual derivation of observability matrix is given by:
     4

System is not observable using the given observable parameters
______________________________________________________________________
Rank using Matlab function of observability is given as(after substituting variables:
     4

System is not observable using the given observable parameters
______________________________________________________________________
```

```
Observability of system when only x and theta_2 are observed is given by Rank of the observability matrix
Rank using manual derivation of observability matrix is given by:
     6

System is observable since matrix is of full rank
----------------------------------------------------------------------
Rank using Matlab function of observability is given as(after substituting variables:
     6

System is observable since matrix is of full rank
----------------------------------------------------------------------
Observability of system when only x, theta_1 and theta_2 are observed is given by Rank of the observabilit
Rank using manual derivation of observability matrix is given by:
     6

System is observable since matrix is of full rank
----------------------------------------------------------------------
Rank using Matlab function of observability is given as(after substituting variables:
     6

System is observable since matrix is of full rank
----------------------------------------------------------------------

######PART F: MATLAB CODE FOR OBTAINING LUENBERGER OBSERVER######
clc
clear all
syms  m_1 m_2 l_2 l_1 g M
%% %Case 1: When x of the system is only measurable or observable
A=[0 1 0 0 0 0;
   0 0 (-g*m_1)/M 0 (-g*m_2)/M 0;
   0 0 0 1 0 0;
   0 0 (-g*(M+m_1))/(l_1*M) 0 (-g*m_2)/(M*l_1) 0;
   0 0 0 0 0 1;
   0 0 (-g*m_1)/(M*l_2) 0 (-g*(M+m_2))/(M*l_2) 0];
B=[0;
    1/M;
    0;
    1/(M*l_1);
    0;
    1/(M*l_2)];
C_1=[1 0 0 0 0 0;
    0 0 0 0 0 0;
    0 0 0 0 0 0];
D = 0;
%Initial Conditions
X0 = [0.0; 0; deg2rad(10); 0; deg2rad(5); 0];
%Substituting values of m1 l1 l2 and M in A
A1=double(subs(A,[M m_1 m_2 l_1 l_2 g], [1000 100 100 20 10 9.8]));
B1 = double(subs(B,[M m_1 m_2 l_1 l_2 g], [1000 100 100 20 10 9.8]));
%state space of the system
sys_1 = ss(A1, B1, C_1, D);
%Disturbance and Measurement noise
B_d = 0.00001*eye(6,6);
V_t = 0.00001*eye(3,3);
[L_1,P, E] = lqe(A1, B_d, C_1, B_d,V_t);
%A matrix with augmented state estimation
```

```
Ac_1 = A1 - L_1*C_1;
%state space with output state estimation
sys1 = ss(Ac_1, [B1 L_1], C_1, 0);
%Unit step response
ts = 0:0.01:100;
unitStep = 0*ts;
unitStep(200:length(ts)) = 1;

[Y1,t] = lsim(sys_1,unitStep, ts);
[X1,t] = lsim(sys1,[unitStep;Y1'],ts);

figure();
hold on
plot(t,Y1(:,1),'g','Linewidth',2)
plot(t,X1(:,1),'y--','Linewidth',1)
ylabel('State Variables')
xlabel('time(sec)')
legend('x(t)','Estimated x(t)')
title('Response for output vector at step input: (x(t))')

[t,x1] = ode45(@(t,x)linearsimulationforEstimation_1(t, x  ,L_1, A1, B1, C_1),ts,X0);
figure();
hold on
plot(t,x1(:,1))
plot(t,x1(:,3))
plot(t,x1(:,5))
ylabel('state variables')
xlabel('time (sec)')
title('Linear system Observer when output vector: x(t)')
legend('x','theta_1','theta_2')
hold off

[t,state1] = ode45(@(t,state)nonLinear1_observer(t,state,1.0,L_1),ts,X0);
figure();
hold on
plot(t,state1(:,1))
plot(t,state1(:,3))
plot(t,state1(:,5))
ylabel('state variables')
xlabel('time (sec)')
title('Non-Linear System Observer for output vector: x(t)')
legend('x','theta_1','theta_2')
hold off
%% %Case 2: When x and theta 2 of the system is only measurable or observable
C_3=[1 0 0 0 0 0;
    0 0 0 0 0 0;
    0 0 0 0 1 0];
%Substituting values of m1 l1 l2 and M in A
A3=double(subs(A,[M m_1 m_2 l_1 l_2 g], [1000 100 100 20 10 9.8]));
B3 = double(subs(B,[M m_1 m_2 l_1 l_2 g], [1000 100 100 20 10 9.8]));
%state space of the system
sys_3 = ss(A3, B3, C_3, D);
%Disturbance and Measurement noise
B_d = 0.00001*eye(6,6);
V_t = 0.00001*eye(3,3);
[L_3,P, E] = lqe(A3, B_d, C_3, B_d,V_t);
```

```
%A matrix with augmented state estimation
Ac_3 = A3 - L_3*C_3;
%state space with output state estimation
sys3 = ss(Ac_3, [B3 L_3], C_3, 0);
%Unit step response
ts3 = 0:0.01:100;
unitStep = 0*ts3;
unitStep(200:length(ts3)) = 1;

[Y3,t] = lsim(sys_3,unitStep, ts3);
[X3,t] = lsim(sys3,[unitStep;Y3'],ts3);

figure();
hold on
plot(t,Y3(:,1),'g','Linewidth',2)
plot(t,X3(:,1),'y--','Linewidth',1)
ylabel('State Variables')
xlabel('time(sec)')
legend('x(t)','Estimated x(t)')
title('Response for output vector at step input: (x(t)) and theta2')

[t,x3] = ode45(@(t,x)linearsimulationforEstimation_3(t, x  ,L_3, A3, B3, C_3),ts3,X0);
figure();
hold on
plot(t,x3(:,1))
plot(t,x3(:,3))
plot(t,x3(:,5))
ylabel('state variables')
xlabel('time (sec)')
title('Linear system Observer when output vector: x(t) and theta2')
legend('x','theta_1','theta_2')
hold off

[t,state3] = ode45(@(t,state)nonLinear3_observer(t,state,1.0,L_3),ts3,X0);
figure();
hold on
plot(t,state3(:,1))
plot(t,state3(:,3))
plot(t,state3(:,5))
ylabel('state variables')
xlabel('time (sec)')
title('Non-Linear System Observer for output vector: x(t) and theta2(t)')
legend('x','theta_1','theta_2')
hold off
%% Case3 When x, theta1 and theta2 are observable
C_4=[1 0 0 0 0 0;
    0 0 1 0 0 0;
    0 0 0 0 1 0];
%Substituting values of m1 l1 l2 and M in A
A4=double(subs(A,[M m_1 m_2 l_1 l_2 g], [1000 100 100 20 10 9.8]));
B4 = double(subs(B,[M m_1 m_2 l_1 l_2 g], [1000 100 100 20 10 9.8]));
%state space of the system
sys_4 = ss(A4, B4, C_4, D);
%Disturbance and Measurement noise
B_d = 0.00001*eye(6,6);
V_t = 0.00001*eye(3,3);
```

```
[L_4,P, E] = lqe(A4, B_d, C_4, B_d,V_t);
%A matrix with augmented state estimation
Ac_4 = A4 - L_4*C_4;
%state space with output state estimation
sys4 = ss(Ac_4, [B4 L_4], C_4, 0);
%Unit step response
ts4 = 0:0.01:100;
unitStep = 0*ts4;
unitStep(200:length(ts4)) = 1;

[Y4,t] = lsim(sys_4,unitStep, ts4);
[X4,t] = lsim(sys4,[unitStep;Y4'],ts4);

figure();
hold on
plot(t,Y4(:,1),'g','Linewidth',2)
plot(t,X4(:,1),'y--','Linewidth',1)
ylabel('State Variables')
xlabel('time(sec)')
legend('x(t)','Estimated x(t)')
title('Response for output vector at step input: (x(t)), theta1 and theta2')

[t,x4] = ode45(@(t,x)linearsimulationforEstimation_4(t, x  ,L_4, A4, B4, C_4),ts4,X0);
figure();
hold on
plot(t,x4(:,1))
plot(t,x4(:,3))
plot(t,x4(:,5))
ylabel('state variables')
xlabel('time (sec)')
title('Linear system Observer when output vector: x(t), theta1 and theta2')
legend('x','theta_1','theta_2')
hold off

[t,state4] = ode45(@(t,state)nonLinear4_observer(t,state,1.0,L_4),ts4,X0);
figure();
hold on
plot(t,state4(:,1))
plot(t,state4(:,3))
plot(t,state4(:,5))
ylabel('state variables')
xlabel('time (sec)')
title('Non-Linear System Observer for output vector: x(t), theta1 and theta2(t)')
legend('x','theta_1','theta_2')
hold off

%% Functions used in the code for case1
%Linear Estimation Simulation function for x(t) observable case
function dX1 = linearsimulationforEstimation_1(t,x1,L1,A1,B1,C1)
    y1 = [x1(1); 0; 0];
%     Q = [10 0 0 0 0 0;
%     0 10 0 0 0 0;
%     0 0 1000 0 0 0;
%     0 0 0 1000 0 0;
%     0 0 0 0 1000 0;
%     0 0 0 0 0 1000];
```

```
%     R= 0.0001;
    K1 = [316.227766016840 1113.60854371561 2393.64061272093 3023.36108957550 3512.76362080512
766.302444401543];
    dX1 =  (A1-B1*K1)*x1+L1*(y1 - C1*x1);
end
%Non-Linear Estimation Simulation function for x(t) observable case
function ds1 = nonLinear1_observer(t,x,f,l1)
    m1 = 100; m2 = 100; M=1000; L1 = 20; L2 = 10; g = 9.81;
    X_ = x(1);
    X_dot = x(2);
    theta_1 = x(3);
    theta_dot1 = x(4);
    theta_2 = x(5);
    theta_dot2 = x(6);
    ds1 = zeros(6,1);
    y1 = [X_;0;0];
    c_1 = [1 0 0 0 0 0; 0 0 0 0 0 0; 0 0 0 0 0 0];
    iter = l1*(y1-c_1*x);
    ds1(1) = X_dot+iter(1);
    ds1(2) = (f - m1*L1*sin(theta_1)*theta_dot1^2 - (m2*L2*sin(theta_2)*theta_dot2^2 - m1*g*sin(theta_1)*c
    ds1(3) = theta_dot1+iter(3);
    ds1(4) = cos(theta_1)*ds1(2)/L1 - (g*sin(theta_1)/L1)+iter(4);
    ds1(5) = theta_dot2+iter(5);
    ds1(6) = cos(theta_2)*ds1(2)/L2 - (g*sin(theta_2)/L2)+iter(6);
end
%% %% Functions used in the code for case2
%Linear Estimation Simulation function for x(t) observable case
function dX1 = linearsimulationforEstimation_3(t,x1,L1,A1,B1,C1)
    y1 = [x1(1); 0; 0];
%     Q = [10 0 0 0 0 0;
%     0 10 0 0 0 0;
%     0 0 1000 0 0 0;
%     0 0 0 1000 0 0;
%     0 0 0 0 1000 0;
%     0 0 0 0 0 1000];
%     R= 0.0001;
    K1 = [316.227766016840 1113.60854371561 2393.64061272093 3023.36108957550 3512.76362080512
766.302444401543];
    dX1 =  (A1-B1*K1)*x1+L1*(y1 - C1*x1);
end
%Non-Linear Estimation Simulation function for x(t) observable case
function ds1 = nonLinear3_observer(t,x,f,l1)
    m1 = 100; m2 = 100; M=1000; L1 = 20; L2 = 10; g = 9.81;
    X_ = x(1);
    X_dot = x(2);
    theta_1 = x(3);
    theta_dot1 = x(4);
    theta_2 = x(5);
    theta_dot2 = x(6);
    ds1 = zeros(6,1);
    y1 = [X_;0;0];
    c_1 = [1 0 0 0 0 0; 0 0 0 0 0 0; 0 0 0 0 0 0];
    iter = l1*(y1-c_1*x);
    ds1(1) = X_dot+iter(1);
    ds1(2) = (f - m1*L1*sin(theta_1)*theta_dot1^2 - (m2*L2*sin(theta_2)*theta_dot2^2 - m1*g*sin(theta_1)*c
    ds1(3) = theta_dot1+iter(3);
```

```
    ds1(4) = cos(theta_1)*ds1(2)/L1 - (g*sin(theta_1)/L1)+iter(4);
    ds1(5) = theta_dot2+iter(5);
    ds1(6) = cos(theta_2)*ds1(2)/L2 - (g*sin(theta_2)/L2)+iter(6);
end
%% Functions used in the code for case2
%Linear Estimation Simulation function for x(t) observable case
function dX1 = linearsimulationforEstimation_4(t,x1,L1,A1,B1,C1)
    y1 = [x1(1); 0; 0];
%     Q = [10 0 0 0 0 0;
%     0 10 0 0 0 0;
%     0 0 1000 0 0 0;
%     0 0 0 1000 0 0;
%     0 0 0 0 1000 0;
%     0 0 0 0 0 1000];
%     R= 0.0001;
    K1 = [316.227766016840 1113.60854371561 2393.64061272093 3023.36108957550 3512.76362080512
766.302444401543];
    dX1 =  (A1-B1*K1)*x1+L1*(y1 - C1*x1);
end
%Non-Linear Estimation Simulation function for x(t) observable case
function ds1 = nonLinear4_observer(t,x,f,l1)
    m1 = 100; m2 = 100; M=1000; L1 = 20; L2 = 10; g = 9.81;
    X_ = x(1);
    X_dot = x(2);
    theta_1 = x(3);
    theta_dot1 = x(4);
    theta_2 = x(5);
    theta_dot2 = x(6);
    ds1 = zeros(6,1);
    y1 = [X_;0;0];
    c_1 = [1 0 0 0 0 0; 0 0 0 0 0 0; 0 0 0 0 0 0];
    iter = l1*(y1-c_1*x);
    ds1(1) = X_dot+iter(1);
    ds1(2) = (f - m1*L1*sin(theta_1)*theta_dot1^2 - (m2*L2*sin(theta_2)*theta_dot2^2 - m1*g*sin(theta_1)*c
    ds1(3) = theta_dot1+iter(3);
    ds1(4) = cos(theta_1)*ds1(2)/L1 - (g*sin(theta_1)/L1)+iter(4);
    ds1(5) = theta_dot2+iter(5);
    ds1(6) = cos(theta_2)*ds1(2)/L2 - (g*sin(theta_2)/L2)+iter(6);
end
####OUTPUT OF THE CODE#####
PLOTS DISPLAYED EARLIER

#####PART G: LQG DESIGN #####
clc
clear all
syms  m_1 m_2 l_2 l_1 g M
%Initial Conditions
X0 = [1; 0; deg2rad(0); 0; deg2rad(0); 0]
A=[0 1 0 0 0 0;
   0 0 (-g*m_1)/M 0 (-g*m_2)/M 0;
   0 0 0 1 0 0;
   0 0 (-g*(M+m_1))/(l_1*M) 0 (-g*m_2)/(M*l_1) 0;
   0 0 0 0 0 1;
   0 0 (-g*m_1)/(M*l_2) 0 (-g*(M+m_2))/(M*l_2) 0];
B=[0;
    1/M;
```

```
    0;
    1/(M*l_1);
    0;
    1/(M*l_2)];
D=0;
C_1=[1 0 0 0 0 0;
    0 0 0 0 0 0;
    0 0 0 0 0 0];

Q = [1000 0 0 0 0 0;
    0 0 0 0 0 0;
    0 0 0 0 0 0;
    0 0 0 0 0 0;
    0 0 0 0 0 0;
    0 0 0 0 0 0];
R= 0.01;
%Substituting values of m1 l1 l2 and M in A
A1=double(subs(A,[M m_1 m_2 l_1 l_2 g], [1000 100 100 20 10 9.8]));
B1 = double(subs(B,[M m_1 m_2 l_1 l_2 g], [1000 100 100 20 10 9.8]));

[K1,S,P] = lqr(A1, B1, Q, R);

sys = ss(A1-B1*K1,B1,C_1,D);

%Kalman Estimator Design
Bd = 0.01*eye(6); %disturbance
Vn = 0.001;     %Gaussian White Noise
[L,P,E] = lqe(A1,Bd,C_1,Bd,Vn*eye(3)); %Considering vector output: x(t)
Ac1 = A1-(L*C_1);
Xf = [20;0;0;0;0;0]
e_sys1 = ss(Ac1,[B1 L],C_1,0);
ts = 0:0.01:100;
[t,state1] = ode45(@(t,state)nonLinear1_LQG(t,state,-K1*(state-Xf),L),ts,X0);
figure();
hold on
plot(t,state1(:,1))
plot(t,state1(:,3))
plot(t,state1(:,5))
ylabel('state variables')
xlabel('time (sec)')
title('Non-Linear LQG for output vector: x(t)')
legend('x','theta_1','theta_2')
hold off



%% LQG Non Linear
function ds1 = nonLinear1_LQG(t,x,f,l1)
    m1 = 100; m2 = 100; M=1000; L1 = 20; L2 = 10; g = 9.81;
    X_ = x(1);
    X_dot = x(2);
    theta_1 = x(3);
    theta_dot1 = x(4);
    theta_2 = x(5);
    theta_dot2 = x(6);
```

```
    ds1 = zeros(6,1);
    y1 = [X_;0;0];
    c_1 = [1 0 0 0 0 0; 0 0 0 0 0 0; 0 0 0 0 0 0];
    iter = l1*(y1-c_1*x);
    ds1(1) = X_dot+iter(1);
    ds1(2) = (f - m1*L1*sin(theta_1)*theta_dot1^2 - (m2*L2*sin(theta_2)*theta_dot2^2 - m1*g*sin(theta_1)*c
    ds1(3) = theta_dot1+iter(3);
    ds1(4) = cos(theta_1)*ds1(2)/L1 - (g*sin(theta_1)/L1)+iter(4);
    ds1(5) = theta_dot2+iter(5);
    ds1(6) = cos(theta_2)*ds1(2)/L2 - (g*sin(theta_2)/L2)+iter(6);
end
```