Section B (b) [Mathematica file → "Normal vector to surface of spheroid."]

NORMAL VECTOR CALCULATION TO SURFACE OF SPHEROID →

→ The position vector to surface of spheroid can be written as,

$$\vec{v}(\theta,\varphi) = r(\theta,\varphi)\,\hat{r} \quad \text{—— (1)b}$$

where $r(\theta,\varphi)$ is parametrization of the surface.

for spheroid, $r(\theta,\varphi) = r(\theta) = R - Rs\cos(2\theta)$
which is given in eq 15a.

→ normal vector to any arbitrary surface with position vector $\vec{v}(\theta,\varphi)$ is given by

$$\vec{n} = \frac{\partial\vec{v}(\theta,\varphi)}{\partial\theta} \times \frac{\partial\vec{v}(\theta,\varphi)}{\partial\varphi} \quad \text{—— (2)b}$$

$$\|\vec{n}\| = \left| \frac{\partial\vec{v}(\theta,\varphi)}{\partial\theta} \times \frac{\partial\vec{v}(\theta,\varphi)}{\partial\varphi} \right| \quad \text{—— (2)b}$$

as we know, $\hat{n} = \dfrac{\vec{n}}{\|\vec{n}\|}$

[Ref :- Vector Calculus, 6th Edition
by Jarrold E. Marsden]

Calculating normal vector,

$$\frac{\partial \vec{v}(\theta,\varphi)}{\partial \theta} = \frac{\partial r(\theta,\varphi)}{\partial \theta}\hat{r} + r(\theta,\varphi)\frac{\partial \hat{r}}{\partial \theta} \quad \text{--- (4) b}$$

$$\frac{\partial \vec{v}(\theta,\varphi)}{\partial \varphi} = \frac{\partial r(\theta,\varphi)}{\partial \varphi}\hat{r} + r(\theta,\varphi)\frac{\partial \hat{r}}{\partial \varphi} \quad \text{--- (5) b}$$

as we know,

$$\frac{\partial \hat{r}}{\partial \theta} = \hat{\theta} \quad , \quad \frac{\partial \hat{r}}{\partial \varphi} = \sin\theta\,\hat{\varphi} \quad \text{--- (6) b}$$

$$\frac{\partial \vec{v}(\theta,\varphi)}{\partial \theta} = \frac{\partial r(\theta,\varphi)}{\partial \theta}\hat{r} + r(\theta,\varphi)\hat{\theta} \quad \text{--- (7) b}$$

$$\frac{\partial \vec{v}(\theta,\varphi)}{\partial \varphi} = \frac{\partial r(\theta,\varphi)}{\partial \varphi}\hat{r} + r(\theta,\varphi)\sin\theta\,\hat{\varphi} \quad \text{--- (8) b}$$

$$\vec{n} = \frac{\partial \vec{v}(\theta,\varphi)}{\partial \theta} \times \frac{\partial \vec{v}(\theta,\varphi)}{\partial \varphi}$$

$$\left[\begin{aligned} \vec{n} &= -r(\theta,\varphi)\sin\theta\frac{\partial r(\theta,\varphi)}{\partial \theta}\hat{\theta} - r(\theta,\varphi)\frac{\partial r(\theta,\varphi)}{\partial \varphi}\hat{\varphi} \\ &\quad + (r(\theta,\varphi))^2\sin\theta\,\hat{r} \end{aligned}\right] \text{--- (9) b}$$

for Spheroid, $r(\theta,\varphi) = r(\theta) = R - Rs\cos(2\theta)$

$$\left[\begin{aligned} \frac{\partial r(\theta,\varphi)}{\partial \theta} &= \frac{\partial r(\theta)}{\partial \theta} = +Rs\sin(2\theta)(2) \\ \frac{\partial r(\theta,\varphi)}{\partial \varphi} &= \frac{\partial r(\theta)}{\partial \varphi} = 0 \end{aligned}\right] \text{--- (10) b}$$

Substituting the variables in eq10b to eq9b, normal vector can be given as,

$$\vec{n}_{spheroid} = (r(\theta))^2 \sin\theta \, \hat{r} - r(\theta) 2 \sin\theta \sin(2\theta) R s \, \hat{\theta} + 0 \, \hat{\phi} \quad \text{—— (11) b}$$

$$\vec{n}_{spheroid} = (R - Rs\cos(2\theta))^2 \sin\theta \, \hat{r} - 4 Rs \sin^2\theta \cos\theta \, (R - Rs\cos(2\theta)) \, \hat{\theta} \quad \text{—— (12) b}$$

$$\hat{n} = \frac{\vec{n}}{\|n\|} = \frac{(R - Rs\cos(2\theta))^2 \sin\theta \, \hat{r} - 4Rs\cos\theta \sin^2\theta \, (R - Rs\cos(2\theta)) \, \hat{\theta}}{\sqrt{(R - Rs\cos(2\theta))^4 \sin^2\theta + 16 R^2 s^2 \cos^2\theta \sin^4\theta \, (R - Rs\cos(2\theta))^2}} \quad \text{—— (13) b}$$

$$\hat{n} = \frac{(R - Rs\cos(2\theta))^2 \sin\theta \, \hat{r} - 4Rs\cos\theta \sin^2\theta \, (R - Rs\cos(2\theta)) \, \hat{\theta}}{\sqrt{(R - Rs\cos(2\theta))^4 \sin^2\theta + 16 R^2 s^2 \cos^2\theta \sin^4\theta \, (R - Rs\cos(2\theta))^2}} \quad \text{—— (14) b}$$

$$\hat{n} = \frac{(R - Rs\cos(2\theta))^2 \sin\theta \, \hat{r} - 4Rs\cos\theta \sin^2\theta \, (R - Rs\cos(2\theta))^2 \, \hat{\theta}}{(R - Rs\cos(2\theta)) \sin\theta \sqrt{(R - Rs\cos(2\theta))^2 + 16 R^2 s^2 \cos^2\theta \sin^2\theta}} \quad \text{—— (15) b}$$

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$$\hat{n} = \frac{(R - Rs\cos(2\theta))\,\hat{r} - 4Rs\cos\theta\sin\theta\,\hat{\theta}}{\sqrt{(R - Rs\cos(2\theta))^2 + 4R^2s^2\sin^2(2\theta)}} \quad \text{— (16) b}$$

$\hat{n}$ represents the unit normal vector.