Bout CALCULATION - FOR z-displacement

Defining the parameters to be used :-

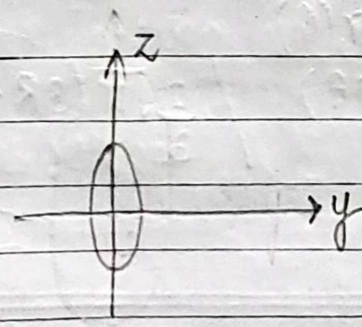
$b_z \rightarrow$ Magnetic field gradient (applied)
$dz \rightarrow$ Displacement in z-direction
$R \rightarrow$ average radius of Sphere.

Spheroid is defined as given in eq. 15a.

$$\boxed{\text{rspheroid}[\theta]}\ r(\theta) = R - \epsilon \cos(2\theta) \quad \text{———} \quad ①d \qquad [\epsilon = RS]$$
$$= R - RS \cos(2\theta)$$
$$\frac{\epsilon}{R} = S \ll 1$$

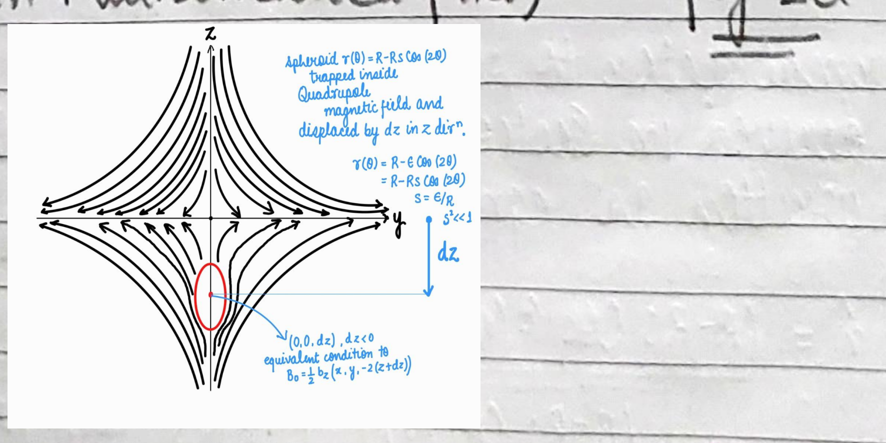
for $\epsilon = -0.1$ or $s = -0.1$ & $R = 1$
the shape of spheroid can be shown as.

$$r(\theta) = R - RS\cos(2\theta) \Rightarrow 1 - 0.1\cos(2\theta)$$



yz plane.

(Plotted in Mathematica file) — fig. 1d



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Position vector of Spheroid is given as

$$\vec{v}(\theta,\varphi) = r(\theta)\,\hat{r} \qquad \text{②d} \qquad \boxed{\text{vsph}[\theta,\varphi]}$$

$$\vec{v}(\theta,\varphi) = (R - Rs\cos(2\theta))\,\hat{r}$$

The above function of $\vec{v}(\theta,\varphi)$ can be expressed in terms of spherical harmonics expansion as,

$$\vec{u}(\theta,\varphi) = \sum h_{l,k}\, Y_l^k(\theta,\varphi)\,\hat{r} + 0\,\hat{\theta} + 0\,\hat{\varphi} \quad \text{— ③d}$$

$\boxed{h_{ij}}$

$$h_{l,k} = \int_0^{2\pi}\int_0^{\pi} r(\theta)\sin\theta\, Y_l^{*k}(\theta,\varphi)\, d\theta\, d\varphi \quad \text{— ④d}$$

the non-zero $f_{l,k}$'s are only $f_{0,0}$ & $f_{2,0}$

Eqn. 4d is referenced from,

[Ref:- Quantum Theory of Angular Momentum, (World Scientific) 1988]

$$\sum h_{l,k}\, Y_l^k(\theta,\varphi) = r(\theta)$$

$$\boxed{\begin{aligned} h_{0,0} &= \tfrac{2}{3}\sqrt{\pi}\,(3R+2\epsilon) \;/\; \tfrac{2}{3}\sqrt{\pi}\,(3R+2Rs) \\ h_{2,0} &= -\tfrac{16}{3}\sqrt{\tfrac{\pi}{5}}\,Rs \end{aligned}} \quad \text{— ⑤d}$$

$$\boxed{r(\theta) = h_{0,0}\, Y_0^0(\theta,\varphi) + h_{2,0}\, Y_2^0(\theta,\varphi)} \quad \text{— ⑥d}$$

Unit Normal vector to surface of spheroid can be written as derived in Section B(b) and final equation can be written as eqn. 15b

$\boxed{\text{Simplified normal}[\theta,\varphi]}$

$$\hat{n} = \frac{(R - Rs\cos(2\theta))\,\hat{r} \; - \; 4Rs\sin\theta\cos\theta\,\hat{\theta}}{\sqrt{(R - Rs\cos(2\theta))^2 + 4R^2s^2\sin^2(2\theta)}} \quad \text{— ⑦d}$$

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Defining the magnetic field B_0. (Applied magnetic field)

$$\vec{B_0} = \frac{1}{2} b_z (x, y, -2(z+dz)) \quad — (8)d \qquad \boxed{B(x,y,z)}$$

$$\nabla \cdot \vec{B_0} = \frac{1}{2} b_z \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} - 2\frac{\partial z}{\partial z} - 2\cancel{\frac{\partial (dz)}{\partial z}}^{\,0} \right) \quad — (9)d$$

$$= \frac{1}{2} b_z (1 + 1 - 2) = 0$$

$\nabla \cdot \vec{B_0} = 0$ (follows maxwell's equations)

$$\underbrace{\vec{B}_{0,sp}}_{\boxed{\text{BOspherical}(r,\theta,\varphi)}} (r, \theta, \varphi) = \underbrace{\begin{pmatrix} \text{cartesian to} \\ \text{spherical co-ordinates} \\ \text{conversion matrix.} \end{pmatrix}}_{M_{c\to s}} \begin{pmatrix} x \\ y \\ -2(z+dz) \end{pmatrix} \frac{1}{2} b_z \quad — (10)d$$

$$\left.\begin{aligned} x &= r \sin\theta \cos\varphi \\ y &= r \sin\theta \sin\varphi \\ z &= r \cos\theta \end{aligned}\right\} — (11)d$$

$$M_{c\to s} = \begin{bmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{bmatrix} \quad — (12)d$$

$$\vec{B}_{0,sp} (r, \theta, \varphi) = \frac{1}{2} b_z \begin{bmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{bmatrix} \begin{bmatrix} r\sin\theta\cos\varphi \\ r\sin\theta\sin\varphi \\ -2(r\cos\theta + dz) \end{bmatrix}$$

$$= \left(\frac{-1}{4} b_z (r + 4dz\cos\theta + 3r\cos(2\theta)), \frac{1}{2} b_z (2dz + 3r\cos\theta)\sin\theta, 0 \right)$$

— (13)d (calculated via Mathematica)

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Applying the boundary condition, given by the eq. 6c.

- Calculating LHS & expressing it in terms of Spherical harmonics.

$\overrightarrow{B_{0,sp}}(r,\theta,\varphi)\cdot\hat{n}(\theta,\varphi)\Big|_{r=r(\theta)} =$

$\boxed{\text{LHSBCVsph}}$

$$\left[-\frac{1}{4}b_z(r+4dz\cos(\theta)+3\cos(2\theta)r)\right]\left[\frac{R-Rs\cos(2\theta)}{\sqrt{(R-Rs\cos(2\theta))^2+4R^2s^2\sin^2(2\theta)}}\right]$$

$$+$$

$$\left[\frac{1}{2}b_z\sin\theta(2dz+3r\cos\theta)\right]\left[\frac{-4Rs\sin\theta\cos\theta}{\sqrt{(R-Rs\cos(2\theta))^2+4R^2s^2\sin^2(2\theta)}}\right]$$

14 d (LHSBCVsph(θ,φ))

Now, we simplify the eq. 14d by taylor expanding the whole expression as a function of s about 0 up to first order as given in eq. 14a.

$\overrightarrow{B_{0,sp}}(r,\theta,\varphi)\cdot\hat{n}(\theta,\varphi)\Big|_{r=r(\theta)}$ (after simplification)

$\boxed{\text{AppLHSBCVsph}(\theta,\varphi)}$

$$=\frac{1}{8}b_z\big(-2R-3Rs-8dz(1+s)\cos\theta + 2R(-3+s)\cos(2\theta) + 8dz\,s\cos(3\theta) + 9Rs\cos(4\theta)\big)$$

15 d

call this fn as $f_{LHS}(\theta,\varphi)$

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Now we express this in terms of spherical harmonics expansion.

Test of eqn 7c is verified in eqn 15.1 d for eqn 15 d.

$$f_{LHS}(\theta,\varphi) = \sum b_{\ell k}\, Y_\ell^k(\theta,\varphi)$$

$$b_{\ell k} = \int_0^{2\pi}\int_0^{\pi} f_{LHS}(\theta,\varphi)\sin\theta\, Y_\ell^{*k}(\theta,\varphi)\, d\theta\, d\varphi \quad \text{— 16 d}$$

$b_{\ell,k}$ are calculated in mathematica file which came out to be

$\boxed{b[l,k]}$

$$b_{0,0} = -\frac{16}{15} b_z \sqrt{\pi}\, R s, \quad b_{1,0} = -\frac{2}{5} b_z\, dz \sqrt{\frac{\pi}{3}}\,(5+8s)$$

$$b_{2,0} = -\frac{2}{21} b_z \sqrt{\frac{\pi}{5}}\, R(21+11s), \quad b_{3,0} = \frac{16}{5} b_z\, dz \sqrt{\frac{\pi}{7}}\, s$$

$$b_{4,0} = \frac{48}{35} b_z \sqrt{\pi}\, R s$$

Now, we are done expressing LHS in terms of spherical harmonics expansion.

→ Calculating RHS now,

$\boxed{\text{PhiOut}}$

$$\Phi_{out}(r,\theta,\varphi) = \sum_{n=0}^{N_{max}} r^{-(n+1)} \sum_{m=-n}^{n} g_{n,m}\, Y_n^m(\theta,\varphi) \quad \text{— 17 d}$$

$$\vec{\nabla}\Phi_{out}(r,\theta,\varphi) = \text{given in eq. 5c.} \quad \text{— 18 d}$$

$\boxed{\text{gradph}}$

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$\boxed{rhsbcvsph(\theta,\varphi)}$

$$\rightarrow \vec{\nabla}\Phi_{out}(r,\theta,\varphi)\cdot\hat{n}(\theta,\varphi)\Big|_{r=r(\theta)} =$$

$$\left[-\sum r^{-(n+2)}(n+1)\sum g_{n,m}Y_n^m(\theta,\varphi)\right]\left[\frac{R-RS\cos(2\theta)}{\sqrt{(R-RS\cos(2\theta))^2+4R^2s^2\sin^2(2\theta)}}\right]$$

$$+\left[\sum r^{-(n+2)}\sum g_{n,m}\frac{\partial Y_n^m(\theta,\varphi)}{\partial\theta}\right]\left[\frac{-4Rs\sin\theta\cos\theta}{\sqrt{(R-Rs\cos(2\theta))^2+4R^2s^2\sin^2(2\theta)}}\right]\Bigg|_{r=r(\theta)}$$

19 d

$\boxed{rhsbcvsph(\theta,\varphi)}$

$$\rightarrow \vec{\nabla}\Phi_{out}(r,\theta,\varphi)\cdot\hat{n}(\theta,\varphi)\Big|_{r=r(\theta)} = \quad \text{— } 20\,d$$

$$\Bigg[-\sum_{n=0}^{N_{max}}(R-RS\cos(2\theta))^{-(n+1)}(n+1)\sum_{m=-n}^{n}g_{n,m}Y_n^m(\theta,\varphi)$$

$$-\sum_{n}(R-RS\cos(2\theta))^{-(n+2)}(2Rs\sin(2\theta))\sum_{m}g_{n,m}\Big(m\cot\theta\, Y_n^m$$

$$+e^{-i\varphi}\frac{\Gamma(n+1-m)\Gamma(n+m+2)}{\Gamma(n+m+1)\Gamma(-m+n)}Y_n^{m+1}\Bigg]$$

$$\overline{\sqrt{(R-Rs\cos(2\theta))^2+4R^2s^2\sin^2(2\theta)}}$$

20 d

$\boxed{\text{Simplified RHS Bc}}$, $\boxed{\text{more Simplified RHS bc}}$

The eqn. 19d-20d are combinations of three equations given on this page.

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Now since LHS only contains the coefficients with $m=0$ and we also know that our spheroid possesses azimuthal symmetry. Hence, we can deduce that RHS should also be independent of $m \neq 0$ terms or independent of phi (φ) terms.

$\boxed{\text{AppRHSBC}(\theta, \varphi, N_{max})}$ (Verified via Mathematica)

$$\vec{\nabla}\Phi_{out} \cdot \hat{n}\Big|_{r=r(\theta)} = \frac{\left(\sum_{n=0}^{N_{max}} -(R - Rs\cos(2\theta))^{-(n+1)} (n+1)\, g_{n,0}\, Y_n^0(\theta,\varphi) - \sum_n 2Rs\sin(2\theta)\,(R - Rs\cos(2\theta))^{-(n+2)} g_{n,0}\, e^{-i\varphi} \frac{\Gamma(n+2)}{\Gamma(n)} Y_n^1(\theta,\varphi)\right)}{\sqrt{(R - Rs\cos(2\theta))^2 + 4R^2 s^2 \sin^2(2\theta)}}$$

21 d

+1 Now, we simplify the eqn. 21d, we do the simplification by taylor expanding it w.r.t s about 0 upto first order.

$$f_{RHS}(s) = f_{RHS}(0) + f'_{RHS}(s)\, s + O(s^2)$$

$$f_{RHS}(0) = \frac{-\sum R^{-(n+1)} (n+1)\, g_{n,0}\, Y_n^0}{R}$$

$$f'_{RHS}(s) = \frac{(1)(2) - (3)(4)}{(5)} \quad \text{——— (22) d}$$

$$(1) = \sum_n (R - Rs\cos(2\theta))^{-(n+2)} (n+1)^2 g_{n,0} Y_n^0(\theta,\varphi)(-R\cos(2\theta))$$

$$- \sum 2R\sin(2\theta)(R - Rs\cos(2\theta))^{-(n+2)} g_{n,0} e^{-i\varphi} \frac{\Gamma(n+2)}{\Gamma(n)} Y_n^1(\theta,\varphi)$$

$$+ \sum 2Rs\sin(2\theta)(n+2)(R - Rs\cos(2\theta))^{-(n+3)} g_{n,0} e^{-i\varphi} \frac{\Gamma(n+2)}{\Gamma(n)} Y_n^1 (-R\cos(2\theta))$$

$$(3) = \frac{2(R - Rs\cos(2\theta))(-R\cos(2\theta)) + 8R^2 s\sin^2(2\theta)}{2\sqrt{(R - Rs\cos(2\theta))^2 + 4R^2s^2\sin^2(2\theta)}}$$

$$(2) = \sqrt{(R - Rs\cos(2\theta))^2 + 4R^2s^2\sin^2(2\theta)}$$

$$(4) = \sum -(R - Rs\cos(2\theta))^{-(n+1)} (n+1) g_{n,0} Y_n^0(\theta,\varphi)$$

$$- \sum 2Rs\sin(2\theta)(R - Rs\cos(2\theta))^{-(n+2)} g_{n,0} e^{-i\varphi} \frac{\Gamma(n+2)}{\Gamma(n)} Y_n^1(\theta,\varphi)$$

$$(5) = (R - Rs\cos(2\theta))^2 + 4R^2s^2\sin^2(2\theta)$$

$$f'_{RHS}(0) = \left\{ \left[\sum_n R^{-n+2} (n+1)^2 (-\cancel{R}\cos(2\theta)) g_{n,0} Y_n^0(\theta,\varphi) - \sum 2\cancel{R}\sin(2\theta)(R)^{-(n+2)} g_{n,0} e^{-i\varphi} \frac{\Gamma(n+2)}{\Gamma(n)} Y_n^1(\theta,\varphi)\right]\left[\cancel{R}\right] - \left[\frac{\cancel{R}(-\cancel{R}\cos(2\theta))}{\cancel{R}}\right]\left[-\sum \frac{R^{-(n+1)}}{R} (n+1) g_{n,0} Y_n^0\right] \middle/ \cancel{R^2} \right\} \quad \text{— (23) d}$$

$$\left. \begin{aligned} f'_{RHS}(0) = {} & \sum R^{-(n+2)} (n+1)^2 (-\cos(2\theta)) g_{n,0} Y_n^0(\theta,\varphi) \\ & - \sum 2\sin(2\theta) R^{-(n+2)} e^{-i\varphi} g_{n,0} \frac{\Gamma(n+2)}{\Gamma(n)} Y_n^1(\theta,\varphi) \\ & - \sum R^{-(n+2)} (n+1) g_{n,0} Y_n^0 \cos(2\theta) \end{aligned} \right\} \text{(24) d}$$

Now we substitute the eq 24d in eq of taylor series expansion,

$$f_{RHS}(S) = f_{RHS}(0) + f_{RHS}'(S)\, s + O(s^2)$$

$$f_{RHS}(S) = -\sum R^{-(n+2)} (n+1)\, g_{n,0}\, Y_n^0(\theta,\varphi)$$

$$+ s\Big(\sum_n R^{-(n+2)} (n+1)^2 (-\cos(2\theta))\, g_{n,0} Y_n^0 - 2 \sum_n \sin(2\theta) R^{-(n+2)} e^{-i\varphi}\, g_{n,0} \frac{\Gamma(n+2)}{\Gamma(n)} Y_n^1(\theta,\phi)$$

$$+ \sum_n R^{-(n+2)} (n+1)\, g_{n,0}\, Y_n^0 \cos(2\theta) \Big) \quad \text{(25) d}$$

After Taylor

Verified via Mathematica.

Eqn 25 d should now be expressed in terms of spherical harmonic expansion. for finding the coefficient of RHS of spherical harmonics expansion we need to put value of N_{max}. After careful observation, $N_{max} = 8$ turns out to be a good number where the coefficients remain unchanged even after increasing the number.

$$Z_{n,m} = \int_0^{2\pi}\int_0^{\pi} Y_n^{*m}(\theta,\varphi) \sin(\theta)\, f_{RHS}(\theta,\varphi)\, d\theta\, d\varphi \quad - \text{(26) d}$$

$Z_{n,m}$

The coefficients are obtained for $m=0$, and varying n. which is given in eq. 26 d.

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RHS coefficients are named as

$$\boxed{zcoffRHS} \quad \text{——} \quad (27)d$$

LHS coefficients are named as $\boxed{b\,coff\,LHS}$ —— (28)d

Now, we compare each coefficients with their respective basis coefficients to evaluate the value of g's.

The RHS is expressed of the form,

$$RHS = \sum_{n,m} Z_{n,m} Y_n^m(\theta, \varphi) \quad \text{——} \quad (29)d$$

$$\left.\begin{aligned}
Z_{0,0} &= f_1(g_{0,0}, g_{2,0}) \\
Z_{1,0} &= f_2(g_{1,0}, g_{3,0}) \\
Z_{2,0} &= f_3(g_{00}, g_{2,0}, g_{4,0}) \\
Z_{3,0} &= f_4(g_{10}, g_{30}, g_{50}) \\
Z_{4,0} &= f_5(g_{20}, g_{40}, g_{60}) \\
Z_{5,0} &= f_6(g_{30}, g_{50}, g_{70}) \\
Z_{60} &= f_7(g_{4,0}, g_{6,0}, g_{8,0})
\end{aligned}\right\} \quad \text{——} \quad (30)d$$

for LHS, we have non zero coefficients only for (0,0), (1,0) (2,0), (3,0), (4,0).

$$\left.\begin{aligned}
f_1(g_{00}, g_{20}) &= b_{00} \\
f_2(g_{10}, g_{30}) &= b_{10} \\
f_3(g_{20}, g_{40}, g_{60}) &= b_{20} \\
f_4(g_{10}, g_{30}, g_{50}) &= b_{30} \\
f_5(g_{4,0}, g_{60}, g_{80}) &= b_{40} \\
f_6(g_{3,0}, g_{50}, g_{70}) &= b_{50} \\
f_7(g_{4,0}, g_{6,0}, g_{80}) &= b_{60}
\end{aligned}\right\} \quad \text{——} \quad (31)d$$

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Refer to Mathematica file

Imp

we have 7 equations but we have 9 variables
upon detailed analysis for eq. 27d (given in mathematica file)
the coefficients of g_{70} & g_{80} contains the terms $\frac{S}{R^9}$ & $\frac{S}{R^{10}}$, respectively. We ignore these two terms of $\frac{S}{R^9}$ & $\frac{S}{R^{10}}$, making us left with 7 equations and 7 variables as $g_{7,0}$ & $g_{8,0}$ can be removed from last two eqn's of eqn. 31d.

$\boxed{\text{zcoeffRHS}}$

As we know have 7 equations & 7 variables. Now, we can get solutions for $g_{n,m}$'s for different n, m.

→ Eqn. 32d, 33d are given in mathematica where we have obtained value of all the g-coefficients.

→ Eqn 34d states all the coefficients obtained at one place. ($g_{n,m}$)

$\boxed{\text{glist}}$ → Eqn. 35d denotes the coefficient stored in list format

→ Eqn 36d denotes the simplified coefficients after taylor expanding each coefficient w.r.t s about s=0 upto first order.

↳ $\boxed{\text{gs}}$

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test → Eqn 37d denotes the coefficients approach the coefficient for the case of sphere calculated by J·Hofer for $s=0$ of our case.

→ Eqn 38d denotes the simplified coefficients also pass the test for sphere.

The simplified coefficients are given as.

$$\left.\begin{aligned}
&\rightarrow g_{0,0} = 0 \\
&g_{1,0} = \frac{1}{5} b_z\, dz \sqrt{\frac{\pi}{3}}\, R^3 (5+9s) \\
&g_{2,0} = \frac{2}{21} b_z \sqrt{\frac{\pi}{5}}\, R^5 (7+5s) \\
&g_{3,0} = -\frac{12}{5} b_z\, dz \sqrt{\frac{\pi}{7}}\, R^5 s \\
&g_{4,0} = -\frac{16}{21} b_z \sqrt{\pi}\, R^7 s
\end{aligned}\right\} \text{— (37) d}$$

→ Calculating Bout :-

(38) d

$$\Phi_{out}(r,\theta,\varphi) = \sum_{n}^{6} r^{-(n+1)} \sum_{m=-n}^{n} g_{n,m} Y_n^m(\theta,\varphi)$$

we substitute the values of $g_{n,m}$ (non-simplified) into the equation & calculate

$\nabla \Phi_{out}(r,\theta,\varphi)$ → with value of coefficients substituted in it.

(39) d

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$$\left[B_{out}(r,\theta,\varphi) = B_0(r,\theta,\varphi) - \nabla\Phi_{out}(r,\theta,\varphi)\right] - \text{eq } 40\text{d}$$

Now, we obtained

$$B_{out}(r,\theta,\varphi)\Big|_{r=r(\theta)} \longrightarrow \text{eq: } 41\text{ d}$$

$\boxed{\text{Bout SpheroidSurface}}$

Then, we simplify the above expression by taylor expanding eqn. 41 d to get.

$\boxed{\text{Bout SurfaceSimplified}}$

$$\left[\begin{aligned} & B_{out}\Big|_{r=r(\theta)} = \\ & \Big(2 b_z s \cos\theta \cdot (3dz + 5R\cos(\theta))\sin^2\theta, \\ & -\frac{1}{140} b_z \big(25R(-14+25s)\cos\theta + 42dz\,(6s-5+10s\cos(2\theta)) \\ & \qquad + 525Rs\cos(3\theta)\big)\sin\theta, \\ & 0\Big) \end{aligned}\right] \text{ — } 42\text{ d}$$

The curve of Bamp V/s θ is plotted in mathematica file as well in fig. 2d.

$$\left[\text{Bamp} = \sqrt{\underbrace{B_r^2 + B_\theta^2 + B_\varphi^2}_{\text{Bout}}}\right] \longrightarrow 43\text{ d}$$