

Section D (d)

[Mathematica file
- Displacement
 $\frac{dz}{dz}$]

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Bout CALCULATION - for z-displacement

Defining the parameters to be used :-

$b_z \rightarrow$ Magnetic field gradient (applied)

$dz \rightarrow$ Displacement in z-direction

$R \rightarrow$ average radius of sphere.

mention condition on "s"

Spheroid is defined as given in eq. 15a.

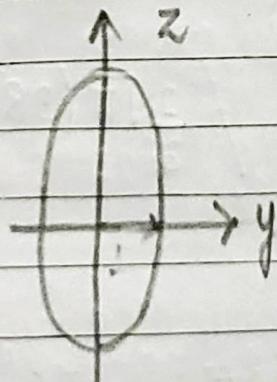
$$[rspheroid[0]] r(\theta) = R - C \cos(\theta) \quad \text{--- ①d} \quad [C = RS]$$

$$= R - RS \cos(\theta)$$

for $C = -0.1$ or $s = -0.1$ & $R = 1$

the shape of spheroid can be shown as,

$$r(\theta) = R - RS \cos(\theta) \Rightarrow 1 - 0.1 \cos(\theta)$$



yz plane.

(Plotted in Mathematica file) - fig. 1d

Position vector of Spheroid is given as

$$\vec{r}(0, \varphi) = r(0) \hat{r} \quad \text{--- (2d)} \quad [v_{\text{sph}}(0, \varphi)]$$

$$\vec{r}(0, \varphi) = (R - R_s \cos(2\theta)) \hat{\theta} \quad \text{--- (2d)}$$

The above function of $\vec{r}(0, \varphi)$ can be expressed in terms of spherical harmonics expansion as,

$$\vec{r}(0, \varphi) = \sum_{l=0}^{\infty} h_{l,k} Y_l^k(0, \varphi) \hat{r} + 0\hat{\theta} + 0\hat{\varphi} \quad \text{--- (3d)}$$

h_{ij}

$$h_{l,k} = \iint_0^{2\pi} r(0) \sin\theta \cdot Y_l^k(0, \varphi) d\theta d\varphi \quad \text{--- (4d)}$$

the non-zero f_{l,k}'s are only f_{0,0} & f_{2,0}

Eqn. 4d is referenced from,

[Ref: - Quantum Theory of Angular Momentum,
(World Scientific) 1988]

$$\sum h_{l,k} Y_l^k(0, \varphi) = r(0)$$

$$h_{0,0} = \frac{2}{3} \sqrt{\pi} (3R+2r) \quad / \quad \frac{2}{3} \sqrt{\pi} (3R+2R_s) \quad \text{--- (5d)}$$

$$h_{2,0} = -\frac{16}{3} \sqrt{\frac{\pi}{5}} R_s$$

$$r(0) = h_{0,0} Y_0^0(0, \varphi) + h_{2,0} Y_2^0(0, \varphi) \quad \text{--- (6d)}$$

Unit Normal vector to surface of spheroid can be written as derived in Section B(b) and final equation can be written as eqn. 16 b

Simplified
normal (0,φ)

$$\hat{n} = \frac{(R - R_s \cos(2\theta)) \hat{\theta}}{\sqrt{(R - R_s \cos(\theta))^2 + 4R_s^2 \sin^2(2\theta)}} - 4R_s \sin\theta \cos\theta \hat{\theta} \quad \text{--- (7d)}$$

Defining the magnetic field B_0 . (Applied magnetic field)

$$B_0 = \frac{1}{2} b_z (x, y, -2(z+dz)) \quad - \textcircled{8} d \quad B(x, y, z)$$

$$\nabla \cdot B_0 = \frac{1}{2} b_z \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} - 2 \frac{\partial z}{\partial z} - 2 \frac{\partial (dz)}{\partial z} \right) - \textcircled{9} d$$

$$= \frac{1}{2} b_z (1 + 1 - 2) = 0$$

$\nabla \cdot B_0 = 0$ (follows maxwell's equations)

$$B_{0,sp}(r, \theta, \phi) = \begin{pmatrix} \text{cartesian to} \\ \text{spherical co-ordinates} \\ \text{conversion matrix.} \end{pmatrix} \begin{pmatrix} x \\ y \\ -2(z+dz) \end{pmatrix} \frac{1}{2} b_z$$

$M_{c \rightarrow s}$

$$\begin{cases} x = r \sin \theta \cos \phi \\ y = r \sin \theta \sin \phi \\ z = r \cos \theta \end{cases} \quad - \textcircled{10} d \quad \textcircled{11} d$$

$$M_{c \rightarrow s} = \begin{bmatrix} \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\ \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\ -\sin \phi & \cos \phi & 0 \end{bmatrix} \quad - \textcircled{12} d$$

$$B_{0,sp}(r, \theta, \phi) = \frac{1}{2} b_z \begin{bmatrix} \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\ \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\ -\sin \phi & \cos \phi & 0 \end{bmatrix} \begin{bmatrix} r \sin \theta \cos \phi \\ r \sin \theta \sin \phi \\ -2(r \cos \theta + dz) \end{bmatrix}$$

$$= \left(-\frac{1}{4} b_z (r + 4dz \cos \theta + 3r \cos(2\theta)), \frac{1}{2} b_z (2dz + 3r \cos \theta) \sin \theta, 0 \right)$$

red box

$- \textcircled{13} d$ (calculated via
Mathematica)

Applying the boundary condition, given by the eq. 6c.

- calculating LHS & expressing it in terms of spherical harmonics.

make sure that all vector notations are consistent (B)

$$B_{0,sp}(r, \theta, \phi) \cdot \hat{n}(0, \phi) \Big|_{r=r(0)}$$

LHSBC V_{sph}

$$\left[-\frac{1}{4} b_z (r + 4dz \cos(0) + 3\cos(2\theta)r) \right] \left[\frac{R - R_s \cos(2\theta)}{\sqrt{(R - R_s \cos(2\theta))^2 + 4R^2 s^2 \sin^2(2\theta)}} \right]$$

+

$$\left[\frac{1}{2} b_z \sin \theta (2dz + 3r \cos \theta) \right] \left[\frac{-4R_s \sin \theta \cos \theta}{\sqrt{(R - R_s \cos(2\theta))^2 + 4R^2 s^2 \sin^2(2\theta)}} \right]$$

⑭ d (LHSBC V_{sph}(0, φ))

Now, we simplify the eq. 14d by Taylor expanding the whole expression as a function of s about 0 up to first order, as given in eq. 14a.

$$B_{0,sp}(r, \theta, \phi) \cdot \hat{n}(0, \phi) \Big|_{r=r(0)} \text{ (after simplification)}$$

AppLHSBC V_{sph}(0, φ)

$$\begin{aligned} &= \frac{1}{8} b_z (-2R - 3R_s - 8dz(HS) \cos \theta \\ &\quad + 2R(-3+s) \cos(2\theta) \\ &\quad + 8dz s \cos(3\theta) \\ &\quad + 9R_s \cos(4\theta)) \end{aligned}$$

call this fn as

$$f_{LHS}(0, \phi) \leftarrow$$

⑮ d

Now we express this in terms of spherical harmonics expansion.

mention your test here that let you define based on sp harmonics

$$f_{\text{LHS}}(\theta, \phi) = \sum b_{ek} Y_e^k(\theta, \phi)$$

$$b_{ek} = \int_0^{2\pi} \int_0^\pi f_{\text{LHS}}(\theta, \phi) \sin\theta Y_e^{*k}(\theta, \phi) d\theta d\phi \quad (16d)$$

$b_{e,k}$ are calculated in mathematica file which came out to be

box 11.19

$$b_{0,0} = -\frac{16}{15} b_z \sqrt{\pi} R_s, \quad b_{1,0} = -\frac{2}{5} b_z dz \sqrt{\frac{\pi}{3}} (5+8s)$$

red box

$$b_{2,0} = -\frac{2}{21} b_z \sqrt{\frac{\pi}{5}} R (21+11s), \quad b_{3,0} = \frac{16}{5} b_z dz \sqrt{\frac{\pi}{7}} s$$

$$b_{4,0} = \frac{48}{35} b_z \sqrt{\pi} R s$$

Now, we are done expressing LHS in terms of spherical harmonics expansion.

→ Calculating RHS now,

PhiOut

$$\Phi_{\text{out}}(r, \theta, \phi) = \sum_{n=0}^{N_{\text{max}}} r^{-(n+1)} \sum_{m=-n}^n g_{n,m} Y_n^m(\theta, \phi)$$

— (17d)

$$\nabla \Phi_{\text{out}}(r, \theta, \phi) = \text{given in eq. 5c.} \quad — (18d)$$

graph

$\text{rhsbcvsph}(0, \phi)$

$$\rightarrow \nabla \Phi_{\text{out}}(r, \theta, \phi) \cdot \hat{n}(\theta, \phi) \Big|_{r=r(\theta)} =$$

$$\left[-\sum r^{-(n+2)} (n+1) \sum g_{n,m} Y_n^m(\theta, \phi) \right] \left[\frac{R - RS \cos(2\theta)}{\sqrt{(R - RS \cos(2\theta))^2 + 4R^2 s^2 \sin^2(2\theta)}} \right]$$

$$+ \left[\sum r^{-(n+2)} \sum g_{n,m} \frac{\partial Y_n^m(\theta, \phi)}{\partial \theta} \right] \left[\frac{-4RS \sin \theta \cos \theta}{\sqrt{(R - RS \cos(2\theta))^2 + 4R^2 s^2 \sin^2(2\theta)}} \right] \Big|_{r=r(\theta)}$$

L ⑯d

$\text{rhsbcvsph}(0, \phi)$

$$\rightarrow \nabla \Phi_{\text{out}}(r, \theta, \phi) \cdot \hat{n}(\theta, \phi) \Big|_{r=r(\theta)} = \quad \text{---} \quad ⑯d$$

$$\left[- \sum_{n=0}^{N_{\max}} (R - RS \cos(2\theta))^{-(n+1)} (n+1) \sum_{m=-n}^n g_{n,m} Y_n^m(\theta, \phi) \right]$$

$$- \sum_n (R - RS \cos(2\theta))^{-(n+2)} (2RS \sin(2\theta)) \sum_m g_{n,m} (m \cot \theta Y_n^m$$

$$+ e^{-i\phi} \frac{\Gamma(n+1-m)}{\Gamma(n+m+1)} \frac{\Gamma(n+m+2)}{\Gamma(-m+n)} r^{m+1} \Big|_{r=r(\theta)}$$

$$\Big/ \sqrt{(R - RS \cos(2\theta))^2 + 4R^2 s^2 \sin^2(2\theta)}$$

L ⑰d

Simplified RHS Bc, more simplified RHS bc

The eqn. 19d-20d are combinations of three equations given on this page.

Now since LHS only contains the coefficients with $m=0$ and we also know that our spheroid possesses azimuthal symmetry. Hence we can deduce that RHS should also be independent of $m \neq 0$ terms or independent of ϕ terms.

AppRHSBC(0, q, Nmax) | (verified via Mathematica)

$$\nabla \vec{\Phi}_{\text{out}} \cdot \hat{n} = \sum_{n=0}^{N_{\text{max}}} - (R - R_S \cos(\theta))^{-(n+1)} (n+1) g_{n,0} Y_n^0(0, q) - \sum_n 2 R_S \sin(\theta) (R - R_S \cos(\theta))^{-(n+2)} g_{n,0} e^{-iq} \frac{\Gamma(n+2)}{\Gamma(n)} Y_n^1(0, q)$$

$$\int (R - R_S \cos(\theta))^2 + 4R^2 S^2 \sin^2(\theta)$$

(21) d

Now, we simplify the eqn 21d, we do the simplification by Taylor expanding it w.r.t s about 0 up to first order.

$$\text{frhs}(s) = \text{frhs}(0) + \text{frhs}'(s)s + O(s^2)$$

$$\text{frhs}(0) = - \underbrace{\sum_R R^{-(n+1)} (n+1) g_{n,0} Y_n^0}$$

$$f'_{RHS}(s) = (1)(2) - (3)(4) \quad \text{--- (22) d}$$

(5)

$$(1) = \sum_n (R - R_s \cos(2\theta))^2 (n+1)^2 g_{n,0} Y_n^0(\theta, \phi) (-R \cos(2\theta))$$

$$- \sum 2R \sin(2\theta) (R - R_s \cos(2\theta))^{-1} (n+2) g_{n,0} e^{-i\phi} \frac{\Gamma(n+2)}{\Gamma(n)} Y_n^1(\theta, \phi)$$

$$+ \sum 2R s \sin(2\theta) (n+2) (R - R_s \cos(2\theta))^{-1} g_{n,0} e^{-i\phi} \frac{\Gamma(n+2)}{\Gamma(n)} Y_n^1(-R \cos(2\theta))$$

$$(3) = \frac{2(R - R_s \cos(2\theta))(-R \cos(2\theta)) + 8R^2 s^2 \sin^2(2\theta)}{2(R - R_s \cos(2\theta))^2 + 4R^2 s^2 \sin^2(2\theta)}$$

$$(2) = \sqrt{(R - R_s \cos(2\theta))^2 + 4R^2 s^2 \sin^2(2\theta)}$$

$$(4) = \sum - (R - R_s \cos(2\theta))^{-(n+1)} g_{n,0} Y_n^0(\theta, \phi)$$

$$- \sum 2R s \sin(2\theta) (R - R_s \cos(2\theta))^{-(n+2)} g_{n,0} e^{-i\phi} \frac{\Gamma(n+2)}{\Gamma(n)} Y_n^1(\theta, \phi)$$

$$(5) = (R - R_s \cos(2\theta))^2 + 4R^2 s^2 \sin^2(2\theta)$$

$$f'_{RHS}(0) = \left[\sum_n R^{-n+2} (n+1)^2 (-R \cos(2\theta)) g_{n,0} Y_n^0(\theta, \phi) \right]$$

$$- \sum 2R s \sin(2\theta) (R) g_{n,0} e^{-i\phi} \frac{\Gamma(n+2)}{\Gamma(n)} Y_n^1(\theta, \phi) \right] [R] \quad \text{--- (23)}$$

$$- \left[\frac{2R (-R \cos(2\theta))}{R} \right] \left[- \sum \frac{R^{-(n+1)}}{R} (n+1) g_{n,0} Y_n^0 \right] / R^2$$

$$f'_{RHS}(0) = \sum R^{-(n+2)} (n+1)^2 (-\cos(2\theta)) g_{n,0} Y_n^0(\theta, \phi)$$

$$- \sum 2 s \sin(2\theta) R^{-(n+2)} e^{-i\phi} g_{n,0} \frac{\Gamma(n+2)}{\Gamma(n)} Y_n^1(\theta, \phi) \quad \text{--- (24) d}$$

$$- \sum R^{-(n+2)} (n+1) g_{n,0} Y_n^0 \cos(2\theta)$$

Now we substitute the eq 24 d in eq of taylor series expansion,

$$f_{RHS}(s) = f_{RHS}(0) + f'_{RHS}(s)s + O(s^2)$$

$$f_{RHS}(s) = -\sum R^{-(n+2)}(n+1)g_{n,0}Y_n^0(\theta, \varphi)$$

$$+ s \left(\sum_n R^{-(n+2)}(n+1)^2(-\cos(20))g_{n,0}Y_n^0 - 2 \sum_n \sin(20)R^{-(n+2)}e^{-i\varphi} \right)$$

$$g_{n,0} \frac{F(n+2)}{F(n)} Y_n^1(\theta, \varphi)$$

$$- \sum_n R^{-(n+2)}(n+1)g_{n,0}Y_n^0 \cos(20) \Big)$$

After Taylor

L (25) d

Verified via Mathematica.

Eqn 25 d should now be expressed in terms of spherical harmonic expansion. for finding the coefficient of RHS of spherical harmonics expansion we need to put value of Nmax. After careful observation, $N_{max}=8$ turns out to be a good number where the coefficients remain unchanged even after increasing the number.

$$z_{n,m} = \iint_0^{2\pi} \iint_0^\pi Y_n^m(\theta, \varphi) \sin(\theta) f_{RHS}(\theta, \varphi) d\theta d\varphi - (26)d$$

The coefficients are obtained for $m=0$, and varying n . which is given in eq. 26 d.

RHS coefficients are named as

$z_{\text{coeff}} \text{ RHS}$

— (27)d

LHS coefficients are named as

$b_{\text{coeff}} \text{ LHS}$

— (28)d

Now, we compare each coefficients with their respective basis coefficients to evaluate the value of g's.

The RHS is expressed of the form,

$$\text{RHS} = \sum_{n,m} z_{n,m} Y_n^m(\theta, \phi) \quad — (29)d$$

$$z_{0,0} = f_1(g_{0,0}, g_{2,0})$$

$$z_{1,0} = f_2(g_{1,0}, g_{3,0})$$

$$z_{2,0} = f_3(g_{0,0}, g_{2,0}, g_{4,0})$$

$$z_{3,0} = f_4(g_{1,0}, g_{3,0}, g_{5,0})$$

$$z_{4,0} = f_5(g_{2,0}, g_{4,0}, g_{6,0})$$

$$z_{5,0} = f_6(g_{3,0}, g_{5,0}, g_{7,0})$$

$$z_{6,0} = f_7(g_{4,0}, g_{6,0}, g_{8,0})$$

for LHS, we have non zero coefficients only for $(0,0), (1,0)$,
 $(2,0), (3,0), (4,0)$.

$$f_1(g_{0,0}, g_{2,0}) = b_{00}$$

$$f_2(g_{1,0}, g_{3,0}) = b_{10}$$

$$f_3(g_{2,0}, g_{4,0}, g_{6,0}) = b_{20}$$

$$f_4(g_{1,0}, g_{3,0}, g_{5,0}) = b_{30}$$

$$f_5(g_{4,0}, g_{6,0}, g_{8,0}) = b_{40}$$

$$f_6(g_{3,0}, g_{5,0}, g_{7,0}) = b_{50}$$

$$f_7(g_{4,0}, g_{6,0}, g_{8,0}) = b_{60}$$

— (31)d

Report to Mathematica file

SMP

we have 7 equations but we have 9 variables

upon detailed analysis for eq. 27d (given in mathematica file)

the coefficients of g_{70} & g_{80} contains the terms $\frac{s}{R^9}$ & $\frac{s}{R^{10}}$,

respectively. We ignore these two terms of $\frac{s}{R^9}$ & $\frac{s}{R^{10}}$, making

us left with 7 equations and 7 variables as $g_{7,0}$ & $g_{8,0}$ can be removed from last two eqn's of eqn. 31d.

coeff RHS

As we know have 7 equations & 7 variables. Now, we can get solutions for $g_{n,m}$'s for different n, m.

→ Eqn. 32d, 33d are given in mathematica where we have obtained value of all the g-coefficients.

→ Eqn. 34d states all the coefficients obtained at one place. ($g_{n,m}$)

list → Eqn. 35d denotes the coefficient stored in list format.

→ Eqn. 36d denotes the simplified coefficients after taylor expanding each coefficient w.r.t s about s=0 upto first order.

↳ gs

g_{tot} → Eqn 37 d denotes the coefficients approach the coefficient for the case of sphere calculated by J. Hofer for $s=0$ of our case.

→ Eqn 38 d denotes the simplified coefficients also pass the test for sphere.

The simplified coefficients are given as.

$$\left. \begin{aligned} g_{0,0} &= 0 \\ g_{1,0} &= \frac{1}{5} b_z dz \sqrt{\frac{\pi}{3}} R^3 (5+9s) \\ g_{2,0} &= \frac{2}{21} b_z dz \sqrt{\frac{\pi}{5}} R^5 (7+5s) \\ g_{3,0} &= -\frac{12}{5} b_z dz \sqrt{\frac{\pi}{7}} R^5 s \\ g_{4,0} &= -\frac{16}{21} b_z dz \sqrt{\frac{\pi}{9}} R^7 s \end{aligned} \right\} \quad \text{--- (37) d}$$

→ Calculating B_{out} :-

$$\Phi_{\text{out}}(r, \theta, \phi) = \sum_{n=0}^{6} r^{-(n+1)} \sum_{m=-n}^n g_{n,m} Y_n^m(\theta, \phi) \quad \text{--- (38) d}$$

We substitute the values of $g_{n,m}$ & (non-simplified) into the equation & calculate

$\nabla \bar{\Phi}_{\text{out}}(r, \theta, \phi)$ → with value of coefficients substituted in it.

--- (39) d

$$[B_{\text{out}}(r, \theta, \phi) = B_0(r, \theta, \phi) - \nabla \Phi_{\text{out}}(r, \theta, \phi)] - \text{eq. (40)d}$$

Now, we obtained

$$B_{\text{out}}(r, \theta, \phi) \Big|_{r=r(\theta)} \rightarrow \text{eq. (41)d}$$

B_{out} Spheroid Surface

Then, we simplify the above expression by Taylor expanding eqn 41 d to get.

B_{out} Surface Simplified

$$B_{\text{out}} \Big|_{r=r(\theta)} =$$

$$\begin{aligned} & (2b_z s \cos \theta (3dz + 5R \cos(\theta)) \sin^2 \theta, \\ & -\frac{1}{140} b_z (25R(-14 + 25s) \cos \theta + 42dz(6s - 5 + 10s \cos(2\theta)) \\ & \quad + 525Rs \cos(3\theta)) \sin \theta, \\ & 0) \end{aligned} \quad \text{L} \quad \text{eq. (42)d}$$

The curve of B_{amp} V/s θ is plotted in mathematica file as well in fig. 2d.

$$\left[B_{\text{amp}} = \sqrt{B_r^2 + B_\theta^2 + B_\phi^2} \right] - \rightarrow \text{eq. (43)d}$$

B_{out}