(5) Our aim is to maximize $f^T C f$ with the constraints that f is a unit vector, i.e., $f^T f = 1$ and $f \perp e$, i.e., $f^T e = e^T f = 0$

Here C is $n \times n$ symmetric matrix e and f are both unit vectors and e is an eigen-vector of C corresponding to the largest eigenvalue (lets call it λ_e).

Let's express the goal using lagrange multipliers

$$\max_f \; f^T C f - \lambda_1 (f^T f - 1) - \lambda_2 (e^T e)$$

Differentiating wrt f and equaling zero,

$$Cf = \lambda_1 f + \lambda_2 e$$

multiply both sides by e^T (pre-multiply)

$$\Rightarrow e^T C f = \underbrace{\lambda_1 e^T f}_{=0} + \underbrace{\lambda_2 e^T e}_{=\lambda_2}$$

$$\begin{aligned}\Rightarrow \lambda_2 &= e^T C f \\ &= e^T C^T f \quad (\because C \text{ is symmetric}) \\ &= (Ce)^T f \\ &= (\lambda_e e)^T f \quad (\because e \text{ is eigenvector of } C) \\ &= \lambda_e e^T f = 0 \quad (\because e \perp f)\end{aligned}$$

$\Rightarrow Cf = \lambda_1 f \Rightarrow f$ is eigenvector of C with eigenvalue $= \lambda_1$

Premultiply both sides with f^T

$$\Rightarrow f^T C f = \lambda_1 f^T f = \lambda_1$$

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Since our original goal was to maximize $f^T C f$ and $f^T C f = \lambda_1$

$\Rightarrow$ we must maximize λ_1

Ideally we should have λ_1 equal to the largest eigenvalue of C. But since e is the eigenvector corresponding to that, it would mean that e and f are equal (upto the sign). Since, we are given that e and f should be $\perp$, we will have to choose the second-largest eigenvalue for λ_1.

$\Rightarrow$ f is the eigenvector of C corresponding to the second largest eigenvalue of C.

$\square$