PROBLEM 2

2.1

(a) $P(q_{t+1}=k \mid q_t = j, O_1, O_2, \ldots, O_T)$

$= P(q_{t+1}=k, q_t=j, O_1 \ldots O_T) \,/\, P(q_t=j, O_1 \ldots O_T)$

$= \alpha_t(j)\, a_{jk}\, b_k(O_{t+1})\, \beta_{t+1}(k) \,/\, \alpha_t(j)\beta_t(j)$

(b) $P(q_{t-1}=i, q_t=j, q_{t+1}=k \mid O_1, O_2 \ldots O_T)$

$= P(q_{t-1}=i, q_t=j, q_{t+1}=k, O_1 \ldots O_T) \,/\, P(O_1 \ldots O_T)$

$= \alpha_{t-1}(i)\, a_{ij}\, b_j(O_t)\, a_{jk}\, b_k(O_{t+1})\, \beta_{t+1}(k) \,/\, \sum_{i=1}^{n} \alpha_t(i)\beta_t(i)$

2.2

$$V_t(j) = \max_{i=1}^{n} V_{t-1}(i)\, a_{ij}\, b_j(O_t)$$

$$= b_j(O_t)\left[\max_{i=1}^{n} V_{t-1}(i)\, a_{ij}\right]$$

$$= b_j(O_t)\left[\max\left(q \max_{\substack{i=1 \\ i \neq}}^{n} V_{t-1}(i),\ p V_{t-1}(j)\right)\right]$$

$$= b_j(O_t)\left[\max\left(q \max_{i=1}^{n} V_{t-1}(i),\ p V_{t-1}(j)\right)\right]$$

[This is possible only because $p > q$]

$\Rightarrow$ At each time t, // $O(T)$

calculate $V_t = \max_{i=1}^{n} V_{t-1}(i)$ // $O(N)$

for each state j, // $O(N)$

$V_t(j) = \max\left(q \times V_t,\ p \times V_{t-1}(j)\right) b_j(O_t)$

$\Rightarrow O(T \times (N+N)) = O(NT)$

2.3 Let $V_t(j)$ be the original quantity as in the viterbi algorithm. Let $W_t(j)$ be the quantity required in the problem.

$$V_1(j) = \pi_j \, b_j(O_1)$$

$$V_t(j) = \max_{i=1}^{n} V_{t-1}(i)\, a_{ij}\, b_j(O_t)$$

$$W_k(j) = \pi_j \, a_{jj}^{k-1} \prod_{i=1}^{k} b_j(O_i)$$

$$W_t(j) = \max\left(\max_{i=1}^{n} W_{t-1}(i)\, a_{ij}\, b_j(O_t),\; V_{t-k+1}(j)\, a_{jj}^{k-1} \prod_{i=t-k+2}^{t} b_j(O_i) \right)$$

Intuition ⇒ You can either take t steps by choosing a path of t-1 steps with at least k repetions and take 1 optimal step, or choosing a path of t-k+1 steps till state j and staying at state j for k-1 more steps.