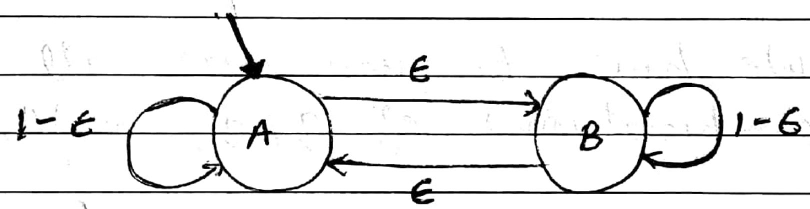
PROBLEM 1

- Input sequence, $X = X_1 X_2 \cdots X_n$
- Input sequence after encoding, $X' = X_1X_1X_2X_2 \cdots X_nX_n$
- Let the output of the noisy channel C,
  $Z = Z_1Z_2 \cdots Z_{2n}$
- Output sequence after decoding Z,
  $Y = Y_1Y_2 \cdots Y_n$

- The noisy channel can be modeled using the following Markov chain. Here, $\epsilon = \epsilon_0 = 1-\epsilon_1$. State A represents that the previous symbol was not erased whereas state B represents that the previous symbol was erased.



State A will be the start state in our model. Let the sequence of states be represented by $q = q_0 q_1 \cdots q_{2n}$.
q_0 is the initial state and $= A$.

- $q_i = A \iff Z_i = X_i$
- $q_i = B \iff Z_i = \square$

This notation is followed throughout the solution.

1.1

$P(y_1=\square) =$ [redacted] $P(Z_1=\square, Z_2=\square)$

$= P(q_1=B, q_2=B) = P(q_2=B \mid q_1=B)\, P(q_1=B)$

$= \underbrace{(1-\epsilon)}_{B\to B}\ \underbrace{(\epsilon)}_{A\to B} \Rightarrow P(y_1=\square) = \underline{(1-\epsilon)(\epsilon)}$

1.2

$P(y_2=\square) = P(Z_3=\square, Z_4=\square)$

$= P(q_3=B, q_4=B) = P(q_4=B \mid q_3=B)\, P(q_3=B)$

$= (1-\epsilon)\, P(q_3=B)$

$P(q_3=B) = \sum_{q_1 q_2} P(q_1, q_2, q_3=B)$

$\Rightarrow$ We have to sum over all combinations of $q_1 q_2 = \{AA, AB, BA, BB\}$

$$\Rightarrow P(q_3=B) = \begin{array}{l|l} (1-\epsilon)(1-\epsilon)(\epsilon) + & (A\to A\to A\to B) \\ (1-\epsilon)(\epsilon)(1-\epsilon) + & (A\to A\to B\to B) \\ (\epsilon)(\epsilon)(\epsilon) + & (A\to B\to A\to B) \\ (\epsilon)(1-\epsilon)(1-\epsilon) & (A\to B\to B\to B) \end{array}$$

$\Rightarrow \left[3(1-\epsilon)^2(\epsilon) + \epsilon^3\right](1-\epsilon)$ Ans

1.3

$P(y_n=\square) = P(Z_{2n-1}=\square, Z_{2n}=\square)$

$= P(q_{2n-1}=B, q_{2n}=B) = P(q_{2n}=B \mid q_{2n-1}=B)\, P(q_{2n-1}=B)$

$= (1-\epsilon)\, P(q_{2n-1}=B)$

Notation $A_t = P(q_t=A)$

$B_t = P(q_t=B)$

$\Rightarrow$ We need to find $\lim_{n\to\infty} B_{2n-1}$.

$$A_t = (1-\epsilon) A_{t-1} + (\epsilon) B_{t-1}$$
$$B_t = (1-\epsilon) B_{t-1} + (\epsilon) A_{t-1}$$

$$\Rightarrow \begin{bmatrix} A_t \\ B_t \end{bmatrix} = \begin{bmatrix} 1-\epsilon & \epsilon \\ \epsilon & 1-\epsilon \end{bmatrix} \begin{bmatrix} A_{t-1} \\ B_{t-1} \end{bmatrix} = \begin{bmatrix} 1-\epsilon & \epsilon \\ \epsilon & 1-\epsilon \end{bmatrix}^2 \begin{bmatrix} A_{t-2} \\ B_{t-2} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} A_t \\ B_t \end{bmatrix} = \begin{bmatrix} 1-\epsilon & \epsilon \\ \epsilon & 1-\epsilon \end{bmatrix}^t \begin{bmatrix} A_0 \\ B_0 \end{bmatrix} = \begin{bmatrix} 1-\epsilon & \epsilon \\ \epsilon & 1-\epsilon \end{bmatrix}^t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\Rightarrow$ We need to find $\begin{bmatrix} 1-\epsilon & \epsilon \\ \epsilon & 1-\epsilon \end{bmatrix}^t$.

Let $R = \begin{bmatrix} 1-\epsilon & \epsilon \\ \epsilon & 1-\epsilon \end{bmatrix} \Rightarrow R = I + \epsilon M$

where $M = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ and I is 2×2 identity matrix.

- Claim $\Rightarrow$ $M^x = (2)^{x-1} \begin{bmatrix} (-1)^x & (-1)^{x+1} \\ (-1)^{x+1} & (-1)^x \end{bmatrix}$, $x \in I_+$

- Proof [by induction]
  - for $x=1$ $\Rightarrow$ $2^0 \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ $\Rightarrow$ correct
  - Let it be true for $x=k$ $\Rightarrow$ $M^k = 2^{k-1} \begin{bmatrix} (-1)^k & (-1)^{k+1} \\ (-1)^{k+1} & (-1)^k \end{bmatrix}$

$$\Rightarrow M^{k+1} = 2^{k-1} \begin{bmatrix} (-1)^k & (-1)^{k+1} \\ (-1)^{k+1} & (-1)^k \end{bmatrix} \begin{bmatrix} (-1) & (1) \\ (1) & (-1) \end{bmatrix}$$

$$= 2^{k-1} \begin{bmatrix} (-1)^{k+1} + (-1)^{k+1} & (-1)^k + (-1)^{k+2} \\ (-1)^k + (-1)^{k+2} & (-1)^{k+1} + (-1)^{k+1} \end{bmatrix}$$

$$= 2^k \begin{bmatrix} (-1)^{k+1} & (-1)^{k+2} \\ (-1)^{k+2} & (-1)^{k+1} \end{bmatrix} = 2^k \begin{bmatrix} (-1)^{k+1} & (-1)^k \\ (-1)^k & (-1)^{k+1} \end{bmatrix}$$

$\rightarrow$ claim is proved.

$$\Rightarrow R^t = (I + \epsilon M)^t = \sum_{i=0}^{t} {}^nC_i (\epsilon M)^t$$

R^t will be of the form $\begin{bmatrix} a_t & b_t \\ b_t & a_t \end{bmatrix}$

since all constituent matrices are of this kind.

$$\star \quad a_t = \sum_{i=0}^{t} {}^nC_i \epsilon^t M^t[0,0] = 1 + \sum_{i=1}^{t} {}^nC_i \epsilon^t 2^{t-1} (-1)^t$$

$$= 1 + \frac{1}{2} \sum_{i=1}^{t} {}^nC_i (-2\epsilon)^t = 1 + \frac{1}{2} \left[\sum_{i=0}^{t} {}^nC_i (-2\epsilon)^t - 1 \right].$$

$+I$

$$= \frac{1}{2} \left[1 + (1 - 2\epsilon)^t \right]$$

$$\star \quad b_t = \sum_{i=0}^{t} {}^nC_i \epsilon^t M^t[0,1] = 0 + \sum_{i=1}^{t} {}^nC_i \epsilon^t 2^{t-1} (-1)^{t+1}$$

$$= \frac{-1}{2} \sum_{i=1}^{t} {}^nC_i (-2\epsilon)^t = \frac{1}{2} \left[1 - \sum_{i=0}^{t} {}^nC_i (-2\epsilon)^t \right]$$

$$= \frac{1}{2} \left[1 - (1 - 2\epsilon)^t \right]$$

Page No.
Date

$$R^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a_t \\ b_t \end{bmatrix} = \begin{bmatrix} A_t \\ B_t \end{bmatrix}$$

$$\Rightarrow B_t = b_t \Rightarrow B_{2n-1} = b_{2n-1}$$

$$= \frac{1}{2}\left[1 - (1-2\epsilon)^{2n-1}\right]$$

$$\lim_{n\to\infty} B_{2n-1} = \lim_{n\to\infty} \frac{1}{2}\left[1-(1-2\epsilon)^{2n-1}\right] = l$$

$l = 0$ if $\epsilon = 0$
$l = 1$ if $\epsilon = 1$
$l = 1/2$ if $\epsilon \in (0,1)$

$\Rightarrow$ Final answer $= (1-\epsilon)\, l$

$= 0$ if $\epsilon \in \{0,1\}$
$= \frac{1}{2}(1-\epsilon)$ if $\epsilon \in (0,1)$

Note: Yeah, I know I could've done this in a much simpler way, but sadly, it occurred to me too late. :-(