Business Analytics

Session 12a. Non-Linear (Quadratic) Programming

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Spring 2019

Final Project Timeline

- Final project presentation on Session 14, May 13 or May 14.
  - Everyone should participate.
  - Everyone should ask at least one question.
- Final project due at 10:00PM, Monday, May 20.
  - Each group should submit a zip file of a report, code, data set(s), and presentation slides on NYU Classes.
  - Each student should submit an evaluation form of your group members on NYU Classes.
  - The project report should be self-contained.
  - You can continue working on the project after the presentation.

Exercise 1: Duality

- What is the dual of the following LP:

$$\text{(Primal)} \; \max 2x_1 - 3x_2 + x_3$$

$$\text{subject to}$$

$$\begin{cases} 2x_1 + x_2 - 4x_3 & \leq 5 \\ x_1 - x_3 & \geq 3 \\ x_2 + 3x_3 & = -1 \\ x_1 \geq 0, x_2 \leq 0, x_3 \in \mathbb{R} & \end{cases}$$

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(Dual) $\min 5y_1 + 3y_2 - y_3$

subject to

$$\begin{cases} 2y_1 + y_2 + 0y_3 & \geq 2 \\ y_1 + 0y_2 + y_3 & \leq -3 \\ -4y_1 - y_2 + 3y_3 & = 1 \\ y_1 \geq 0, y_2 \leq 0, y_3 \in \mathbb{R} & \end{cases}$$

Exercise 2: Reformulation

- Is the following optimization model a linear one?

$$\text{Max } 3x_1 + 5x_2 + 4x_3$$

Subject to

$$x_1(x_2 + x_3) \leq 1$$
$$x_1, x_2, x_3 \in \{0, 1\}$$

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- No. But it can be transformed into a linear model:

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Non-Linear Models

Non-Linear Models

- In a lot of applications, we cannot reformulate the model as a linear one.
  - Production cost is often not a linear function of production quantity (economy of scale).
  - Ordering cost is often not a linear function of order quantity (quantity discount).
  - Price of a stock option is not a linear function of the price of the underlying stock.
  - Revenue is non-linear in price (revenue is equal to price multiplied to demand, which is a function of price itself).
  - Optimization models raising from statistics and data science are often non-linear (e.g., ordinary least squares).
- How can we model and solve models with non-linear objective function and/or constraints?

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- How can we model and solve models with non-linear objective function and/or constraints?
- A new approach: Non-linear programming.

A Simple Pricing Problem

Pricing with Heterogeneous Customers

- Starbucks is trying to price a new product: Shanghai Latte.
  - Cost of the product: $c = 10$
  - The price $p \in [15, 50]$

- Hourly demand of the Shanghai Latte: $D = 160 - 3p$ (estimated from historical data)

- Question: How to price Shanghai Latte to maximize the total profit?

Price Optimization with Non-Linear Programming

- Decision variable: $p =$ price charged for the Shanghai Latte.
- Objective function: $\pi(p) = (p - c)D = (p - 10)(160 - 3p)$.
- Constraint: $p \in [15, 50]$
- Use the package "cvxopt" in Python to solve this model.
  - This is a called quadratic program.
- Optimal pricing: $p^* = 31.67$; optimal profit: $\pi(p^*) = 1408.3$

Markowitz Portfolio Optimization

Portfolio Optimization

- When building an investment portfolio, we seek to maximize the return and minimize the risk.

- More specifically, we want to maximize the following objective (mean-variance utility):

$$\text{Expected Annual Return} - \lambda * \text{Variance of Annual Return}$$

- $\lambda \geq 0$: A parameter that captures how we weight return and risk.
  - A large λ means we care more about minimizing risk.
  - A small λ means we care more about maximizing return.

Data

- 5 stocks:
  - Apple Inc. (AAPL)
  - Amazon.com, Inc. (AMZN)
  - Walt Disney Co. (DIS)
  - Whole Foods Market, Inc. (WFM)
  - Wal-Mart Stores, Inc. (WMT)

- The monthly returns of these 5 stocks over 10 years.
  - Average monthly return
  - Standard deviation of monthly return
  - Correlation between different stocks

Descriptive Analytics of Past Data

- Annual return of the 6 assets:

Index	Stock	Expected Annual Return	Standard Deviation of Annual Return
1	AAPL	0.114	0.039
2	AMZN	0.103	0.030
3	DIS	0.092	0.032
4	WFM	0.085	0.029
5	WMT	0.078	0.022
6	Bond	0.05	0

- Correlations between different stocks:

	AAPL	AMZN	DIS	WFM	WMT	Bond
AAPL	1	0.160	0.163	-0.260	0.399	0
AMZN	0.160	1	0.029	0.272	-0.193	0
DIS	0.163	0.029	1	0.173	0.124	0
WFM	-0.260	0.272	0.173	1	0.125	0
WMT	0.399	-0.193	0.124	0.125	1	0
Bond	0	0	0	0	0	1

Ms. Liu's Portfolio Optimization

Non-Linear Programming Formulation

- Goal of Ms. Liu: Maximize the risk-adjusted return (also called the mean-variance utility).
- Model primitives:
  - μ_i = expected annual return of Stock i
  - σ_i = standard deviation of annual return of Stock i
  - ρ_{ij} = correlation between Stock i and Stock j
- Decision variables:
  - x_i = quantity of Stock i (Stock 6 is the bond)
- Objective function:

$$MV(x) = \underbrace{\sum_{i=1}^{6} \mu_i x_i}_{\text{Mean Return}} - \lambda \underbrace{\left(\sum_{i=1}^{6}\sum_{j=1}^{6} \sigma_i \sigma_j \rho_{ij} x_i x_j\right)}_{\text{Variance of Return}}$$

- Constraint:
  - $B(x) = \sum_{i=1}^{6} x_i = 1$: Budget constraint
  - $x_i \geq 0$

Quadratic Program

- Define matrix $P_{6\times6}$ as $P_{ij} = -2\lambda\sigma_i\sigma_j\rho_{ij}$ and vector $q = (\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6)'$

- The Markowitz portfolio optimization problem can be formulated as a quadratic program:

$$\max \left[\frac{1}{2} \cdot x' \cdot P \cdot x + q' \cdot x\right],$$

subject to $\sum_{i=1}^{6} x_i = 1$ and $x_i \geq 0$ ($i = 1, 2, 3, \cdots, 6$).

Solving the Quadratic Program

- For different values of λ, we have:

λ	Mean Return	SD Return	x_1^*	x_2^*	x_3^*	x_4^*	x_5^*	x_6^*
0	11.4%	3.9%	1	0	0	0	0	0
1	11.4%	3.9%	1	0	0	0	0	0
10	10.9%	2.80%	0.595	0.405	0	0	0	0
50	9.87%	1.79%	0.265	0.312	0.151	0.197	0.0746	0
100	8.90%	1.40%	0.134	0.263	0.133	0.140	0.215	0.116
500	5.78%	0.28%	0.0267	0.0527	0.0265	0.0280	0.0429	0.823
1,000	5.39%	0.14%	0.0134	0.0263	0.0133	0.0140	0.0215	0.912

Table 1: Mean and Standard Deviation of Annual Return

- Alternative formulation:
  - Given mean return, minimize the variance of return.
  - Given return variance, maximize the mean of return.
  - Given minimum return, maximize the Sharp ratio=Mean Return/SD of Return.