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Duality and the Fundamental Theorem of Asset Pricing

BUSF-SHU 210: Business Analytics (Spring 2019)

Duality in Linear Programming

We give an example about establishing the dual of a linear program. Consider the following linear programming:

$$
\begin{aligned}
(\textsc{Primal})\ \max\ 2x_1 + 3x_2 + x_3 + 3x_4 + 5x_5 + 4x_6 \\
\text{subject to} \\
x_1 + x_2 + x_3 \le 4 \\
x_1 + 2x_3 - x_5 \le 2 \\
x_2 - x_6 \ge 0 \\
2x_3 - 3x_4 + x_5 \ge 1 \\
x_2 + x_3 + 2x_5 = -1 \\
x_1 + 3x_2 + 2x_6 = 3 \\
x_1, x_2 \ge 0, x_3, x_4 \le 0, x_5, x_6 \in \mathbb{R}
\end{aligned}
$$

The constraint matrix of this linear program is:

$$
A = \begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 2 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 & 0 & -1 \\
0 & 0 & 2 & -3 & 1 & 0 \\
0 & 1 & 1 & 0 & 2 & 0 \\
1 & 3 & 0 & 0 & 0 & 2
\end{pmatrix}
$$

And the right-hand-side of the constraints are: $(4, 2, 0, 1, -1, 3)'$. Therefore, for the dual of this linear program, the coefficient vector should be $(4, 2, 0, 1, -1, 3)$, the constraint matrix should be

$$
A' = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 3 \\
1 & 2 & 0 & 2 & 1 & 0 \\
0 & 0 & 0 & -3 & 0 & 0 \\
0 & -1 & 0 & 1 & 2 & 0 \\
0 & 0 & -1 & 0 & 0 & 2
\end{pmatrix}
$$

And the RHS of the constraints should be $(2, 3, 1, 3, 5, 4)'$. Therefore, by Slide 13 of Session 11a, we can write the dual of the linear program (Primal) as:

$$
\begin{aligned}
(\text{Dual}) \ \min \ 4y_1 + 2y_2 + 0y_3 + y_4 - y_5 + 3y_6 \\
\text{subject to} \\
y_1 + y_2 + 0y_3 + 0y_4 + 0y_5 + y_6 \geq 2 \\
y_1 + 0y_2 + y_3 + 0y_4 + y_5 + 3y_6 \geq 3 \\
y_2 + 2y_2 + 0y_3 + 2y_4 + y_5 + 0y_6 \leq 1 \\
0y_1 + 0y_2 + 0y_3 - 3y_4 + 0y_5 + 0x_6 \leq 3 \\
0y_1 - y_2 + 0y_3 + y_4 + 2y_5 + 0y_6 = 5 \\
0y_1 + 0y_2 - y_3 + 0y_4 + 0y_5 + 2y_6 = 4 \\
y_1, y_2 \geq 0, y_3, y_4 \leq 0, y_5, y_6 \in \mathbb{R}
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
(\text{Dual}) \ \min \ 4y_1 + 2y_2 + y_4 - y_5 + 3y_6 \\
\text{subject to} \\
y_1 + y_2 + y_6 \geq 2 \\
y_1 + y_3 + y_5 + 3y_6 \geq 3 \\
y_2 + 2y_2 + 2y_4 + y_5 \leq 1 \\
-3y_4 \leq 3 \\
-y_2 + y_4 + 2y_5 = 5 \\
-y_3 + 2y_6 = 4 \\
y_1, y_2 \geq 0, y_3, y_4 \leq 0, y_5, y_6 \in \mathbb{R}
\end{aligned}
$$

In linear programming, dual and primal have the following important properties:

- If the primal has a finite solution, so does the dual, and vice versa (i.e., if the dual has a finite solution, so does the primal).
- If either the primal or the dual has a finite solution, the optimal objective function value of the primal is the **same** as that of dual.
- If either the primal or the dual has a finite solution, the optimal solution of the dual is the shadow price of the primal, and vice versa (i.e., the optimal solution of the primal is the shadow price of the dual).
- The dual of the dual is the primal: Solving one obtains the solutions to both.

Fundamental Theorem of Asset Pricing

A notable application of the duality theory of linear programming is the so-called fundamental theorem of asset pricing. Arbitrage is an important notion in finance, meaning the trading strategy that yields a risk-free strictly positive return. More specifically, the arbitrage strategy would give rise to a **strictly positive** cash flow in the current period and a **non-negative** cash flow for **each possible state** in the future.

A Simple Example

Assume that we have a bond, stock, and a call option written on the stock. The strike price of the option is $K = \$100$, i.e., the owner of the option has the right to exercise the option and purchase the stock at the price \$100. In period 0 (the current period), the price of the bond is \$100, the price of the stock is \$100, and the price of the option is p_o. In period 2, we have two possible states, each with probability 0.5. In each state, the cash flow of the bond is \$105. There is some volatility in the stock price of Period 1, i.e., the price of the stock is \$150 in State 1, and is \$70 in State 2. The cash flow and current price information is summarized in the following table:

	Bond	Stock	Call Option	Probability
State 1	\$105	\$150	max(150-100,0)=\$50	0.5
State 2	\$105	\$70	max(70-100,0)=\$0	0.5
Current Price	\$100	\$100	$p_o =?$	

How can we price the call option in period 0 (i.e., to determine the value of p_o), so that there is free of arbitrage?

We use (x_b, x_s, x_o) to denote the trading strategy, where $x_i > 0$ means buying this asset and $x_i < 0$ means selling this asset in period 0. Then, we can calculate the cash flow in period 0 is:

$$-100x_b - 100x_s - p_o x_o$$

The cash flow of State 1 in period 1 is

$$105x_b + 150x_s + 50x_o$$

The cash flow of State 2 in period 2 is

$$105x_b + 70x_s + 0x_o = 105x_b + 70x_s$$

Consider the following linear program:

$$
\begin{aligned}
(\mathcal{P}) \;\; & \max[-100x_b - 100x_s - p_o x_o] \text{ (current cash flow)} \\
\text{Subject to} \quad & \\
& 105x_b + 150x_s + 50x_o \geq 0 \text{ (cash flow of State 1 is non-negative)} \\
& 105x_b + 70x_s + 0x_o \geq 0 \text{ (cash flow of State 2 is non-negative)}
\end{aligned}
$$

The linear program $(\mathcal{P})$ has a feasible solution $(x_b, x_s, x_o) = (0, 0, 0)$, which yields 0 cash flow in period 0. Hence, there is no arbitrage opportunity, if and only if $(x_b, x_s, x_o) = (0, 0, 0)$ is the optimal solution to $(\mathcal{P})$.

The dual of $(\mathcal{P})$ can be written as

$$
\begin{aligned}
&(\mathcal{D})\ \min[0y_1 + 0y_2]\\
&\text{Subject to}\\
&\qquad 105y_1 + 105y_2 = -100\\
&\qquad 150y_2 + 70y_2 = -100\\
&\qquad 50y_1 + 0y_2 = -p_o\\
&\qquad y_1 \le 0, y_2 \le 0
\end{aligned}
$$

Recall that, if $(x_b, x_s, x_o) = (0, 0, 0)$ is the optimal solution to $(\mathcal{P})$, the dual linear program $(\mathcal{D})$ also bears **a finite optimal solution**. Since the objective function of $(\mathcal{D})$ is a constant 0, any finite feasible solution to $\mathcal{D}$ is also optimal. A feasible solution (y_1^*, y_2^*) satisfies that:

$$
\begin{cases}
105y_1^* + 105y_2^* = -100\\
150y_1^* + 70y_2^* = -100\\
50y_1^* + 0y_2^* = -p_o\\
y_1^* \le 0, y_2^* \le 0
\end{cases}
$$

Hence, if and only if $y_1^* = -0.417$, $y_2^* = -0.536$, $p_o = -50y_1^* = 20.83$, the dual $(\mathcal{D})$ has a finite feasible (and thus optimal) solution. Therefore, the non-arbitrage price of the call option is $p_o = 20.83$. It is clear from the derivation above that the non-arbitrage price p_o is independent of the true probabilities $(0.5, 0.5)$.

Based on the solution to the dual (y_1^*, y_2^*), we derive the risk-neutral probabilities for any asset written on the stock. Specifically, define $P_i^* = y_i^*/(y_1^* + y_2^*)$ as the risk-neutral probability of State i. Then, (P_1^*, P_2^*) can be used to price any asset written on the stock in a non-arbitrage manner:

$$\text{The non-arbitrage price is } p^* = (P_1^* C_1 + P_2^* C_2)/1.05,$$

where C_i is the cash flow of the asset in state i $(i = 1, 2)$ and 1.05 is the risk-free return (of the bond).

Fundamental Theorem of Asset Pricing: General Form

We can generalize the argument above to a setting with more assets and more states. Supose we have N assets. In period 0, the price of asset j is p_j. We define the price vector $p := (p_1, p_2, \cdots, p_N)$. In period 1, we have M states, with cash flow C_{ij} defined as the cash flow of asset j in state i. Each state i occurs with probability $P_i > 0$. We define the probability vector $P := (P_1, P_2, \cdots, P_M)$.

For a trading strategy $x = (x_1, x_2, \cdots, x_N)$ ($x_j > 0$ means buying x_i units of asset j, and $x_j < 0$ means selling $-x_j$ units of asset j), there is no arbitrage opportunity if the optimal objective function value of the following linear program is non-positive:

$$(\mathcal{P})\ \max[-\sum_{j=1}^{N} p_j x_j]$$

Subject to

$$\sum_{j=1}^{N} C_{ij} x_j \geq 0 \text{ for each } 1 \leq i \leq M$$

Fundamental Theorem of Asset Pricing

There is no arbitrage if and only if there exists a linear pricing rule for the assets, i.e., there exists a vector $(y_1^*, y_2^*, \cdots, y_M^*)$ ($y_i^* > 0$ for each i), such that for each asset j, $p_j = \sum_{i=1}^{M} C_{ij} y_i^*$. Moreover, y^* is independent of the true probabilities P.

Based on the Fundamental Theorem of Asset Pricing, we can derive the risk-neutral probability for each state i:

$$P_i^* = \frac{y_i^*}{\sum_{j=1}^{M} y_j^*}$$

Then, for a new asset written on assets 1 through N, the non-arbitrage price of this asset is

$$\text{The non-arbitrage price is } p^* = \frac{1}{1+r}\left[\sum_{i=1}^{M} P_i^* C_i\right],$$

where r is the interest rate of the risk-free asset (e.g., bond), and C_i is the cash flow of the new asset in state i in period 1.