New York University Shanghai

Problem Set 9

Solutions

BUSF-SHU 210: Business Analytics (Spring 2019)

1. United Airlines Revenue Management

United Airlines is trying to optimize the seat allocation for its connecting flight from Newark to San Francisco, via Chicago. The regular price for the entire 2-leg flight from Newark to San Francisco is \$642, whereas the discount price is \$224. The regular price for the flight from Newark to Chicago (leg 1) is \$428, whereas the discount price is \$140. The regular price for the flight from Chicago to San Francisco (leg 2) is \$512, whereas the discount price for this leg is \$190. The airplane capacity is 166 for each of the two legs. United estimates that the demand for the regular-price ticket of the entire flight from Newark to San Francisco is 80, and that for the discount-price ticket of this flight is 120. For the flight from Newark to Chicago, the demand for the regular price ticket is 75, and that for the discount-price ticket is 120. Finally, for the flight from Chicago to San Francisco, the estimated demand for the regular-price ticket is 60, whereas that for the discount-price ticket is 110. We summarize the necessary data in the following table:

	Price	Demand	Flight Leg (capacity 166 on each)
EWR-SFO Regular	\$642	80	1&2
EWR-SFO Discount	\$224	120	1&2
EWR-ORD Regular	\$428	75	1
EWR-ORD Discount	\$140	120	1
ORD-SFO Regular	\$512	60	2
ORD-SFO Discount	\$190	110	2

United Airlines' objective is to allocate the seats to customers of the six segments (EWR-SFO Regular, EWR-SFO Discount, EWR-ORD Regular, EWR-ORD Discount, ORD-SFO Regular, and ORD-SFO Discount) in order to maximize the total revenue. What is the optimal seat allocation policy for United Airline?

Decision Variables

- R_{12}: Number of seats allocated to regular price ticket from Newark to San Francisco
- D_{12}: Number of seats allocated to discount price ticket from Newark to San Francisco
- R_1: Number of seats allocated to regular price ticket from Newark to Chicago
- D_1: Number of seats allocated to discount price ticket from Newark to Chicago
- R_2: Number of seats allocated to regular price ticket from Chicago to San Francisco

- D_2: Number of seats allocated to discount price ticket from Chicago to San Francisco

Objective Function:

Just use the price of each ticket multiplied by the number of seats allocated to the ticket to calculate the total revenue of United:

$$642R_{12} + 224D_{12} + 428R_1 + 140D_1 + 512R_2 + 190D_2$$

Constraints:

- $R_{12} \leq 80$: Number of seats allocated to regular price ticket from Newark to San Francisco should not exceed its demand
- $D_{12} \leq 120$: Number of seats allocated to discount price ticket from Newark to San Francisco should not exceed its demand
- $R_1 \leq 75$: Number of seats allocated to regular price ticket from Newark to Chicago should not exceed its demand
- $D_1 \leq 120$: Number of seats allocated to discount price ticket from Newark to Chicago should not exceed its demand
- $R_2 \leq 60$: Number of seats allocated to regular price ticket from Chicago to San Francisco should not exceed its demand
- $D_2 \leq 110$: Number of seats allocated to discount price ticket from Chicago to San Francisco should not exceed its demand
- $R_{12} + D_{12} + R_1 + D_1 \leq 166$: Total number of seats allocated to leg 1 (Newark to Chicago) should not exceed airplane capacity
- $R_{12}+D_{12}+R_2+D_2 \leq 166$: Total number of seats allocated to leg 2 (Chicago to San Francisco) should not exceed airplane capacity
- $R_{12}, D_{12}, R_1, D_1, R_2, D_2 \geq 0$: The number of seats for each ticket is non-negative

Putting everything together, we formulate the linear program for the airline revenue management problem as

$$\max 642R_{12} + 224D_{12} + 428R_1 + 140D_1 + 512R_2 + 190D_2$$

Subject to

$$
\begin{aligned}
&R_{12} \leq 80 \\
&D_{12} \leq 120 \\
&R_1 \leq 75 \\
&D_1 \leq 120 \\
&R_2 \leq 60 \\
&D_2 \leq 110 \\
&R_{12} + D_{12} + R_1 + D_1 \leq 166 \\
&R_{12} + D_{12} + R_2 + D_2 \leq 166 \\
&R_{12}, D_{12}, R_1, D_1, R_2, D_2 \geq 0
\end{aligned}
$$

2. Funding a Project

It is January 1, 2016 now. Director of Special Projects Rakesh Parameshwar has a planned project, which will require the following expected cash flows between 2016 and 2018:

Date	Cash Requirement
Jul 1, 2016	\$7.5M
Jan 1, 2017	\$4.5M
Jul 1, 2017	\$1.0M
Jan 1, 2018	\$1.0M

Rakesh turns to his Director of Financial Planning, Christine Reyling, and asks her to ensure that funding is available for the project. Christine is considering buying a portfolio of bonds, with cash flows from the bonds arranged to coincide with the needs of Rakesh's project. The following four bonds are available, and can be purchased in any quantity:

Maturity Date	Coupon	Price
Jul 1, 2016	7.00%	\$1.00
Jan 1, 2017	7.50%	\$1.10
Jul 1, 2017	6.75%	\$0.90
Jan 1, 2018	10.00%	\$1.15

Every 6 months, starting 6 months from the current date (Jan 1, 2016) and ending at the maturity date, each bond pays

$$0.5 \times (\text{coupon rate})$$

At the maturity date, the face value is also paid. For example, each of the second bond pays

- $0.5 \times 7.5\% = \$0.0375$ in July 1, 2016
- $0.5 \times 7.5\% + 1 = \$1.0375$ in January 1, 2017

Christine's job is to find a portfolio of these bonds to purchase **today** that will result in the minimum total cost, but will meet the project's cash-flow requirements. Assume that any cash can be reinvested at an annual interest rate of 4% (so the half-year interest rate is 2%), and don't worry about discounting. What is the minimum cost to meet Rakesh's cashflow requirement?

Decision Variables

- X_1: Quantity of the first bond to purchase in Jan 1, 2016 (in millions)
- X_2: Quantity of the second bond to purchase in Jan 1, 2016 (in millions)
- X_3: Quantity of the second bond to purchase in Jan 1, 2016 (in millions)
- X_4: Quantity of the second bond to purchase in Jan 1, 2016 (in millions)
- C_1: Cash flow in Jul 1, 2016 (in millions of USD)

- C_2: Cash flow in Jan 1, 2017 (in millions of USD)
- C_3: Cash flow in Jul 1, 2017 (in millions of USD)
- C_4: Cash flow in Jan 1, 2018 (in millions of USD)

Objective Function:

The objective function is the total cost of the bond portfolio, which equals

$$X_1 + 1.1X_2 + 0.9X_3 + 1.15X_4$$

Constraints:

All the constraints boil down to that the cash flow can meet the requirement of the project. The cash flow in July 1, 2016 is

$$C_1 = 1.035X_1 + 0.0375X_2 + 0.03375X_3 + 0.05X_4$$

The cash flow in Jan 1, 2017 is

$$C_2 = 1.02(C_1-7.5)+1.0375X_2+0.03375X_3+0.05X_4 = 1.02C_1+1.0375X_2+0.03375X_3+0.05X_4-7.65$$

The cash flow in Jul 1, 2017 is

$$C_3 = 1.02(C_2 - 4.5) + 1.03375X_3 + 0.05X_4 = 1.02C_2 + 1.03375X_3 + 0.05X_4 - 4.59$$

The cash flow in Jan 1, 2018 is

$$C_4 = 1.02(C_3 - 1) + 1.05X_4 = 1.02C_3 + 1.05X_4 - 1.02$$

Hence, we can summarize the constraints as follows:

- $C_1 = 1.035X_1 + 0.0375X_2 + 0.03375X_3 + 0.05X_4$: The cash flow in July 1, 2016
- $C_1 \geq 7.5$: The cash flow requirement in July 1, 2016 is satisfied
- $C_2 = 1.02C_1 + 1.0375X_2 + 0.03375X_3 + 0.05X_4 - 7.65$: The cash flow in Jan 1, 2017
- $C_2 \geq 4.5$: The cash flow requirement in Jan 1, 2017 is satisfied
- $C_3 = 1.02C_2 + 1.03375X_3 + 0.05X_4 - 4.59$: The cash flow in Jul 1, 2017
- $C_3 \geq 1$: The cash flow requirement in Jul 1, 2017 is satisfied
- $C_4 = 1.02C_3 + 1.05X_4 - 1.02$: The cash flow in Jan 1, 2018
- $C_4 \geq 1$: The cash flow requirement in Jan 1, 2018 is satisfied
- $X_1, X_2, X_3, X_4 \geq 0$: The quantity of each bond is non-negative

Putting everything together, we formulate the linear program for the project funding problem as

$$\min X_1 + 1.1X_2 + 0.9X_3 + 1.15X_4$$

Subject to

$$\begin{aligned}
&1.035X_1 + 0.0375X_2 + 0.03375X_3 + 0.05X_4 - C_1 = 0 \\
&C_1 \geq 7.5 \\
&1.0375X_2 + 0.03375X_3 + 0.05X_4 + 1.02C_1 - C_2 = 7.65 \\
&C_2 \geq 4.5 \\
&1.03375X_3 + 0.05X_4 + 1.02C_2 - C_3 = 4.59 \\
&C_3 \geq 1 \\
&1.05X_4 + 1.02C_3 - C_4 - 1.02 \\
&C_4 \geq 1 \\
&X_1, X_2, X_3, X_4 \geq 0
\end{aligned}$$

3. Steakhouse Staff Scheduling

Western Family Steakhouse offers a variety of low-cost meals and quick service. Other than management, the steakhouse operates with two full-time employees who work 8 hours per day. The rest of the employees are part-time employees who are scheduled for 4-hour shifts during peak meal times[1]. On Saturdays the steakhouse is open from 11:00am to 10:00pm. Management wants to develop a schedule for part-time employees that will minimize labor costs and still provide excellent customer service. The average wage rate for the part-time employee is $7.60 per hour. The total number of full-time and par-time employees needed varies with the time of day as shown in the following table

Time	Total Number of Employees Needed
11am-noon	9
noon-1pm	9
1pm-2pm	9
2pm-3pm	3
3pm-4pm	3
4pm-5pm	3
5pm-6pm	6
6pm-7pm	12
7pm-8pm	12
8pm-9pm	7
9pm-10pm	7

One full-time employee comes on duty at 11am, works 4 hours, takes an hour off, and returns for another 4 hours. The other full-time employee comes to work at 1pm and works the same 4-hours-on, 1-hour-off, 4-hours-on pattern.

(a) What is the minimum-cost schedule for part-time employees? Please formulate this problem as a linear program and solve it using Python. Clearly state the decision variables, the objective function, and the constraints for the linear program.

We define the number of part-time employees who start working at the beginning of each hour as follows:

- x_1 = Number of employees from 11am to 3pm
- x_2 = Number of employees from noon to 4pm
- x_3 = Number of employees from 1pm to 5pm
- x_4 = Number of employees from 2pm to 6pm
- x_5 = Number of employees from 3pm to 7pm
- x_6 = Number of employees from 4pm to 8pm
- x_7 = Number of employees from 5pm to 9pm
- x_8 = Number of employees from 6pm to 10pm

[1] 4-hour shifts mean that the part-time employee works for a consecutive period of 4 hours.

The objective function is to minimize the total cost of recruiting part-time employees:

min $7.6 \times 4(x_1+x_2+x_3+x_4+x_5+x_6+x_7+x_8) = 30.2(x_1+x_2+x_3+x_4+x_5+x_6+x_7+x_8)$

The constraints are that, in each hour, the total number of employees meet the requirements of the Steakhouse. Hence, we have the following constraints:

- $x_1 + 1 \geq 9$: 11am-noon
- $x_1 + x_2 + 1 \geq 9$: noon-1pm
- $x_1 + x_2 + x_3 + 2 \geq 9$: 1pm-2pm
- $x_1 + x_2 + x_3 + x_4 + 2 \geq 3$: 2pm-3pm
- $x_2 + x_3 + x_4 + x_5 + 1 \geq 3$: 3pm-4pm
- $x_3 + x_4 + x_5 + x_6 + 2 \geq 3$: 4pm-5pm
- $x_4 + x_5 + x_6 + x_7 + 1 \geq 6$: 5pm-6pm
- $x_5 + x_6 + x_7 + x_8 + 2 \geq 12$: 6pm-7pm
- $x_6 + x_7 + x_8 + 2 \geq 12$: 7pm-8pm
- $x_7 + x_8 + 1 \geq 7$: 8pm-9pm
- $x_8 + 1 \geq 7$: 9pm-10pm
- $x_i \geq 0$ for all i: Non-negativity constraint

Putting everything together, we have the linear program can be written as:

$$
\begin{aligned}
& \min\ 30.4(x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8) \\
\text{Subject to} \quad & \\
& x_1 \geq 8 \\
& x_1 + x_2 \geq 8 \\
& x_1 + x_2 + x_3 \geq 7 \\
& x_1 + x_2 + x_3 + x_4 \geq 1 \\
& x_2 + x_3 + x_4 + x_5 \geq 2 \\
& x_3 + x_4 + x_5 + x_6 \geq 1 \\
& x_4 + x_5 + x_6 + x_7 \geq 5 \\
& x_5 + x_6 + x_7 + x_8 \geq 10 \\
& x_6 + x_7 + x_8 \geq 10 \\
& x_7 + x_8 \geq 6 \\
& x_8 \geq 6 \\
& x_i \geq 0 \text{ for all } i
\end{aligned}
$$

Then, we use Python to solve this linear program. The optimal recruiting plan is $x_1^* = 8$, $x_2^* = 0$, $x_3^* = 0$, $x_4^* = 0$, $x_5^* = 2$, $x_6^* = 4$, $x_7^* = 0$, and $x_8^* = 6$, with the minimum cost equal to \$608.

(b) Assume that part-time employees can be assigned either a 3-hour or a 4-hour shift. What is the cost savings compared to the previous schedule you give in part (a)?

Hint: Note that you will stop recruiting part-time 4-hour-shift employees after 6:00PM, and you will stop recruiting part-time 3-hour-shift employees after 7:00PM.

We define the number of part-time employees (in different shifts) who start working at the beginning of each hour as follows:

- y_1 = Number of employees from 11am to 3pm
- y_2 = Number of employees from noon to 4pm
- y_3 = Number of employees from 1pm to 5pm
- y_4 = Number of employees from 2pm to 6pm
- y_5 = Number of employees from 3pm to 7pm
- y_6 = Number of employees from 4pm to 8pm
- y_7 = Number of employees from 5pm to 9pm
- y_8 = Number of employees from 6pm to 10pm
- y_9 = Number of employees from 11am to 2pm
- y_{10} = Number of employees from noon to 3pm
- y_{11} = Number of employees from 1pm to 4pm
- y_{12} = Number of employees from 2pm to 5pm
- y_{13} = Number of employees from 3pm to 6pm
- y_{14} = Number of employees from 4pm to 7pm
- y_{15} = Number of employees from 5pm to 8pm
- y_{16} = Number of employees from 6pm to 9pm
- y_{17} = Number of employees from 7pm to 10pm

The objective function is to minimize the total cost of recruiting part-time employees:

$$\min\ 30.4(y_1+y_2+y_3+y_4+y_5+y_6+y_7+y_8)+22.8(y_9+y_{10}+y_{11}+y_{12}+y_{13}+y_{14}+y_{15}+y_{16}+y_{17})$$

The constraints are that, in each hour, the total number of employees meet the requirements of the Steakhouse. Hence, we have the following constraints:

- $y_1 + y_9 + 1 \geq 9$: 11am-noon
- $y_1 + y_2 + y_9 + y_{10} + 1 \geq 9$: noon-1pm
- $y_1 + y_2 + y_3 + y_9 + y_{10} + y_{11} + 2 \geq 9$: 1pm-2pm
- $y_1 + y_2 + y_3 + y_4 + y_{10} + y_{11} + y_{12} + 2 \geq 3$: 2pm-3pm
- $y_2 + y_3 + y_4 + y_5 + y_{11} + y_{12} + y_{13} + 1 \geq 3$: 3pm-4pm
- $y_3 + y_4 + y_5 + y_6 + y_{12} + y_{13} + y_{14} + 2 \geq 3$: 4pm-5pm
- $y_4 + y_5 + y_6 + y_7 + y_{13} + y_{14} + y_{15} + 1 \geq 6$: 5pm-6pm

- $y_5 + y_6 + y_7 + y_8 + y_{14} + y_{15} + y_{16} + 2 \geq 12$: 6pm-7pm
- $y_6 + y_7 + y_8 + y_{15} + y_{16} + y_{17} + 2 \geq 12$: 7pm-8pm
- $y_7 + y_8 + y_{16} + y_{17} + 1 \geq 7$: 8pm-9pm
- $y_8 + y_{17} + 1 \geq 7$: 9pm-10pm
- $y_i \geq 0$ for all i: Non-negativity constraint

Putting everything together, we have the linear program can be written as:

$$
\begin{aligned}
\min\ & 30.4(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8) \\
& + 22.8(y_9 + y_{10} + y_{11} + y_{12} + y_{13} + y_{14} + y_{15} + y_{16} + y_{17}) \\
\text{Subject to}\ \ & \\
& y_1 + y_9 \geq 8 \\
& y_1 + y_2 + y_9 + y_{10} \geq 8 \\
& y_1 + y_2 + y_3 + y_9 + y_{10} + y_{11} \geq 7 \\
& y_1 + y_2 + y_3 + y_4 + y_{10} + y_{11} + y_{12} \geq 1 \\
& y_2 + y_3 + y_4 + y_5 + y_{11} + y_{12} + y_{13} \geq 2 \\
& y_3 + y_4 + y_5 + y_6 + y_{12} + y_{13} + y_{14} \geq 1 \\
& y_4 + y_5 + y_6 + y_7 + y_{13} + y_{14} + y_{15} \geq 5 \\
& y_5 + y_6 + y_7 + y_8 + y_{14} + y_{15} + y_{16} \geq 10 \\
& y_6 + y_7 + y_8 + y_{15} + y_{16} + y_{17} \geq 10 \\
& y_7 + y_8 + y_{16} + y_{17} \geq 6 \\
& y_8 + y_{17} \geq 6 \\
& y_i \geq 0 \text{ for all } i
\end{aligned}
$$

Then, we use Python to solve this linear program. The optimal recruiting plan is $y_1^* = 0$, $y_2^* = 0$, $y_3^* = 0$, $y_4^* = 0$, $y_5^* = 0$, $y_6^* = 0$, $y_7^* = 0$, $y_8^* = 6$, $y_9^* = 8$, $y_{10}^* = 0$, $y_{11}^* = 1$, $y_{12}^* = 0$, $y_{13}^* = 1$, $y_{14}^* = 0$, $y_{15}^* = 4$, $y_{16}^* = 0$, and $y_{17}^* = 0$ with the minimum cost equal to \$501.6. Hence, the cost reduction is 608-500.4=\$107.6.