Business Analytics

Session 11a. Linear Programming

(Sensitivity Analysis and Duality)

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Exercises

- What are the three essential components of linear programming (and optimization models in general)?

- (T/F) The optimal solution to a linear program always exists.

- (T/F) It is possible to formulate all the linear programs as a maximization problem.

Google's AdWords Problem

- **Decisions**:

Advertiser	Query 1 "4G LTE"	Query 2 "largest LTE"	Query 3 "best LTE network"
AT&T	X_{A1}	X_{A2}	X_{A3}
T-Mobile	X_{T1}	X_{T2}	X_{T3}
Verizon	X_{V1}	X_{V2}	X_{V3}

- **Revenue**:

$$0.5X_{A1} + 0.5X_{A2} + 1.6X_{A3} + X_{T1} + 0.75X_{T2} + 2X_{T3} + 0.5X_{V1} + 4X_{V2} + 5X_{V3}$$

- **Constraints**:
  - Budget for AT&T: $0.5X_{A1} + 0.5X_{A2} + 1.6X_{A3} \leq 170$
  - Budget for T-Mobile: $X_{T1} + 0.75X_{T2} + 2X_{T3} \leq 100$
  - Budget for Verizon: $0.5X_{V1} + 4X_{V2} + 5X_{V3} \leq 160$
  - Number of Q_1: $X_{A1} + X_{T1} + X_{V1} \leq 140$
  - Number of Q_2: $X_{A2} + X_{T2} + X_{V2} \leq 80$
  - Number of Q_3: $X_{A3} + X_{T3} + X_{V3} \leq 80$
  - Non-negativity: $X_{A1}, X_{A2}, X_{A3}, X_{T1}, X_{T2}, X_{T3}, X_{V1}, X_{V2}, X_{V3} \geq 0$

Linear Programming (LP) for Google AdWords

$$\max\ 0.5X_{A1} + 0.5X_{A2} + 1.6X_{A3} + X_{T1} + 0.75X_{T2} + 2X_{T3} + 0.5X_{V1} + 4X_{V2} + 5X_{V3}$$

Subject to

$$\begin{aligned}
0.5X_{A1} + 0.5X_{A2} + 1.6X_{A3} &\le 170 \\
X_{T1} + 0.75X_{T2} + 2X_{T3} &\le 100 \\
0.5X_{V1} + 4X_{V2} + 5X_{V3} &\le 160 \\
X_{A1} + X_{T1} + X_{V1} &\le 140 \\
X_{A2} + X_{T2} + X_{V2} &\le 80 \\
X_{A3} + X_{T3} + X_{V3} &\le 80 \\
X_{A1}, X_{A2}, X_{A3}, X_{T1}, X_{T2}, X_{T3}, X_{V1}, X_{V2}, X_{V3} &\ge 0
\end{aligned}$$

Google AdWords Solution

- Use the Python package "cvxopt" to obtain the optimal solution.

- Optimal Ad Display Strategy:

Advertiser	Query 1 "4G LTE"	Query 2 "largest LTE"	Query 3 "best LTE network"
AT&T	$X^*_{A1} = 40$	$X^*_{A2} = 40$	$X^*_{A3} = 80$
T-Mobile	$X^*_{T1} = 100$	$X^*_{T2} = 0$	$X^*_{T3} = 0$
Verizon	$X^*_{V1} = 0$	$X^*_{V2} = 40$	$X^*_{V3} = 0$

- Optimal Revenue=$428

Sensitivity Analysis

Sensitivity Analysis

- What if questions:
  - What if the budget for T-Mobile is 10 dollars higher?
  - What if the number of displays for Q_1 decreases to 135?
- How would the optimal decision variables and optimal objective function value change if the constraints or the coefficients change?
  - You can change the constraint(s)/coefficient(s) and re-run the optimization.
- Alternatively,
  - "GoogleAd['z']": The shadow price (the change in the optimal objective function value associated with a unit change in this constraint).

Sensitivity Analysis: Shadow Price

- Shadow prices for the budget and query number constraints:

	A	T	V	Q_1	Q_2	Q_3
Shadow price	0	0.5	0.875	0.5	0.5	1.6

- Shadow prices for the non-negativity constraints:

	x_{A1}	x_{A2}	x_{A3}	x_{T1}	x_{T2}	x_{T3}	x_{V1}	x_{V2}	x_{V3}
Shadow price	0	0	0	0	0.125	0.6	0.437	0	0.975

- If the budget for T-mobile increases to $\$110$, the revenune of Google will be $\$428 + 0.5 \times (110 - 100) = \$433$.
- If the number of Q_1 decreases to $\$135$, the revenue of Google will be $\$428 + 0.5 \times (135 - 140) = \$425.5$.
- To use the shadow price, we assume that only one RHS of the constraints is changed.

Duality in Linear Programming

Primal and Dual

$$
\text{(Primal)} \quad \max_{(x_1, x_2, \cdots, x_n)} C_1 x_1 + C_2 x_2 + \cdots C_n x_n
$$

subject to

$$
\begin{cases}
A_{1,1}x_1 + A_{1,2}x_2 + \cdots + A_{1,n}x_n & \leq B_1 \\
A_{2,1}x_1 + A_{2,2}x_2 + \cdots + A_{2,n}x_n & \leq B_2 \\
\cdots\cdots & \\
A_{m,1}x_1 + A_{m,2}x_2 + \cdots + A_{m,n}x_n & \leq B_m \\
x_1 \geq 0, x_2 \geq 0, \cdots, x_n \geq 0 &
\end{cases}
$$

$$
\text{(Dual)} \quad \min_{(y_1, y_2, \cdots, y_m)} B_1 y_1 + B_2 y_2 + \cdots B_m y_m
$$

subject to

$$
\begin{cases}
A_{1,1}y_1 + A_{2,1}y_2 + \cdots + A_{m,1}y_m & \geq C_1 \\
A_{1,2}y_1 + A_{2,2}y_2 + \cdots + A_{m,2}y_m & \geq C_2 \\
\cdots\cdots & \\
A_{1,n}y_1 + A_{2,n}y_2 + \cdots + A_{m,n}y_m & \geq C_n \\
y_1 \geq 0, y_2 \geq 0, \cdots, y_m \geq 0 &
\end{cases}
$$

Dual for Google AdWords

$$\min\ 170y_1 + 100y_2 + 160y_3 + 140y_4 + 80y_5 + 80y_6$$

Subject to

$$\begin{aligned}
0.5y_1 + y_4 &\geq 0.5 \\
0.5y_1 + y_5 &\geq 0.5 \\
1.6y_1 + y_6 &\geq 1.6 \\
y_2 + y_4 &\geq 1 \\
0.75y_2 + y_5 &\geq 0.75 \\
2y_2 + y_6 &\geq 2 \\
0.5y_3 + y_4 &\geq 0.5 \\
4y_3 + y_5 &\geq 4 \\
5y_3 + y_6 &\geq 5 \\
y_1, y_2, y_3, y_4, y_5, y_6 &\geq 0
\end{aligned}$$

Primal and Dual

- As long as either the primal or the dual has a finite optimal solution, the other one also has a finite optimal solution.
- If either the primal or the dual has a finite optimal solution, the optimal objective function values of the primal and the dual are the same.
- If either the primal or the dual has a finite optimal solution, the shadow price of the primal is an optimal solution to the dual.
- If either the primal or the dual has a finite optimal solution, the shadow price of the dual is an optimal solution to the primal.
- Dual of dual is primal.
  - Solving one problem obtains solutions to both.

Duality in LP

- A is a constraint matrix. $\mathbf{a}_i'$ is the ith row vector, and A_j is the jth column vector.
  - Primal on the left; Dual on the right:

$$
\begin{array}{llllll}
\text{minimize} & \mathbf{c}'\mathbf{x} & & \text{maximize} & \mathbf{p}'\mathbf{b} & \\
\text{subject to} & \mathbf{a}_i'\mathbf{x} \geq b_i, & i \in M_1, & \text{subject to} & p_i \geq 0, & i \in M_1, \\
& \mathbf{a}_i'\mathbf{x} \leq b_i, & i \in M_2, & & p_i \leq 0, & i \in M_2, \\
& \mathbf{a}_i'\mathbf{x} = b_i, & i \in M_3, & & p_i \text{ free}, & i \in M_3, \\
& x_j \geq 0, & j \in N_1, & & \mathbf{p}'\mathbf{A}_j \leq c_j, & j \in N_1, \\
& x_j \leq 0, & j \in N_2, & & \mathbf{p}'\mathbf{A}_j \geq c_j, & j \in N_2, \\
& x_j \text{ free}, & j \in N_3, & & \mathbf{p}'\mathbf{A}_j = c_j, & j \in N_3.
\end{array}
$$

Fundamental Theorem of Asset Pricing

- Arbitrage: Positive cash flow now, non-negative cash flow for every possible state in the future.
- Option pricing: How should we price a call option to avoid arbitrage?

	Bond	Stock	Call Option	Probability
State 1	\$105	\$150	max(150-100,0)=\$50	0.5
State 2	\$105	\$70	max(70-100,0)=\$0	0.5
Current Price	\$100	\$100	$p_o = ?$	

- Buying x_b units of bond, x_s units of stock, and x_o units of option.
  - $x_i < 0$ means selling.
- No arbitrage means the following LP has a finite solution $(0, 0, 0)$:

$(\mathcal{P})$ $\max[-100x_b - 100x_s - p_o x_o]$ (current cash flow)

Subject to

$105x_b + 150x_s + 50x_o \geq 0$ (cash flow of State 1 is non-negative)

$105x_b + 70x_s + 0x_o \geq 0$ (cash flow of State 2 is non-negative)

Fundamental Theorem of Asset Pricing

- The dual of $(\mathcal{P})$:

$$(\mathcal{D})\ \min[0y_1 + 0y_2]$$

Subject to

$$105y_1 + 105y_2 = -100$$
$$150y_1 + 70y_2 = -100$$
$$50y_1 + 0y_2 = -p_o$$
$$y_1 \leq 0, y_2 \leq 0$$

- Non-arbitrage is equivalent to that the dual $(\mathcal{D})$ has a feasible solution.
  - $y_1^* = -0.417$, $y_2^* = -0.536$, $p_o = -50y_1^* = 20.83$.
  - Risk-neutral probabilities: $(P_1^*, P_2^*) = (0.438, 0.562)$, independent of "true probabilities".
- For any asset written on the stock, where C_1 is the cash flow in State 1 and C_2 is the cash flow in State 2.

The non-arbitrage price is $p^* = (P_1^* C_1 + P_2^* C_2)/1.05$.

- No arbitrage $\Leftrightarrow$ $(\mathcal{P})$ has a finite solution $\Leftrightarrow$ $(\mathcal{D})$ has a feasible solution $\Leftrightarrow$ existence of risk-neutral probabilities

Fundamental Theorem of Asset Pricing: General Form

- M states (in period 1) and N assets, with cash flow C_{ij} for state i and asset j.
- Each state i occurs with probability $P_i > 0$. $P := (P_1, P_2, \cdots, P_M)$
- The price of asset j in period 0: p_j. $p := (p_1, p_2, \cdots, p_N)$

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Fundamental Theorem of Asset Pricing

There is no arbitrage (for the cash flow matrix C and the price vector p) if and only if there exists a linear pricing rule for the assets, i.e., there exists a vector $(y_1^*, y_2^*, \cdots, y_M^*)$ ($y_i^* > 0$ for each i), such that for each asset j, $p_j = \sum_{i=1}^{M} C_{ij} y_i^*$. Moreover, y^* is independent of the true probabilities P.