BLM3620 Digital Signal Processing

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Lecture #4 – Sampling and Aliasing

- Sampling

- Principal of Aliasing

- Spectrum of a Discrete-Time Signal

- Over-Sampling & Under-Sampling

- Stroboscopic effect

Course Materials

Important Materials:

- James H. McClellan, R. W. Schafer, M. A. Yoder, *DSP First Second Edition*, Pearson, 2015.

- Lizhe Tan, Jean Jiang, *Digital Signal Processing: Fundamentals and Applications*, Third Edition, Academic Press, 2019.

Auxilary Materials:

- Prof. Sarp Ertürk, *Sayısal İşaret İşleme*, Birsen Yayınevi.

- Prof. Nizamettin Aydin, DSP Lecture Notes.

- J. G. Proakis, D. K. Manolakis, *Digital Signal Processing Fourth Edition*, Peason, 2014.

- J. K. Perin, *Digital Signal Processing*, *Lecture Notes,* Standford University, 2018.

Syllabus

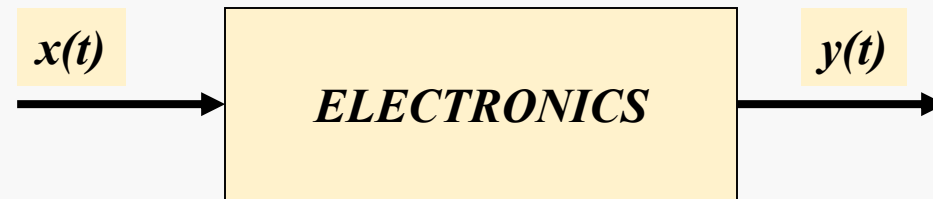| Week	Lectures
1	Introduction to DSP and MATLAB
2	Sinuzoids and Complex Exponentials
3	Spectrum Representation
4	Sampling and Aliasing
5	Discrete Time Signal Properties and Convolution
6	Convolution and FIR Filters
7	Frequency Response of FIR Filters
8	Midterm Exam
9	Discrete Time Fourier Transform and Properties
10	Discrete Fourier Transform and Properties
11	Fast Fourier Transform and Windowing
12	z- Transforms
13	FIR Filter Design and Applications
14	IIR Filter Design and Applications
15	Final Exam

For more details -> Bologna page: http://www.bologna.yildiz.edu.tr/index.php?r=course/view&id=5730&aid=3

- ## ANALOG/ELECTRONIC:
  - ### Circuits: resistors, capacitors, op-amps

$x(t)$ → **ELECTRONICS** → $y(t)$

- ## DIGITAL/MICROPROCESSOR
  - ### Convert x(t) to numbers stored in memory

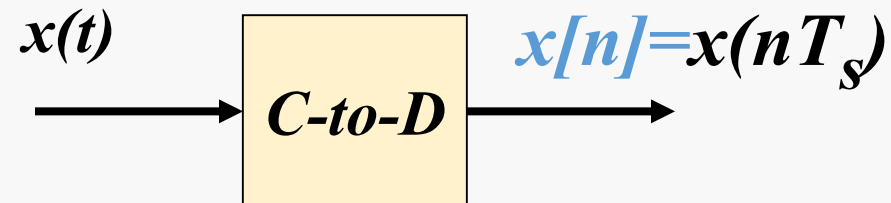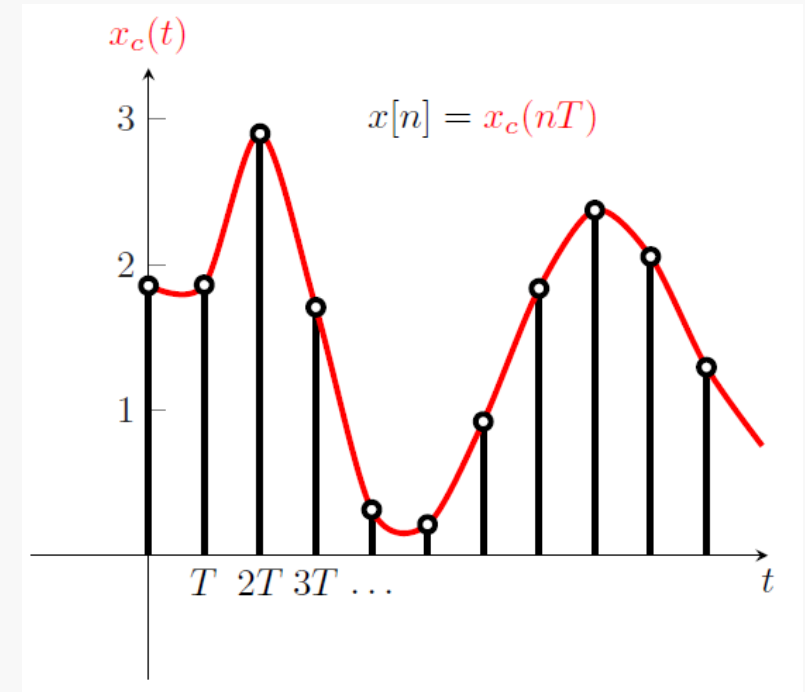$x(t)$ → **A-to-D** → $x[n]$ → **COMPUTER / ES** → $y[n]$ → **D-to-A** → $y(t)$

- ## SAMPLING PROCESS
    - Convert x(t) to numbers x[n]
    - "n" is an <u>integer index</u>; x[n] is a sequence of values
    - Think of "n" as the storage address in memory

- ## UNIFORM SAMPLING at t = nT_s
    - IDEAL: x[n] = x(nT_s)

$$x(t) \longrightarrow \boxed{C\text{-}to\text{-}D} \longrightarrow x[n]$$
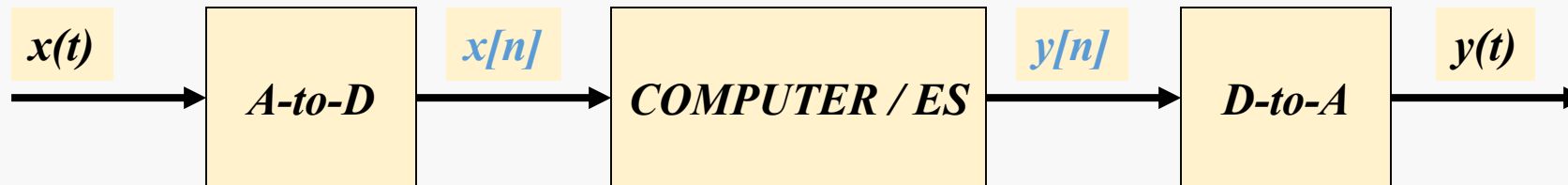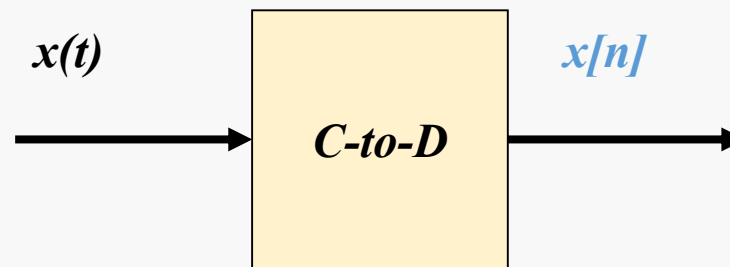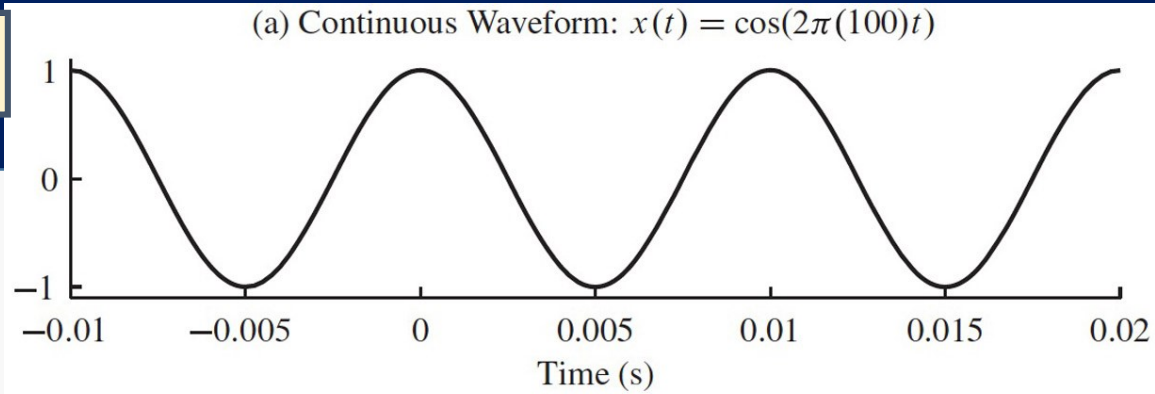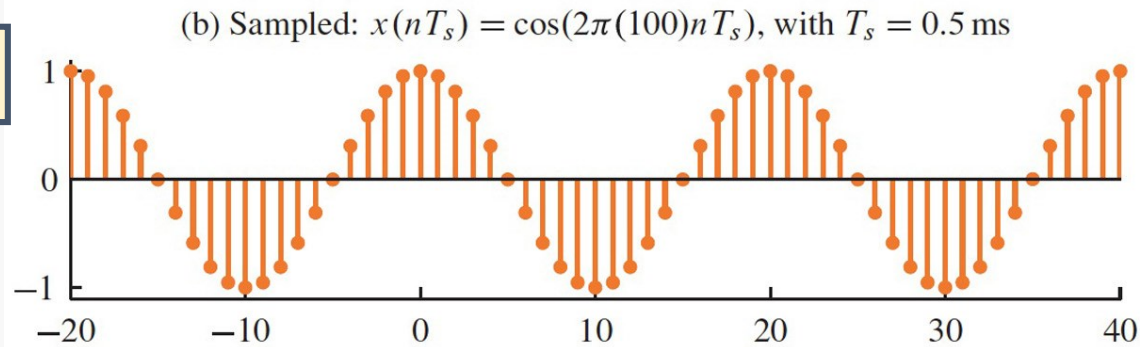
Sampling of Analog Signals

- SAMPLING RATE (f_s)
  - $f_s = 1/T_s$
    - NUMBER of SAMPLES PER SECOND
  - $T_s = 125$ microsec $\rightarrow f_s = 8000$ samples/sec
    - UNITS of f_s ARE HERTZ: 8000 Hz

- UNIFORM SAMPLING at $t = nT_s = n/f_s$
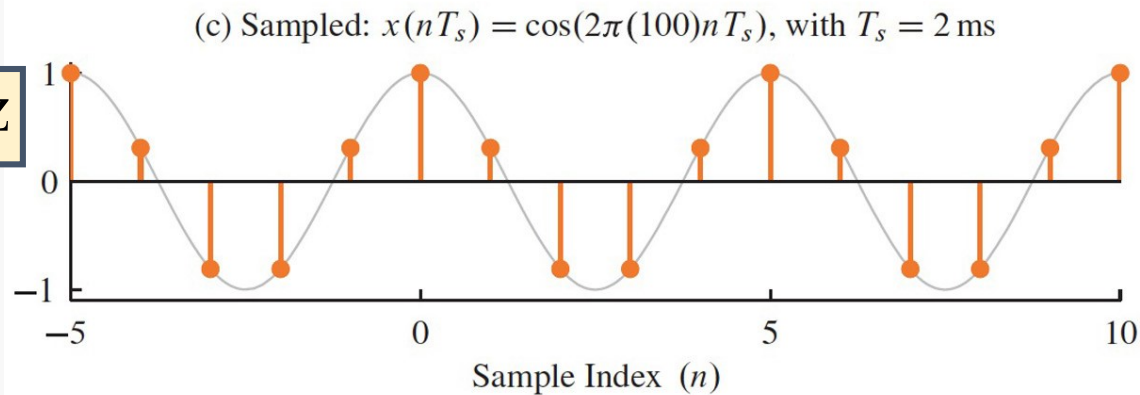 - IDEAL: $x[n] = x(nT_s) = x(n/f_s)$

$x(t) \rightarrow \boxed{C\text{-}to\text{-}D} \rightarrow x[n]=x(nT_s)$

$f = 100 \text{Hz}$

(a) Continuous Waveform: $x(t) = \cos(2\pi(100)t)$

$f_s = 2\,\text{kHz}$

(b) Sampled: $x(nT_s) = \cos(2\pi(100)nT_s)$, with $T_s = 0.5\,\text{ms}$

$f_s = 500 \text{Hz}$

(c) Sampled: $x(nT_s) = \cos(2\pi(100)nT_s)$, with $T_s = 2\,\text{ms}$

Sample Index (n)

Which one provides the most accurate representation of x(t)?

- HOW OFTEN DO WE NEED TO SAMPLE?
  - DEPENDS on FREQUENCY of SINUSOID
  - ANSWERED by SHANNON/NYQUIST Theorem
  - ALSO DEPENDS on "**RECONSTRUCTION**"

Shannon Sampling Theorem

A continuous-time signal $x(t)$ with frequencies no higher than f_{max} can be reconstructed exactly from its samples $x[n] = x(nT_s)$, if the samples are taken at a rate $f_s = 1/T_s$ that is greater than $2f_{max}$.

- "**Nyquist Rate**" Sampling
  - f_s > **TWICE** the HIGHEST Frequency in x(t)
  - "Sampling above the Nyquist rate"

- **BANDLIMITED SIGNALS**
  - DEF: HIGHEST FREQUENCY COMPONENT in SPECTRUM of x(t) is finite
  - NON-BANDLIMITED EXAMPLE
    - TRIANGLE WAVE is NOT BANDLIMITED

- Change x(t) into x[n]   **DERIVATION**

$$x(t) = A\cos(\omega t + \varphi)$$

$$x[n] = x(nT_s) = A\cos(\omega n T_s + \varphi)$$

$$x[n] = A\cos((\omega T_s)n + \varphi)$$
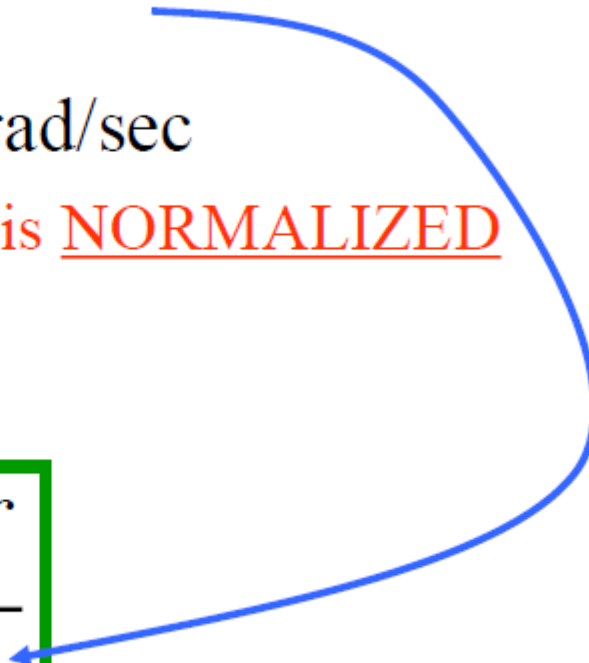
$$x[n] = A\cos(\hat{\omega} n + \varphi)$$

$$\hat{\omega} = \omega T_s = \frac{\omega}{f_s}$$ **DEFINE DIGITAL FREQUENCY**
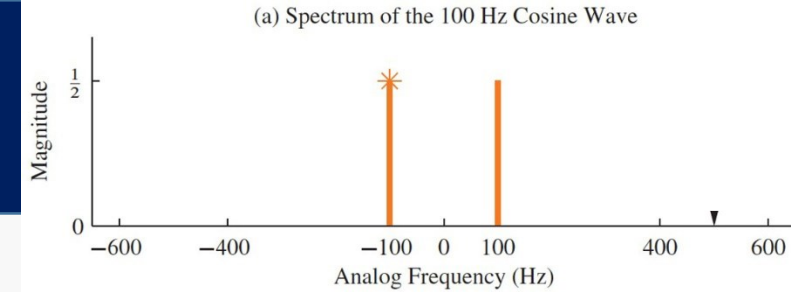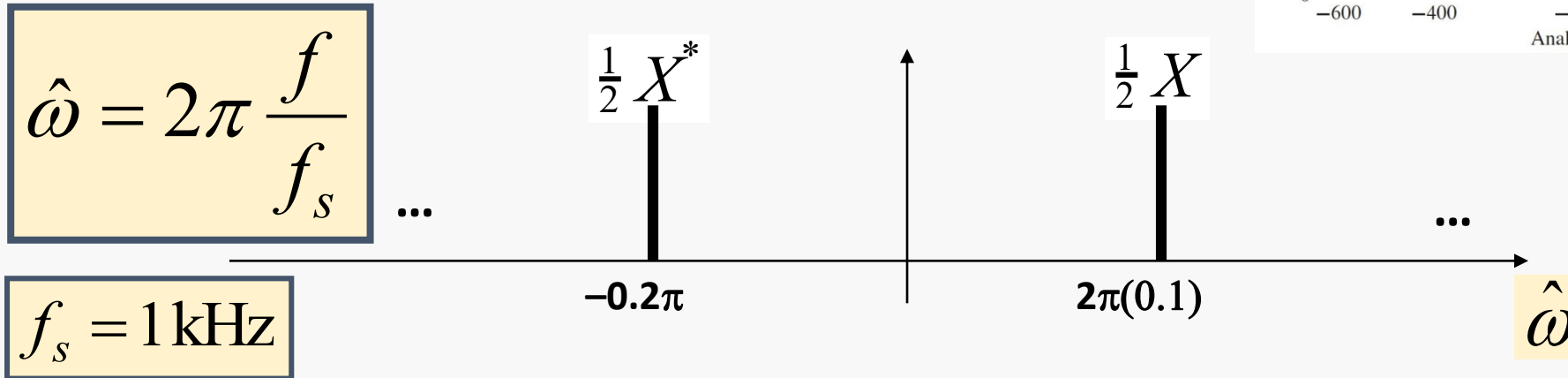
11

- $\hat{\omega}$ VARIES from **0** to **2π**, as f varies from 0 to the sampling frequency
- UNITS are radians, **not** rad/sec
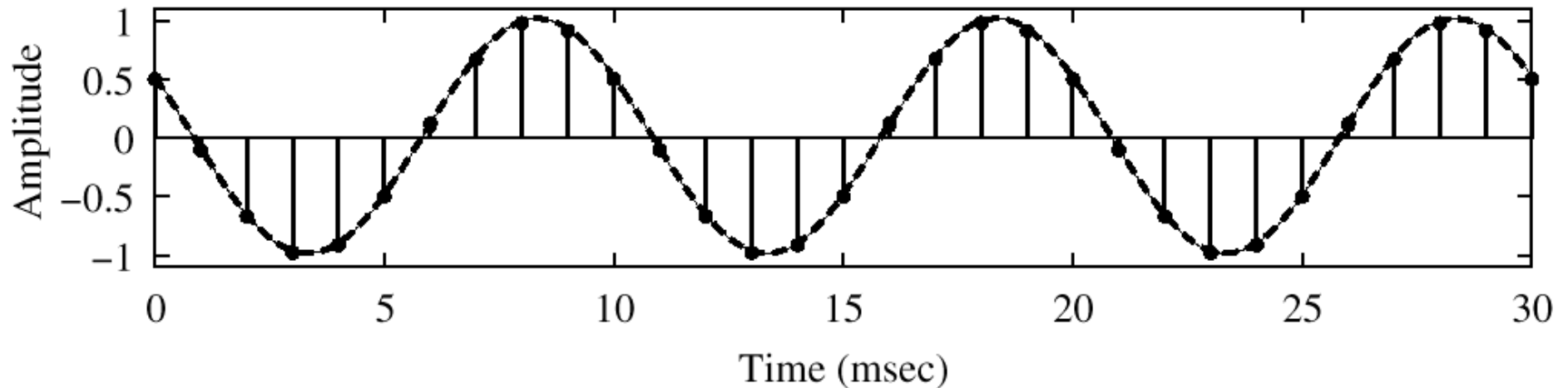 - DIGITAL FREQUENCY is NORMALIZED

$$\hat{\omega} = \omega T_s = \frac{2\pi f}{f_s}$$

Spectrum of Digital Signal

Magnitude

$\frac{1}{2}$

0

-600 -400 -100 0 100 400 600

Analog Frequency (Hz)

$$\hat{\omega} = 2\pi \frac{f}{f_s}$$

$$f_s = 1\,\text{kHz}$$

$\frac{1}{2}X^*$

$\frac{1}{2}X$

...

...

-0.2π

$2\pi(0.1)$

$\hat{\omega}$

$$x[n] = A\cos(2\pi(100)(n/1000) + \varphi)$$

100-Hz Cosine Wave: Sampled with $T_s = 1$ msec (1000 Hz)

Amplitude

1
0.5
0
-0.5
-1

0   5   10   15   20   25   30

Time (msec)

$$\hat{\omega} = 2\pi \frac{f}{f_s}$$

$$f_s = 100 \text{ Hz}$$

$$\tfrac{1}{2}X^* \qquad ? \qquad \tfrac{1}{2}X$$

$$-2\pi \qquad \qquad 2\pi(1)$$

$$\hat{\omega}$$
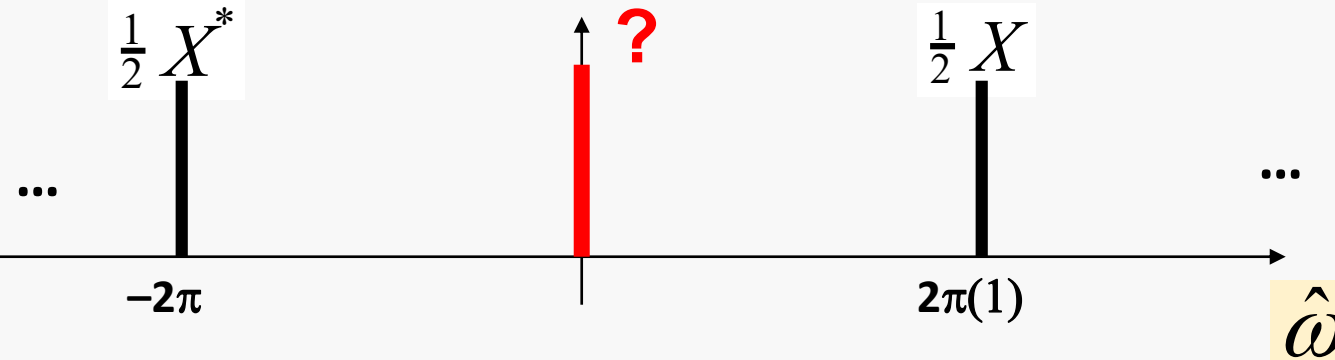
$$x[n] = A\cos(2\pi(100)(n/100) + \varphi)$$

100-Hz Cosine Wave: Sampled with $T_s = 10$ msec (100 Hz)
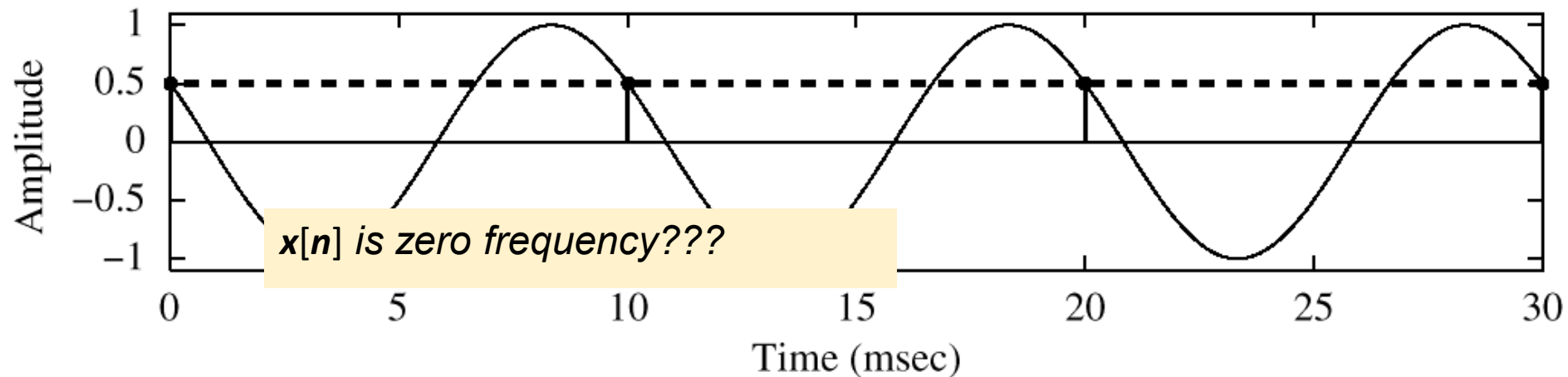
x[n] is zero frequency???

- Spectrum of x[n] has more than one line for each complex exponential
    - Called **ALIASING**
    - **MANY SPECTRAL LINES**

- SPECTRUM is PERIODIC with period = **2π**
    - Because

$$A\cos(\hat{\omega}n + \varphi) = A\cos((\hat{\omega} + 2\pi\ell)n + \varphi)$$

- Other Frequencies give the same

$$\hat{\omega}$$

$$x_1(t) = \cos(400\pi t) \quad \text{sampled at } f_s = 1000\text{Hz}$$

$$x_1[n] = \cos(400\pi \tfrac{n}{1000}) = \cos(0.4\pi n)$$

$$x_2(t) = \cos(2400\pi t) \quad \text{sampled at } f_s = 1000\text{Hz}$$

$$x_2[n] = \cos(2400\pi \tfrac{n}{1000}) = \cos(2.4\pi n)$$

$$x_2[n] = \cos(2.4\pi n) = \cos(0.4\pi n + 2\pi n) = \cos(0.4\pi n)$$

$$\Rightarrow x_2[n] = x_1[n]$$

$$2400\pi - 400\pi = 2\pi(1000)$$

GIVEN x[n], we CAN'T KNOW whether it came from a sinusoid at f_o or ($f_o + f_s$) or ($f_o + 2f_s$) …

- Other Frequencies give the same $\hat{\omega}$

$$\text{If } x(t) = A\cos(2\pi(f + \ell f_s)t + \varphi)$$

$$t \leftarrow \frac{n}{f_s}$$

$$\text{and we want : } x[n] = A\cos(\hat{\omega}n + \varphi)$$

$$\text{then : } \hat{\omega} = \frac{2\pi(f + \ell f_s)}{f_s} = \frac{2\pi f}{f_s} + \frac{2\pi \ell f_s}{f_s}$$

$$\hat{\omega} = \omega T_s = \frac{2\pi f}{f_s} + 2\pi \ell$$

Normalized Frequency

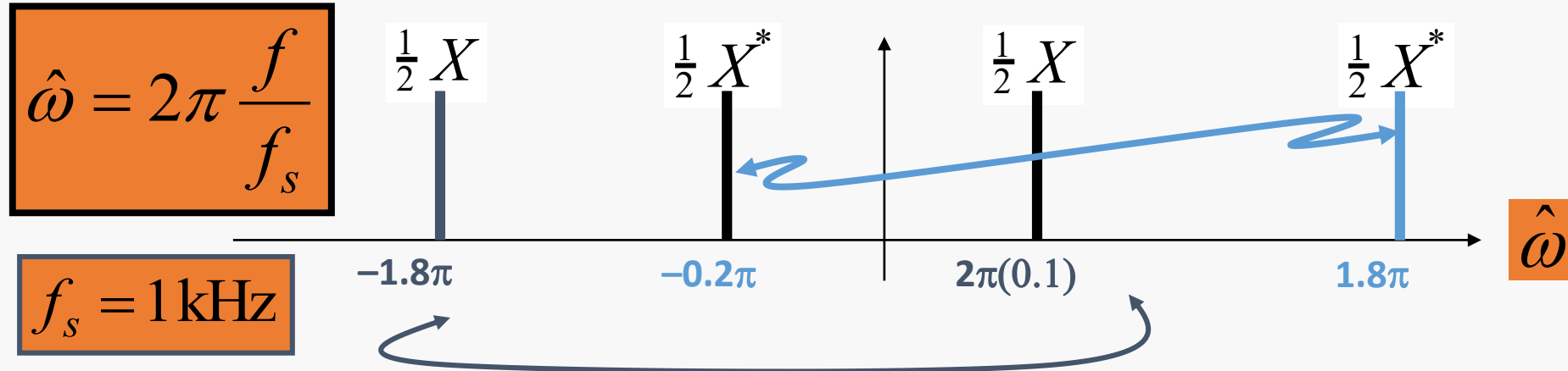$$\hat{\omega}$$

Normalized Radian Frequency

$$\boxed{\hat{\omega} = \omega T_s = \frac{2\pi f}{f_s}} + 2\pi\ell$$

ALIASING

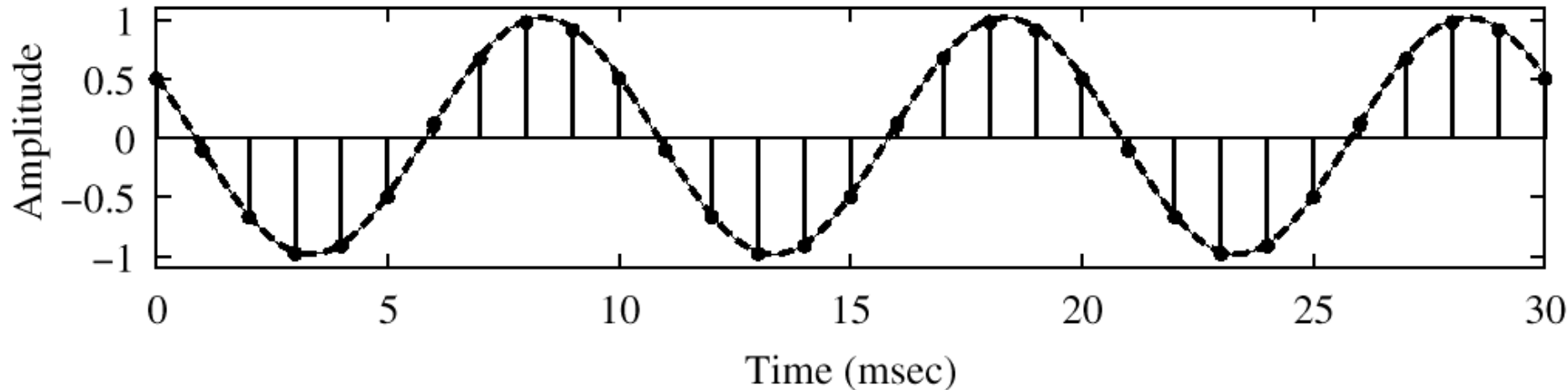$$\hat{\omega} = \omega T_s = -\frac{2\pi f}{f_s} + 2\pi\ell$$

FOLDED ALIAS

$$\hat{\omega} = 2\pi \frac{f}{f_s}$$

$$f_s = 1\,\text{kHz}$$

$\frac{1}{2}X$ $\frac{1}{2}X^*$ $\frac{1}{2}X$ $\frac{1}{2}X^*$

$\hat{\omega}$

−1.8π   **−0.2π**   **2π(0.1)**   **1.8π**

$$x[n] = A\cos(2\pi(100)(n/1000) + \varphi)$$

100-Hz Cosine Wave: Sampled with T_s = 1 msec (1000 Hz)

Principal alias :

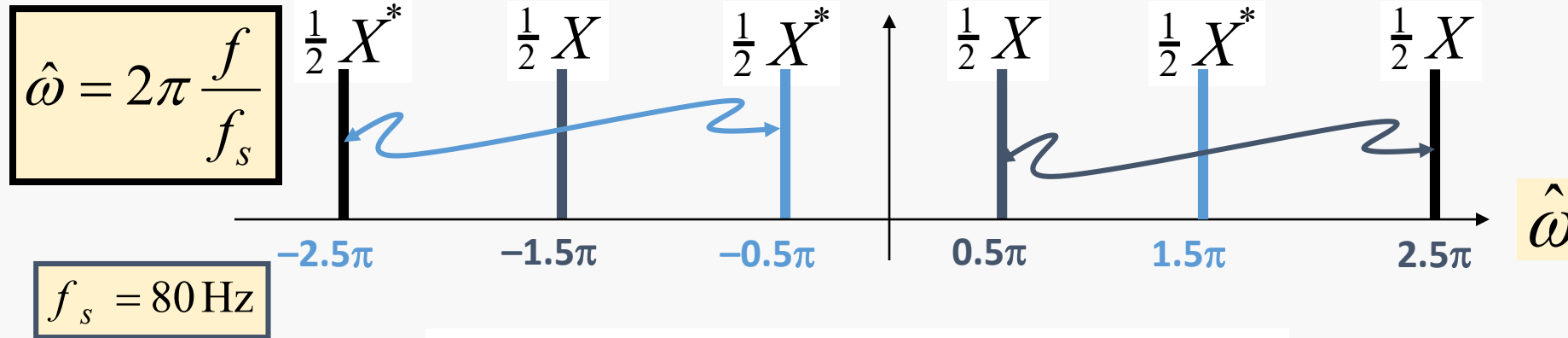$$f = \frac{\hat{\omega} f_s}{2\pi} = 0.1(1000) = 100\,\text{Hz}$$

$$x(t) = A\cos(2\pi 100t + \varphi)$$

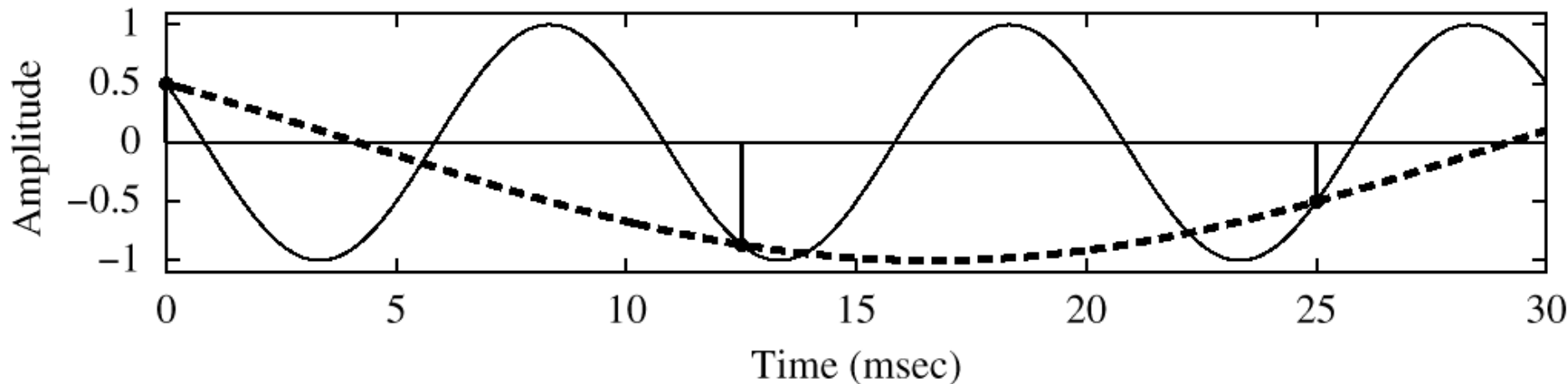Principal alias is always between $-\pi \leq \hat{\omega} \leq \pi$

$$\hat{\omega} = 2\pi \frac{f}{f_s}$$

$\frac{1}{2}X^*$ $\frac{1}{2}X$ $\frac{1}{2}X^*$ $\frac{1}{2}X$ $\frac{1}{2}X^*$ $\frac{1}{2}X$

$\hat{\omega}$

−2.5π −1.5π −0.5π 0.5π 1.5π 2.5π

$f_s = 80\,\mathrm{Hz}$

$$x[n] = A\cos(2\pi(100)(n/80) + \varphi)$$

100-Hz Cosine Wave: Sampled with T_s = 12.5 msec (80 Hz)

Principal alias :

$$f = \frac{\hat{\omega} f_s}{2\pi} = 0.25(80) = 20\,\mathrm{Hz}$$

$$x(t) = A\cos(2\pi 20 t + \varphi)$$

(a) Spectrum of the 100 Hz Cosine Wave

(b) Sampled at $T_s = 12.5$ ms ($f_s = 80$ Hz)

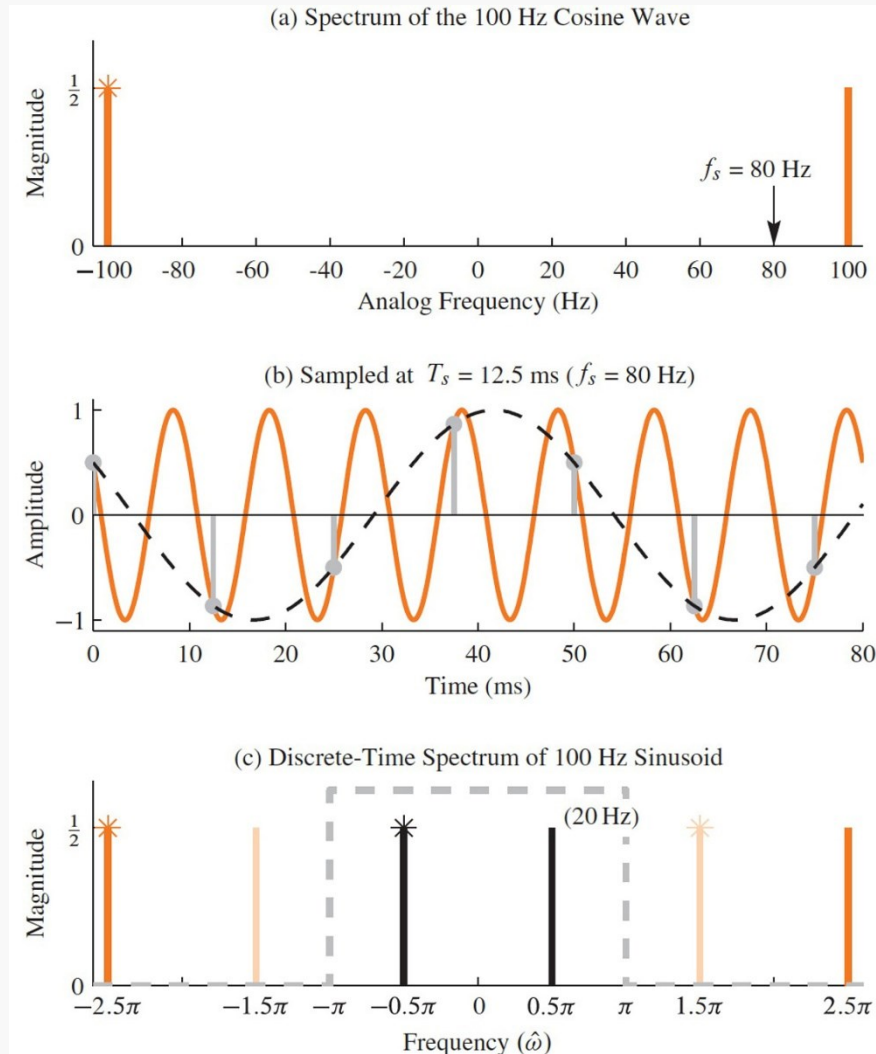(c) Discrete-Time Spectrum of 100 Hz Sinusoid

Figure 4-9 *Under-sampling* a 100 Hz sinusoid at f_s = 80 samples/s.

(a) Continuous-time spectrum;

(b) time-domain plot, showing the samples $x[n]$ as gray dots, the original signal $x(t)$ as a continuous **orange** line, and the reconstructed signal $y(t)$ as a dashed black line, which is a 20 Hz sinusoid passing through the same sample points; and

(c) discrete-time spectrum plot, showing the positive and negative frequency components of the original sinusoid at = ±2.5π rad, along with two sets of alias components.

FOLDING (a type of ALIASING)

- EXAMPLE: 3 different x(t); same x[n]

$$\hat{\omega} = 2\pi \frac{100}{1000} = 2\pi(0.1)$$

$$f_s = 1000$$

$$\cos(2\pi(100)t) \rightarrow \cos[2\pi(0.1)n]$$

$$\cos(2\pi(1100)t) \rightarrow \cos[2\pi(1.1)n] = \cos[2\pi(0.1)n]$$

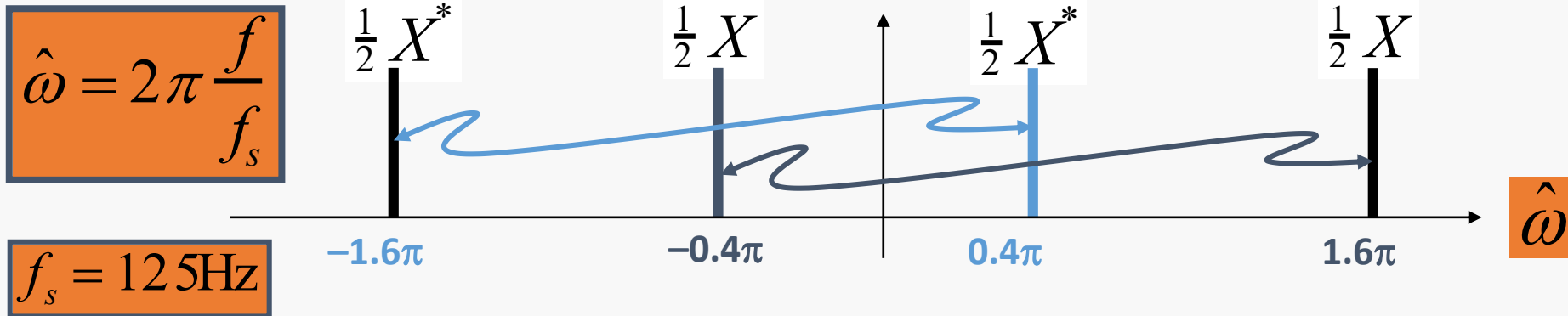$$\cos(2\pi(900)t) \rightarrow \cos[2\pi(0.9)n]$$

$$= \cos[2\pi(0.9)n - 2\pi n] = \cos[2\pi(-0.1)n] = \cos[2\pi(0.1)n]$$

- 900 Hz "folds" to 100 Hz when f_s=1kHz

$$\hat{\omega} = 2\pi \frac{f}{f_s}$$

$$f_s = 125\,\text{Hz}$$

$\frac{1}{2}X^*$ $\frac{1}{2}X$ $\frac{1}{2}X^*$ $\frac{1}{2}X$

-1.6π -0.4π 0.4π 1.6π $\hat{\omega}$

$$x[n] = A\cos(2\pi(100)(n/125) + \varphi)$$

100-Hz Cosine Wave: Sampled with $T_s = 8$ msec (125 Hz)
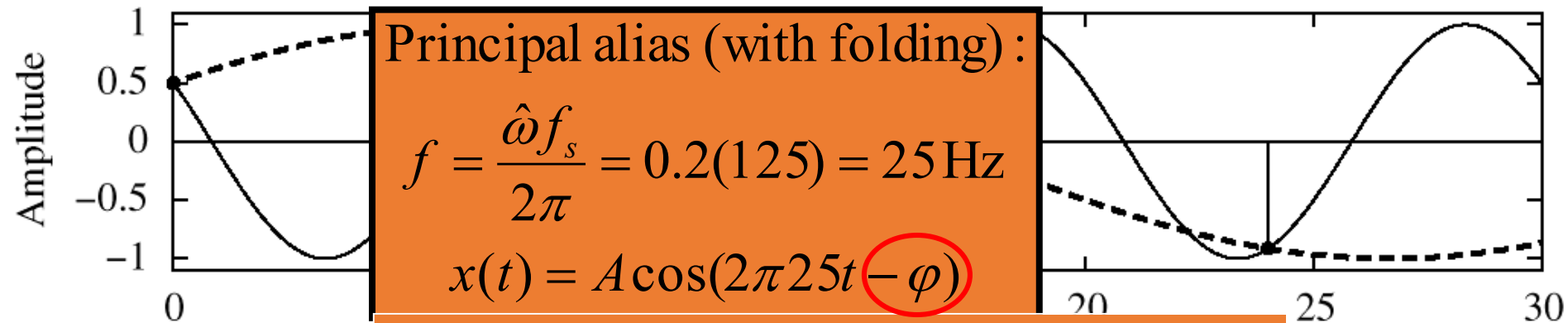
Principal alias (with folding) :

$$f = \frac{\hat{\omega}f_s}{2\pi} = 0.2(125) = 25\,\text{Hz}$$

$$x(t) = A\cos(2\pi 25 t - \varphi)$$

Principal alias is always between $-\pi \le \hat{\omega} \le \pi$

- How do we prevent aliasing?

- Guarantee original signal is principal alias:

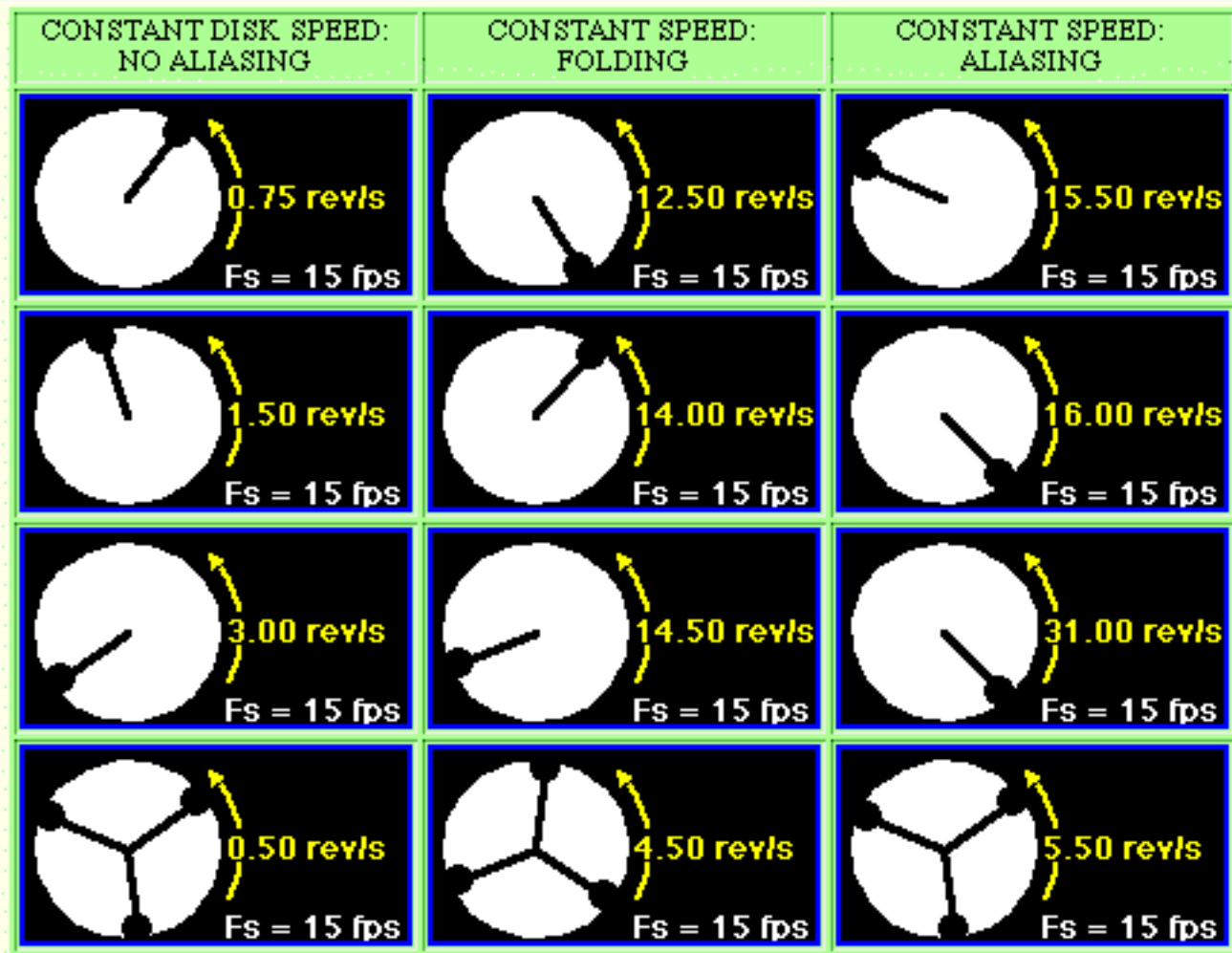$$\hat{\omega}_0 - 2\pi < -\hat{\omega}_0 \Rightarrow \hat{\omega}_0 < \pi$$

$$\hat{\omega}_0 = \frac{2\pi f_0}{f_s} < \pi \Rightarrow f_0 < \frac{f_s}{2}$$

Be Careful:

https://www.youtube.com/watch?v=qgvuQGY946g

Strobe Demo

https://dspfirst.gatech.edu/chapters/04samplin/demos/strobe/index.html

https://dspfirst.gatech.edu/chapters/04samplin/demos/synstrob/index.html
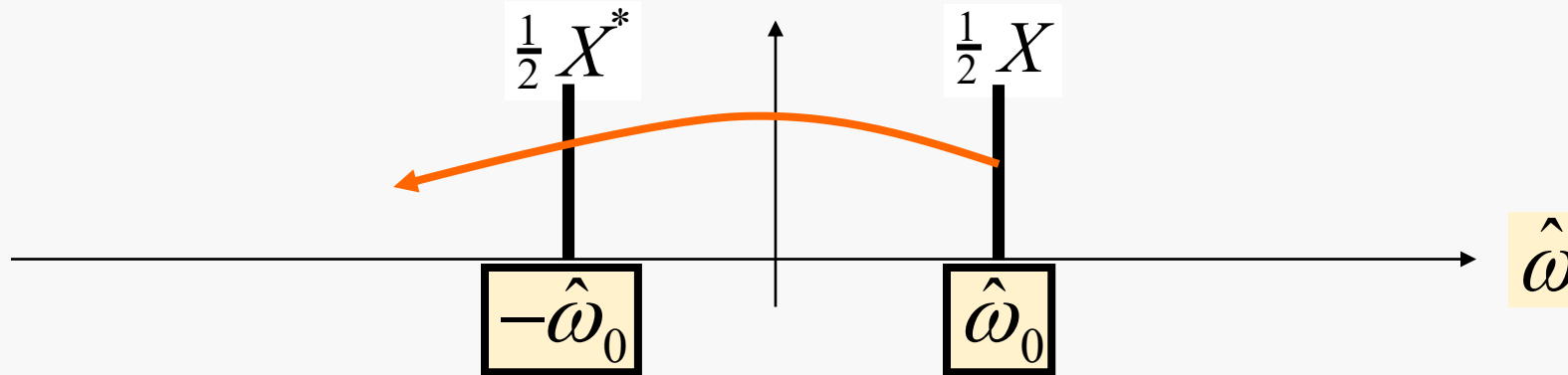
Digital to Analog Reconstruction

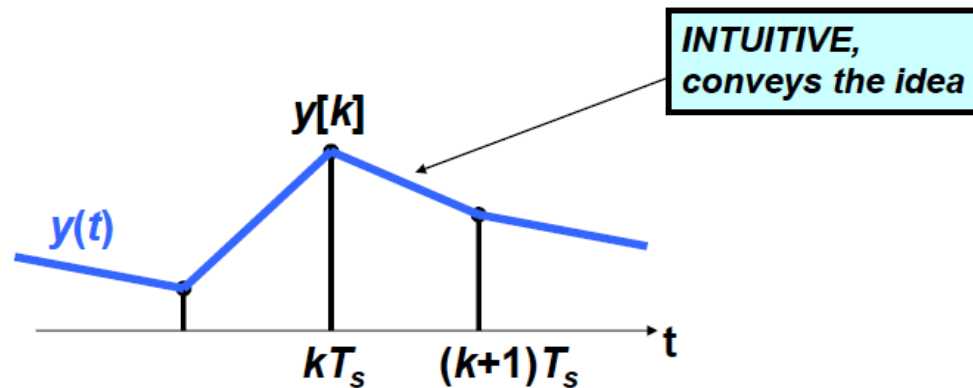x(t) → **A-to-D** → x[n] → **COMPUTER** → y[n] → **D-to-A** → y(t)

- Create continuous y(t) from y[n]
   - **IDEAL D-to-A:**
        - If you have formula for y[n]
    - Invert sampling ($t = nT_s$) by $\boxed{n = f_s t}$
    - $y[n] = A\cos(0.2\pi n + \phi)$ with $f_s = 8000$ Hz
    - $y(t) = A\cos(0.2\pi(8000t) + \phi) = A\cos(2\pi(\mathbf{800})t + \phi)$

- RECONSTRUCT THE SMOOTHEST ONE
  - THE LOWEST FREQ, if $y[n]$ = sinusoid

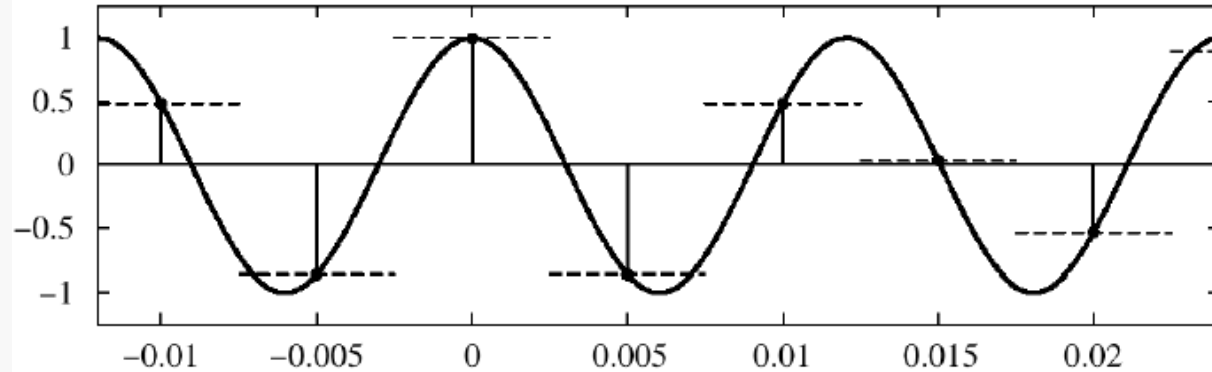- CONVERT STREAM of NUMBERS to $x(t)$
- "CONNECT THE DOTS"
- INTERPOLATION

INTUITIVE, conveys the idea

$y[k]$

$y(t)$

kT_s $(k+1)T_s$ t

- CONVERT $y[n]$ to $y(t)$
  - $y[k]$ should be the value of $y(t)$ at $t = kT_s$
  - Make $y(t)$ equal to $y[k]$ for
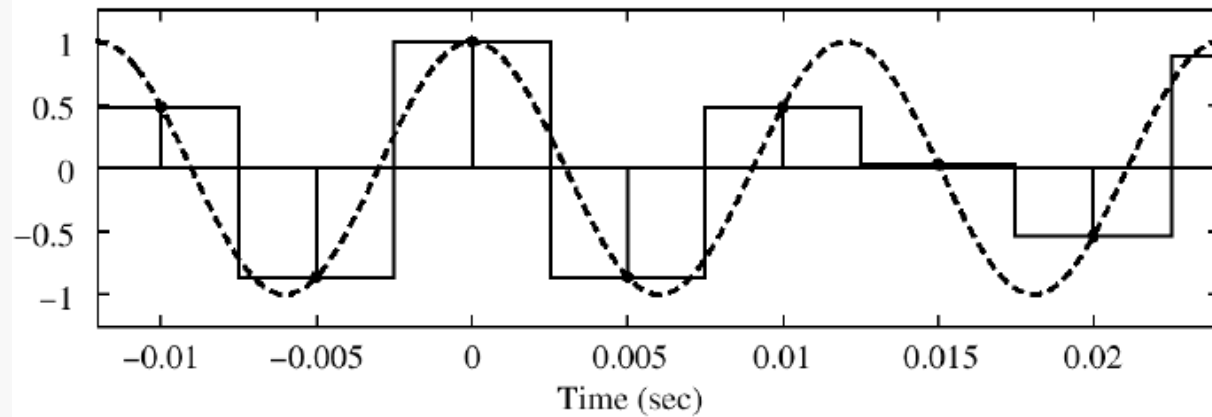   - $kT_s - 0.5T_s < t < kT_s + 0.5T_s$

$y[k]$

$y(t)$
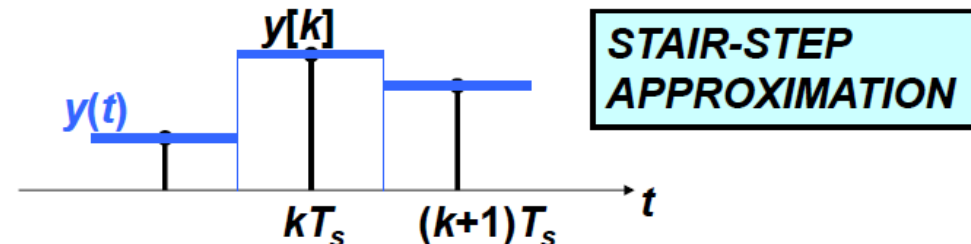
STAIR-STEP APPROXIMATION

kT_s $(k+1)T_s$ t

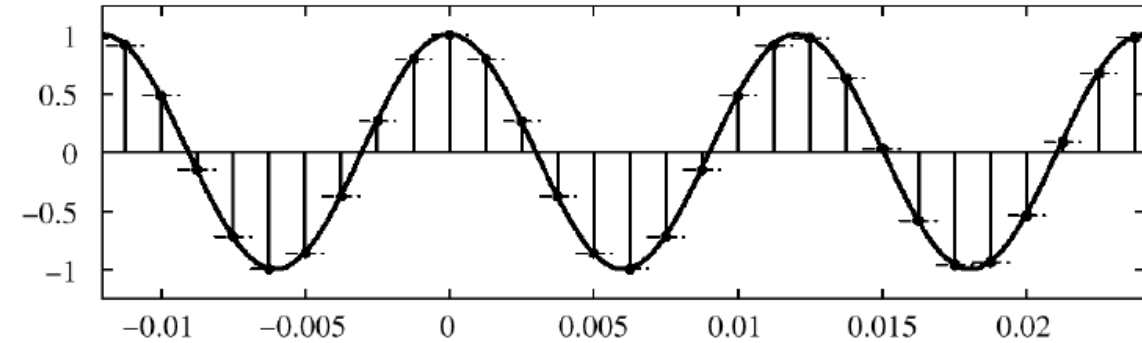Sampling and Zero-Order Reconstruction: $f_0 = 83$ $f_s = 200$
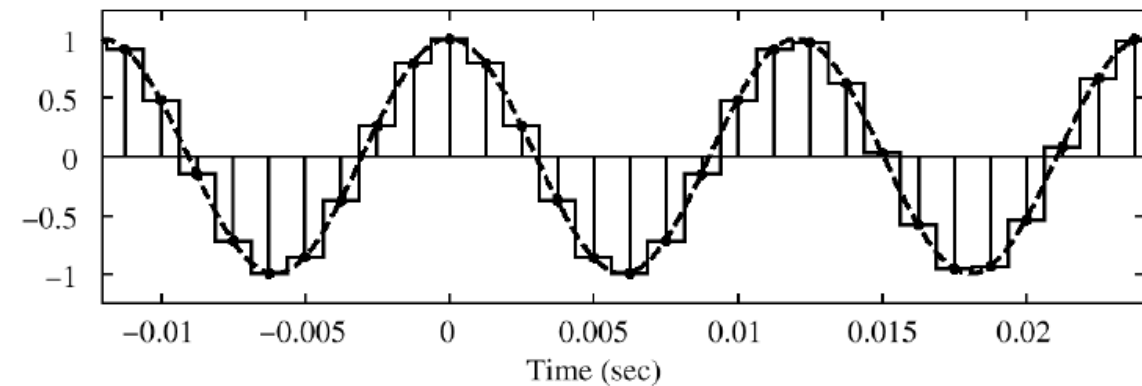
Original and Reconstructed Waveforms

Time (sec)

OVER-SAMPLING CASE

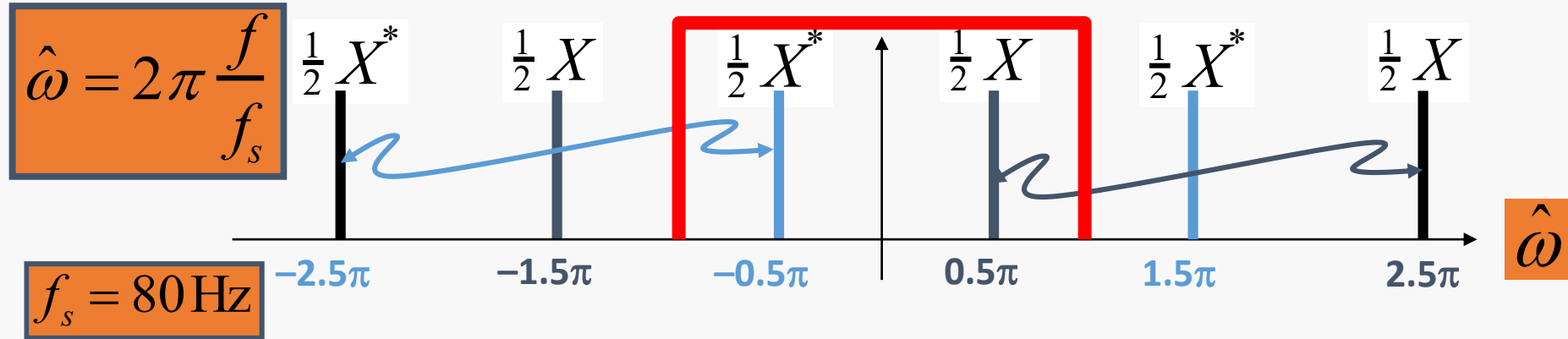Sampling and Zero-Order Reconstruction: $f_0 = 83$ $f_s = 800$

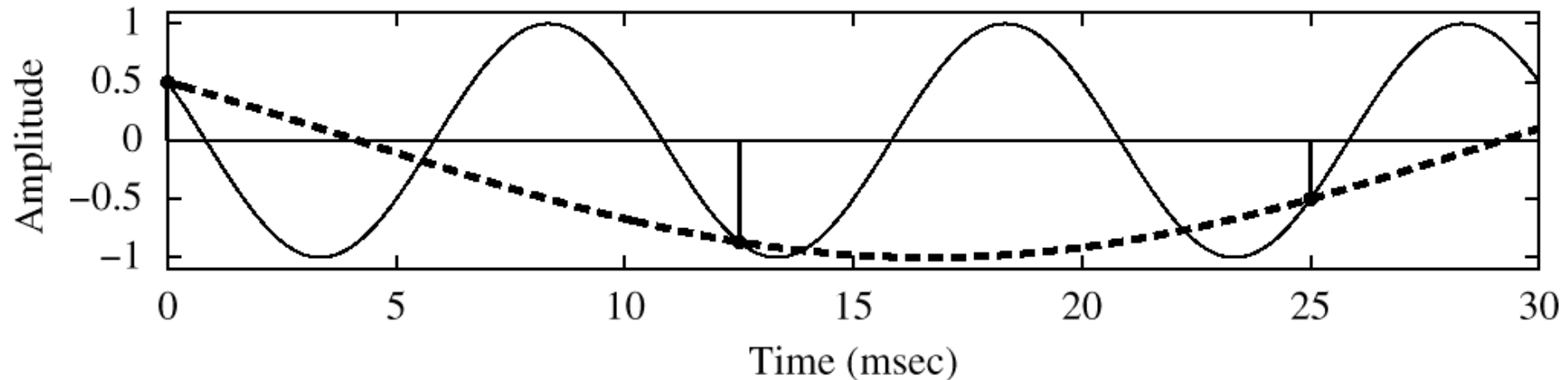**EASIER TO RECONSTRUCT**

Original and Reconstructed Waveforms

Time (sec)

$$\hat{\omega} = 2\pi \frac{f}{f_s}$$

$$f_s = 80\,\text{Hz}$$

$$\tfrac{1}{2}X^* \qquad \tfrac{1}{2}X \qquad \tfrac{1}{2}X^* \qquad \tfrac{1}{2}X \qquad \tfrac{1}{2}X^* \qquad \tfrac{1}{2}X$$

$$-2.5\pi \qquad -1.5\pi \qquad -0.5\pi \qquad 0.5\pi \qquad 1.5\pi \qquad 2.5\pi$$

$$\hat{\omega}$$
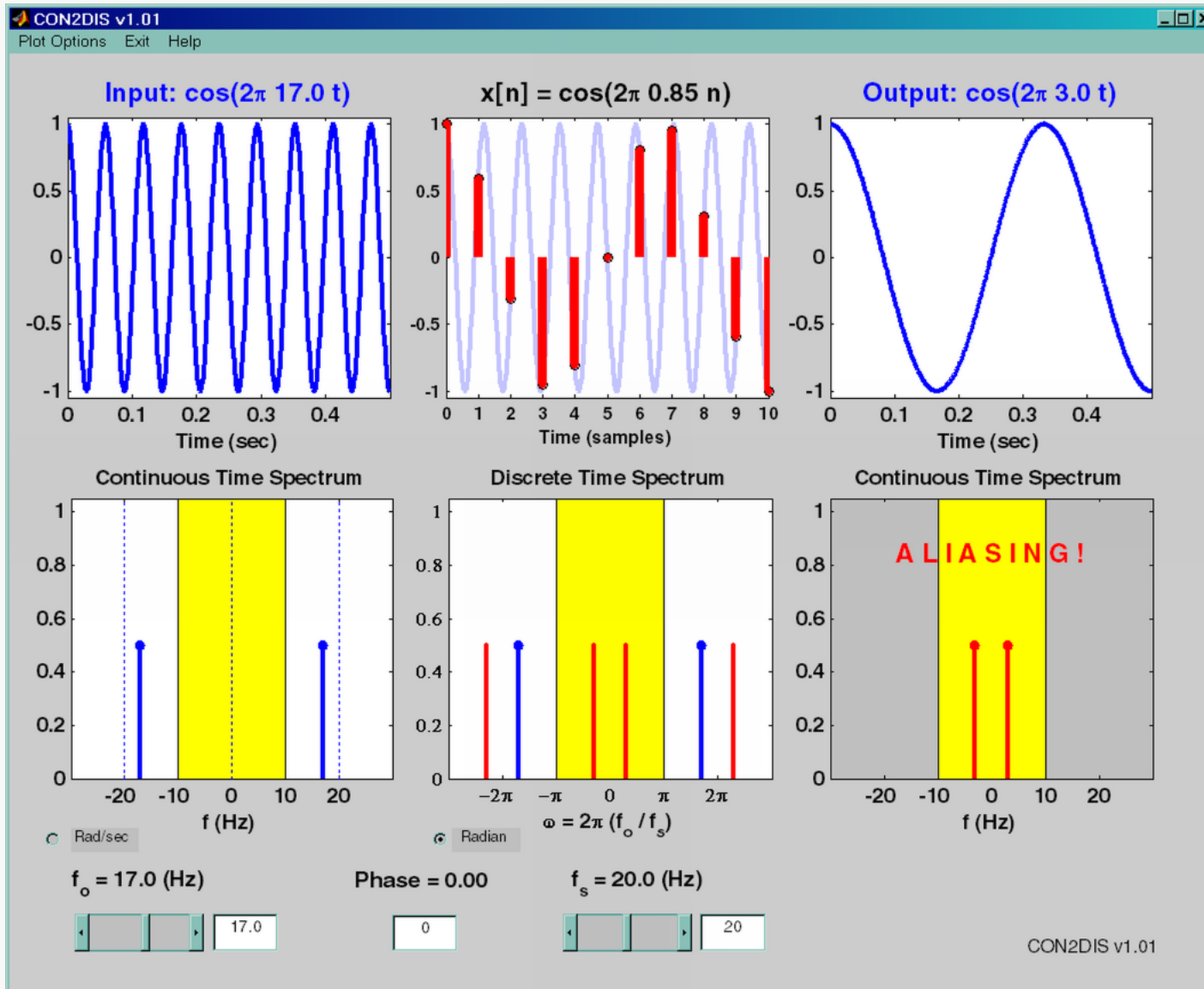
$$x[n] = A\cos(2\pi(100)(n/80) + \varphi)$$

100-Hz Cosine Wave: Sampled with $T_s = 12.5$ msec (80 Hz)

Let's Analyze with MATLAB

https://dspfirst.gatech.edu/matlab/#con2dis
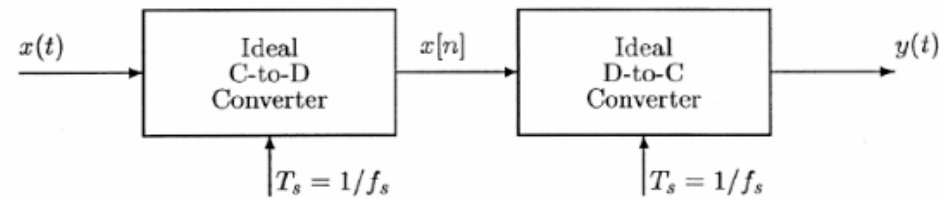
Example - 1

Suppose that the [output] of the D-to-C converter in the system above is found to be

$$y(t) = 2 + 10\cos(2\pi(150)t + \pi/3)$$

when the sampling rate is $f_s = 1/T_s = 400$ samples/second.

(a) Give an equation for $x[n]$ in terms of cosine functions. Write your answer on the line below.

With no aliasing, going through a D/C converter is the inverse of going through the C/D. Therefore, we can get $x[n]$ by passing $y(t)$ through a C/D converter.

Answer: $x[n] = $ $\dfrac{2 + 10\cos\left(2\pi(150)n/400 + \pi/3\right)}{= \quad 2 + 10\cos\left(3n\pi/4 + \pi/3\right)}$
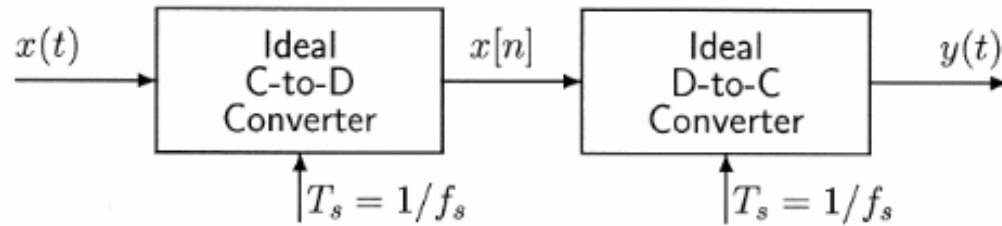
(b) Determine two *different* input signals $x(t) = x_1(t)$ and $x(t) = x_2(t)$ that could have produced the given output of the D-to-C converter. **All of the frequencies in your answers must be positive and less than 400 Hz.** Write your answers for both inputs on the lines below.

Answer: $x_1(t) = $ $\dfrac{2 + 10\cos\left(2\pi(150)t + \pi/3\right)}{(\text{no aliasing})}$

Answer: $x_2(t) = $ $2 + 10\cos\left(2\pi(250)t - \pi/3\right)$

We have folding in the second case.

Example-2

Suppose that the continuous-time input $x(t)$ to the above system is given as

$$x(t) = \cos(14000\pi t) + \cos(2000\pi t) + \cos(1000\pi t).$$

$f_{max} = 7,000 \; Hz$

(a) What sampling rate is required such that no aliasing occurs for $x(t)$? $f_s = 14,000 \; Hz$

(c) Given that $x(t) = \cos(25000\pi t)$ and $f_s = 10000$ samples/second, write a simplified expression for the output $y(t)$ in terms of cosine functions.

$$x[n] = \cos\left(\frac{2500\pi n}{10000}\right) = \cos(0.5\pi n)$$

$$y(t) = \cos(5000\pi t)$$

NOTE THAT ALIASING OCCURS.

$$y(t) = \cos(0.5\pi t 10000) = \cos(5000\pi t)$$

Example-3

PROBLEM:

The "spectrum" diagram gives the frequency content of a signal.

(a) Draw a sketch of the spectrum of $x(t)$ which is "cosine-times-sine"
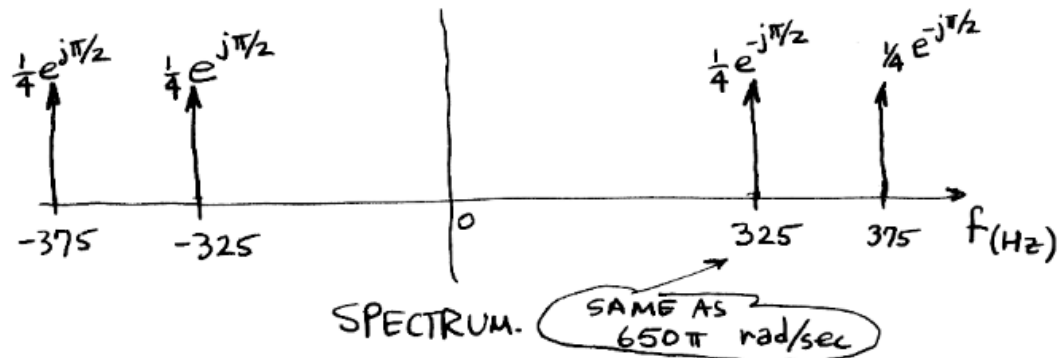
$$x(t) = \cos(50\pi t)\ \sin(700\pi t)$$

Label the frequencies and complex amplitudes of each component.

(b) Determine the minimum sampling rate that can be used to sample $x(t)$ without any aliasing.

$$x(t) = \cos(50\pi t)\ \sin(700\pi t)$$

(a) $x(t) = \left(\frac{1}{2} e^{j50\pi t} + \frac{1}{2} e^{-j50\pi t} \right) \left(\frac{1}{2j} e^{j700\pi t} - \frac{1}{2j} e^{-j700\pi t} \right)$

$= \frac{1}{4j} e^{j750\pi t} + \frac{1}{4j} e^{j650\pi t} - \frac{1}{4j} e^{-j650\pi t} - \frac{1}{4j} e^{-j750\pi t}$

SAME AS $\frac{1}{4} e^{-j\pi/2}$ SAME AS $\frac{1}{4} e^{+j\pi/2}$

$\frac{1}{4} e^{j\pi/2}$ $\frac{1}{4} e^{j\pi/2}$ $\frac{1}{4} e^{-j\pi/2}$ $\frac{1}{4} e^{-j\pi/2}$

-375   -325   0   325   375 $f_{(Hz)}$

SPECTRUM.   SAME AS 650π rad/sec

(b) Sampling Thm says sample at a rate greater than [two] times the highest freq.
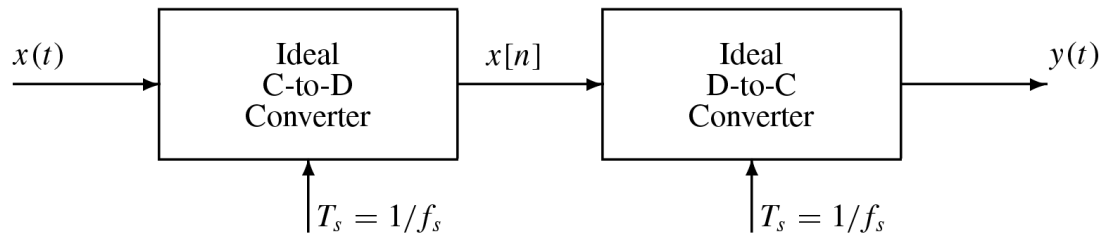
HIGHEST FREQ = 375 Hz

$\Rightarrow f_s \geq 750$ Hz.

Example-4

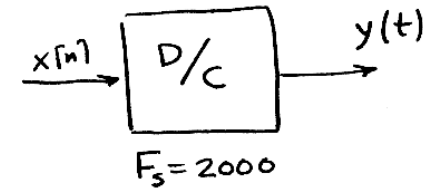# Example-4

PROBLEM:

Consider the following system.

Suppose that the output of the C-to-D converter is

$$x[n] = 5 + 8\cos(0.4\pi n) + 4\cos(0.8\pi n + \pi/3)$$

when the sampling rate is $f_s = 1/T_s = 2000$ samples/second. Determine the output $y(t)$ of the ideal D-to-C converter.

$$x[n] = 5 + 8\cos\left(0.4\pi n\right) + 4\cos\left(0.8\pi n + \pi/3\right)$$

$$F_s = 2000$$

For discrete to continuous, we replace "n" with $F_s t$

$$y(t) = x[n]\Big|_{n=F_s t}$$

$$= 5 + 8\cos\left(0.4\pi(2000)t\right) + 4\cos\left(0.8\pi(2000)t + \pi/3\right)$$

$$= 5 + 8\cos\left(2\pi(400)t\right) + 4\cos\left(2\pi(800)t + \pi/3\right)$$