BLM2041 Signals and Systems

Syllabus

The Instructors:

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Sampling & Aliasing

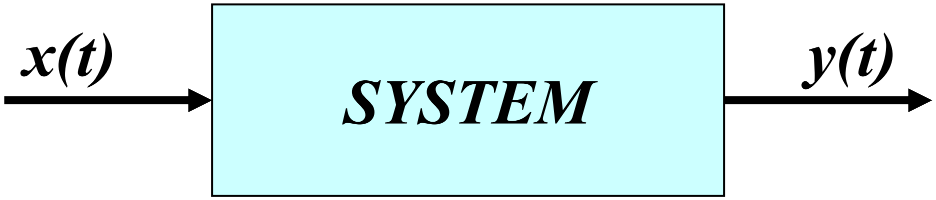
LECTURE OBJECTIVES

- SAMPLING can cause ALIASING
  - **Sampling Theorem**
    - Sampling Rate > 2(Highest Frequency)
- Spectrum for digital signals, $x[n]$
  - Normalized Frequency

$$\hat{\omega} = \omega T_s = \frac{2\pi f}{f_s} + 2\pi\ell$$

ALIASING

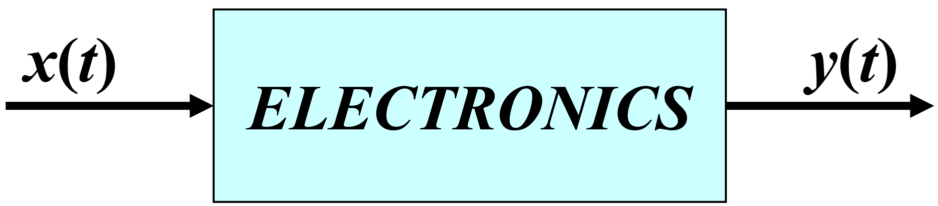
SYSTEMS Process Signals



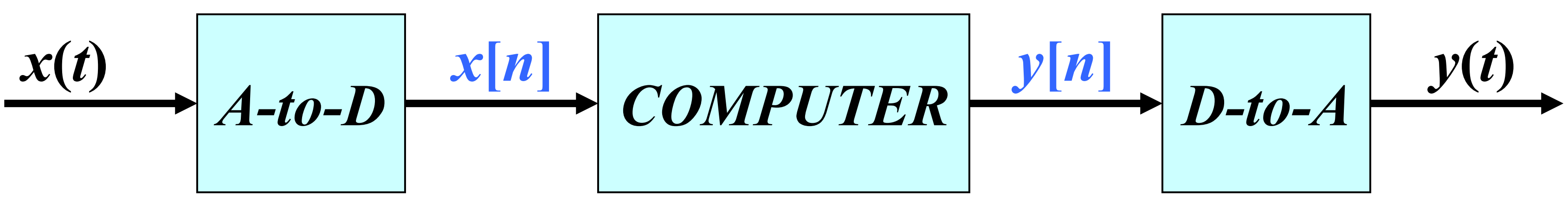
- PROCESSING GOALS:
  - Change $x(t)$ into $y(t)$
    - For example, more BASS
  - Improve $x(t)$,
    - e.g., image deblurring
  - Extract information from $x(t)$

System IMPLEMENTATION

- ANALOG/ELECTRONIC:
  - Circuits: resistors, capacitors, op-amps

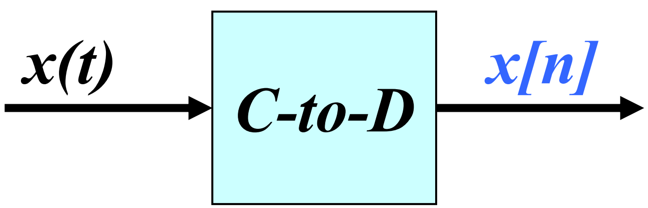


- DIGITAL/MICROPROCESSOR
  - Convert $x(t)$ to numbers stored in memory



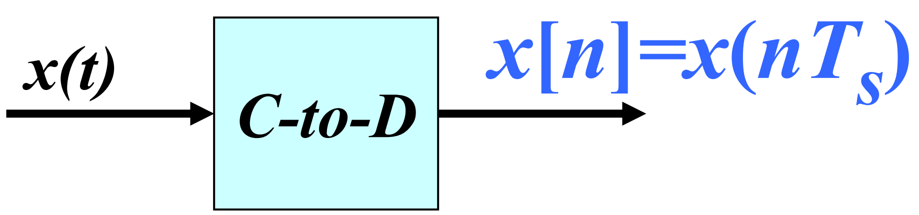
SAMPLING $x(t)$

- SAMPLING PROCESS
  - Convert $x(t)$ to numbers $x[n]$
  - "n" is an integer;
  - $x[n]$ is a sequence of values
  - Think of "n" as the storage address in memory
- UNIFORM SAMPLING at $t = nT_s$
  - IDEAL: $x[n] = x(nT_s)$



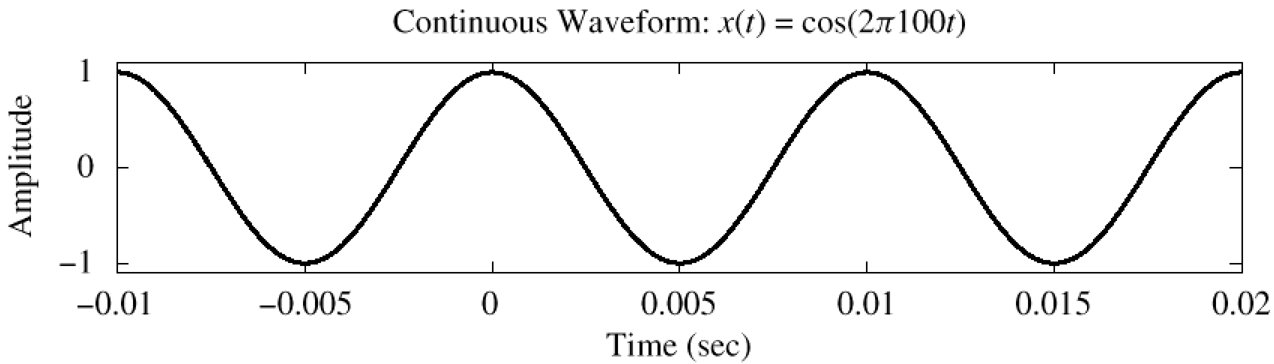
SAMPLING RATE, f_s

- SAMPLING RATE (f_s)
  - $f_s = 1/T_s$
    - NUMBER of SAMPLES PER SECOND
      - T_s = 125 microsec → f_s = 8000 samples/sec
      - UNITS ARE HERTZ: 8000 Hz
- UNIFORM SAMPLING at $t = nT_s = n/f_s$
  - IDEAL: $x[n] = x(nT_s) = x(n/f_s)$

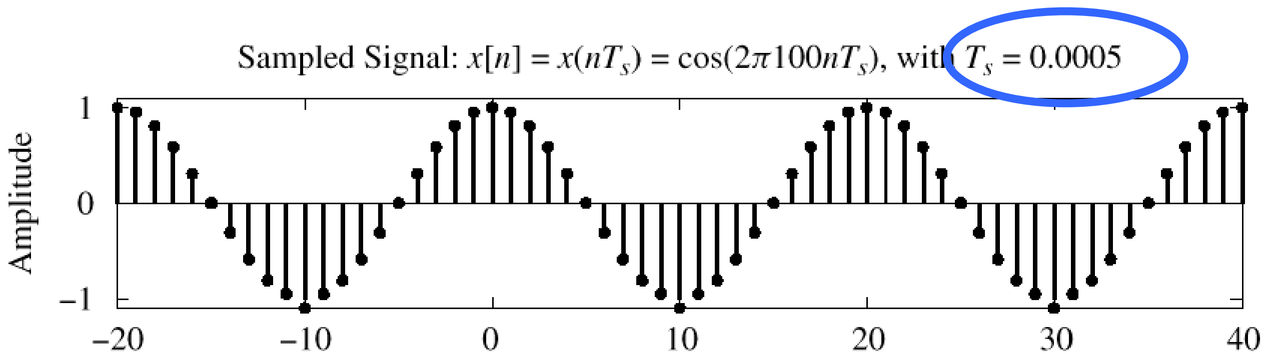


SAMPLING RATE, f_s

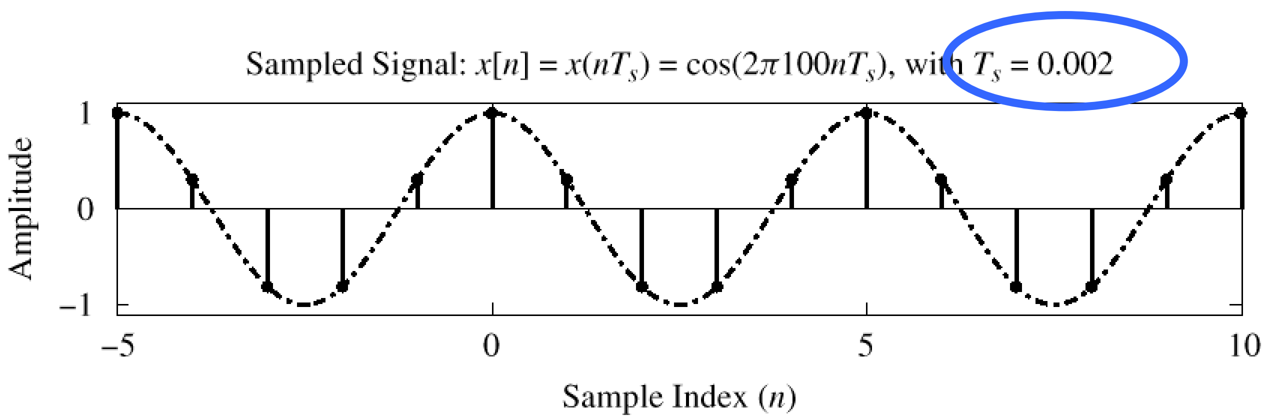
$f = 100\text{Hz}$



$f_s = 2\,\text{kHz}$



$f_s = 500\text{Hz}$



SAMPLING THEOREM

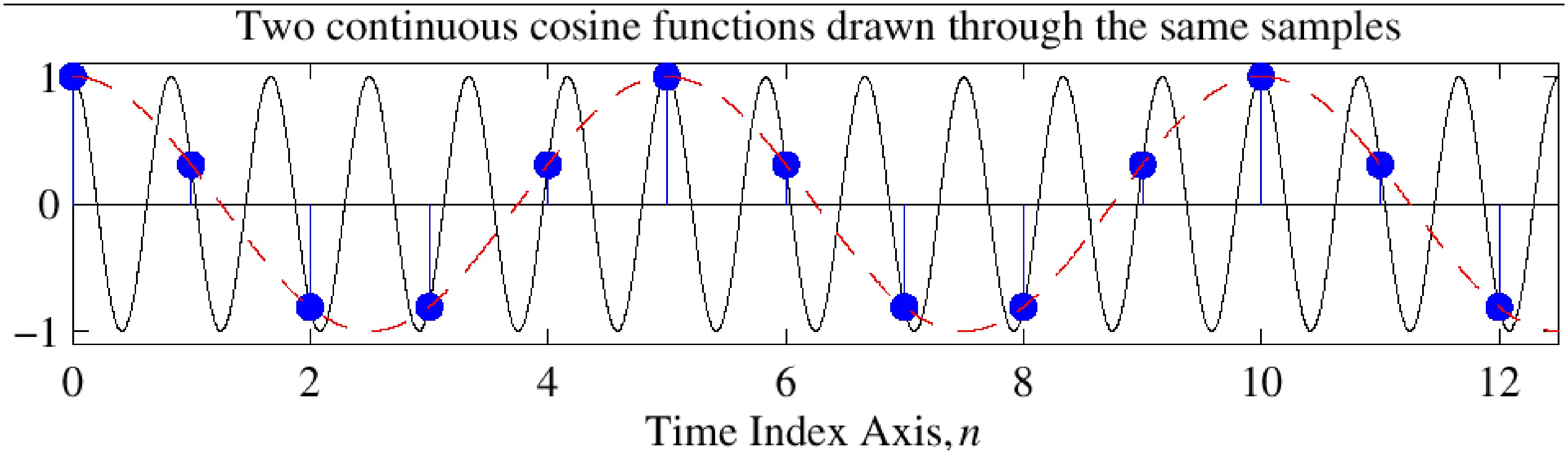
- HOW OFTEN ?
  - DEPENDS on FREQUENCY of SINUSOID
  - ANSWERED by SHANNON/NYQUIST Theorem
  - ALSO DEPENDS on "**RECONSTRUCTION**"

Shannon Sampling Theorem

A continuous-time signal $x(t)$ with frequencies no higher than $f_{\max}$ can be reconstructed exactly from its samples $x[n] = x(nT_s)$, if the samples are taken at a rate $f_s = 1/T_s$ that is greater than $2f_{\max}$.

Reconstruction? Which One?

Given the samples, draw a sinusoid through the values



$$x[n] = \cos(0.4\pi n)$$

When n is an integer
$\cos(0.4\pi n) = \cos(2.4\pi n)$

STORING DIGITAL SOUND

- ***x*[*n*]** is a SAMPLED SINUSOID
  - A list of numbers stored in memory
- EXAMPLE: audio CD
- CD rate is 44,100 samples per second
  - 16-bit samples
  - Stereo uses 2 channels
- Number of bytes for 1 minute is
  - $2 \times (16/8) \times 60 \times 44100 = 10.584$ Mbytes

DISCRETE-TIME SINUSOID

- Change x(t) into x[n]
- **DERIVATION**

$$x(t) = A\cos(\omega t + \varphi)$$

$$x[n] = x(nT_s) = A\cos(\omega n T_s + \varphi)$$

$$x[n] = A\cos((\omega T_s)n + \varphi)$$

$$x[n] = A\cos(\hat{\omega} n + \varphi)$$

$$\hat{\omega} = \omega T_s = \frac{\omega}{f_s}$$

DEFINE DIGITAL FREQUENCY

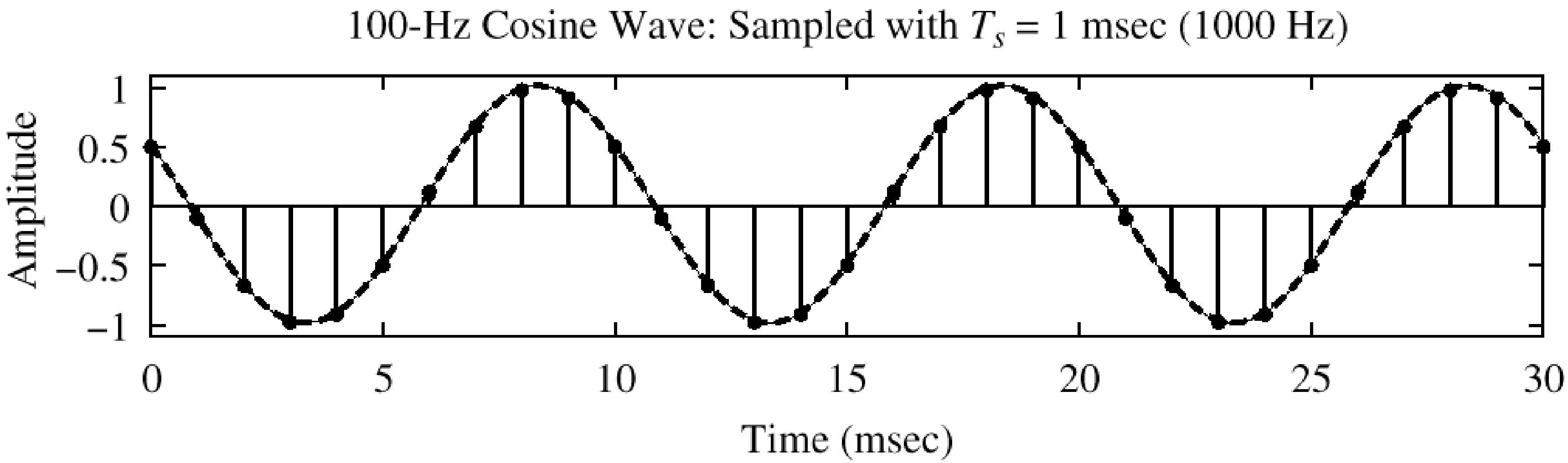
DIGITAL FREQUENCY

- Digital frequency $\hat{\omega}$ VARIES from **0** to **2π**, as f varies from 0 to the sampling frequency
- UNITS are radians, **not** rad/sec
  - DIGITAL FREQUENCY is NORMALIZED

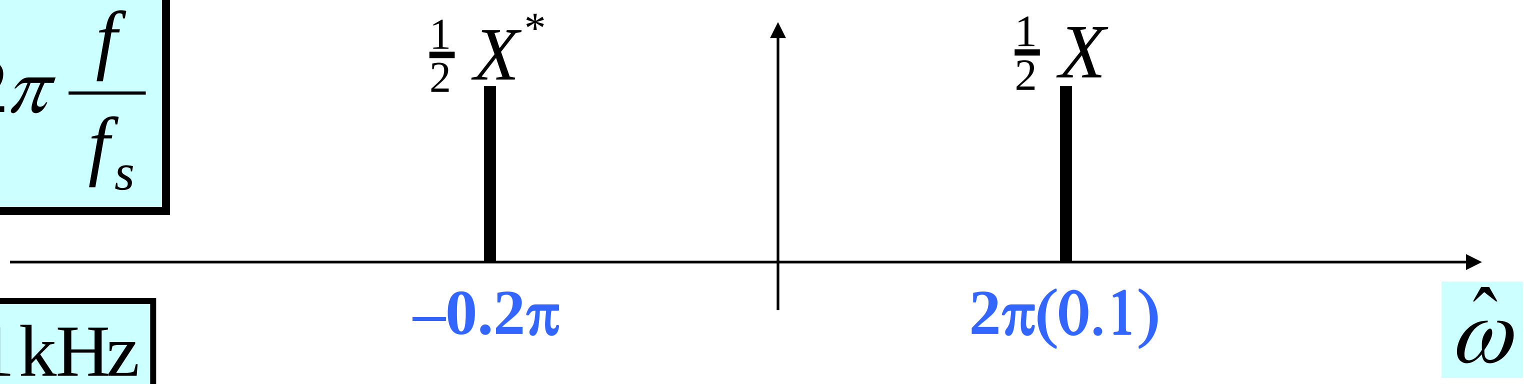
$$\boxed{\hat{\omega} = \omega T_s = \frac{2\pi f}{f_s}}$$

SPECTRUM (DIGITAL)

$$x[n] = A\cos(2\pi(100)(n/1000) + \varphi)$$



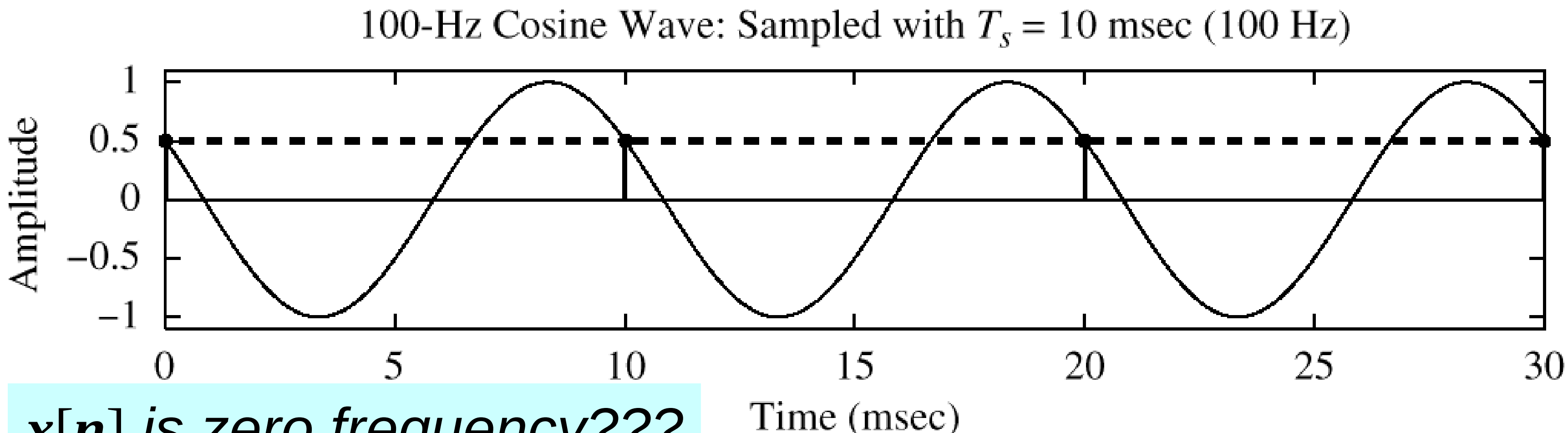
$$\boxed{\hat{\omega} = 2\pi \frac{f}{f_s}}$$



$$\boxed{f_s = 1\,\text{kHz}}$$

SPECTRUM (DIGITAL) ???

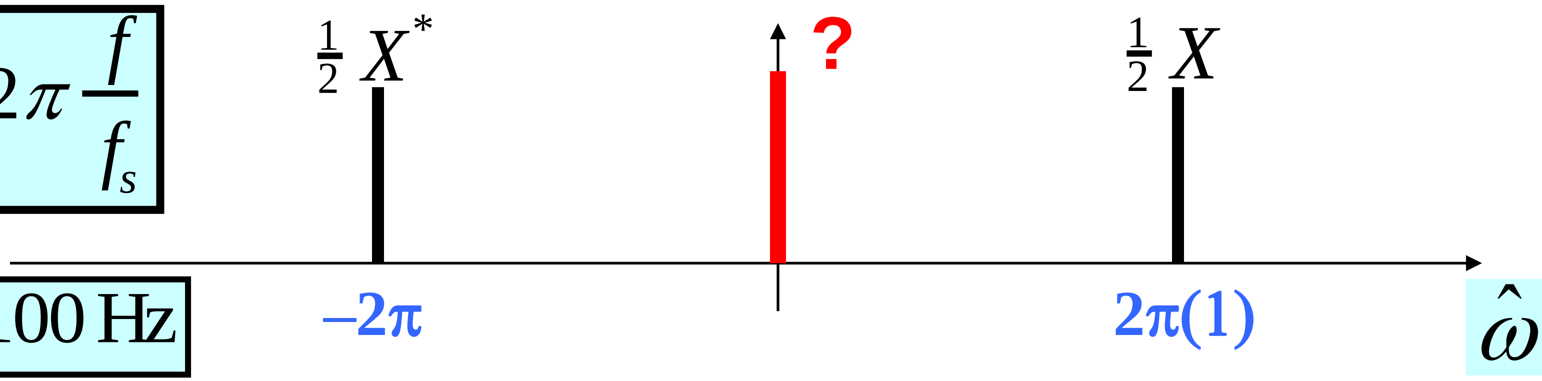
$$x[n] = A\cos(2\pi(100)(n/100) + \varphi)$$



***x*[*n*]** *is zero frequency???*

$$\hat{\omega} = 2\pi \frac{f}{f_s}$$

$$f_s = 100\ \text{Hz}$$



The REST of the STORY

- Spectrum of x[n] has more than one line for each complex exponential
  - Called **ALIASING**
  - **MANY SPECTRAL LINES**

- SPECTRUM is PERIODIC with period = **2π**
  - Because

$$A\cos(\hat{\omega}n+\varphi)=A\cos((\hat{\omega}+2\pi)n+\varphi)$$

ALIASING DERIVATION

- Other Frequencies give the same $\hat{\omega}$

$$x_1(t) = \cos(400\pi t) \quad \text{sampled at } f_s = 1000\,\text{Hz}$$

$$x_1[n] = \cos(400\pi \tfrac{n}{1000}) = \cos(0.4\pi n)$$

$$x_2(t) = \cos(2400\pi t) \quad \text{sampled at } f_s = 1000\,\text{Hz}$$

$$x_2[n] = \cos(2400\pi \tfrac{n}{1000}) = \cos(2.4\pi n)$$

$$x_2[n] = \cos(2.4\pi n) = \cos(0.4\pi n + 2\pi n) = \cos(0.4\pi n)$$

$$\Rightarrow x_2[n] = x_1[n]$$

$$2400\pi - 400\pi = 2\pi(1000)$$

ALIASING DERIVATION–2

- Other Frequencies give the same $\hat{\omega}$

If $x(t) = A\cos(2\pi\underline{(f + \ell f_s)}t + \varphi)$ $\quad \boxed{t \leftarrow \frac{n}{f_s}}$

and we want : $x[n] = A\cos(\hat{\omega}n + \varphi)$

$$\text{then}: \hat{\omega} = \frac{2\pi(f + \ell f_s)}{f_s} = \frac{2\pi f}{f_s} + \frac{2\pi \ell f_s}{f_s}$$

$$\boxed{\hat{\omega} = \omega T_s = \frac{2\pi f}{f_s}} \boxed{+2\pi\ell}$$

ALIASING CONCLUSIONS

- ADDING f_s or $2f_s$ or $-f_s$ to the FREQ of $x(t)$ gives exactly the same $x[n]$
  - The samples, $x[n] = x(n/f_s)$ are EXACTLY THE SAME VALUES

- GIVEN $x[n]$, WE CANNOT DISTINGUISH f_o FROM $(f_o + f_s)$ or $(f_o + 2f_s)$

NORMALIZED FREQUENCY

- DIGITAL FREQUENCY

Normalized Radian Frequency

$$\boxed{\hat{\omega} = \omega T_s = \frac{2\pi f}{f_s}}\ \boxed{+2\pi\ell}$$

Normalized Cyclic Frequency

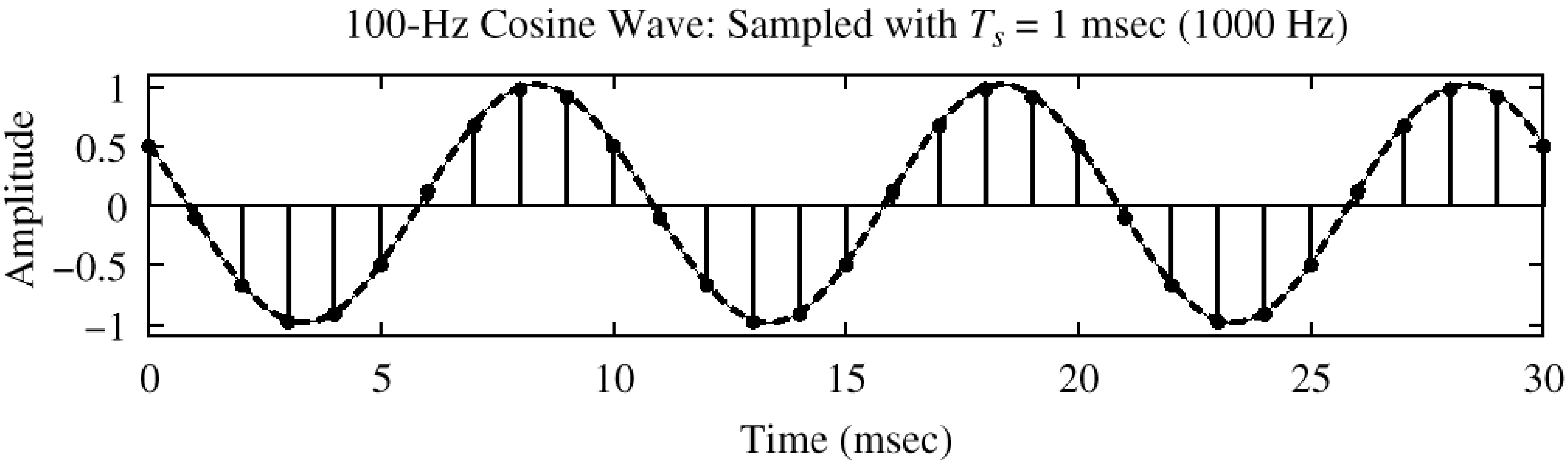
$$\hat{f} = \hat{\omega}/(2\pi) = f\,T_s = f/f_s$$

SPECTRUM for $x[n]$

- PLOT versus NORMALIZED FREQUENCY
- INCLUDE **ALL** SPECTRUM LINES
  - ALIASES
    - ADD MULTIPLES of 2π
    - SUBTRACT MULTIPLES of 2π
  - FOLDED ALIASES
    - ALIASES of NEGATIVE FREQS

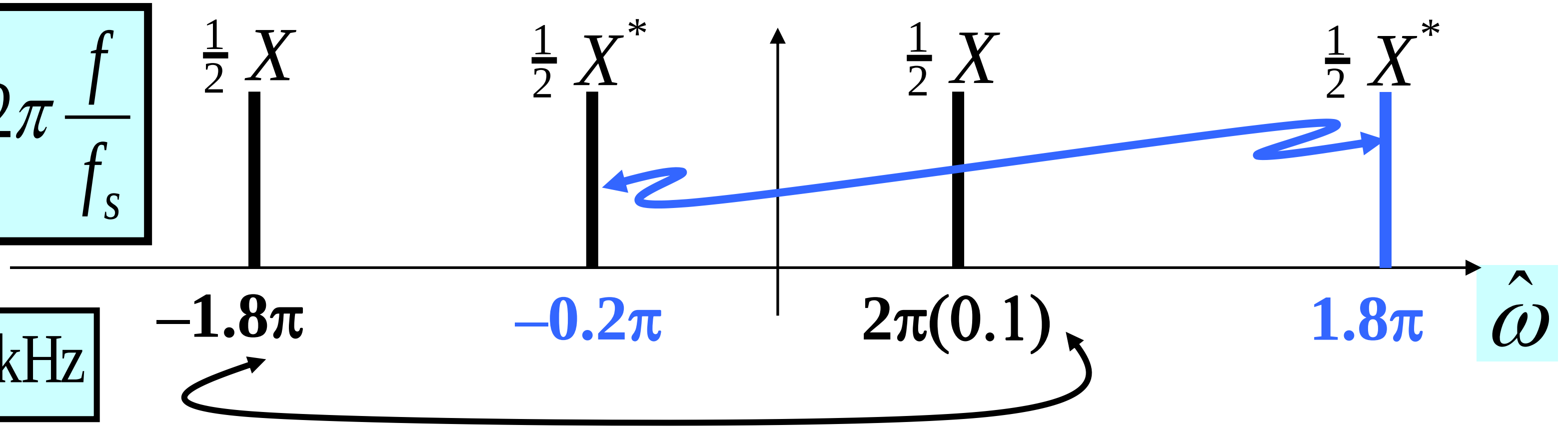
SPECTRUM (MORE LINES)

$$x[n] = A\cos(2\pi(100)(n/1000) + \varphi)$$



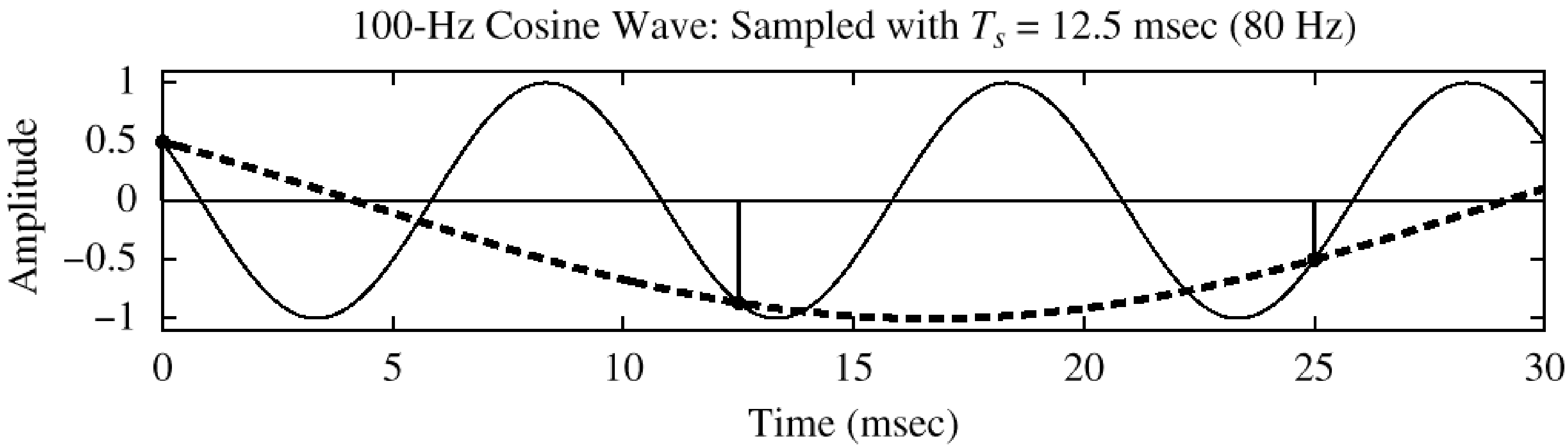
$$\boxed{\hat{\omega} = 2\pi \frac{f}{f_s}}$$

$$\boxed{f_s = 1\,\mathrm{kHz}}$$

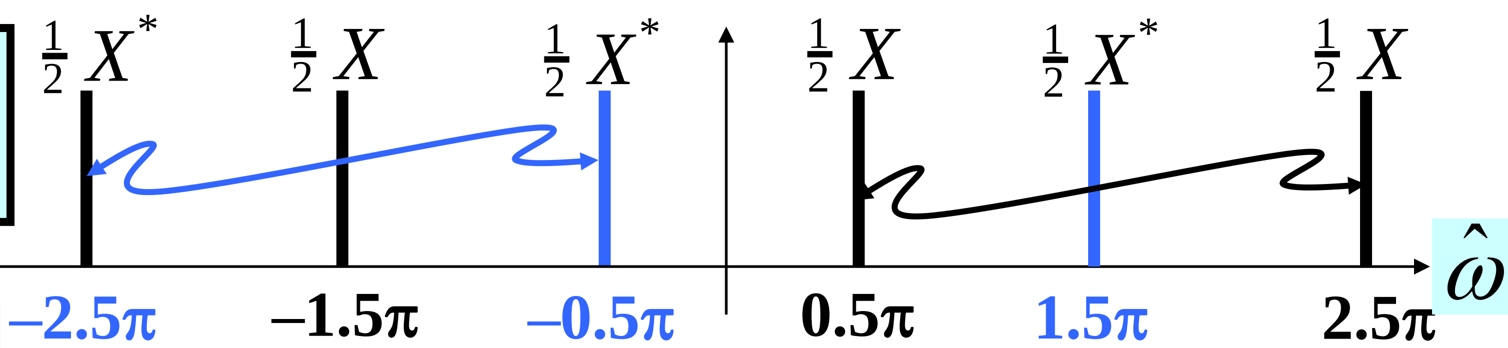


SPECTRUM (ALIASING CASE)

$$x[n] = A\cos(2\pi(100)(n/80) + \varphi)$$



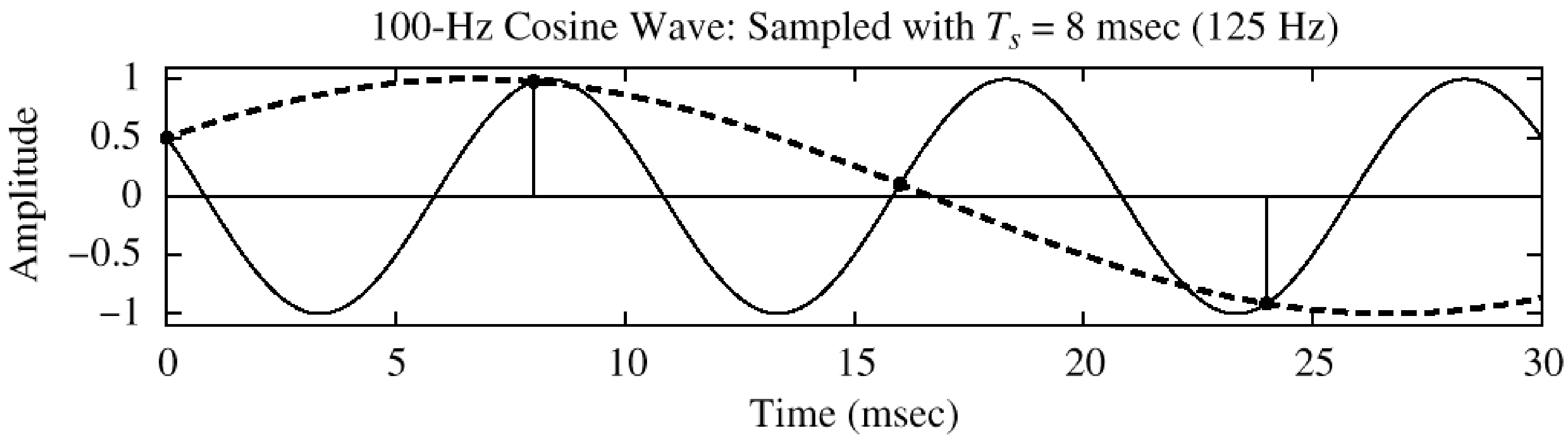
$$\boxed{\hat{\omega} = 2\pi \frac{f}{f_s}}$$



$$\boxed{f_s = 80\,\text{kHz}}$$

SPECTRUM (FOLDING CASE)

$$x[n] = A\cos(2\pi(100)(n/125) + \varphi)$$



$$\boxed{\hat{\omega} = 2\pi \frac{f}{f_s}}$$

$$\boxed{f_s = 125\text{Hz}}$$