BLM2041 Signals and Systems

Week 3

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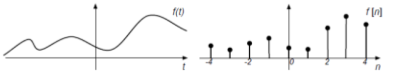
Signals and Systems Properties

Continuous and Discrete Time Signals

- Most of the signals we will talk about are functions of time.
- There are many ways to classify signals. This class is organized according to whether the signals are continuous in time, or discrete.
- A *continuous-time* signal has values for all points in time in some (possibly infinite) interval.
- A *discrete time* signal has values for only discrete points in time.



- Signals can also be a function of space (images) or of space and time (video), and may be continuous or discrete in each dimension.

Signals and Systems Properties

Types of Systems

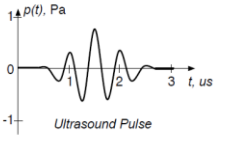
Systems are classified according to the types of input and output signals

- *Continuous-time system* has continuous-time inputs and outputs.
  - AM or FM radio
  - Conventional (all mechanical) car
- *Discrete-time system* has discrete-time inputs and outputs.
  - PC computer game
  - Matlab
  - Your mortgage
- Hybrid systems are also very important (A/D, D/A converters).
  - You playing a game on a PC
  - Modern cars with ECU (electronic control units)
  - Most commercial and military aircraft

Signals and Systems Properties

Continuous Time Signals

- Function of a time variable, something like t, τ, t_1.
- The entire signal is denoted as v, $v(.)$, or $v(t)$, where t is a dummy variable.
- The value of the signal at a particular time is $v(1.2)$, or $v(t)$, $t = 2$.



Signals and Systems Properties

Discrete Time Signals

- Fundamentally, a discrete-time signal is sequence of samples, written

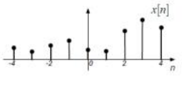
$$x[n]$$

where n is an integer over some (possibly infinite) interval.

- Often, at least conceptually, samples of a continuous time signal

$$x[n] = x(nT)$$

where n is an integer, and T is the *sampling period.*

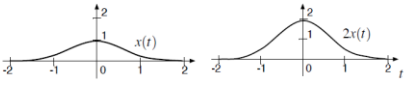


- Discrete time signals may not represent uniform time samples

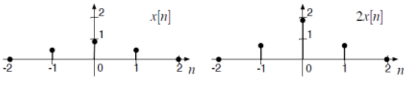
Operations on Signals

Amplitude Scaling

- The scaled signal $ax(t)$ is $x(t)$ multiplied by the constant a



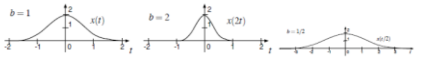
- The scaled signal $ax[n]$ is $x[n]$ multiplied by the constant a



Signals and Systems Properties

Time Scaling, Continuous Time

A signal $x(t)$ is scaled in time by multiplying the time variable by a positive constant b, to produce $x(bt)$. A positive factor of b either expands ($0 < b < 1$) or compresses ($b > 1$) the signal in time.



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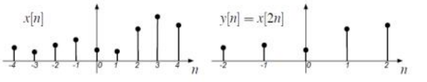
Signals and Systems Properties

Time Scaling, Discrete Time

The discrete-time sequence $x[n]$ is *compressed* in time by multiplying the index n by an integer k, to produce the time-scaled sequence $x[nk]$.

- This extracts every k^{th} sample of $x[n]$.
- Intermediate samples are lost.
- The sequence is shorter.



Called *downsampling*, or *decimation*.

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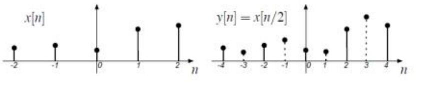
Signals and Systems Properties

The discrete-time sequence $x[n]$ is *expanded* in time by dividing the index n by an integer m, to produce the time-scaled sequence $x[n/m]$.

- This specifies every m^{th} sample.
- The intermediate samples must be synthesized (set to zero, or interpolated).
- The sequence is longer.



Called *upsampling*, or *interpolation*.

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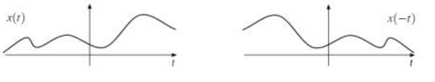
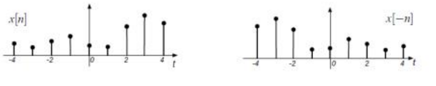
Signals and Systems Properties

Time Reversal

- Continuous time: replace t with $-t$, time reversed signal is $x(-t)$



- Discrete time: replace n with $-n$, time reversed signal is $x[-n]$.



- Same as time scaling, but with $b = -1$.

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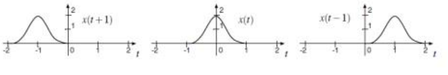
Signals and Systems Properties

Time Shift

For a continuous-time signal $x(t)$, and a time $t_1 > 0$,

- Replacing t with $t - t_1$ gives a *delayed* signal $x(t - t_1)$
- Replacing t with $t + t_1$ gives an *advanced* signal $x(t + t_1)$



- May seem counterintuitive. Think about where $t - t_1$ is zero.

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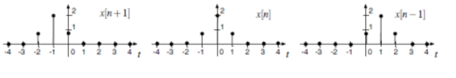
Signals and Systems Properties

For a discrete time signal $x[n]$, and an integer $n_1 > 0$

- $x[n - n_1]$ is a delayed signal.
- $x[n + n_1]$ is an advanced signal.
- The delay or advance is an integer number of sample times.



- Again, where is $n - n_1$ zero?

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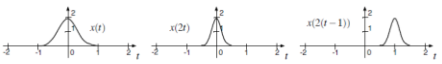
Signals and Systems Properties

Combinations of Operations

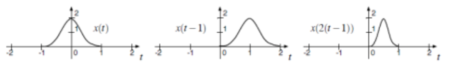
- Time scaling, shifting, and reversal can all be combined.
- Operation can be performed in any order, but care is required.
- This *will* cause confusion.
- Example: $x(2(t-1))$

Scale first, then shift
Compress by 2, shift by 1

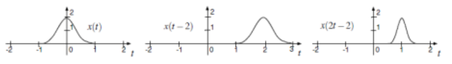


Signals and Systems Properties

Example $x(2(t-1))$, continued
Shift first, then scale
Shift by 1, compress by 2 *Incorrect*



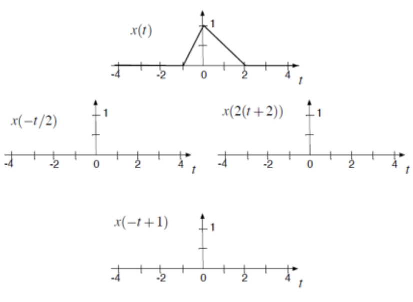
Shift first, then scale
Rewrite $x(2(t-1)) = x(2t-2)$
Shift by 2, scale by 2 *Correct*



Where is $2(t-1)$ equal to zero?

Signals and Systems Properties

Try these yourselves ....



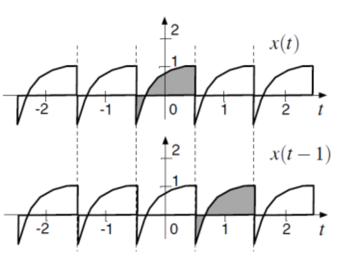
Signals and Systems Properties

Periodic Signals

- Very important in this class.
- Continuous time signal is periodic if and only if there exists a $T_0 > 0$ such that
$$x(t+T_0) = x(t) \quad \text{for all t}$$
T_0 is the period of $x(t)$ in time.
- A discrete-time signal is periodic if and only if there exists an integer $N_0 > 0$ such that
$$x[n+N_0] = x[n] \quad \text{for all } n$$
N_0 is the period of $x[n]$ in sample spacings.
- The smallest T_0 or N_0 is the *fundamental period* of the periodic signal.

Signals and Systems Properties

Example:



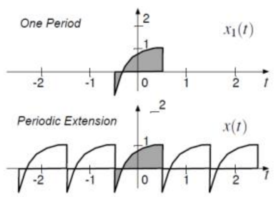
Shifting $x(t)$ by 1 time unit results in the same signal.

- Common periodic signals are sines and cosines

Signals and Systems Properties

Periodic Extension

- Periodic signals can be generated by *periodic extension* by any segment of length one period T_0 (or a multiple of the period).



- We will often take a signal that is defined only over an interval T_0 and use periodic extension to make a periodic signal.

Signals and Systems Properties

Complex Signals

- So far, we have only considered real (or integer) valued signals.
- Signals can also be complex

$$z(t) = x(t) + jy(t)$$

  where $x(t)$ and $y(t)$ are each real valued signals, and $j = \sqrt{-1}$.
- Arises naturally in many problems
  - Convenient representation for sinusoids
  - Communications
  - Radar, sonar, ultrasound

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Signals and Systems Properties

Review of Complex Numbers

Complex number in Cartesian form: $z = x + jy$

- $x = \Re z$, the *real part* of z
- $y = \Im z$, the *imaginary part* of z
- x and y are also often called the *in-phase* and *quadrature* components of z.
- $j = \sqrt{-1}$ (engineering notation)
- $i = \sqrt{-1}$ (physics, chemistry, mathematics)

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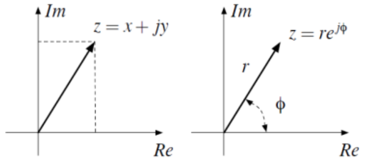
Signals and Systems Properties

Complex number in polar form: $z = re^{j\phi}$

- r is the *modulus* or *magnitude* of z
- ϕ is the *angle* or *phase* of z
- $\exp(j\phi) = \cos\phi + j\sin\phi$



- complex exponential of $z = x + jy$:

$$e^z = e^{x+jy} = e^x e^{jy} = e^x(\cos y + j\sin y)$$

Know how to add, multiply, and divide complex numbers, and be able to go between representations easily.

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Signals and Systems Properties

Signal Energy and Power

If $i(t)$ is the current through a resistor, then the energy dissipated in the resistor is

$$E_R = \lim_{T\to\infty} \int_{-T}^{T} i^2(t)\ R\ dt$$

This is energy in *Joules*.

The signal energy for $i(t)$ is defined as the energy dissipated in a 1 Ω resistor

$$E_i = \lim_{T\to\infty} \int_{-T}^{T} i^2(t) dt$$

The *signal energy* for a (possibly complex) signal $x(t)$ is

$$E_x = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt.$$

In most applications, this is not an actual energy (most signals aren't actually applied to 1Ω resistor).

The average of the signal energy over time is the *signal power*.

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Signals and Systems Properties

Properties of Energy and Power Signals

An energy signal $x(t)$ has zero power

$$\begin{aligned} P_x &= \lim_{T\to\infty} \frac{1}{2T} \underbrace{\int_{-T}^{T} |x(t)|^2\ dt}_{\to E_x < \infty} \\ &= 0 \end{aligned}$$

A power signal has infinite energy

$$\begin{aligned} E_x &= \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2\ dt \\ &= \lim_{T\to\infty} 2T \underbrace{\frac{1}{2T} \int_{-T}^{T} |x(t)|^2\ dt}_{\to P_x > 0} = \infty. \end{aligned}$$

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Signals and Systems Properties

Sinusoidal Signals

- A sinusoidal signal is of the form

$$x(t) = \cos(\omega t + \theta).$$

  where the *radian frequency* is ω, which has the units of radians/s.
- Also very commonly written as

$$x(t) = A\cos(2\pi f t + \theta).$$

  where f is the frequency in Hertz.
- We will often refer to ω as the frequency, but it must be kept in mind that it is really the *radian frequency*, and the *frequency* is actually f.

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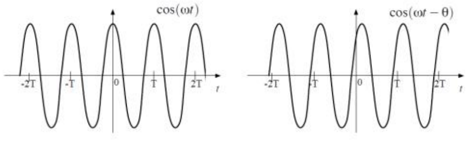
Signals and Systems Properties

- The period of the sinuoid is

$$T = \frac{1}{f} = \frac{2\pi}{\omega}$$

with the units of seconds.

- The *phase* or *phase angle* of the signal is θ, given in radians.



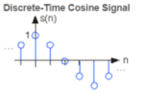
Signals and Systems Properties

Sinusoids

One of the most important elemental signal that you will deal with is the real-valued sinusoid. In its discrete-time form, we write the general expression as

$$A\cos(\omega n + \varphi)$$

where A is the amplitude, ω is the frequency, and φ is the phase. Because n only takes integer values, the resulting function is only periodic if $\frac{2\pi}{\omega}$ is a rational number.

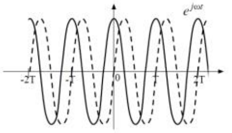


Signals and Systems Properties

Complex Sinusoids

- The Euler relation defines $e^{j\phi} = \cos\phi + j\sin\phi$.
- A complex sinusoid is

$$Ae^{j(\omega t+\theta)} = A\cos(\omega t + \theta) + jA\sin(\omega t + \theta).$$



- Real sinusoid can be represented as the real part of a complex sinusoid

$$\Re\{Ae^{j(\omega t+\theta)}\} = A\cos(\omega t + \theta)$$

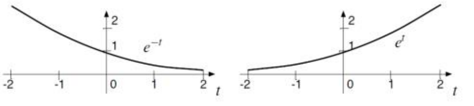
Signals and Systems Properties

Exponential Signals

- An exponential signal is given by

$$x(t) = e^{\sigma t}$$

- If $\sigma < 0$ this is *exponential decay*.
- If $\sigma > 0$ this is *exponential growth*.



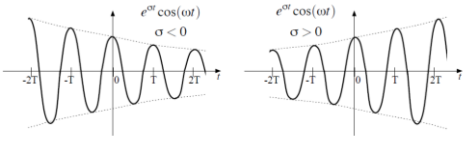
Signals and Systems Properties

Damped or Growing Sinusoids

- A damped or growing sinusoid is given by

$$x(t) = e^{\sigma t}\cos(\omega t + \theta)$$

- Exponential growth ($\sigma > 0$) or decay ($\sigma < 0$), modulated by a sinusoid.



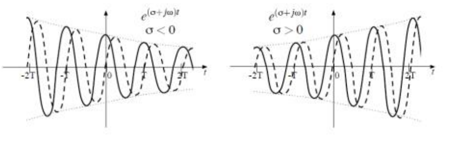
Signals and Systems Properties

Complex Exponential Signals

- A complex exponential signal is given by

$$e^{(\sigma+j\omega)t+j\theta} = e^{\sigma t}(\cos(\omega t + \theta) + i\sin(\omega t + \theta))$$

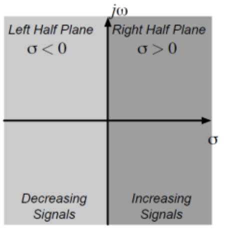
- A exponential growth or decay, modulated by a complex sinusoid.
- Includes all of the previous signals as special cases.



Signals and Systems Properties

Complex Plane

Each complex frequency $s = \sigma + j\omega$ corresponds to a position in the complex plane.



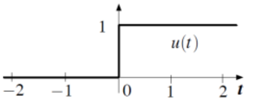
Signals and Systems Properties

Unit Step Functions

- The *unit step function* $u(t)$ is defined as

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

- Also known as the *Heaviside step function*.
- Alternate definitions of value exactly at zero, such as $1/2$.



Signals and Systems Properties

Unit Step

Another very basic signal is the **unit-step function** defined as

$$u[n] = \begin{cases} 0 & \text{if } n < 0 \\ 1 & \text{if } n \geq 0 \end{cases}$$

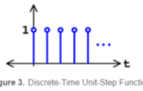


Figure 3. Discrete-Time Unit-Step Function

The step function is a useful tool for testing and for defining other signals. For example, when different shifted versions of the step function are multiplied by other signals, one can select a certain portion of the signal and zero out the rest.

Signals and Systems Properties

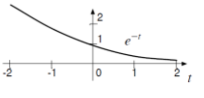
Uses for the unit step:

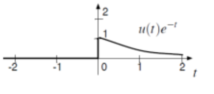
- Extracting part of another signal. For example, the piecewise-defined signal

$$x(t) = \begin{cases} e^{-t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

can be written as

$$x(t) = u(t)e^{-t}$$





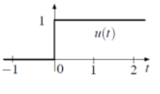
Signals and Systems Properties

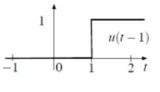
- Combinations of unit steps to create other signals. The offset rectangular signal

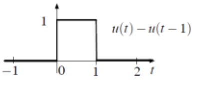
$$x(t) = \begin{cases} 0, & t \geq 1 \\ 1, & 0 \leq t < 1 \\ 0, & t < 0 \end{cases}$$

can be written as

$$x(t) = u(t) - u(t-1).$$





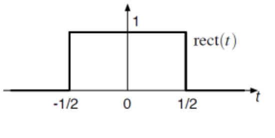


Signals and Systems Properties

Unit Rectangle

Unit rectangle signal:

$$\text{rect}(t) = \begin{cases} 1 & \text{if } |t| \leq 1/2 \\ 0 & \text{otherwise.} \end{cases}$$



Signals and Systems Properties

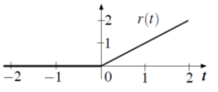
Unit Ramp

- The *unit ramp* is defined as

$$r(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

- The unit ramp is the integral of the unit step,

$$r(t) = \int_{-\infty}^{t} u(\tau)d\tau$$

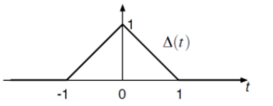


Signals and Systems Properties

Unit Triangle

Unit Triangle Signal

$$\Delta(t) = \begin{cases} 1 - |t| & \text{if } |t| < 1 \\ 0 & \text{otherwise.} \end{cases}$$

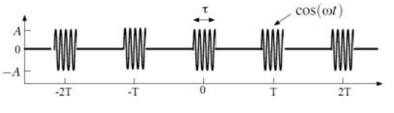


Signals and Systems Properties

More Complex Signals

Many more interesting signals can be made up by combining these elements.

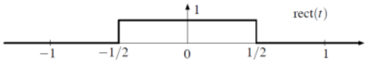
Example: Pulsed Doppler RF Waveform (we'll talk about this later!)



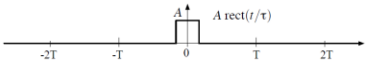
RF cosine gated on for τ μs, repeated every T μs, for a total of N pulses.

Signals and Systems Properties

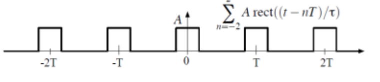
Start with a simple rect(t) pulse



Scale to the correct duration and amplitude for one subpulse



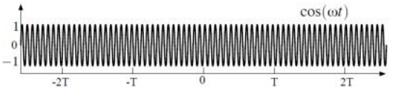
Combine shifted replicas



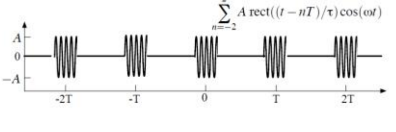
This is the *envelope* of the signal.

Signals and Systems Properties

Then multiply by the RF carrier, shown below



to produce the pulsed Doppler waveform



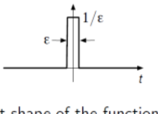
Signals and Systems Properties

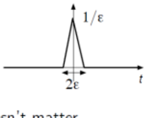
Impulsive signals

(Dirac's) **delta function** or **impulse** δ is an *idealization* of a signal that

- is very large near $t = 0$
- is very small away from $t = 0$
- has integral 1

for example:

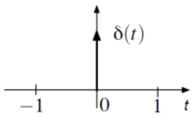


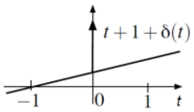


- the exact shape of the function doesn't matter
- ϵ is small (which depends on context)

Signals and Systems Properties

On plots δ is shown as a solid arrow:





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Signals and Systems Properties

Unit Impulses

The second-most important discrete-time signal is the **unit sample**, which is defined as

$$\delta[n] = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

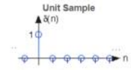

Figure 2. The unit sample.

More detail is provided in the section on the discrete time impulse function. For now, it suffices to say that this signal is crucially important in the study of discrete signals, as it allows the sifting property to be used in signal representation and signal decomposition.

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Signals and Systems Properties

Formal properties

Formally we **define** δ by the property that

$$\int_{-\infty}^{\infty} f(t)\delta(t)\, dt = f(0)$$

provided f is continuous at $t = 0$

idea: δ acts over a time interval very small, over which $f(t) \approx f(0)$

- $\delta(t)$ is not really defined for any t, only its behavior in an integral.
- Conceptually $\delta(t) = 0$ for $t \neq 0$, infinite at $t = 0$, but this doesn't make sense mathematically.

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Signals and Systems Properties

Scaled impulses

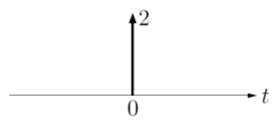
$\alpha\delta(t)$ is an impulse at time T, with *magnitude* or *strength* α

We have

$$\int_{-\infty}^{\infty} \alpha\delta(t) f(t)\, dt = \alpha f(0)$$

provided f is continuous at 0

On plots: write area next to the arrow, *e.g.*, for $2\delta(t)$,



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Signals and Systems Properties

Multiplication of a Function by an Impulse

- Consider a function $\phi(x)$ multiplied by an impulse $\delta(t)$,

  $$\phi(t)\delta(t)$$

  If $\phi(t)$ is continuous at $t = 0$, can this be simplified?
- Substitute into the formal definition with a continuous $f(t)$ and evaluate,

  $$\begin{aligned} \int_{-\infty}^{\infty} f(t)\,[\phi(t)\delta(t)]\, dt &= \int_{-\infty}^{\infty} [f(t)\phi(t)]\,\delta(t)\, dt \\ &= f(0)\phi(0) \end{aligned}$$

- Hence

  $$\phi(t)\delta(t) = \phi(0)\delta(t)$$

  is a scaled impulse, with strength $\phi(0)$.

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Signals and Systems Properties

Sifting property

- The signal $x(t) = \delta(t - T)$ is an impulse function with impulse at $t = T$.
- For f continuous at $t = T$,

  $$\int_{-\infty}^{\infty} f(t)\delta(t - T)\, dt = f(T)$$

- Multiplying by a function $f(t)$ by an impulse at time T and integrating, extracts the value of $f(T)$.
- This will be important in modeling sampling later in the course.

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Signals and Systems Properties

Limits of Integration

The integral of a δ is non-zero only if it is in the integration interval:

- If $a < 0$ and $b > 0$ then
$$\int_a^b \delta(t)\,dt = 1$$
because the δ is within the limits.
- If $a > 0$ or $b < 0$, and $a < b$ then
$$\int_a^b \delta(t)\,dt = 0$$
because the δ is outside the integration interval.
- **Ambiguous** if $a = 0$ or $b = 0$

Signals and Systems Properties

Our convention: to avoid confusion we use limits such as $a-$ or $b+$ to denote whether we include the impulse or not.

$$\int_{0+}^{1} \delta(t)\,dt = 0, \quad \int_{0-}^{1} \delta(t)\,dt = 1, \quad \int_{-1}^{0-} \delta(t)\,dt = 0, \quad \int_{-1}^{0+} \delta(t)\,dt = 1$$

example:

$$\begin{aligned}
&\int_{-2}^{3} f(t)(2 + \delta(t+1) - 3\delta(t-1) + 2\delta(t+3))\,dt \\
&= 2\int_{-2}^{3} f(t)\,dt + \int_{-2}^{3} f(t)\delta(t+1)\,dt - 3\int_{-2}^{3} f(t)\delta(t-1)\,dt \\
&\quad + 2\int_{-2}^{3} f(t)\delta(t+3))\,dt \\
&= 2\int_{-2}^{3} f(t)\,dt + f(-1) - 3f(1)
\end{aligned}$$

Signals and Systems Properties

Physical interpretation

Impulse functions are used to model physical signals

- that act over short time intervals
- whose effect depends on integral of signal

example: hammer blow, or bat hitting ball, at $t = 2$

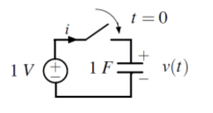
- force f acts on mass m between $t = 1.999$ sec and $t = 2.001$ sec
- $\int_{1.999}^{2.001} f(t)\,dt = I$ (mechanical impulse, $\mathrm{N}\cdot\mathrm{sec}$)
- blow induces change in velocity of

$$v(2.001) - v(1.999) = \frac{1}{m}\int_{1.999}^{2.001} f(\tau)\,d\tau = I/m$$

For most applications, model force as impulse at $t = 2$, with magnitude I.

Signals and Systems Properties

example: rapid charging of capacitor



assuming $v(0) = 0$, what is $v(t)$, $i(t)$ for $t > 0$?

- $i(t)$ is very large, for a very short time
- a unit charge is transferred to the capacitor 'almost instantaneously'
- $v(t)$ increases to $v(t) = 1$ 'almost instantaneously'

To calculate i, v, we need a more detailed model.

Signals and Systems Properties

In conclusion,

- large current i acts over very short time between $t = 0$ and ϵ
- total charge transfer is $\int_0^\epsilon i(t)\,dt = 1$
- resulting change in $v(t)$ is $v(\epsilon) - v(0) = 1$
- can approximate i as impulse at $t = 0$ with magnitude 1

Modeling current as impulse

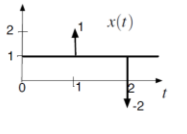
- obscures details of current signal
- obscures details of voltage change during the rapid charging
- preserves total change in charge, voltage
- is reasonable model for time scales $\gg \epsilon$

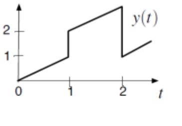
Signals and Systems Properties

Integrals of impulsive functions

Integral of a function with impulses has jump at each impulse, equal to the magnitude of impulse

example: $x(t) = 1 + \delta(t-1) - 2\delta(t-2)$; define $y(t) = \int_0^t x(\tau)\,d\tau$





Signals and Systems Properties

Derivatives of discontinuous functions

Conversely, derivative of function with discontinuities has impulse at each jump in function

- Derivative of unit step function $u(t)$ is $\delta(t)$
- Signal y of previous page

$$y'(t) = 1 + \delta(t-1) - 2\delta(t-2)$$