Random experiment: Roll a die, HT

outcome: Result of random experiment roll a die - HT

events: is a collection of outcomes

sample space: set of all possible outcomes

3 Axioms

1- $P(A) \geq 0$

2- $P(\Omega) = P(S) = 1$

3- $A_1, A_2 \cdots A_n$ are disjoint events

$P(A_1 \cup A_2 \cdots \cup A_n) = P(A_1) + P(A_2) \cdots P(A_n)$

Rules

1-) $P(A) = 1 - P(A^c)$

$A \cup A^c = \Omega$

$P(A \cup A^c) = 1$

$P(A) + P(A^c) = 1$

2-) $P(\emptyset) = 0$

$P(\emptyset) = P(\Omega^c) \rightarrow 1 - P(\Omega) = 0$

3-) $P(A) \leq 1$

$P(A) = 1 - \underbrace{P(A^c)}_{\geq 0} \rightarrow P(A) \leq 1$

4-) $P(A - B) = P(A) - P(A \cap B)$

$A = (A \cap B) \cup (A - B)$ $\quad P(A) = P(A \cap B) + P(A - B)$

$P(A - B) = P(A) - P(A \cap B)$

5-) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

6-) $A \subset B \overset{if}{\Longrightarrow} P(A) \leq P(B)$

Axioms are important

Rules of prob. are obtained using these axioms.

Any function that ... these axioms define a prob. measure

Calculating prob.

4-) Equally likely outcomes.

- Rolling a die

$\Omega = \{1, 2, 3, 4, 5, 6\} \rightarrow P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = \frac{1}{|\Omega|}$

Rolling a fair die and get an odd numbers?

$P(odd) = \overset{\{1,3,5\}}{\{1\} + \{3\} + \{5\}} = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$

Conditional probabilities (kosullu olasılık) (condition yoksa)

$P(rain) = 0.4$ $\quad P(rain) = 0.4$

→ Clouds in the sky → $P(rain) = 0.85$

$P(rain \mid \underbrace{clouds}_{given}) = 0.75$

Definition: The conditional probability prob. that A occured given B has occured is: $P(A|B) = \frac{P(A \cap B)}{P(B)}$

$\frac{P(A \cap B)}{P(B)}$ $\quad P(B) \neq 0$

Ex: Roll a fair die

prob. of getting an even number?

given the outcome is less than or equal to 3 → $P(even \mid outcome \leq 3)$

$\Omega = \{1, 2, 3\}$ $\quad P(even) = \{2\} \rightarrow \frac{1}{3}$

Flight we know that 70% of the flight depends on time

80% of the flights arrive on time

70% of the flights both arrive and depart on time

a) We know that flight departed on time. Prop that it will arrive on time

$P(A \cap B) = 70\%$ $\quad P(A) = 80\%$ $\quad P(B|A) = \frac{70}{80} = \frac{7}{8}$

b) " " " " arrived " " " " " depart " "?

$P(B) = 70\% \rightarrow P(A|B) = \frac{70}{80} = \frac{7}{8}$

if there are more than 2 events?

$P(rain \mid clouds, windy) = ?$

Chain rule

$P(A \cap B) = P(A|B) \cdot P(B)$ → cond. prob.

$P(A \cap B \cap C) = P(C) \cdot P(B|C) \cdot P(A|B,C)$

$P(A|B,C)$

$P(A \cap B \cap C)$

independence of events

definition → 2 events A and B are independent iff $:\Longleftrightarrow$

$P(A|B) = P(A)$

ie is independent of B

$P(A \cap B) = P(A) \cdot P(B)$

A: rain → Dest No

B: 4th → coin → H/T

$P(rain \mid 7)$ A B

$P(A|B) = \frac{P(A \cap B)}{P(B)} \rightarrow P(A) \cdot P(B) = P(A \cap B)$

Bağımsız olaylar ayrık olmak zorunda değil

independent events are not disjoint

Ex: I roll a die and obtain the number x. Let A be an event that x is an even number and let B be an event that $x > 4$

Are A and B independent?

$A\ \{2, 4, 6\}$ $\quad B\ \{5, 6\}$ $\quad P(A) = \frac{1}{2}$ $\quad P(B) = \frac{1}{3}$

$P(A \cap B) = \frac{1}{6}$ $\quad \frac{1}{6} = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6} \rightarrow$ Events A an B independent

$P(A \cap B) \overset{?}{=} P(A) \cdot P(B)$

Flight example

$P(A) = 80\%$ $\quad P(B) = 90\%$ $\quad P(A \cap B) = 70\%$ Are A and B independent?

$\frac{4}{5} \cdot \frac{9}{10} \overset{?}{=} \frac{7}{10} \rightarrow \frac{18}{25} \neq \frac{7}{10}$ events A and B are not independent

or $P(A|B) \overset{?}{=} P(A) \rightarrow \frac{70\%}{90\%} \neq 80\%$ " " " " " "

or $P(B|A) \overset{?}{=} P(B) \rightarrow \frac{70\%}{80\%} \neq 90\%$

independence of 3 events

$P(A \cap B) = P(A) \cdot P(B)$ $\quad P(A \cap C) = P(C) \cdot P(A)$ $\quad P(B \cap C) = P(B) \cdot P(C)$

$P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$ ise A, B and C are independent

Conditional independence

Two event A and B are conditionally independent given C iff

$P(A \cap B \mid C) = P(B|C) \cdot P(A|C)$

law of total probability

Let $B_1, B_2, \ldots, B_n$ be a partition of Ω and $P(B_i) > 0$

$B_1 \cup B_2 \cdots B_n = \Omega$

$B_1 \cap B_2 = \emptyset$

For any event A we have

$P(A) = \sum P(A \cap B_i) = \sum P(A|B_i) \cdot P(B_i)$

$(A \cap B_1) \cup (A \cap B_2) \cup (A \cap B_3) = P(A)$

Ex: 3 urns exist we know that - for coin 1 $P(H) = 0.4$

" " " 2 $P(H) = 0.6$

" " " 3 $P(H) = 0.3$

we draw a coin randomly and we flip $P(H) = ?$

$P(H) = \sum P(A \cap B_i) = \sum P(A|B_i) \cdot P(B_i)$

$= 0.4 \cdot \frac{1}{3} + 0.6 \cdot \frac{1}{3} + 0.3 \cdot \frac{1}{3} = 0.4$

Bayes Rule

$P(A|B) = \frac{P(A \cap B)}{P(B)}$ $\quad P(B|A) = \frac{P(A \cap B)}{P(A)}$

Bayes Law

$\rightarrow P(A \cap B) \cdot P(B) = P(B|A) \cdot P(A) \rightarrow P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$

Ex: if we get H, what is the prob. we draw the first coin?

$P(C_1 / H) = \frac{\overbrace{P(H/C_1)}^{\frac{0.4}{0.4}} \cdot \overbrace{P(C_1)}^{\frac{1}{3}}}{\underbrace{P(H)}_{\frac{1}{3}}} \rightarrow \frac{1}{2}$