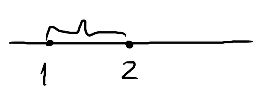
Continuous Probabilities

- Discrete r.v.
  - $\hookrightarrow R_X \to$ countable set
- Continuous r.v.

$R_X \to$ an interval in the real line

$R_X \in [1,2]$ infinite ∞



Ex $\to P(1 < X < 1.2) = \frac{1.2-1}{2-1} = \frac{0.2}{1} = \%20$

$P_X(x) = 0 \to \frac{1}{\infty} = 0$

discrete $\to$ summation

$E[X] = \sum_x x \cdot P(x)$ — PMF X

$E[X] = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$ — PDF $f(x)$ — $\hookrightarrow$ integration

Discrete R.V. Rules $\approx$ Continuous R.V. Rules

Continuous r.v.: X with CDF $F_X(x)$ is said to be continuous if $F_X(x)$ is a continuous function for all $x \in R$.

- Also assume $F_X(x)$ is differentiable almost everywhere.

Ex: If we will to choose a real number uniformly at random in the interval $[a,b]$ and call it X. ← continuous r.v.

$F_X(x) = ?$

$F_X(x) = P(X \le x)$

if $x < a$ $\quad F_X(x) = 0$

if $x > b$ $\quad F_X(x) = 1$

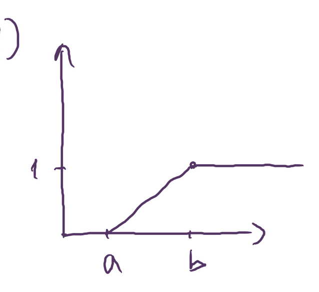
if $a < x \le b$ $\quad F_X(x) = \frac{x-a}{b-a} = P(X \le x) = P(X \in [a,x])$

$$F_X(x) = \begin{cases} 0 & \text{for } x < a \\ \frac{x-a}{b-a} & \text{for } a \le x \le b \\ 1 & \text{for } b < x \end{cases}$$

$< \approx \le$ for continuous prob

$\frac{1}{b-a}$ for $a < x < b$, 0 otherwise

Graph CDF



Probability density function (PDF)

PDF of an absolutely continuous r.v. X is $\to$ (PDF of X on x)

$f_X(x) = \frac{d}{dx}$ ← differentiation

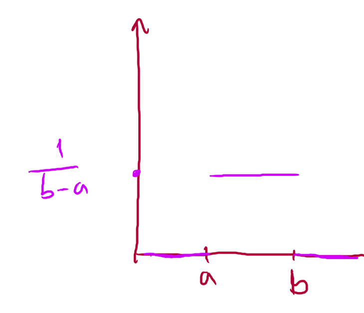
Ex: continued

$$F_X(x) = \begin{cases} 0 & \text{for } x < a \\ \frac{x-a}{b-a} & \text{for } a \le x \le b \\ 1 & \text{for } x > b \end{cases}$$

$$f_X(x) = \frac{d}{dx} F_X(x) = \begin{cases} \frac{d0}{dx} = 0 & \text{for } x < a \\ \frac{d1}{dx} = 0 & \text{for } x > b \end{cases}$$

$\frac{d}{dx}\left(\frac{x-a}{b-a}\right) = \frac{1}{b-a}$ for $a \le x \le b$

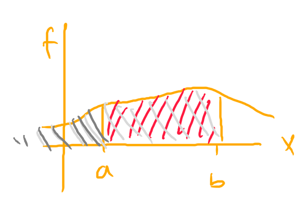
Graph of PDF



PDF and CDF

$f_X(x) = \frac{d}{dx} F_X(x)$ $\qquad F_X(x) = \int_{-\infty}^{x} f_X(u)\,du$

Prob of a range



$P(a < x \le b) = F_X(b) - F_X(a) = \int_a^b f_X(u)\,du$

Note:

1) $f_X(x) \ge 0$ for all $x \in R$

2) $f_X(x)$ can be larger than 1

Total Prob is 1 $\to \int_{-\infty}^{\infty} f_X(u)\,du = 1$

Prob of a set

$P(X \in A) = \int_A f_X(x)\,dx$

Ex: $P(X \in [0,1] \cup [3,4]) = \int_0^1 f_X(x)\,dx + \int_3^4 f_X(x)\,dx$

Range of C.R.V.

↓ set of possible values of X

If X is continuous, R_X can be defined in terms of set of real numbers whose density is larger than "0"

$R_X = \{x \mid f_X(x) > 0\}$

Ex: Lifetime in years of some electronic component is a continuous r.v. with

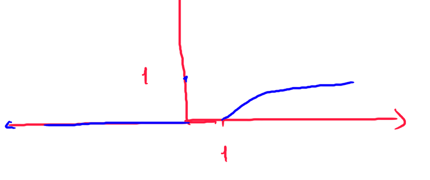
$$f_X(x) = \begin{cases} \frac{k}{x^3} & \text{for } x > 1 \\ 0 & \text{for } x < 1 \end{cases}$$

- $F_X(x) = ?$ → Graph a
- $k = ?$ b
- $P\{$lifetime exceeding 5 years$\} = ?$ c

b) $\int_{-\infty}^{\infty} f_X(x)\,dx = \int_1^{\infty} f_X(x)\,dx = \frac{-k}{x^2}\Big|_1^{\infty} \to \frac{+k}{2} = 1$ $\quad \underline{k=2}$ $\quad$ total prob = 1

a) $F_X(x) = \int_{-\infty}^{x} f_X(u)\,du \to \int_1^x f_X(u)\,du = \frac{-2}{2u^2}\Big|_1^x \to 1 - \frac{1}{x^2} \to$

$$F_X(x) = \begin{cases} 0 & \text{for } x \le 1 \\ 1 - \frac{1}{x^2} & \text{for } x > 1 \end{cases}$$



c-) $\int_{5}^{\infty} f_X(x)\,dx = -\frac{1}{x^2}\Big|_5^{\infty} \to \frac{1}{25}$

Expected value of C.R.V.

$EX = E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx$

Expected value of Function of C.R.V.

$Y = g(X)$

$EY = \int_{-\infty}^{\infty} g(x) \cdot f_X(x)\,dx$

ex: $f_X(x) = 2x^{-3}$ $\quad x \ge 1$

$\int_{-\infty}^{\infty} x \cdot f_X(x)\,dx = E[X] = \int_1^{\infty} 2x^{-2}\,dx = \frac{-2}{x}\Big|_1^{\infty} = 2$

Ex: $E[X^2] = \int_1^{\infty} x^2 \cdot f_X(x)\,dx = 2 \cdot \ln|x|\Big|_1^{\infty} = \infty$

EXPECTATION IS LINEAR

- $E(aX+b) = a \cdot EX + b$ for all $a,b \in R$
- $E(X_1+X_2 \dots X_n) = EX_1 + EX_2 + \dots + EX_n$
  - $\hookrightarrow$ Gama, Negative binomial..

Variance of a C.R.V.

$Var(X) = E((X-\mu_X)^2) = \int_{-\infty}^{\infty} (x-\mu_X)^2 \cdot f_X(x)\,dx$ ← form I

$\hookrightarrow EX^2 - \mu_X^2 = -\left(\int_{-\infty}^{\infty} x \cdot f_X(x)\,dx\right)^2 + \int_{-\infty}^{\infty} x^2 f_X(x)\,dx$ ← form II

$\to EX^2 - (EX)^2$

EX: continued

$f_X(x) = 2x^{-3}$ for $x \ge 1$ $\qquad Var(X) = ?$

$EX = 2$ $\qquad EX^2 = \infty$ $\qquad Var(X) = EX^2 - \mu_X^2 = \infty - 2^2 = \infty$

Properities of Variance

- $Var(aX+b) = a^2\,Var(X)$
- For independent X and Y $\quad Var(X+Y) = Var(X) + Var(Y)$

Functions of Continuous Random Variables

if X is a C.R.V. and $Y = g(X)$ then Y itself is a r.v.

- we can find the $F_Y(y)$, $f_Y(y)$ ....
- ① find $R_Y = \{g(x) \mid f_X(x) \ge 0\}$

② $F_Y(y) = P(Y \le y) = P(g(X) \le y)$

③ $f_Y(y) = \frac{d}{dy} F_Y(y)$

Ex: Let $X \sim$ uniform $(0,1)$ (a = 0, b = 1)

Let $Y = e^X$

a) $F_Y(y) = ?$ $\qquad$ b) $f_Y(y) = ?$ $\qquad$ c) EY

$$F(x) = \begin{cases} 0 & x < 0 \\ x & 0 < x < 1 \\ 1 & x > 1 \end{cases} \quad a = 0, b = 1$$

$R_Y = \{g(x) \mid f_X(x) \ge 0\} = [g(0), g(1)] = [1, e]$

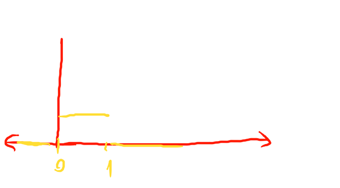
$$F_X(x) = \begin{cases} \frac{x-a}{b-a} & a \le x \le b \\ 0 & 0 > x \\ 1 & 1 < x \end{cases}$$

$\ln y = 0$ $\qquad \ln y = 1$ $\quad e$

$$\begin{cases} 0 & y < 1 \\ \ln y & 1 < y < e \\ 1 & y > e \end{cases} \qquad \begin{cases} 0 & y < 1 \\ \frac{1}{y} & 1 < y < e \\ 0 & y > e \end{cases}$$

$\int_1^e \frac{1}{y} \cdot y\,dy = \underline{e-1}$

$R_X = [0,1]$



$$F_Y(y) = \begin{cases} 0 & y < 1 \\ 1 & e < y \\ \frac{y-1}{e-1} & 1 < y < e \to ? \end{cases}$$

for $1 \le y \le e$

$F_Y(y) = P(e^X \le y)$

$= P(X \le \ln y)$

$= F_X(\ln y) =$ from $F_X(x)$

$$F_Y(y) = \begin{cases} 0 & , y < 1 \\ \ln y & , 1 \le y \le e \\ 1 & , y > e \end{cases}$$

$f_Y(y) = \frac{d}{dy} F_Y(y)$