Optimization Techniques
Section 2

M. Fatih Amasyalı

Steepest Descent

- Exact step size
- x_new = x_old - eps * df;
- Eps is calculated as:
  - z(eps)=x – eps*df
  - Find eps point where f’(z(eps))=0

Steepest Descent

- Find the minimum point of f(x)=x^2
- z(eps)=x-eps*2*x=x(1-2*eps) = search direction
- f(z(eps))= the value of f at a point on the search direction
- Find eps value which is minimum of f(z(eps)), f'(z(eps))=0
- f(z(eps))=(x(1-2*eps))^2=x^2*(1-2*eps)^2
- f(z(eps))=x^2*(1-4*eps+4*eps^2)
- f'(z(eps))=x^2*(-4+8*eps)=0
- eps=1/2
- $X_{n+1}=X_n$-eps*df=X_n-eps*2*X_n=X_n-X_n=0
- Wherever you start, the minimum point is found at one iteration !

Steepest Descent

- Find the minimum point of f(x)=x^4
- z(eps)=x-eps*4*x^3= search direction
- If we know that the minimum point of f(z(eps)) is 0 (But, we do not know, in reality)
- eps*4*x^3=x than,
- eps=1/(4*x^2)
- $X_{n+1}=X_n$-epx*4*x^3=X_n-*(1/(4*X_n^2))*4*X_n^3=
- X_n-X_n=0
- Wherever you start, the minimum point is found at one iteration !

Steepest Descent in 2 dims.

- Find the iteration equation to find the minimum of f(x1,x2)=x1^2+3*x2^2
- df=[2*x1 ; 6*x2]
- t=eps
- z(t) have 2 dims as x
- z(t)=[x1 ; x2]-t*df
- z(t)=[x1;x2]-[2*x1*t; 6*x2*t]
- z(t)=[x1*(1-2*t) ; x2*(1-6*t)]
- f(z(t))= (x1^2)*(1-2*t) ^2+3*(x2^2)(1-6*t)^2

Steepest Descent in 2 dims.

- f(z(t))= (x1^2)*(1-2*t) ^2+3*(x2^2)(1-6*t)^2
- df(z(t))/dt=f'(z(t))=
- =(x1^2)*2*(1-2*t)*(-2)+3*(x2^2)*2*(1-6*t)*(-6)
- =(x1^2)*(-4)*(1-2*t)-36*(x2^2)*(1-6*t)
- =(x1^2)*(-4+8*t)- (x2^2)*(36-216*t)
- =-4*(x1^2)+(x1^2)*8*t-36*(x2^2)+216*t*(x2^2)
- =0 because f'(z(t))=0
- (x1^2)*8*t+216*t*(x2^2)=4*(x1^2)+36*(x2^2)
- t= (4*(x1^2)+36*(x2^2)) / ((x1^2)*8+216*(x2^2))
- t= ((x1^2)+9*(x2^2)) / (2*(x1^2)+54*(x2^2))

Steepest Descent in 2 dims.

- So the iteration equation is
- X_{n+1}=X_n-t*[2*x1; 6*x2] where

t= ((x1^2)+9*(x2^2)) / (2*(x1^2)+54*(x2^2))

Steepest Descent Examples

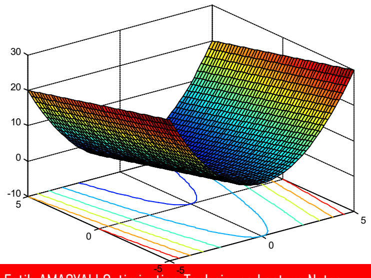
- f(x)=x1^2+x2^2
- df=[2*x1 ; 2*x2]
- z(t)=[x1 ; x2]-t*df
- z(t)=[x1;x2]-[2*x1*t; 2*x2*t]
- z(t)=[x1*(1-2*t) ; x2*(1-2*t)]
- f(z(t))= (x1^2)*(1-2*t) ^2+(x2^2)(1-2*t)^2
- f(z(t))=((1-2*t) ^2)(x1^2+ x2^2)
- f'(z(t))=(x1^2 + x2^2)*(8*t - 4)=0
- t=1/2
- z(t)=[x1 ; x2]-t* [2*x1 ; 2*x2]
- z(t)=[x1 ; x2]-[x1 ; x2]= [0 0]
- Wherever you start, the minimum point is found at one iteration !

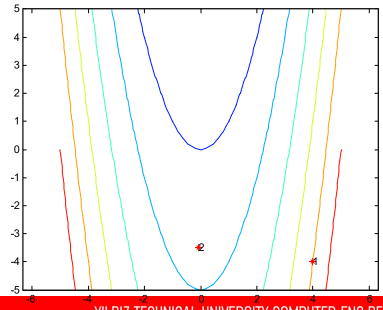
Steepest Descent Examples

- f(x)=x1^2+x2^2, t=1/2
- f(x) =x1^2-x2, t= 0.5+1/(8*x1^2)

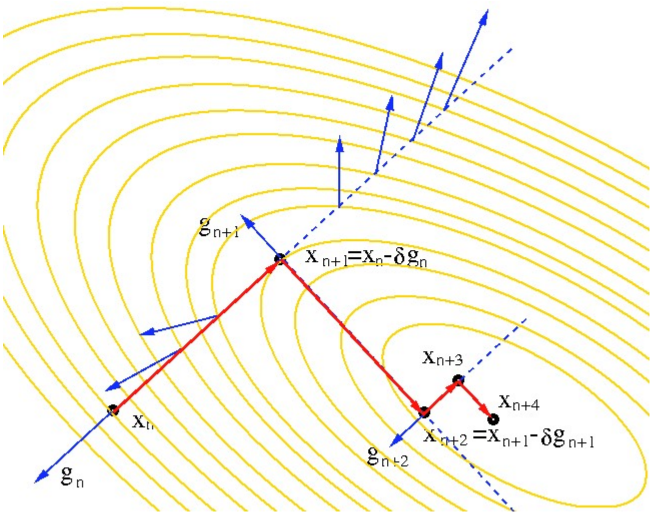
steepest_desc_2dim_2.m

where is the 3rd,4th, iterations…?





the new search direction will always be perpendicular to the previous direction.



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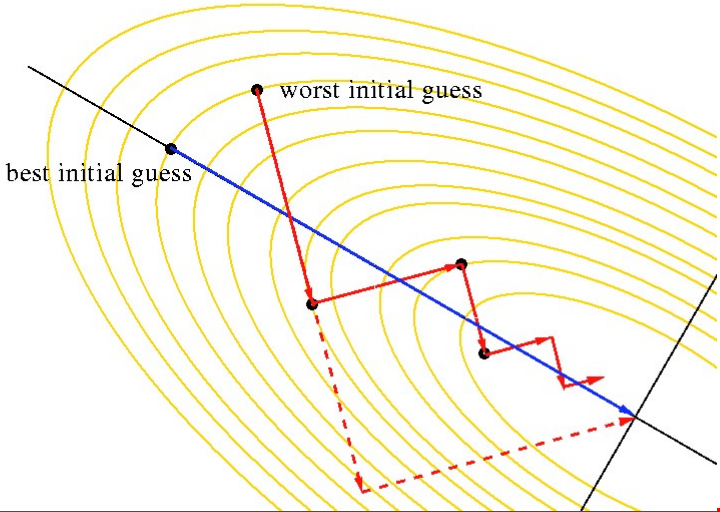
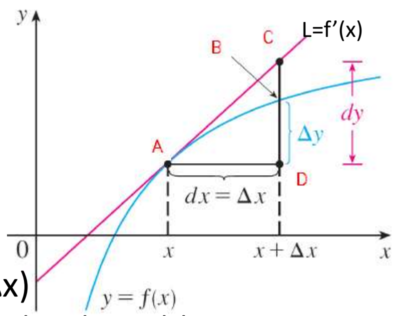


Linear Approximation

- Assuming the function is linear at a point.



f(x+Δx) ≈ L(x+Δx)
$\lim_{(\Delta x \to 0)}$ (f(x+Δx) - L(x+Δx)) = 0
L(x+Δx) =f(x) + dy = f(x) + Δx f'(x) since f'(x)=dy/dx
New point: domx=x+Δx,
Δx=domx-x, f(domx) ≈ f(x)+(domx-x)f'(x)

Linear Approximation Example 1

- $(1.0002)^{50} \approx ?$
- $f(x)=x^{50}$
- $f(x+\Delta x) \approx f(x) + \Delta x \, f'(x)$
- domx=1.0002, x=1 , $\Delta x=0.0002$
- $f(1+0.0002) \approx f(1) + 0.0002 \, f'(1)$
- $f(1+0.0002) \approx f(1) + 0.0002 * 50 * 1^{49}$
- $f(1+0.0002) \approx 1 + 0.0002 * 50 * 1 = 1.01$

Linear Approximation Example 2

- Find the linear approximation for x tends to 1 where f(x) = ln x.
- $f(x+\Delta x) \approx f(x) + \Delta x \, f'(x)$
- domx=x+Δx , Δx = domx-x , x=1, f'(x)=1/x
- f(domx) ≈ ln 1 + f'(1) (domx - 1) = domx -1
- ln x ≈ x-1 , for x close to 1

- For x tends to 2
- Δx = domx-x, x=2
- f(domx) ≈ ln 2 + f'(2) (domx - 2) = ln 2 + (domx-2)/2
- ln x ≈ ln 2+ (x-2)/2 , for x close to 2

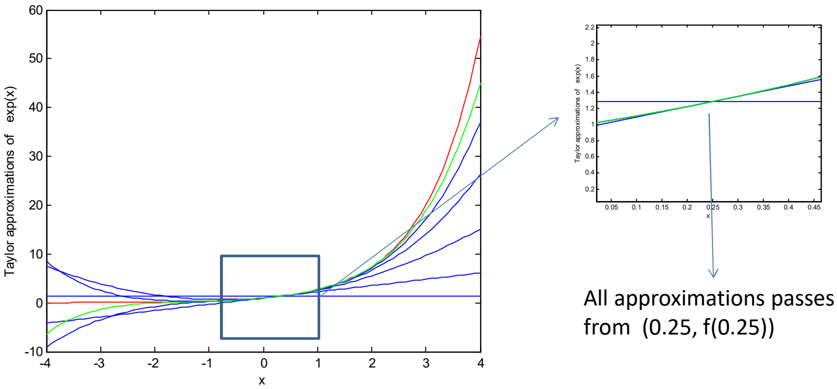
For a better approximation

- 1^{st} order Taylor: (linear approx.)

  $f(x+\Delta x) \approx f(x) + \Delta x\, f'(x)$
- 2^{nd} order Taylor: (non-linear approx.)

  $f(x+\Delta x) \approx f(x) + \Delta x\, f'(x) + ½\, f''(x)\, \Delta x^2$
- …
- N^{th} order Taylor: (non-linear approx.)
- $f(x+\Delta x) \approx \sum \left(f^{(i)'}(x)\, \Delta x^i\right) / i!$ $i=0\ldots N$

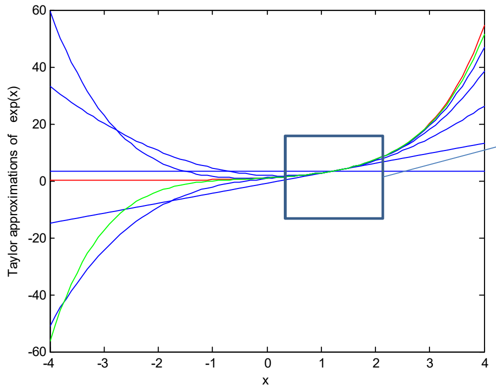
Function approx. with Taylor

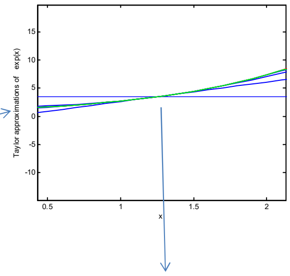


- f(x)=exp(x), N=0:5, X=0.25
- Red: real f, Blues and green: approximations
- Green: The last approx.

approx_taylor.m

Function approx. with Taylor



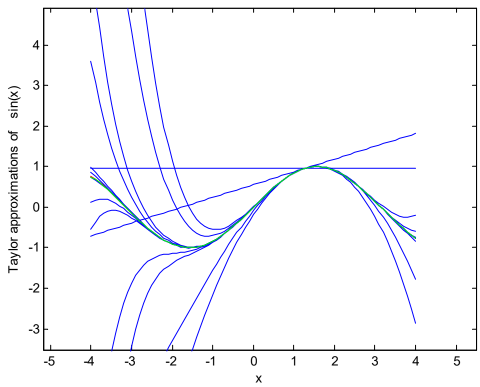


All approximations passes from (1.25, f(1.25))

- f(x)=exp(x), N=0:5, X=1.25
- Red: real f, Blues and green: approximations
- Green: The last approx.

approx_taylor.m

Function approx. with Taylor

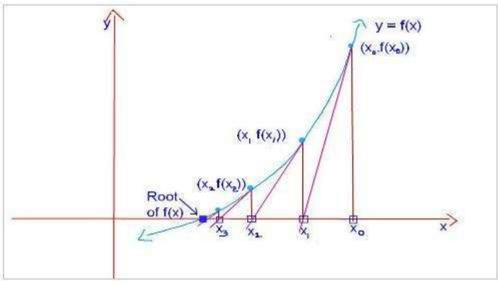


- f(X)=sin(x)
- X=1.25
- N=0:15
- Red: real f
- Blues and green: approximations
- Green: The last approx.

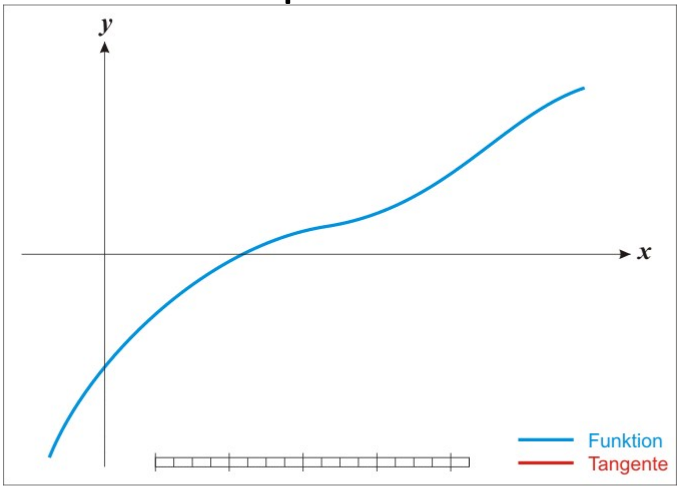
approx_taylor.m

Finding a root of f(x) iteratively (find a point x where f(x)=0) Newton Raphson 1st order

- $f'(x_n) = f(x_n) / (x_n - x_{n+1})$
- $x_n - x_{n+1} = f(x_n) / f'(x_n)$
- $x_{n+1} = x_n - f(x_n) / f'(x_n)$
  n = 0,1,2,3....
- If we require the root correct up to 6 decimal places, we stop when the digits in x_{n+1} and x_n agree till the 6th decimal place.



Newton Raphson- 1st order



Example

- Find sqrt(2)
- Means find the root of $x^2-2=0$
- $x_0=1$
- $x_{n+1}= x_n-f(x_n)/f'(x_n)$
- $x_{n+1}= x_n - (x_n*x_n-2)/(2*x_n)$
- $x_0=1$
  $x_1=1.5$
  $x_2=1.41667$
  $x_3=1.41422$
  $x_4=1.41421$
  if the current improvement (0.00001) is insignificant, we can say sqrt(2) =1.41421, if not go on the iterations.

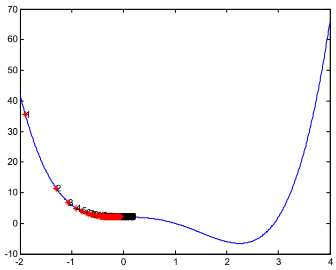
Taylor Series - Newton Raphson 2nd order

- 1st order Taylor:
  $f(x+\Delta x) \approx f(x) + \Delta x\, f'(x)$ (1)
- According to 1st order Taylor, to find $f(x+\Delta x)=0$,
  $\Delta x=-f(x)/f'(x)$
- To find $f(x+\Delta x)'=0$, take derivative of (1)
  $f'(x+\Delta x) \approx f'(x) + \Delta x\, f''(x)$
  $\Delta x=-f'(x)/f''(x) \leftarrow$ Newton Raphson 2nd order
  $x_{n+1} = x_n - f(x_n) / f'(x_n)$
  $\Delta x = x_{n+1} - x_n$

Newton Raphson

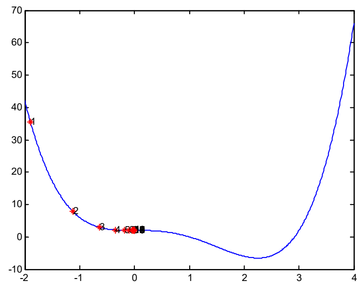
- Generally faster converge, because it uses more information (2nd derivative)
- No explicit step size selection
- 1st order : x_new = x_old - f/df; (f(x)=0)
- 2nd order : x_new = x_old - df/ddf; (f'(x)=0)
- instead of x_new = x_old - eps * df; (gradient descent)

Gradient Descent Step size=0.01 Starting point=-1.9



Does not converge with 200 iterations

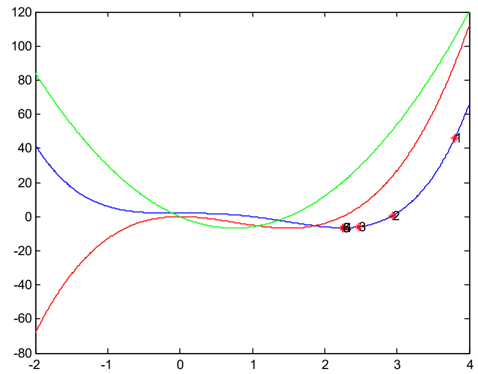
Newton Raphson 2nd order



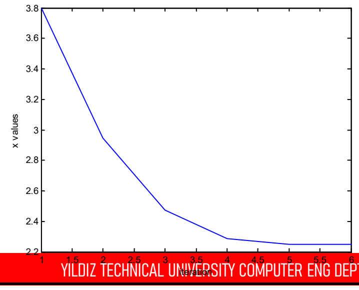
Converged at 20 iterations

newton_raphson_1.m

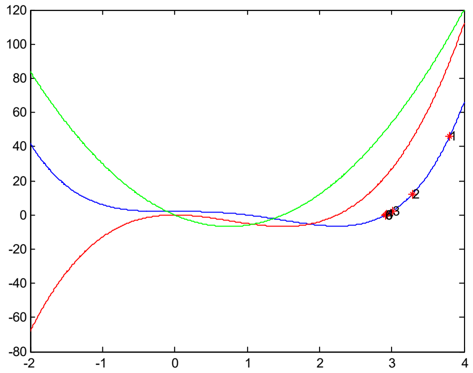
newton_raphson_1.m
2nd order finds f’(x)=0



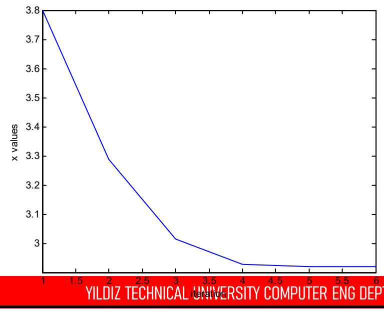
- Blue f
- Red f’
- Green f’’



newton_raphson_1.m
1nd order finds f(x)=0

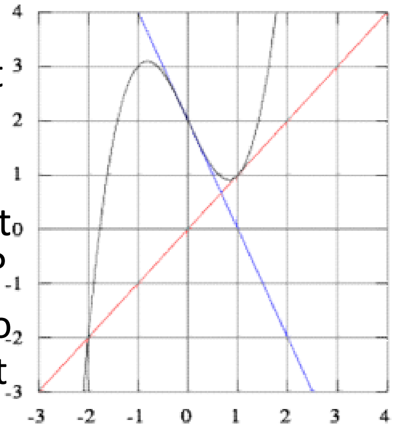


- Blue f
- Red f’
- Green f’’



Newton Raphson 1st order -Cycle problem

- The tangent lines of $x^3 - 2x + 2$ at 0 and 1 intersect the x-axis at 1 and 0 respectively.
- What happened if we start at a point in (0,1) interval?
- If we start at 0.1, it goes to 1, then it goes to 0, then it goes to 1 …



Newton Raphson

- Faster convergence (lower iteration number)
- But, more calculation for each iteration

References

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