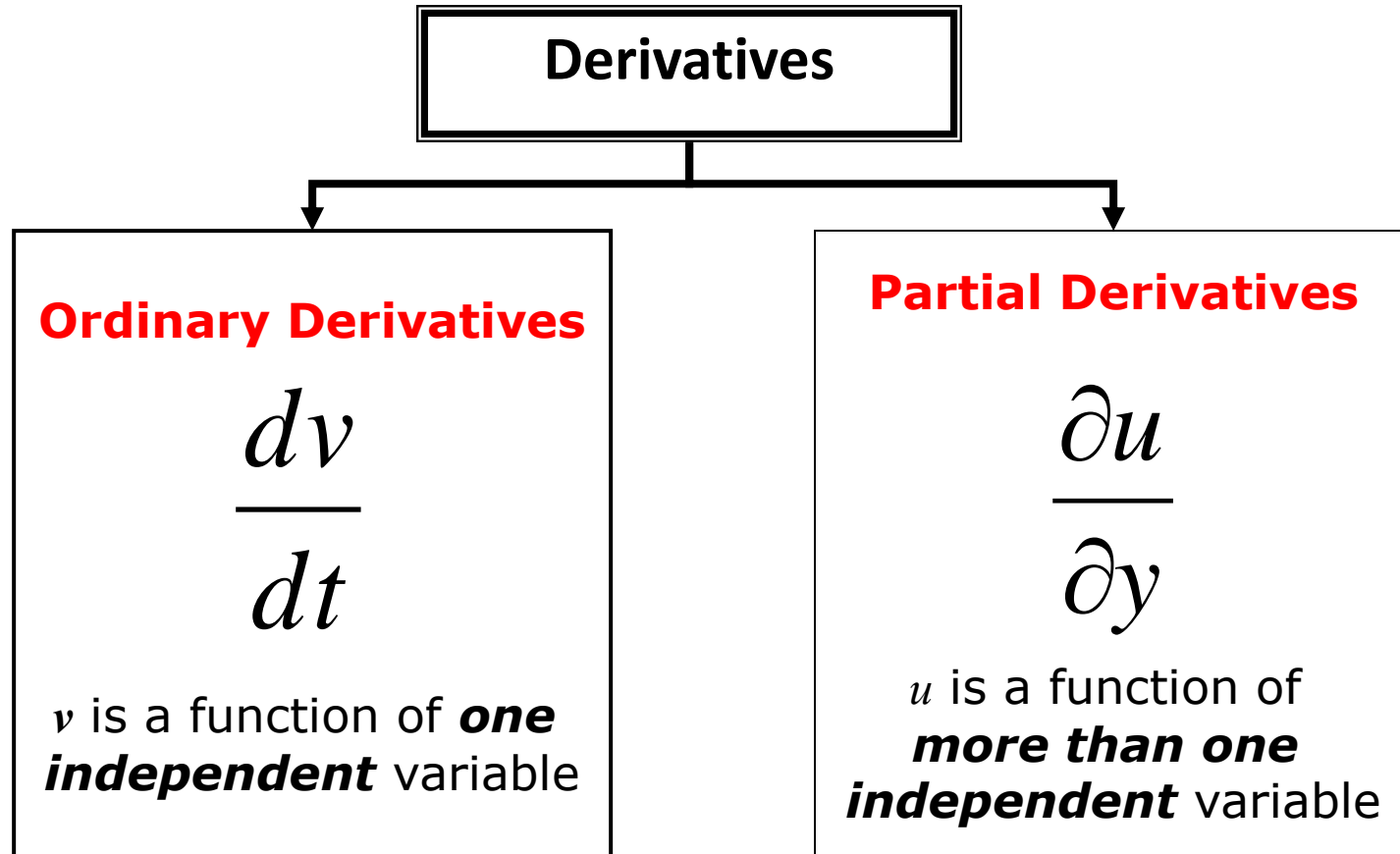


# Ordinary Differential Equations

# Derivatives



# Differential Equations

## Differential Equations

```
graph TD; A[Differential Equations] --> B[Ordinary Differential Equations]; A --> C[Partial Differential Equations];
```

### Ordinary Differential Equations

$$\frac{d^2 v}{dt^2} + 6tv = 1$$

involve one or more  
Ordinary derivatives of  
unknown functions

### Partial Differential Equations

$$\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x^2} = 0$$

involve one or more  
partial derivatives of  
unknown functions

# Ordinary Differential Equations

Ordinary Differential Equations (ODEs) involve one or more **ordinary derivatives of unknown functions with respect to one independent variable**

*Examples:*

$$\frac{dv(t)}{dt} - v(t) = e^t$$

x(t): unknown function

$$\frac{d^2 x(t)}{dt^2} - 5 \frac{dx(t)}{dt} + 2x(t) = \cos(t)$$

t: independent variable

# Example of ODE: Model of Falling Parachutist

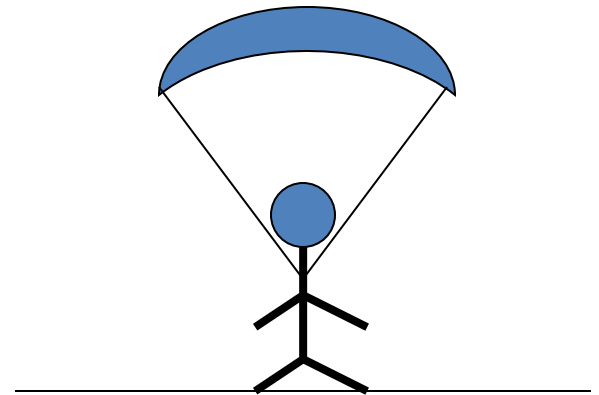
The velocity of a falling  
parachutist is given by:

$$\frac{d v}{d t} = 9.8 - \frac{c}{M} v$$

$M$  : *mass*

$c$  : *drag coefficient*

$v$  : *velocity*



# Model of Falling Parachutist

Drag force :  $F_d = cv$

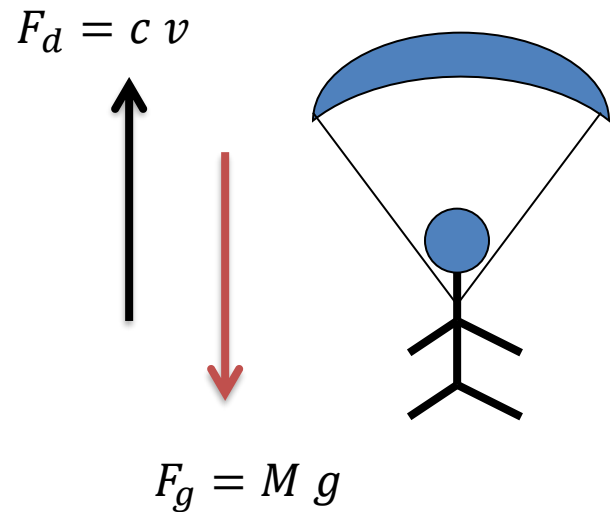
Gravity force:  $F_g = Mg$

Acceleration:  $a = \frac{dv}{dt}$

Net Force:  $F = Ma$  or  $F = F_g - F_d$

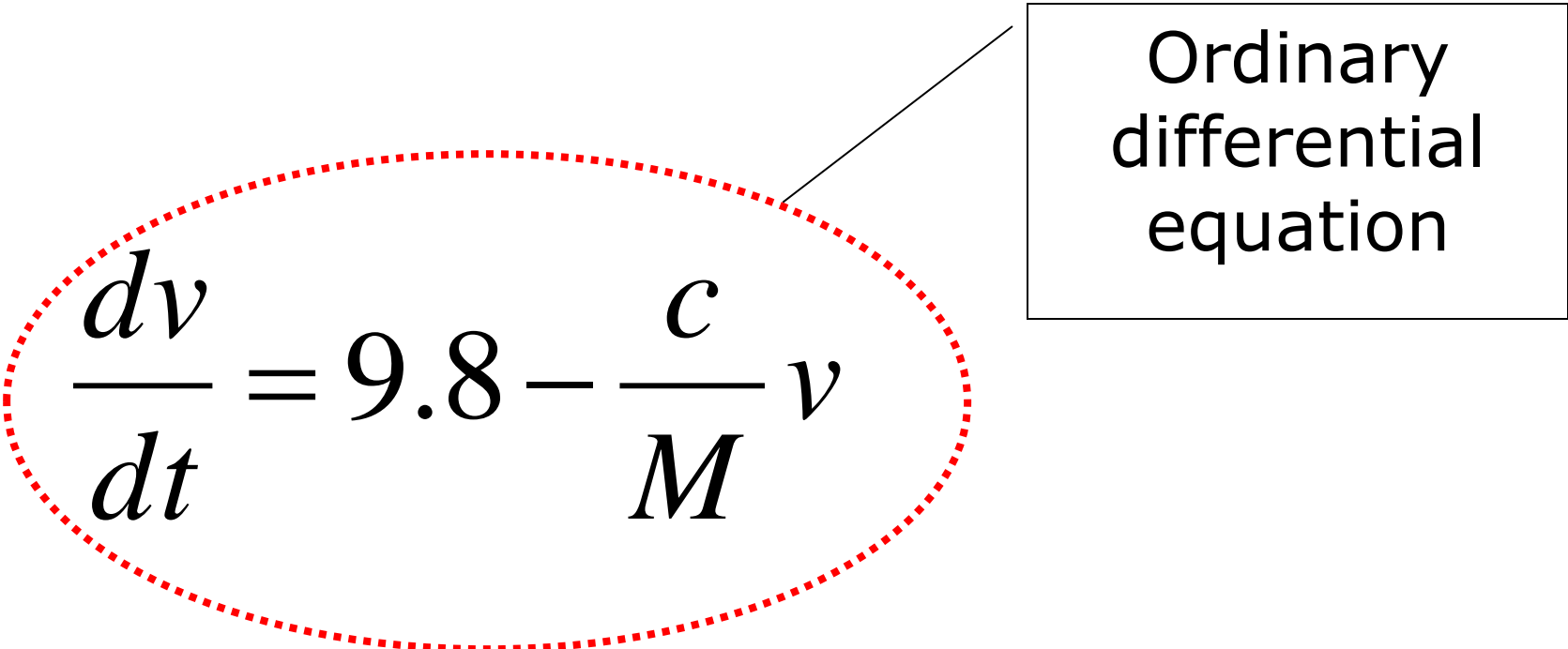
$$M \frac{dv}{dt} = Mg - cv$$

$$\frac{dv}{dt} = g - \frac{c}{M}v$$



# Definitions

Ordinary  
differential  
equation


$$\frac{dv}{dt} = 9.8 - \frac{c}{M} v$$

# Definitions (Cont.)

$$\frac{dv}{dt} = 9.8 - \frac{c}{M}v$$

(Dependent variable)  
unknown  
function to be determined



# Definitions (Cont.)

$$\frac{d v}{d t} = 9.8 - \frac{c}{M} v$$

(independent variable)  
the variable with respect to which  
other variables are differentiated

# Order of a Differential Equation

- The number of the highest derivative in a differential equation. A differential equation of order 1 is called first order, order 2 second order, etc.

Example: The differential equation  $y'' + xy' - x^3y = \sin x$  is second order since the highest derivative is  $y''$  or the second derivative.

# Order of a Differential Equation

The **order** of an ordinary differential equation is the order of the highest order derivative.

*Examples:*

$$\frac{dx(t)}{dt} - x(t) = e^t$$

First order ODE

$$\frac{d^2x(t)}{dt^2} - 5\frac{dx(t)}{dt} + 2x(t) = \cos(t)$$

Second order ODE

$$\left( \frac{d^2x(t)}{dt^2} \right)^3 - \frac{dx(t)}{dt} + 2x^4(t) = 1$$

Second order ODE

# Solution of a Differential Equation

A **solution** to a differential equation is a function that satisfies the equation.

*Example:*

$$\frac{dx(t)}{dt} + x(t) = 0$$

*Solution*  $x(t) = e^{-t}$

Proof:

$$\frac{dx(t)}{dt} = -e^{-t}$$

$$\frac{dx(t)}{dt} + x(t) = -e^{-t} + e^{-t} = 0$$

# Linear and non-linear

Both ordinary and partial differential equations are broadly classified as **linear** and **nonlinear**.

A differential equation is linear if the unknown function and its derivatives appear to the power 1 (products of the unknown function and its derivatives are not allowed) and nonlinear otherwise.

The characteristic property of linear equations is that their solutions form an affine subspace of an appropriate function space, which results in much more developed theory of linear differential equations. **Homogeneous linear** differential equations are a further subclass for which the space of solutions is a linear subspace i.e. the sum of any set of solutions or multiples of solutions is also a solution. The coefficients of the unknown function and its derivatives in a linear differential equation are allowed to be (known) functions of the independent variable or variables; if these coefficients are constants then one speaks of a **constant coefficient linear differential equation**.

# Linear and Affine Function

- $f(x)=2x$  is linear and affine.
- $f(x)=2x+3$  is affine but not linear

# Linear and Non-linear

There are very few methods of solving nonlinear differential equations exactly; those that are known typically depend on the equation having particular [symmetries](#).

Nonlinear differential equations can exhibit very complicated behavior over extended time intervals, characteristic of [chaos](#). Even the fundamental questions of existence, uniqueness, and extendability of solutions for nonlinear differential equations, and well-posedness of initial and boundary value problems for nonlinear PDEs are hard problems.

However, if the differential equation is a correctly formulated representation of a meaningful physical process, then one expects it to have a solution.

Linear differential equations frequently appear as [approximations](#) to nonlinear equations. These approximations are only valid under restricted conditions.

# Linear and Non-linear

A **linear differential equation** is any differential equation that can be written in the following form:

$$a_n(t) y^{(n)}(t) + a_{n-1}(t) y^{(n-1)}(t) + \dots + a_1(t) y'(t) + a_0(t) y(t) = g(t)$$

If a differential equation cannot be written in this form, then it is called **a non-linear differential equation**.



# Linear ODE

An ODE is linear if the unknown function and its derivatives appear to power one.

No product of the unknown function and/or its derivatives

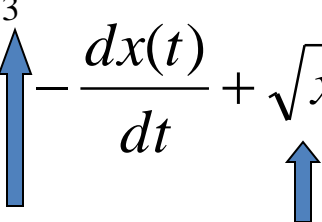
*Examples:*

$$\frac{dx(t)}{dt} - x(t) = e^t$$

Linear ODE

$$\frac{d^2x(t)}{dt^2} - 5\frac{dx(t)}{dt} + 2t^2x(t) = \cos(t)$$

Linear ODE

$$\left(\frac{d^2x(t)}{dt^2}\right)^3 - \frac{dx(t)}{dt} + \sqrt{x(t)} = 1$$


Non-linear ODE

# Nonlinear ODE

Examples of nonlinear ODE :

$$\frac{dx(t)}{dt} - \cos(x(t)) = 1$$

$$\frac{d^2 x(t)}{dt^2} - 5 \frac{dx(t)}{dt} x(t) = 2$$

$$\frac{d^2 x(t)}{dt^2} - \left| \frac{dx(t)}{dt} \right| + x(t) = 1$$

# Solutions of Ordinary Differential Equations

$$x(t) = \cos(2t)$$

is a solution to the ODE

$$\frac{d^2 x(t)}{dt^2} + 4x(t) = 0$$

Is it unique?

All functions of the form  $x(t) = \cos(2t + c)$   
(where  $c$  is a real constant) are solutions.

# Uniqueness of a Solution

In order to uniquely specify a solution to an  $n^{\text{th}}$  order differential equation we need  $n$  conditions.

$$\frac{d^2 x(t)}{dt^2} + 4x(t) = 0$$

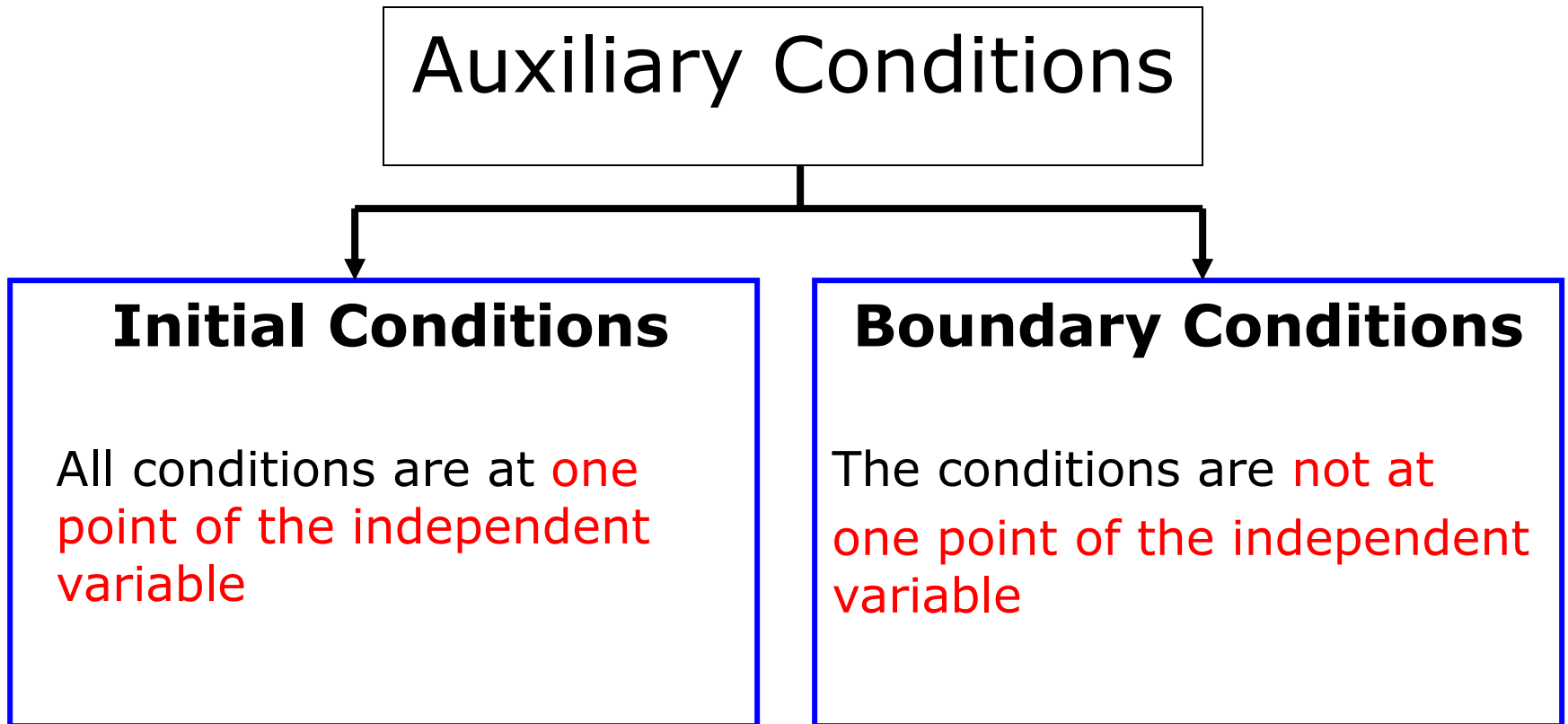
Second order ODE

$$x(0) = a$$

$$\dot{x}(0) = b$$

Two conditions are needed to uniquely specify the solution

# Auxiliary Conditions



# Boundary-Value and Initial value Problems

## Initial-Value Problems

- The auxiliary conditions are at **one point of the independent variable**

$$\ddot{x} + 2\dot{x} + x = e^{-2t}$$

$$x(0) = 1, \quad \dot{x}(0) = 2.5$$

same

## Boundary-Value Problems

The auxiliary conditions are **not at one point of the independent variable**

More difficult to solve than initial value problems

$$\ddot{x} + 2\dot{x} + x = e^{-2t}$$

$$x(0) = 1, \quad x(2) = 1.5$$

different

# Classification of ODEs

ODEs can be classified in different ways:

- Order
  - First order ODE
  - Second order ODE
  - $N^{\text{th}}$  order ODE
- Linearity
  - Linear ODE
  - Nonlinear ODE
- Auxiliary conditions
  - Initial value problems
  - Boundary value problems

# Analytical Solutions

- Analytical Solutions to ODEs are available for linear ODEs and special classes of nonlinear differential equations.



# Numerical Solutions

- Numerical methods are used to obtain a graph or a table of the unknown function.
- Most of the Numerical methods used to solve ODEs are based directly (or indirectly) on the truncated Taylor series expansion.

# Separable Differential Equations

A separable differential equation can be expressed as the product of a function of  $x$  and a function of  $y$ .

$$\frac{dy}{dx} = g(x) \cdot h(y) \quad h(y) \neq 0$$

Example:

$$\frac{dy}{dx} = 2xy^2$$

Multiply both sides by  $dx$  and divide both sides by  $y^2$  to separate the variables. (Assume  $y^2$  is never zero.)

$$\frac{dy}{y^2} = 2x \, dx$$

$$y^{-2} dy = 2x \, dx$$

# Separable Differential Equations

A separable differential equation can be expressed as the product of a function of  $x$  and a function of  $y$ .

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Example:

$$\frac{dy}{dx} = 2xy^2$$

$$\frac{dy}{y^2} = 2x \, dx$$

$$y^{-2} dy = 2x \, dx$$

$$\int y^{-2} dy = \int 2x \, dx$$

$$-y^{-1} + C_1 = x^2 + C_2$$

$$-\frac{1}{y} = x^2 + C$$

$$-\frac{1}{x^2 + C} = y$$

Combined  
constants of  
integration

$$y = -\frac{1}{x^2 + C}$$



# Family of solutions (general solution) of a differential equation

Example

$$\frac{dy}{dx} = \frac{x}{y} \quad \int y dy = \int x dx$$

$$y^2 = x^2 + C$$

The picture on the right shows some solutions to the above differential equation.

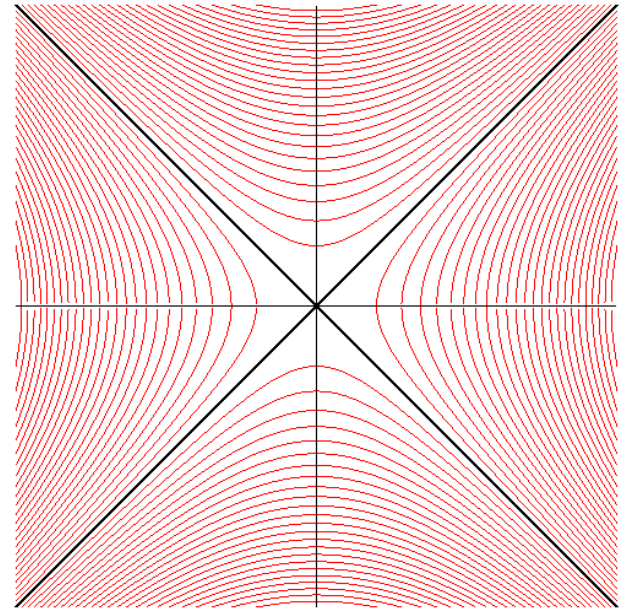
The straight lines

$$y = x \text{ and } y = -x$$

are special solutions.

A unique solution curve goes through any point of the plane different from the origin.

The special solutions  $y = x$  and  $y = -x$  go both through the origin.



# **Ordinary Differential Equations (Initial Value Problem)**

# Initial conditions

- In many physical problems we need to find the particular solution that satisfies a condition of the form  $y(x_0)=y_0$ . This is called an **initial condition**, and the problem of finding a solution of the differential equation that satisfies the initial condition is called an **initial-value problem**.
- *Example (cont.):* Find a solution to  $y' = x^2 + C$  satisfying the initial condition  $y(0) = 2$ .

$$2 = 0^2 + C$$

$$C = 4$$

$$y' = x^2 + 4$$

Example:

$$\frac{dy}{dx} = 2x(1 + y^2)e^{x^2} \leftarrow \text{Separable differential equation}$$

$$\frac{1}{1 + y^2} dy = 2x e^{x^2} dx$$

$$\int \frac{1}{1 + y^2} dy = \int 2x e^{x^2} dx$$

$$\begin{aligned} u &= x^2 \\ du &= 2x dx \end{aligned}$$

$$\int \frac{1}{1 + y^2} dy = \int e^u du$$

$$\tan^{-1} y + C_1 = e^u + C_2$$

$$\tan^{-1} y + C_1 = e^{x^2} + C_2$$

$$\tan^{-1} y = e^{x^2} + C \leftarrow \text{Combined constants of integration}$$



Example (cont.):

$$\frac{dy}{dx} = 2x(1 + y^2)e^{x^2}$$

⋮

$$\tan^{-1} y = e^{x^2} + C \quad \leftarrow \text{We now have } y \text{ as an implicit function of } X.$$

$$\tan(\tan^{-1} y) = \tan(e^{x^2} + C) \quad \text{We can find } y \text{ as an explicit function of } X \text{ by taking the tangent of both sides.}$$

$$y = \tan(e^{x^2} + C)$$



# Law of natural growth or decay

A population of living creatures normally increases at a rate that is proportional to the current level of the population. Other things that increase or decrease at a rate proportional to the amount present include radioactive material and money in an interest-bearing account.

If the rate of change is proportional to the amount present, the change can be modeled by:

$$\frac{dy}{dt} = ky$$



$$\frac{dy}{dt} = ky$$

Rate of change is proportional to the amount present. This equation is the Analytical Model.

$$\frac{1}{y} dy = k dt$$

Divide both sides by  $y$ .

$$\int \frac{1}{y} dy = \int k dt$$

Integrate both sides.  $\longrightarrow \ln|y| = kt + C$

$$e^{\ln|y|} = e^{kt+C}$$

Exponentiate both sides.

$$|y| = e^C \cdot e^{kt} \longrightarrow y = \pm e^C e^{kt}$$

$\pm e^C$  is a constant value which can be represented as **A**

$$y = Ae^{kt}$$

Analytical Solution for the Model

# Logistic Growth Model

Real-life populations do not increase forever. There is some limiting factor such as food or living space.

There is a maximum population, or carrying capacity,  $M$ .

A more realistic model is the **logistic growth model** where growth rate is proportional to both the size of the population ( $y$ ) and the amount by which  $y$  falls short of the maximal size ( $M-y$ ). Then we have the equation:

$$\frac{dy}{dt} = ky(M - y)$$

**Analytical solution** to this differential equation :

$$y = \frac{y_0 M}{y_0 + (M - y_0)e^{-kMt}}, \quad \text{where } y_0 = y(0)$$

# Example - Mixing Problems

A tank has pure water flowing into it at 10 liter/min.

The contents of the tank are kept thoroughly mixed, and the contents flow out at 10 liter/min.

Initially, the tank contains 10 kg of salt in 100 liter of water.

How much salt will be in the tank

(a) after  $t$  minutes;

(b) after 30 minutes?

We are interested in the **amount of salt** at any **time** --

$S$  – amount of salt - is the dependent variable – unknown function to be determined  
 $t$  - time - independent variable

To study such a question, we consider *the rate of change of the amount of salt in the tank* :  $\frac{dS}{dt}$

If we can create an equation relating  $\frac{dS}{dt}$  to  $S$  and  $t$ , then we will have a differential equation which we can, ideally, solve to determine the relationship between  $S$  and  $t$ .

To describe  $\frac{dS}{dt}$ , we can use the concept of concentration (the amount of salt per unit of volume of water).

In this example, **the inflow and outflow rates are the same**, so **the volume of liquid in the tank stays constant at 100 liter**.

Hence, we can describe *the concentration of salt in the tank* by :

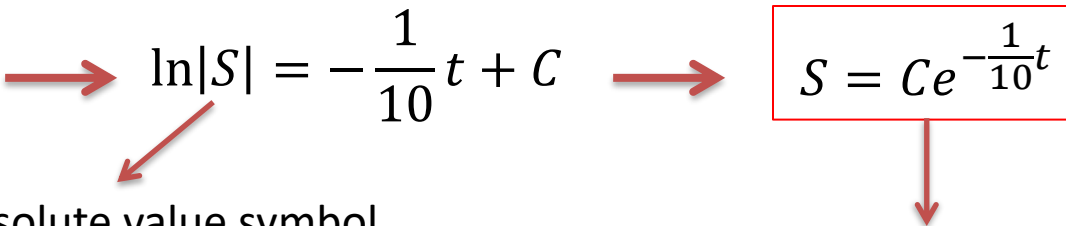
$$\text{Concentration of salt} = \frac{S}{100} \text{ kg/liter}$$

Since mixture **leaves the tank at the rate of 10 liter/min**, salt is leaving the tank at the rate of  $\frac{S}{100} (10 \text{ liter/min}) = \frac{S}{10}$

Salt **leaves** the tank so; the rate of change is  $k = -\frac{1}{10}$

$$\boxed{\frac{dS}{dt} = -\frac{S}{10}}$$

The model of decay in the amount of salt

$$\int \frac{dS}{S} = -\int \frac{1}{10} dt \quad \longrightarrow \quad \ln|S| = -\frac{1}{10}t + C \quad \longrightarrow \quad \boxed{S = Ce^{-\frac{1}{10}t}}$$


We eliminated the absolute value symbol due to the fact that  $S \geq 0$

This is the implicit solution of the ODE

For an **explicit solution** we will use the **initial value** provided :

Since  $S = 10$  when  $t = 0$ , we can find that  $C = 10$

**(a)** After  $t$  minutes, the amount of salt in the tank will be:

$$S = 10e^{-\frac{1}{10}t}$$

Now, this is the **explicit solution** of the ODE

We can see from this that as  $t$  goes to infinity, the amount of salt in the tank goes to zero.

**(b)** After 30 minutes, the amount of salt in the tank will be:

$$S = 10e^{-\frac{1}{10}30} = 10e^{-3} = 0,497870 \text{ kg}$$

# Exercise for Home-1

A tank has pure water flowing into it at 10 liter/min. The contents of the tank are kept thoroughly mixed, and the contents flow out at 10 l/min. Salt is added to the tank at the rate of 0.1 kg/min. Initially, the tank contains 10 kg of salt in 100 liter of water.

How much salt is in the tank after 30 minutes?

**Clue :** The setup is very similar the previous example. The only difference is the addition of 0.1 kg/min of salt to the tank. So modify your differential equation to take this into account.



# Exercise for Home-2

A tank has pure water flowing into it at 12 liter/min. The contents of the tank are kept thoroughly mixed, and the contents flow out at 10 liter/min.

Initially, the tank contains 10 kg of salt in 100 liter of water.

How much salt will be in the tank

- (a) after  $t$  minutes;
- (b) after 30 minutes?

**Clue:** The inflow rate is greater than the outflow rate. As a result, the volume is not constant.

Using the initial conditions and the flow rates, we can say that the volume  $V$  of liquid in the tank is  $V = 100 + 2t$

The concentration of salt after  $t$  minutes is  $\frac{S}{V}$