BLM1612 - Circuit Theory

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Frequency Response

Frequency Response

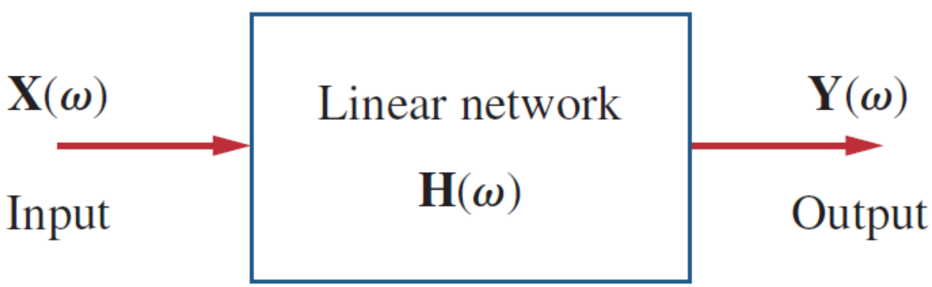
- Objective of Lecture
  - Describe the sinusoidal steady-state behavior of a circuit as a function of frequency.

Frequency Response

- If we let the amplitude of the sinusoidal source remain constant and vary the frequency,
  - we obtain the circuit's frequency response.
- The frequency response of a circuit is the variation in its behavior with change in signal frequency.
- The frequency response of a circuit may also be considered as the variation of the gain and phase with frequency.

Transfer Function

- The transfer function $\mathbf{H}(\omega)$ of a circuit is the frequency-dependent ratio of a phasor output $\mathbf{Y}(\omega)$ (an element voltage or current) to a phasor input $\mathbf{X}(\omega)$ (source voltage or current).



$$\boxed{\mathbf{H}(\omega) = \frac{\mathbf{Y}(\omega)}{\mathbf{X}(\omega)}}$$

Transfer Function

- Since the input and output can be either voltage or current at any place in the circuit,
  - there are four possible transfer functions:

$$\mathbf{H}(\omega) = \text{Voltage gain} = \frac{\mathbf{V}_o(\omega)}{\mathbf{V}_i(\omega)}$$

$$\mathbf{H}(\omega) = \text{Current gain} = \frac{\mathbf{I}_o(\omega)}{\mathbf{I}_i(\omega)}$$

$$\mathbf{H}(\omega) = \text{Transfer Impedance} = \frac{\mathbf{V}_o(\omega)}{\mathbf{I}_i(\omega)}$$

$$\mathbf{H}(\omega) = \text{Transfer Admittance} = \frac{\mathbf{I}_o(\omega)}{\mathbf{V}_i(\omega)}$$

Transfer Function

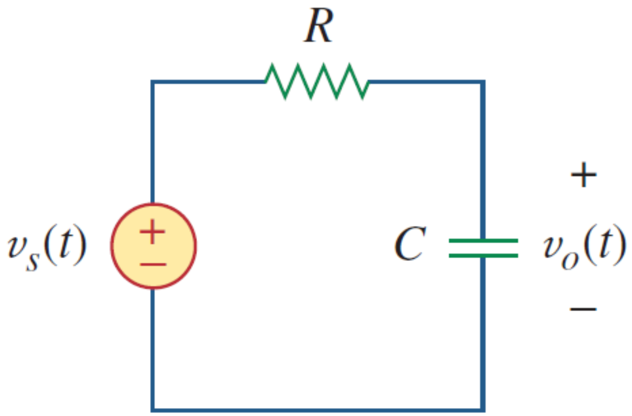
- The transfer function $\mathbf{H}(\omega)$ can be expressed in terms of its numerator polynomial $\mathbf{N}(\omega)$ and denominator polynomial $\mathbf{D}(\omega)$ as

$$\boxed{\mathbf{H}(\omega) = \frac{\mathbf{N}(\omega)}{\mathbf{D}(\omega)}}$$

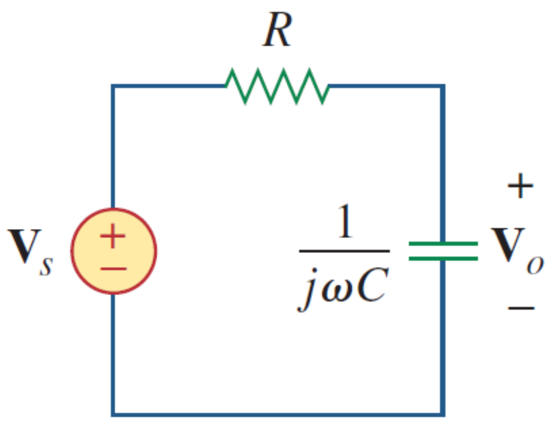
  – The roots of $\mathbf{N}(\omega) = 0$ are called the zeros of $\mathbf{H}(\omega)$ and are usually represented as $j\omega = z_1, z_2, \ldots$.
  – Similarly, the roots of $\mathbf{D}(\omega) = 0$ are the poles of $\mathbf{H}(\omega)$ and are represented as $j\omega = p_1, p_2, \ldots$.
    - A zero is a value that results in a zero value of the function.
    - A pole is a value for which the function is infinite.

Example 01…

- For the following *RC* circuit, obtain the transfer function $\mathbf{V}_o / \mathbf{V}_s$ and its frequency response
  - Let $v_s = V_m \cos \omega t$.



- The frequency-domain equivalent of the circuit:



- The transfer function

$$\mathbf{H}(\omega) = \frac{\mathbf{V}_o}{\mathbf{V}_s} = \frac{1/j\omega C}{R + 1/j\omega C} = \frac{1}{1 + j\omega RC}$$

…Example 01…

- The magnitude and phase of $\mathbf{H}(\omega)$

$$H = \frac{1}{\sqrt{1 + (\omega/\omega_0)^2}}, \qquad \phi = -\tan^{-1}\frac{\omega}{\omega_0}$$

where $\omega_0 = 1 / RC$

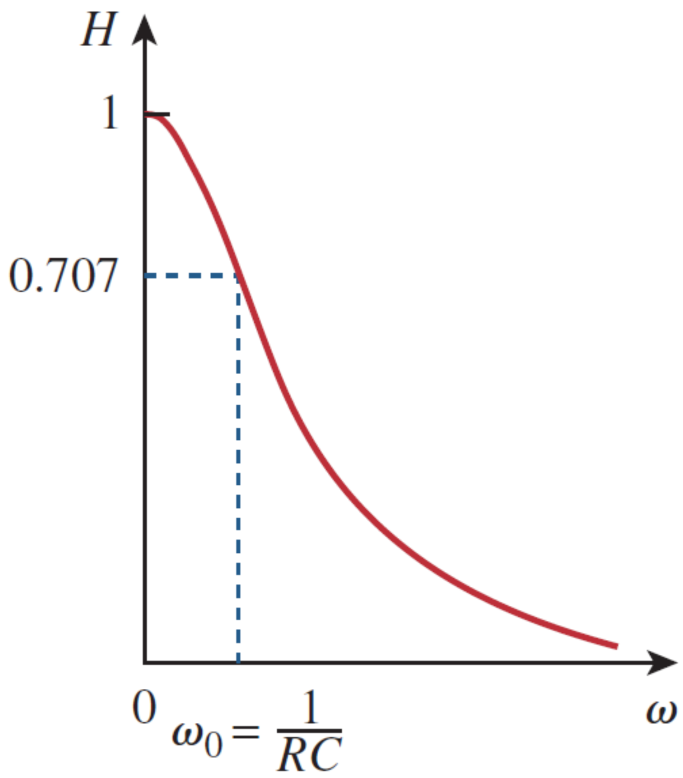
- To plot H and ϕ for $0 < \omega < \infty$ we obtain their values at some critical points and then sketch.

ω/ω_0	H	ϕ	ω/ω_0	H	ϕ
0	1	0	10	0.1	$-84°$
1	0.71	$-45°$	20	0.05	$-87°$
2	0.45	$-63°$	100	0.01	$-89°$
3	0.32	$-72°$	∞	0	$-90°$

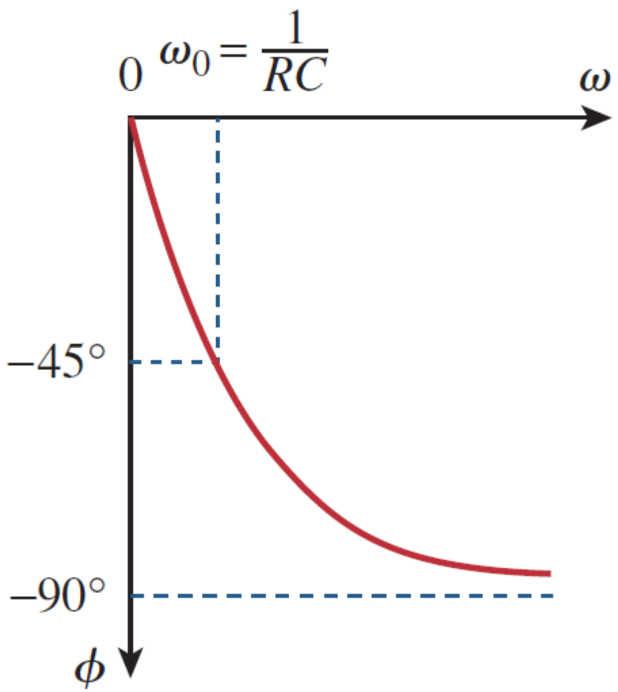
…Example 01

- Frequency response of the *RC* circuit:

amplitude response

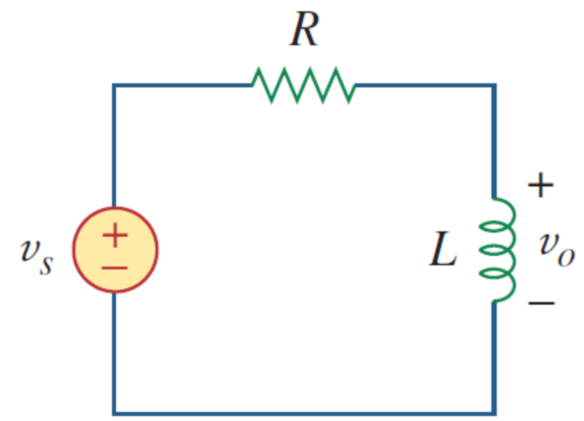


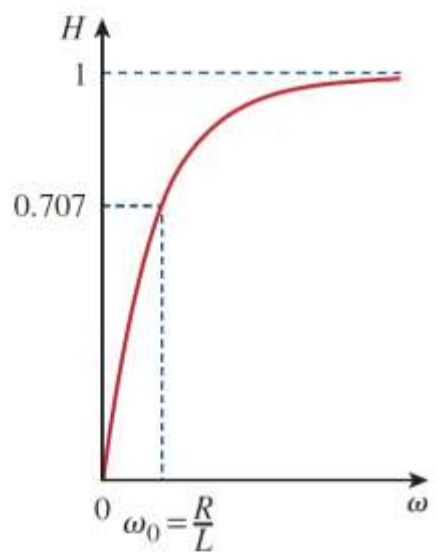
phase response

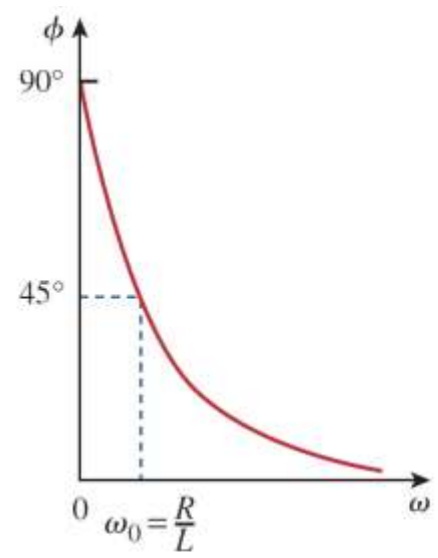


Example 02

- For the following *RL* circuit, obtain the transfer function $\mathbf{V}_o / \mathbf{V}_s$ and its frequency response.
  - Let $v_s = V_m \cos \omega t$.
- Answer: $j\omega L/(R + j\omega L)$

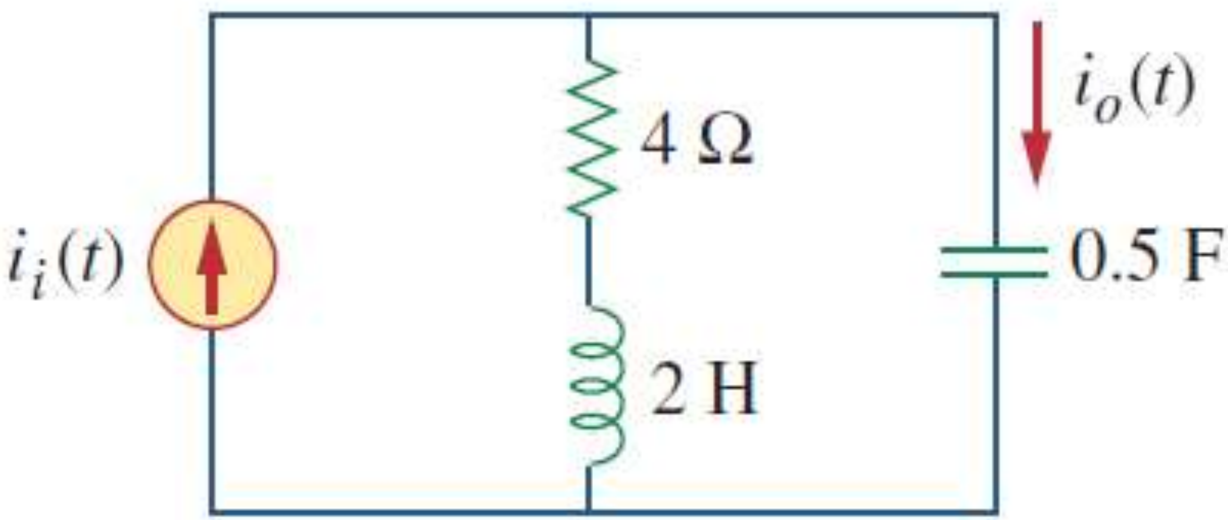






Example 03..

- For the following *RLC* circuit, obtain the current gain $\mathbf{I}_o(\omega)$ / $\mathbf{I}_i(\omega)$ and its poles and zeros.



…Example 03

By current division,

$$\mathbf{I}_o(\omega) = \frac{4 + j2\omega}{4 + j2\omega + 1/j0.5\omega}\mathbf{I}_i(\omega)$$

or

$$\frac{\mathbf{I}_o(\omega)}{\mathbf{I}_i(\omega)} = \frac{j0.5\omega(4 + j2\omega)}{1 + j2\omega + (j\omega)^2} = \frac{s(s + 2)}{s^2 + 2s + 1}, \qquad s = j\omega$$

The zeros are at

$$s(s + 2) = 0 \qquad \Rightarrow \qquad z_1 = 0, z_2 = -2$$

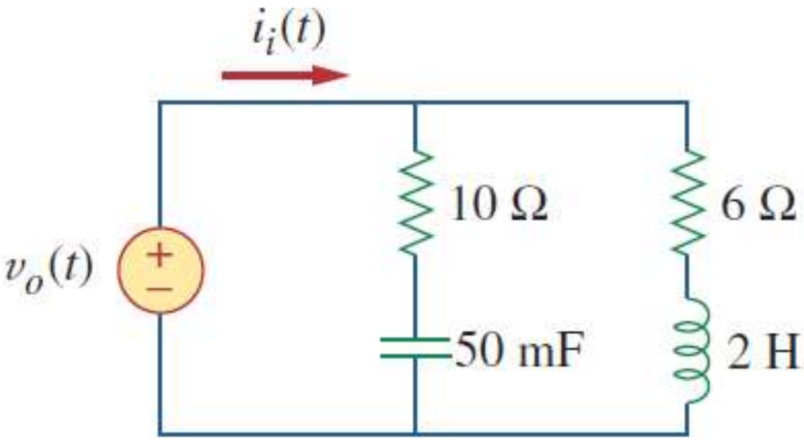
The poles are at

$$s^2 + 2s + 1 = (s + 1)^2 = 0$$

Thus, there is a repeated pole (or double pole) at $p = -1$.

Example 04

- For the following *RLC* circuit, obtain the transfer function $\mathbf{V}_o(\omega)$ / $\mathbf{I}_i(\omega)$ and its zeros and poles.

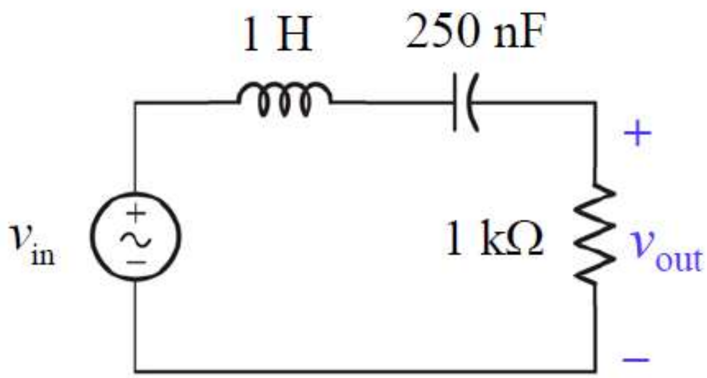


- Answer: $$\frac{10(s+2)(s+3)}{s^2+8s+10}, \; s = j\omega;$$

$$\text{zeros: } -2, -3; \text{ poles: } -1.5505, -6.449.$$

Example 05…

- Plot the amplitude and phase of the voltage transfer funciton for this circuit for
  100 Hz < f < 1 kHz (linear)



- Answer:

```
L = 1;
C = 250e-9;
R = 1000;

f = linspace(100,1000,5e2);
omega = 2*pi*f;

Z_L = j*omega*L;
Z_C = -j./(omega*C);

H = R ./ (R + Z_C + Z_L);
```

```
>> abs(H(1))

   ans =

                                   0.1717

>> angle(H(1))*180/pi

   ans =

                                  80.1138


>> abs(H(length(H)))

   ans =

                                   0.1744

>> angle(H(length(H))) * 180/pi

   ans =

                                 -79.9571
```

$$\mathbf{H}_v(100) = 0.17\angle 80.1^\circ$$
$$\mathbf{H}_v(500) = 0.47\angle -62.4^\circ$$
$$\mathbf{H}_v(1000) = 0.17\angle -80.0^\circ$$

…Example 05…

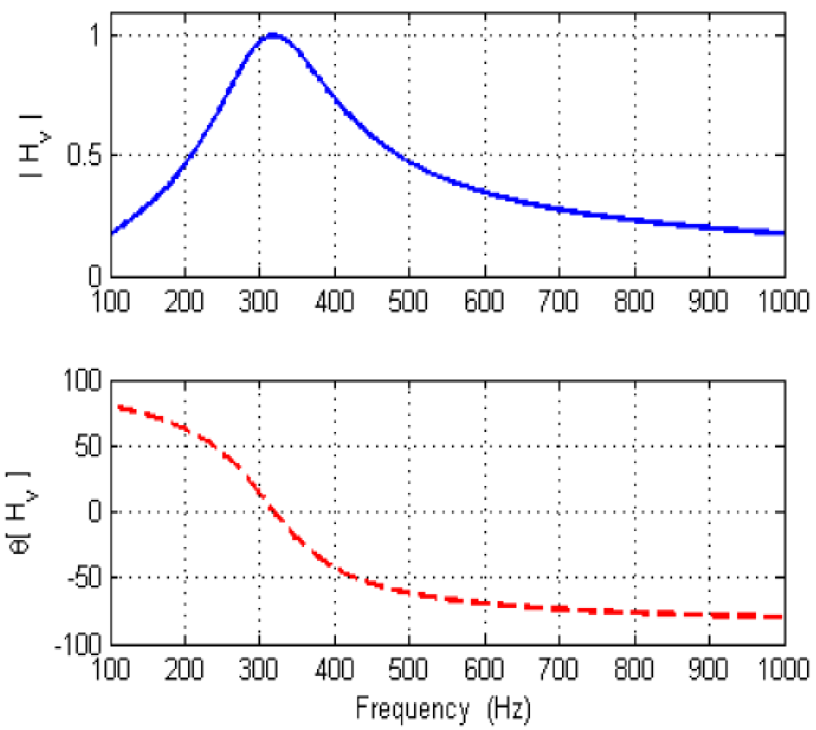
```
L = 1;
C = 250e-9;
R = 1000;

f = linspace(100,1000,5e2);
omega = 2*pi*f;

Z_L = j*omega*L;
Z_C = -j./(omega*C);

H = R ./ (R + Z_C + Z_L);
```

```
figure(1)
subplot(2,1,1)
plot(f,abs(H),'b-','LineWidth',2)
ylabel('| H_v |')
axis([-Inf Inf 0 1.1])
grid
subplot(2,1,2)
plot(f,angle(H)*180/pi,'r--','LineWidth',2)
ylabel('\theta[ H_v ]')
axis([-Inf Inf -100 100])
xlabel('Frequency  (Hz)')
grid
```



…Example 05…

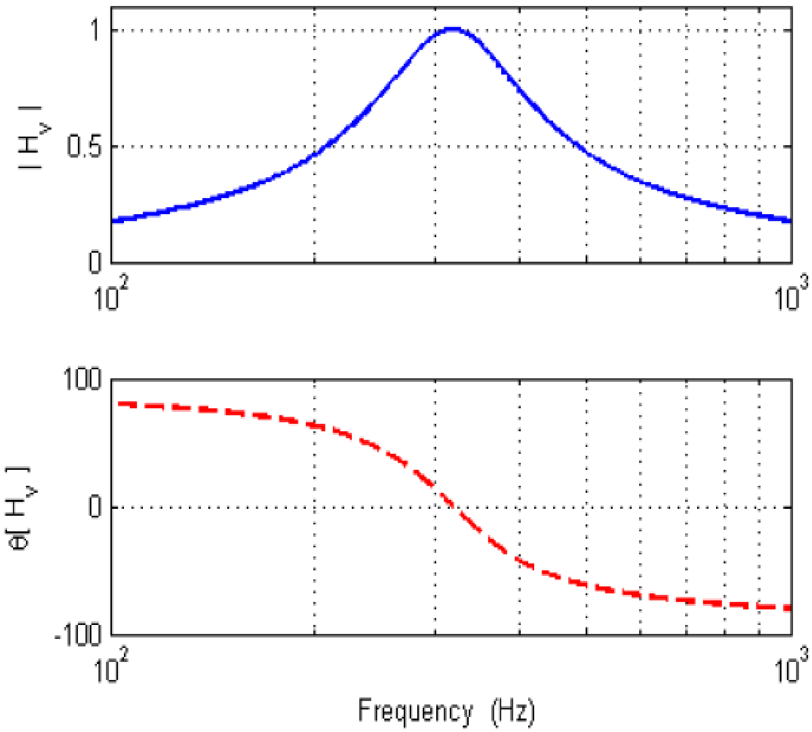
```
L = 1;
C = 250e-9;
R = 1000;

f = logspace(2,3,5e2);
omega = 2*pi*f;

Z_L = j*omega*L;
Z_C = -j./(omega*C);

H = R ./ (R + Z_C + Z_L);
```

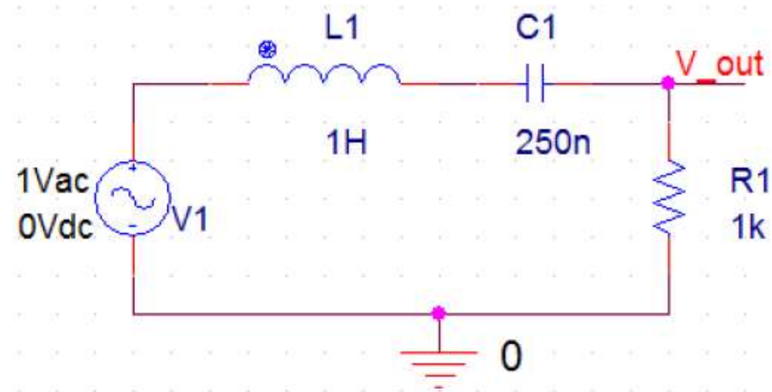
```
figure(1)
subplot(2,1,1)
semilogx(f,abs(H),'b-','LineWidth',2)
ylabel('| H_v |')
axis([1e2 1e3 0 1.1])
grid
subplot(2,1,2)
semilogx(f,angle(H)*180/pi,'r--','LineWidth',2)
ylabel('\theta[ H_v ]')
axis([1e2 1e3 -100 100])
xlabel('Frequency  (Hz)')
grid
```

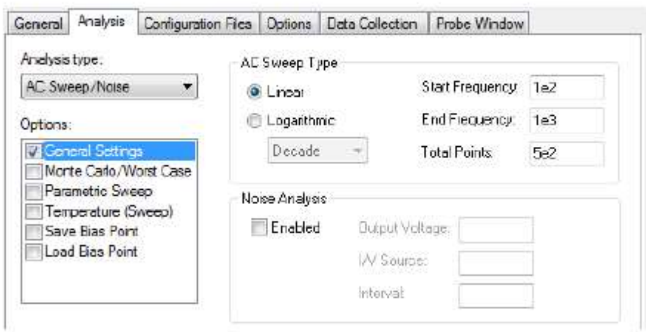


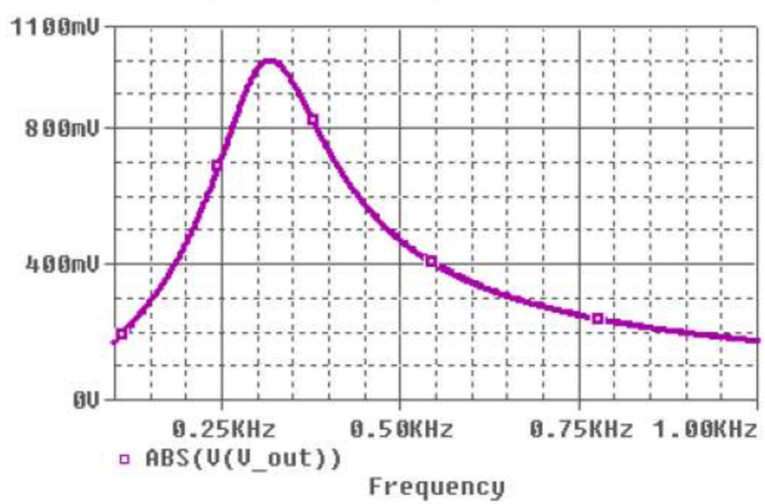
…Example 05…

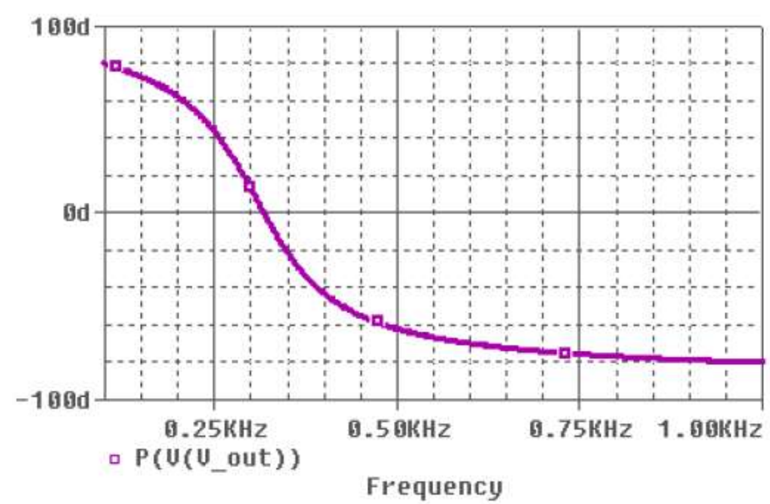
Plot the amplitude and phase of the voltage transfer function for this circuit for

$$100\text{ Hz} < f < 1\text{ kHz} \quad \text{(linear)}$$





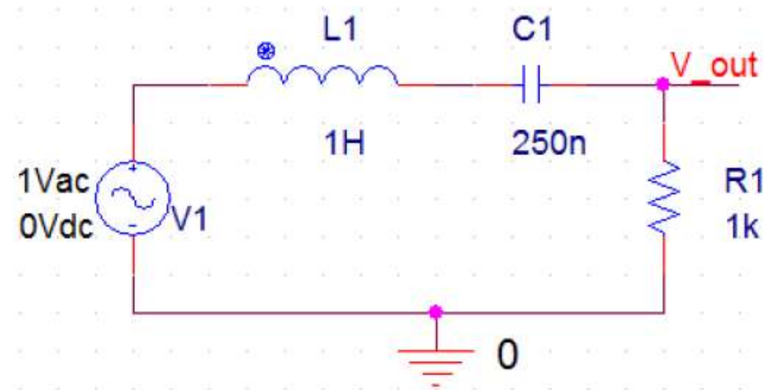


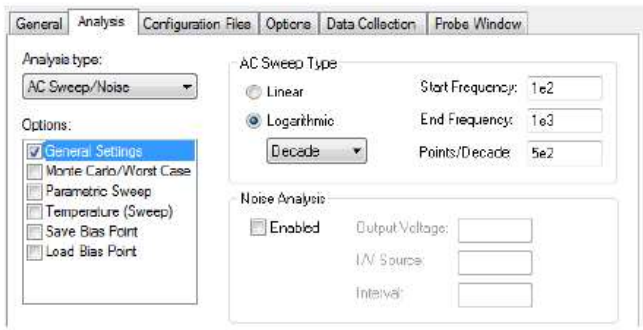


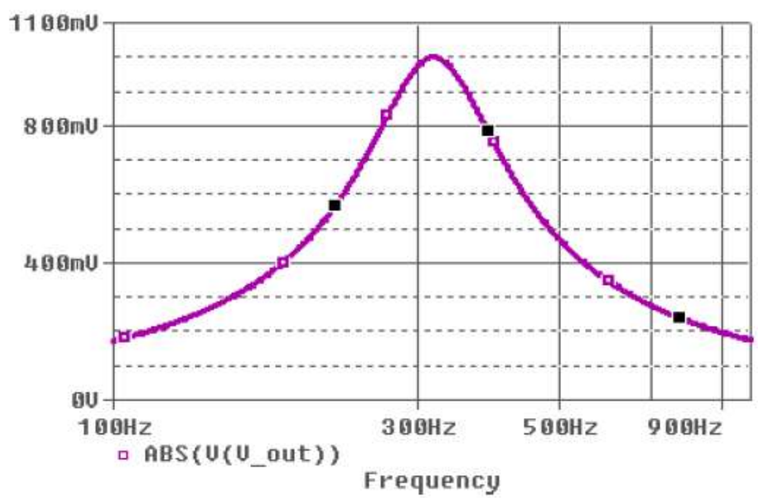
…Example 05

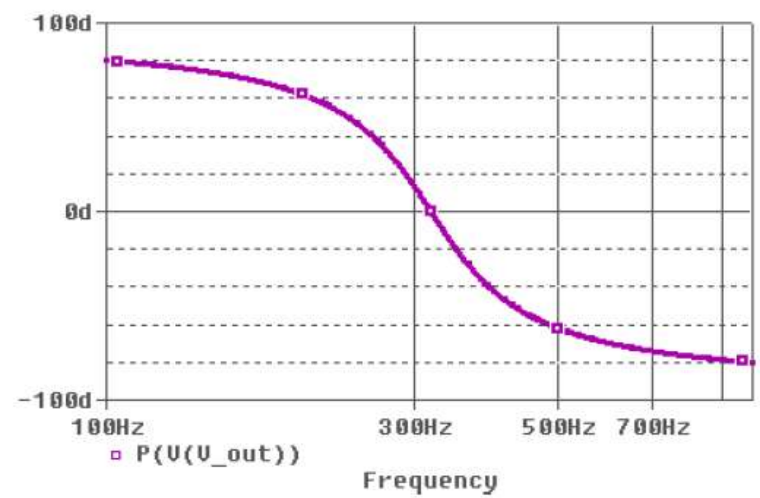
Plot the amplitude and phase of the voltage transfer function for this circuit for

$$100 \text{ Hz} < f < 1 \text{ kHz} \quad \text{(logarithmic)}$$









Example 06…

Consider the circuit shown in Figure 13.2-4*a*. The input to the circuit is the voltage of the voltage source $v_i(t)$. The output is the voltage $v_o(t)$ across the series connection of the capacitor and the 16-kΩ resistor. The network function that represents this circuit has the form

$$\mathbf{H}(\omega) = \frac{\mathbf{V}_o(\omega)}{\mathbf{V}_i(\omega)} = \frac{1 + j\dfrac{\omega}{z}}{1 + j\dfrac{\omega}{p}} \tag{13.2-4}$$

The network function depends on two parameters, z and p. The parameter z is called the zero of the circuit and the parameter p is called the pole of the circuit. Determine the values of z and of p for the circuit in Figure 13.2-4*a*.

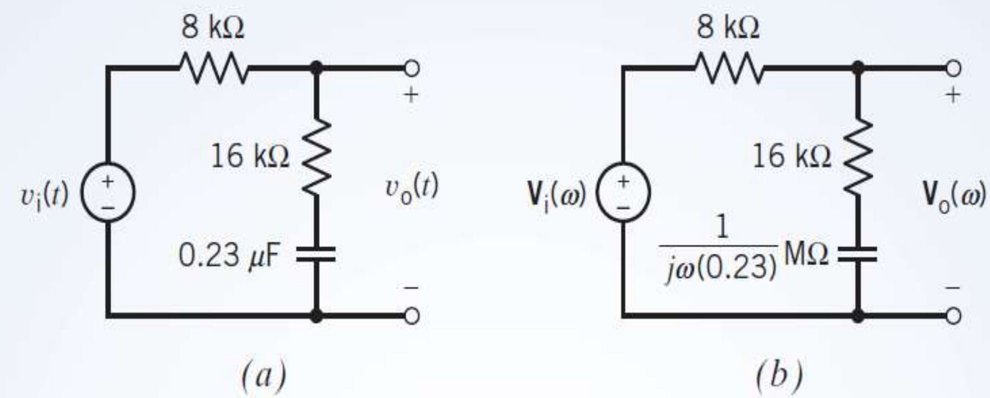


FIGURE 13.2-4 The circuit considered in Example 13.2-1 represented (*a*) in the time domain and (*b*) in the frequency domain.

…Example 06…

We will analyze the circuit to determine its network function and then put the network function into the form given in Eq. 13.2-4. A network function is the ratio of the output phasor to the input phasor. Phasors exist in the frequency domain. Consequently, our first step is to represent the circuit in the frequency domain, using phasors and impedances. Figure 13.2-4*b* shows the frequency-domain representation of the circuit from Figure 13.2-4*a*.

The impedances of the capacitor and the 16-kΩ resistor are connected in series in Figure 13.2-4*b*. The equivalent impedance is

$$\boldsymbol{Z}_{\mathrm{e}}(\omega) = 16,000 + \frac{10^6}{j(0.23)\omega}$$

The equivalent impedance is connected in series with the 8-kΩ resistor. $\mathbf{V}_{\mathrm{i}}(\omega)$ is the voltage across the series impedances, and $\mathbf{V}_{\mathrm{o}}(\omega)$ is the voltage across the equivalent impedance $\mathbf{Z}_{\mathrm{e}}(\omega)$. Apply the voltage division principle to get

$$\mathbf{V}_{\mathrm{o}}(\omega) = \frac{16,000 + \dfrac{10^6}{j(0.23)\omega}}{8000 + 16,000 + \dfrac{10^6}{j(0.23)\omega}}\mathbf{V}_{\mathrm{i}}(\omega) = \frac{10^6 + j(0.23)\omega(16,000)}{10^6 + j(0.23)\omega(24,000)}\mathbf{V}_{\mathrm{i}}(\omega)$$

$$= \frac{\dfrac{10^6 + j(3680)\omega}{10^6}}{\dfrac{10^6 + j(5520)\omega}{10^6}}\mathbf{V}_{\mathrm{i}}(\omega) = \frac{1 + j(0.00368)\omega}{1 + j(0.00552)\omega}\mathbf{V}_{\mathrm{i}}(\omega)$$

…Example 06

Divide both sides of this equation by $\mathbf{V}_\mathrm{i}(\omega)$ to obtain the network function of the circuit

$$\mathbf{H}(\omega) = \frac{\mathbf{V}_\mathrm{o}(\omega)}{\mathbf{V}_\mathrm{i}(\omega)} = \frac{1 + j(0.00368)\omega}{1 + j(0.00552)\omega}$$

Equating the network functions given by Eq. 13.2-4 and 13.2-5 gives

$$\frac{1 + j(0.00368)\omega}{1 + j(0.00552)\omega} = \frac{1 + j\dfrac{\omega}{z}}{1 + j\dfrac{\omega}{p}}$$

Comparing these network functions gives

$$z = \frac{1}{0.00368} = 271.74 \text{ rad/s} \quad \text{and} \quad p = \frac{1}{0.00552} = 181.16 \text{ rad/s}$$

Example 07…

Consider the circuit shown in Figure 13.2-5*a*. The input to the circuit is the voltage of the voltage source $v_i(t)$. The output is the voltage $v_o(t)$ across the series connection of the inductor and the 2-Ω resistor. The network function that represents this circuit is

$$\mathbf{H}(\omega) = \frac{\mathbf{V}_o(\omega)}{\mathbf{V}_i(\omega)} = 0.2\frac{1 + j\frac{\omega}{5}}{1 + j\frac{\omega}{25}} \tag{13.2-6}$$

Determine the value of the inductance L.

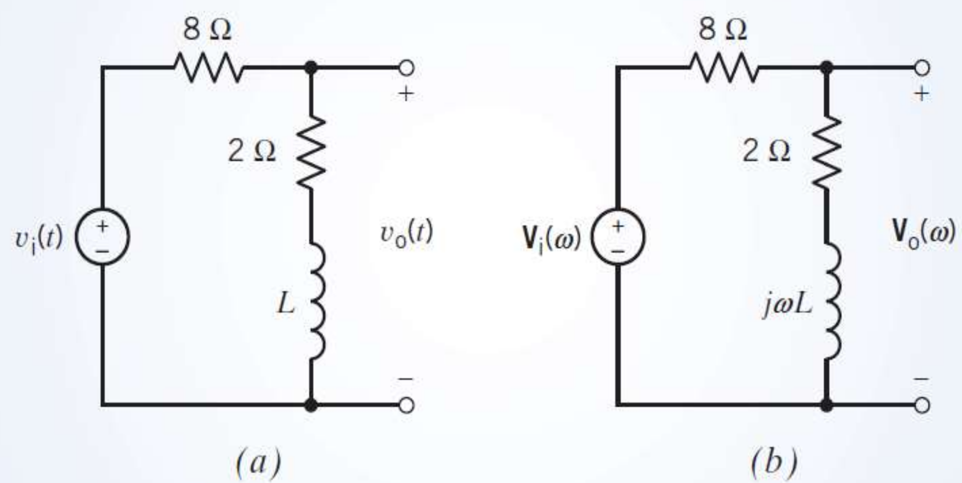


FIGURE 13.2-5 The circuit considered in Example 13.2-2 represented (*a*) in the time domain and (*b*) in the frequency domain.

…Example 07…

The circuit has been represented twice, by a circuit diagram and by a network function. The unknown inductance L appears in the circuit diagram but not in the given network function. We can analyze the circuit to determine its network function. This second network function will depend on the unknown inductance. We will determine the value of the inductance by equating the two network functions.

A network function is the ratio of the output phasor to the input phasor. Phasors exist in the frequency domain. Consequently, our first step is to represent the circuit in the frequency domain, using phasors and impedances. Figure 13.2-5b shows the frequency-domain representation of the circuit from Figure 13.2-5a.

The impedances of the inductor and the 2-Ω resistor are connected in series in Figure 13.2-5b. The equivalent impedance is

$$\mathbf{Z}_{\mathrm{e}}(\omega) = 2 + j\omega L$$

The equivalent impedance is connected in series with the 8-Ω resistor. $\mathbf{V}_{\mathrm{i}}(\omega)$ is the voltage across the series impedances, and $\mathbf{V}_{\mathrm{o}}(\omega)$ is the voltage across the equivalent impedance $\mathbf{Z}_{\mathrm{e}}(\omega)$. Apply the voltage division principle to get

$$\mathbf{V}_{\mathrm{o}}(\omega) = \frac{2 + j\omega L}{8 + 2 + j\omega L}\mathbf{V}_{\mathrm{i}}(\omega) = \frac{2 + j\omega L}{10 + j\omega L}\mathbf{V}_{\mathrm{i}}(\omega)$$

…Example 07

Next, we put the network function into the form specified by Eq. 13.2-6. Factoring 2 out of both terms in the numerator and factoring 10 out of both terms in the denominator, we get

$$\mathbf{H}(\omega) = \frac{2\left(1 + j\omega\frac{L}{2}\right)}{10\left(1 + j\omega\frac{L}{10}\right)} = 0.2\frac{1 + j\omega\frac{L}{2}}{1 + j\omega\frac{L}{10}} \tag{13.2-7}$$

Equating the network functions given by Eqs. 13.2-6 and 13.2-7 gives

$$0.2\frac{1 + j\omega\frac{L}{2}}{1 + j\omega\frac{L}{10}} = 0.2\frac{1 + j\frac{\omega}{5}}{1 + j\frac{\omega}{25}}$$

Comparing these network functions gives

$$\frac{L}{2} = \frac{1}{5} \quad \text{and} \quad \frac{L}{10} = \frac{1}{25}$$

The values of L obtained from these equations must agree, and they do. (If they do not, we've made an error.) Solving each of these equations gives $L = 0.4\,\text{H}$.

The Decibel Scale

- A more systematic way of obtaining the frequency response is to use Bode plots.
  - Two important issues in Bode plots :
    - The use of logarithms and decibels in expressing gain.
- Since Bode plots are based on logarithms, it is important that we keep the following properties of logarithms in mind:
  1. $\log P_1 P_2 = \log P_1 + \log P_2$
  2. $\log P_1/P_2 = \log P_1 - \log P_2$
  3. $\log P^n = n \log P$
  4. $\log 1 = 0$

Logarithms

$$N = b^x \quad \Leftrightarrow \quad \boxed{\log_b N = x}$$

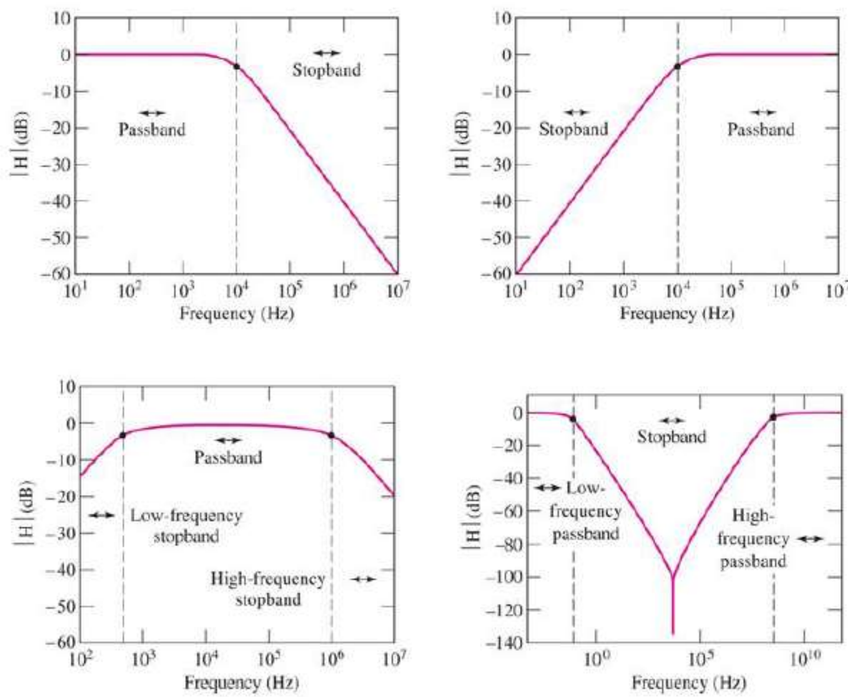
where N = positive number ("linear value")
b = the **base** of the logarithm
x = the **exponent** of the logarithm

-- a way to easily write/compare numbers that are very large and/or very small, simultaneously

-- an alternative to scientific notation

using b = 10 ("base-10")…

N	$\Leftrightarrow$	x
0.000001	$\Leftrightarrow$	–6
0.001	$\Leftrightarrow$	–3
1	$\Leftrightarrow$	0
1,000	$\Leftrightarrow$	3
1,000,000	$\Leftrightarrow$	6



Logarithms

$$N = b^x \quad \Leftrightarrow \quad \boxed{\log_b N = x}$$

where N = positive number ("linear value")
b = the **base** of the logarithm
x = the **exponent** of the logarithm

$b = 2$ ("base-2")			$b = e \approx 2.718$ ("base-e")			$b = 10$ ("base-10")			$b = 16$ ("base-16")		
N	$\Leftrightarrow$	x	N	$\Leftrightarrow$	x	N	$\Leftrightarrow$	x	N	$\Leftrightarrow$	x
2	$\Leftrightarrow$	1	$e^{-5} \approx 1\%$	$\Leftrightarrow$	-5	$10^{-7.5}$	$\Leftrightarrow$	-7.5	16	$\Leftrightarrow$	1
4	$\Leftrightarrow$	2	$e^{-3} \approx 5\%$	$\Leftrightarrow$	-3	$10^{-5.0}$	$\Leftrightarrow$	-5.0	256	$\Leftrightarrow$	2
8	$\Leftrightarrow$	3	$e^{-1} \approx 37\%$	$\Leftrightarrow$	-1	0.5	$\Leftrightarrow$	-0.3	4096	$\Leftrightarrow$	3
16	$\Leftrightarrow$	4	1	$\Leftrightarrow$	0	1	$\Leftrightarrow$	0	64K	$\Leftrightarrow$	4
32	$\Leftrightarrow$	5	e	$\Leftrightarrow$	1	2	$\Leftrightarrow$	0.3	1M	$\Leftrightarrow$	5
64	$\Leftrightarrow$	6	$e^3 \approx 20$	$\Leftrightarrow$	3	$10^{5.0}$	$\Leftrightarrow$	5.0	16M	$\Leftrightarrow$	6
128	$\Leftrightarrow$	7	$e^5 \approx 150$	$\Leftrightarrow$	5	$10^{7.5}$	$\Leftrightarrow$	7.5	256M	$\Leftrightarrow$	7

Logarithms (Arithmetic)

$$N = b^x \quad \Leftrightarrow \quad \log_b N = x$$

where N = positive number ("linear value")
b = the **base** of the logarithm
x = the **exponent** of the logarithm

$$\log_b(x \cdot y) = \log_b x + \log_b y$$

$$\log_b(x/y) = \log_b x - \log_b y$$

$$\log_b(x) = \frac{\log_n(x)}{\log_n(b)}$$

Examples:

$$\log_2(4 \cdot 16) =$$

$$\log_3(81/9) =$$

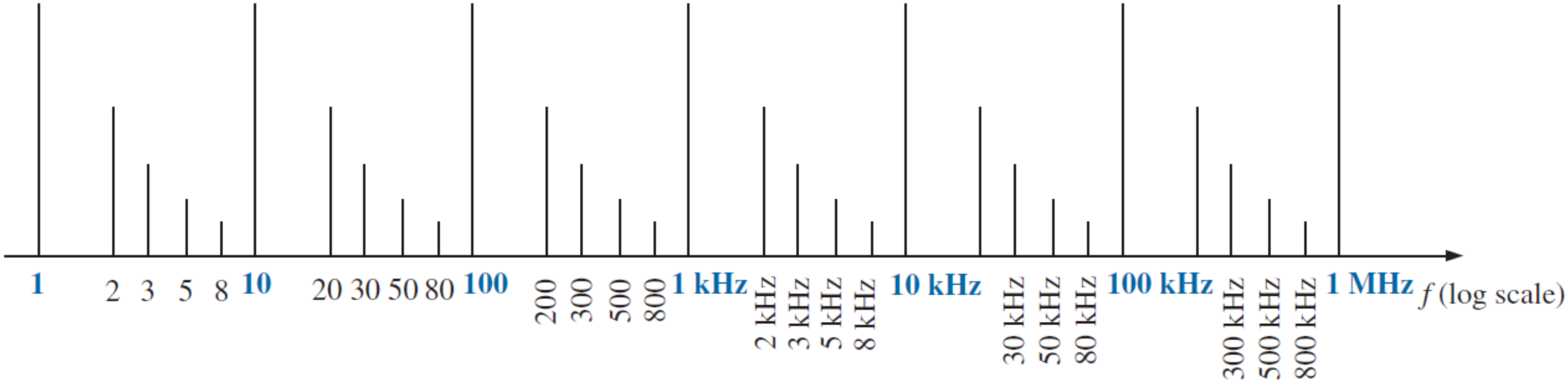
$$10^x = 3 \cdot 10^6$$

Some Areas of Application

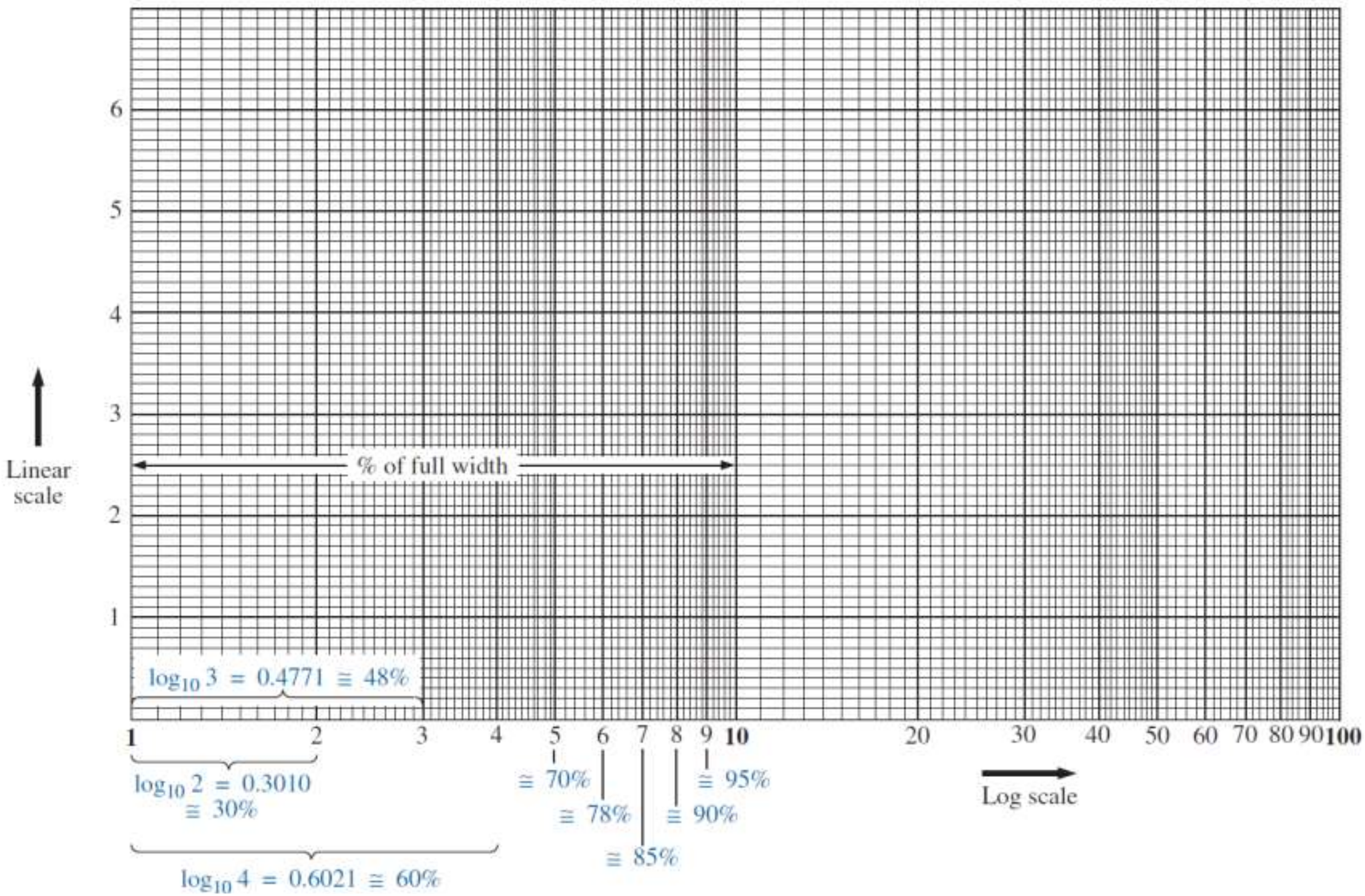
- The response of a system can be plotted for a range of values that may otherwise be impossible or unwieldy with a linear scale.
- Levels of power, voltage, and the like, can be compared without dealing with very large or very small numbers that often cloud the true impact of the difference in magnitudes.
- A number of systems respond to outside stimuli in a nonlinear logarithmic manner.
  - The result is a mathematical model that permits a direct calculation of the response of the system to a particular input signal.
- The response of a cascaded or compound system can be rapidly determined using logarithms if the gain of each stage is known on a logarithmic basis.

Graphs

- Graph paper is available in semilog and log-log varieties.
- Example:
  – Frequency log scale



Graphs



6
5
4
3
2
1
Linear scale
% of full width
log10 3 = 0.4771 ≅ 48%
1
2
3
4
5
6
7
8
9
10
20
30
40
50
60
70
80
90
100
log10 2 = 0.3010
≅ 30%
log10 4 = 0.6021 ≅ 60%
≅ 70%
≅ 78%
≅ 85%
≅ 90%
≅ 95%
Log scale

The Decibel Scale

- In communications systems, gain is measured in bels.
  - The bel is named after Alexander Graham Bell, the inventor of the telephone.
- Historically, the bel is used to measure the ratio of two levels of power or power gain G; that is,

$$G = \text{Number of bels} = \log_{10} \frac{P_2}{P_1}$$

The Decibel Scale

- The decibel (dB) provides us with a unit of less magnitude.
  - It is 1/10th of a bel and is given by

$$G_{dB} = 10log_{10}\frac{P_1}{P_2}$$

    - If $P_2=P_1$ $\qquad G_{dB} = 10log_{10}1 = 0 \text{ dB}$
    - If $P_2=2P_1$ $\qquad G_{dB} = 10log_{10}2 \cong 3 \text{ dB}$
    - If $P_2=0.5P_1$ $\qquad G_{dB} = 10log_{10}1 \cong -3 \text{ dB}$
- The logarithm of the reciprocal of a quantity is simply negative the logarithm of that quantity.

The Decibel Scale

- Alternatively, the gain G can be expressed in terms of voltage or current ratio.
- If P_1 is the input power, P_2 is the output (load) power, R_1 is the input resistance, and R_2 is the load resistance, then $P_1 = 0.5V_1^2/R_1$ and $P_2 = 0.5V_2^2/R_2$

$$G_{\text{dB}} = 10\log_{10}\frac{P_2}{P_1} = 10\log_{10}\frac{V_2^2/R_2}{V_1^2/R_1}$$

$$= 10\log_{10}\left(\frac{V_2}{V_1}\right)^2 + 10\log_{10}\frac{R_1}{R_2}$$

$$G_{\text{dB}} = 20\log_{10}\frac{V_2}{V_1} - 10\log_{10}\frac{R_2}{R_1}$$

when $R_2 = R_1$

$$\boxed{G_{\text{dB}} = 20\log_{10}\frac{V_2}{V_1}}$$

$$G_{\text{dB}} = 20\log_{10}\frac{I_2}{I_1}$$

The Decibel Scale

- Three things are important to note:
  - $10log_{10}$ is used for power, while $20log_{10}$ is used for voltage or current, because of the square relationship between them ($P = V^2/R = I^2R$).
  - The dB value is a logarithmic measurement of the ratio of one variable to another of the same type.
    - Therefore, it applies in expressing the transfer function H in terms of voltage or current gain, which are dimensionless quantities, but not in expressing H in interms of transfer impedance or transfer admitance gain.
  - It is important to note that we only use voltage and current magnitudes.

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The Decibel Scale

- Specific gain and their decibel values.

Magnitude H	**$20 \log_{10} H$ (dB)**
0.001	−60
0.01	−40
0.1	−20
0.5	−6
$1/\sqrt{2}$	−3
1	0
$\sqrt{2}$	3
2	6
10	20
20	26
100	40
1000	60

Human Auditory Response

- One of the most frequent applications of the decibel scale is in the communication and entertainment industries.
- The human ear does not respond in a linear fashion to changes in source power level.
  - A doubling of the audio power level from 1/2 W to 1 W does not result in a doubling of the loudness level for the human ear.
  - In addition, a change from 5 W to 10 W is received by the ear as the same change in sound intensity as experienced from 1/2 W to 1 W.
- The ear responds in a logarithmic fashion to changes in audio power levels.

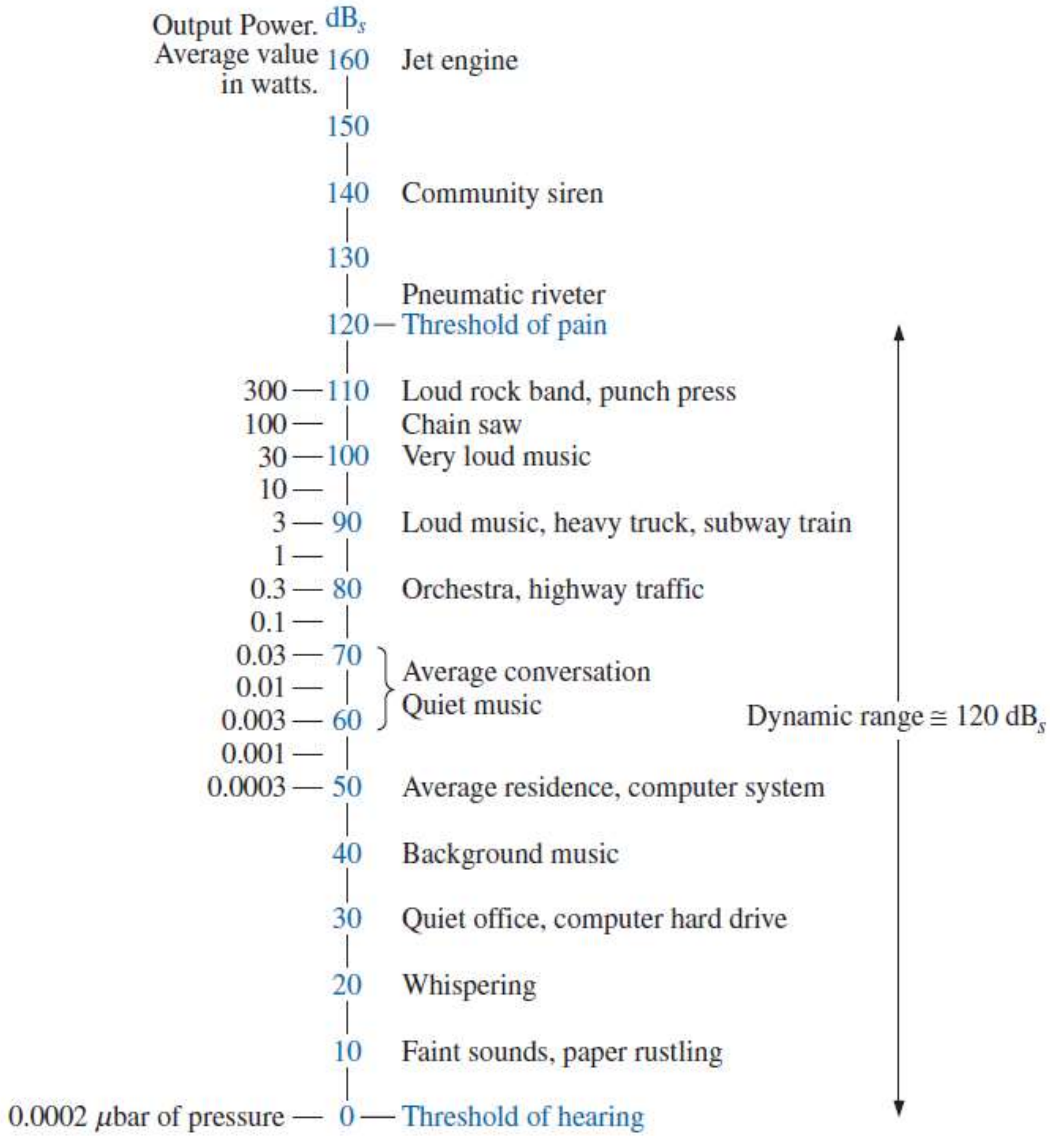
Human Auditory Response

- To establish a basis for comparison between audio levels, a reference level of 0.0002 **microbar** (μbar) was chosen,
  - where 1 μbar is equal to the sound pressure of 1 dyne per square centimeter, or about 1 millionth of the normal atmospheric pressure at sea level.
    - The 0.0002 μbar level is the threshold level of hearing.
- Using this reference level, the sound pressure level in decibels is defined by the following equation:

$$\boxed{\mathrm{dB}_s = 20 \log_{10} \frac{P}{0.0002\ \mu\mathrm{bar}}}$$

where P is the sound pressure in microbars.

Typical sound levels and their decibel levels



- To double the sound level received by the human ear, the power rating of the acoustical source (in watts) must be increased by a factor of 10.