Chapter 1

Introduction

Background:

Mathematics is at the heart of many problems that need to be solved in science and engineering. Applications are myriad and range from weather forecasting, radio-coverage prediction, drug delivery in the body to financial forecasting.

Many mathematical problems can be solved analytically. That is to say that an exact solution can be obtained in terms of the variables associated with the problem.

Examples:

and

Background:

A numerical solution to the problem is performed where the problem cannot be solved analytically or where the number of operations required are too excessive to be done by hand.

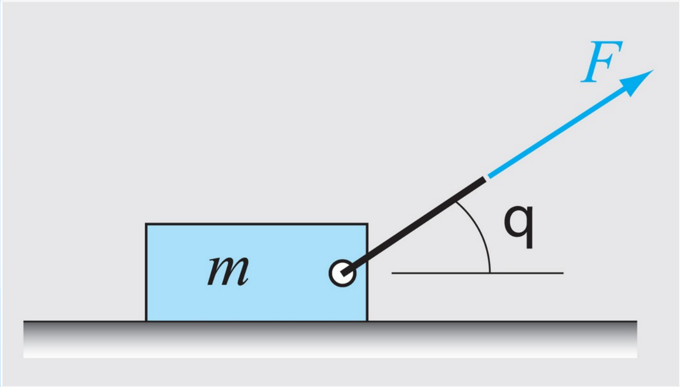
A numerical solution is usually approximate to within a specified interval.

Background:

Consider the following problem:

For a given the angle, , that is required to move the block cannot be solved analytically.

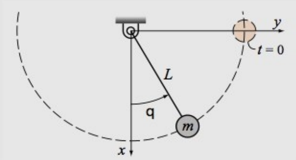
A numerical solution however would allow one to approximately determine for a given .



$$F = \frac{\mu m g}{\cos\theta + \mu \sin\theta} \tag{1.1}$$

Example 1-1: Problem formulation

Consider the following problem statement:
A pendulum of mass m is attached to a rigid rod of length L, as shown in the figure. The pendulum is displaced from the vertical position such that the angle between the rod and the x axis is θ_0, and then the pendulum is released from rest. Formulate the problem for determining the angle θ as a function of time, t, once the pendulum is released. In the formulation include a damping force that is proportional to the velocity of the pendulum.



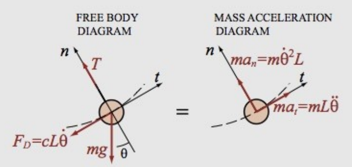
Formulate the solution for two cases:
(*a*) $\theta_0 = 5°$, and (*b*) $\theta_0 = 90°$.

SOLUTION

Physical law

The physical law that is used for solving the problem is Newton's second law of mechanics, according to which, as the pendulum swings back and forth, the sum of the forces that are acting on the mass is equal to the mass times its acceleration.

$$\Sigma\bar{F} = m\bar{a} \tag{1.2}$$



Newton's Second Law

This can be visualized by drawing a free body diagram and a mass acceleration diagram, which are shown on the right. The constant c is the damping coefficient. It should be pointed out that the mass of the rod is neglected in the present solution.

Governing equation

The governing equation is derived by applying Newton's second law in the tangential direction:

$$\Sigma F_t = -cL\frac{d\theta}{dt} - mg\sin\theta = mL\frac{d^2\theta}{dt^2} \tag{1.3}$$

Equation (1.3), which is a second-order, nonlinear, ordinary differential equation, can be written in the form:

$$mL\frac{d^2\theta}{dt^2} + cL\frac{d\theta}{dt} + mg\sin\theta = 0 \tag{1.4}$$

The initial conditions are that when the motion of the pendulum starts ($t = 0$), the pendulum is at angle θ_0 and its velocity is zero (released from rest):

$$\theta(0) = \theta_0 \quad \text{and} \quad \left.\frac{d\theta}{dt}\right|_{t=0} = 0 \tag{1.5}$$

Method of solution

Equation (1.4) is a nonlinear equation and cannot be solved analytically. However, in part (*a*) the initial displacement of the pendulum is $\theta_0 = 5°$, and once the pendulum is released, the angle as the pendulum oscillates will be less than $5°$. For this case, Eq. (1.4) can be linearized by assuming that $\sin\theta \approx \theta$. With this approximation, the equation that has to be solved is linear and can be solved analytically:

$$mL\frac{d^2\theta}{dt^2} + cL\frac{d\theta}{dt} + mg\theta = 0 \tag{1.6}$$

with the initial conditions Eq. (1.5).

In part (*b*), the initial displacement of the pendulum is $\theta_0 = 90°$ and the equation has to be solved numerically. An actual numerical solution for this problem is shown in Example 8-8.