

Perfect Particle Transmission through Duality Defects

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We study wavepackets that propagate across (a) topological interfaces in quantum spin systems exhibiting non-invertible symmetries and (b) duality defects coupling dual theories. We demonstrate that the transmission is always perfect, and that a particle traversing the interface is converted into a nonlocal string-like excitation. We give a systematic way of constructing such a defect by identifying its Hilbert space with the virtual bond dimension of the matrix product operator representing defect lines. Our work both gives an operational meaning to topological interfaces, and provides a lattice analogue of recent results solving the monopole paradox in quantum field theory.

Introduction. Quantum many-body systems often exhibit entirely counterintuitive behavior. One of such examples in quantum field theory is the well-known “monopole paradox” originally proposed by Callan [1, 2]. This paradox arises from a thought experiment: what happens if we throw a charged Weyl fermion at a magnetic monopole? Early studies found that the scattered particle is nothing like a Weyl fermion, possessing different and sometimes fractional quantum numbers. Naively, this suggests that the electron breaks into fractions under the influence of a magnetic monopole, which does not happen in our universe. This led to a long debate about how to interpret this result.

Recent studies have addressed this question [3–6]. They revealed that the magnetic monopole serves as a conformal boundary between incoming and outgoing states shown in Fig. 1(a). This boundary directs the incoming fermion to a twisted sector by attaching a topological string to the scattered particle, which generates the fractional quantum number. This perspective is further supported by the fermion-rotor model [7], which interprets the monopole as a quantum rotor impurity that scatters the fermion as shown in (b). In this case, the Weyl fermion is scattered perfectly by the impurity, but acquires a string and so behaves as a different particle.

While the monopole paradox is a nice example that bridges quantum many-body dynamics and generalized symmetries in quantum field theory, it raises the question of whether this is a specific case. In this Letter, we demonstrate that this phenomenon is ubiquitous in strongly interacting models. We show how to construct more generic and direct examples on a lattice. In these quantum spin chains, incoming particles are perfectly transmitted by the impurity and behave as if the original particle has disappeared from the spectrum. After illustrating this phenomenon with a simple example, we provide an intuitive picture of these seemingly exotic dynamics using the tensor-network framework. From the technical point of view, the results follow from the fact

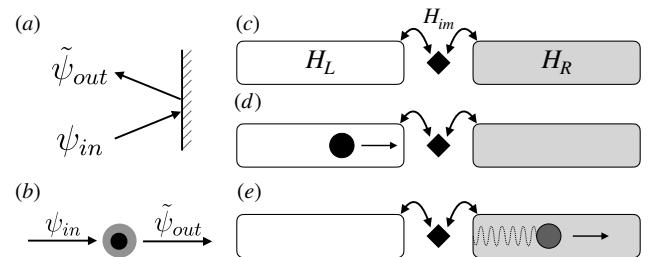


FIG. 1. (a) Chiral fermion scattered by a Dirac monopole. (b) Fermion rotor model. (c) A schematic illustration of the system. Two systems with different Hamiltonians interact through the impurity, denoted by a black diamond. (d) We create a right-moving particle/wave packet in the left medium on top of the many-body ground state. (e) When H_L and H_R are related by duality and separated by a topological impurity, the excitation propagates with perfect transmittance. The outgoing particle on the right appears to be a different particle, but can be described by the same particle with a topological string attached to the impurity.

that both categorical symmetries and duality operators can be represented in the form of matrix product operator (MPO) algebras [8–13], and that those MPOs are sequential unitary quantum circuits [14, 15]; the impurity can then be described utilizing the virtual space of the corresponding MPOs. The fact that particles passing through defect lines in conformal field theories (CFT) are 100% transmitted [16] and can leave a string in their wake was discussed in a series of papers describing CFTs from the point of view of category theory [17–22]. This paper demonstrates that this is also the case for quantum spin chains, and that neither criticality [23] nor integrability [24] is needed to get this effect.

Simple example. A puzzle, in the monopole paradox, is that the Hilbert space of incoming and outgoing state are different in nature. For instance, in a non-anomalous example discussed in Ref. [3, 6], ψ_{in} and ψ_{out} consist of right-moving Weyl fermions with charges 3 and 4, and

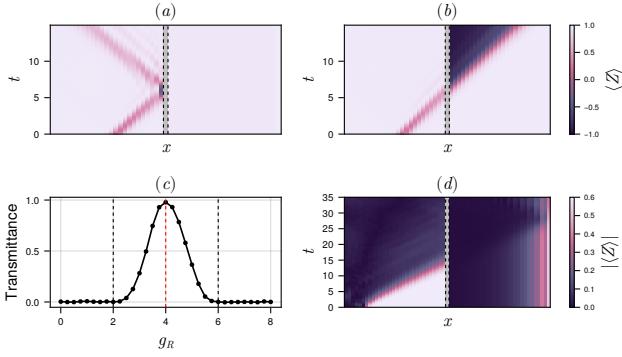


FIG. 2. The time evolution of the local magnetization $\langle Z_x \rangle$ for (a) $(g_L, g_R) = (4.0, 2.0)$ and (b) $(g_L, g_R) = (4.0, 4.0)$ with $(L, x_0, k) = (50, 15, 0.7\pi)$. The impurity site is represented by a gray shaded strip. (c) Transmittance rate of the wave packet. The scattering is purely reflective for $g_R < 2$ and $g_R > 6$ as can be seen from the exact solution.^a(d) Perfect transmission of a domain wall of a ferromagnet to a Haldane chain. The video versions are available in Supplemental Material and https://github.com/dartsushi/Video_scattering.

^a The single particle excitation spectrum is given by $\epsilon_k = 2\sqrt{1 + g^2 - 2\cos(k)}$. This means that there is no overlap of energy spectrum when $|g_L - g_R| > 2$.

left-moving Weyl fermions with charges 5 and 0. This means that even if a charge 3 particle is sent to the monopole, the naive charge 3 fermion is *prohibited* in the scattered state.

Similar situations arise in condensed matter physics. For simplification, we consider two quantum spin chains described by Hamiltonians H_L and H_R , coupled through an impurity H_{im} . The full Hamiltonian H is given by

$$H = H_L + H_R + H_{im}. \quad (1)$$

This model is schematically shown in Fig. 1 (c). As with the fermion-rotor model, we study the scattering problem by creating a wave packet of a quasiparticle on top of the many-body ground state. We utilize the matrix-product-state framework [25, 26] to create a Gaussian wavepacket [27–29] of quasi-particle excitations, and then evolve it using the time-dependent variational principle [30]. Generally speaking, H_L and H_R can be completely different Hamiltonians, and there is no guarantee that the particle can pass through the impurity. In fact, the original excitation on the left may correspond to a high-energy excitation on the right, making it energetically prohibited. However, when the spectrum of the two quantum spin systems match, as is the case for dual systems, an appropriately chosen junction allows the particle to pass completely through the impurity by changing its form. In the following, we see that the scattered particle is indeed the original wave packet dressed with a topological line as illustrated in Fig. 1(d-e).

To elaborate, let us consider the following model:

$$\begin{aligned} H_L &= -\sum_{i=1}^{I-2} X_i X_{i+1} - g_L \sum_{i=1}^{I-1} Z_i, \\ H_R &= -\sum_{i=I+1}^{L-1} X_i - g_R \sum_{i=I+1}^{L-1} Z_i Z_{i+1}, \\ H_{im} &= -X_{I-1} X_I - g_R Z_I Z_{I+1}, \end{aligned}$$

where X and Z are the Pauli matrices and the impurity sits at the site I out of the length L chain. The left (right) chain represents a transverse-field Ising model with ferromagnetic XX (ZZ) couplings and a transverse field along Z (X), joined by a ferromagnetic XX and ZZ link at the interface. The ground states of H_L and H_R are symmetric and symmetry-breaking in the X/Z direction when the coupling parameter $g_{L,R}$ becomes larger than one. In particular, we set $g_L = 4$ to ensure that the system on the left becomes a polarized state in the Z -direction.

We next introduce a wave packet of a Z spin-flip excitation on the left chain, defined as [27]

$$\hat{W} = \sum_{j \in H_L} e^{-\frac{(x-x_0)^2}{2\sigma^2}} e^{ikj} X_j,$$

where x_0 , σ , and k set the packet's center, width, and mean momentum. The initial state is constructed by applying this operator to the nontrivial ground state of H :

$$|\psi(t=0)\rangle = \hat{W}|\psi_{GS}\rangle.$$

The system evolves according to the original Hamiltonian in Eq. (1), which drives the packet toward the interface. When the right chain is tuned to $g_R > g_c = 1$, it resides in the Z -ordered (ferromagnetic) phase, while the left chain with $g_L = 4$ is deep in the field-polarized (disordered) phase. Consequently, the interface connects two distinct gapped phases. Naively, one would expect a propagating spin-flip excitation from the left to be reflected, as a single-spin flip excitation represents a high-energy excitation in the symmetry-breaking phase of H_R .

This is indeed the case for $g_R = 2$ as shown in Fig. 2 (a). The wave packet, illustrated with pink, perfectly reflects at the interface and bounces back to the left edge. On the other hand, the outcome is different when $g_R = g_L$ as shown in Fig. 2(b): The wave packet passes through the impurity with perfect transmittance. The scattered state, however, is no longer a spin-flip but a domain wall. It is a low-energy excitation in the symmetry-breaking phase, but differs from the original spin-flip wave packet before scattering. As we will see, this domain-wall excitation can be understood as the spin-flip excitation dressed with a topological string $\prod_{i=1}^{x(t)} X_i^R$ with $x(t)$ being the center of the wave packet at time t . This behavior is exactly that described in Fig. 1, raising the question: What is special about this point $g_L = g_R$?

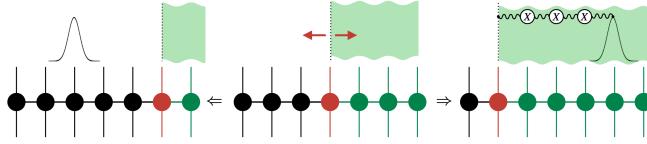


FIG. 3. Equivalence between perfect transmittance and topological defects. Moving the topological impurity is a unitary transformation and does not alter the Hamiltonian spectrum.

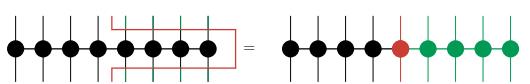
Perfect transmission and Topological defects. A short answer to this question is the following: perfect transmission occurs when two mutually dual Hamiltonian H_L and H_R are separated by a topological “duality” defect. In the Ising example above, H_L and H_R are related by the Kramers-Wannier (KW) transformation when $g_L = g_R$ [31, 32], and the corresponding duality defect has long been known [33, 34]. A convenient way to write is in terms of a unitary quantum circuit/MPO [12, 14] as

$$U_{KW} = \begin{array}{c} \oplus \quad \cdot \quad \oplus \quad \cdot \quad \oplus \quad \cdot \\ | \quad | \quad | \quad | \quad | \quad | \\ \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \end{array},$$

where \oplus and the black dot represent the XOR gate and the Greenberger-Horne-Zeilinger (GHZ) state [35], respectively. Since $U_{KW}^\dagger U_{KW} = I$, as we lifted the virtual degree of freedom to a physical one, the application of these transformations does not alter the energy spectrum. More importantly, its application to the subsystem also does not change the physics. The Hamiltonian (1) with perfect transmittance can be constructed as

$$H = U_{KW}^\dagger (\tilde{H} \otimes \mathbb{1}_{L+1}) U_{KW}, \quad (2)$$

where \tilde{H} is the original uniform Hamiltonian of length L and U_{KW} acts only on the right half of the chain. This can be graphically represented as follows:



The explicit construction of the red and green MPO for a local Hamiltonian basis h_L , h_R , and h_{im} can be done by applying the unitary transformation on the local terms as

$$\begin{array}{ccc} h_R & = & h_L \\ \boxed{\text{---}} & & \boxed{\text{---}} \end{array}, \quad \begin{array}{ccc} h_{im} & = & h_L \\ \boxed{\text{---}} & & \boxed{\text{---}} \end{array}.$$

It is now clear that the impurity site, marked by the red line, originates from the virtual bond of the MPO.

Although changing the length of the unitary N_R from the left boundary alters the position of the impurity site, the energy spectrum of the full Hamiltonian remains unchanged, up to a doubled degeneracy for all energy levels originating from the boundary term $\mathbb{1}_{L+1}$. This invariance allows the topological impurity to move freely from right to left as shown in Fig. 3. This movement is similar to drawing a curtain. When the wave packet propagates to the right, it acts as if no impurity were present, and passing through the impurity resembles hiding behind the green curtain. The topological X string from the interface is just that of the duality MPO. The doubled degeneracy implies that no energy will get stuck on the interface when the wavepacket alters its degrees of freedom.

From this perspective, we can observe the equivalence between perfect transmission and duality. If *any* particle from the left can perfectly pass through the impurity, there must be a duality transformation represented by a specific MPO which leaves the spectrum invariant. The impurity degrees of freedom are then identical to the *gauge* degrees of freedom that exist in the virtual space of this symmetry. More generally, the perfect transmittance of particles in a particular sector, such as the low-energy sector, indicates a non-trivial duality between H_L and H_R . From experimental perspective, it is interesting to observe the breaking of the string by measuring the impurity site.

The transformation Eq. (2) is not restricted to free-fermion models such as Ising, or even integrable models. We apply it here to the spin-1 Heisenberg chain, which realizes the gapped Haldane phase [36, 37]. While this phase is now known as a canonical example of symmetry-protected topological (SPT) order [38–40], it is dual to a usual ferromagnet through Kennedy-Tasaki(KT) transformation [41, 42]. This transformation can be represented with a bond dimension 2 MPO as $T_{ii\alpha\beta} = \tilde{\sigma}_{\alpha\beta}^i$ with $\tilde{\sigma}^i = (\sigma^x, i\sigma^y, \sigma^z)$, where the first and the last two indices act on the physical and virtual space. The resulting Hamiltonian coupling the systems is

$$\begin{aligned} H_L &= - \sum_{i=1}^{I-1} (S_i \cdot S_{i+1}), \\ H_R &= \sum_{i=I+1}^{L-1} (-S_i^x S_{i+1}^x + e^{i\pi S_i^x} S_i^y S_{i+1}^y e^{i\pi S_{i+1}^z} - S_i^z S_{i+1}^z), \\ H_{im} &= S_{I-1}^x \sigma_I^z S_{I+1}^x + i S_{I-1}^y \sigma_I^y S_{I+1}^y e^{i\pi S_{I+1}^z} - S_{I-1}^z \sigma_I^x Z_{I+1}. \end{aligned}$$

It is important to note that the impurity is a qubit distinct from the spin-1 physical space. This qubit is similar to the edge mode of the AKLT ground state, which transforms projectively under the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry that is being gauged by the KT transform [41–44]. The scattering process is illustrated in Fig. 2 (d), with left and right are flipped for clarity. The domain wall, created in

the ferromagnetic phase, propagates through the impurity and becomes invisible. This invisible wave then flips the edge spins of the SPT phase when it reaches the right boundary. The wave packet cannot be invisible in S^z in the SPT phase because it becomes a non-local string operator $\tilde{S}_n^z = e^{i\pi \sum_{j=1}^n S_j^z} S_n^z$ [45]. This string attachment is inferred from correlation functions, where $\langle S_i^z S_j^z \rangle = 0$, when $i < I < j$ but $\langle \tilde{S}_i^z \tilde{S}_j^z \rangle \neq 0$. This is analogous to what was observed in the fermion-rotor model. The similar thing happens for the spin-spin correlation function of the Ising model as reported in Ref. [34, 46].

Topological defects from symmetric matrix product operators. At criticality, the Ising model is self-dual, and the KW operator is promoted to a symmetry of the Hamiltonian ¹. In this scenario, H_L and H_R are identical, resulting in a uniform system with an impurity at the center. This concept can be generalized to any system with MPO symmetry.

We consider the following $\text{Rep}(\mathcal{S}_3)$ model [11, 47, 48]:

$$\begin{aligned} H_L &= \sum_{i=2}^{I-2} (\sigma_{i-1}^x (\mathbb{1} + \sigma^z)_i \sigma_{i+1}^x) - g_L \sum_{i=1}^{I-1} \sigma_i^z, \\ H_R &= \sum_{i=I+2}^{L-1} (\sigma_{i-1}^x (\mathbb{1} + \sigma^z)_i \sigma_{i+1}^x) - g_R \sum_{i=I+1}^L \sigma_i^z, \\ H_{im} &= \frac{1}{2} \sigma_{I-2}^x (\mathbb{1} + \sigma^z)_{I-1} (\sigma^x + \sqrt{3}\sigma^y)_I \sigma_{I+1}^x \\ &\quad + \frac{1}{2} \sigma_{I-1}^x (\sigma^x - \sqrt{3}\sigma^y)_I (\mathbb{1} + \sigma^z)_{I+1} \sigma_{I+2}^x. \end{aligned}$$

The MPO is given by $T_{ii\alpha\beta} = \rho(i)_{\alpha\beta}$, with $\rho(1) = \sigma^x, \rho(2) = (\mathbb{1} + \sqrt{3}\sigma^z)/2$. Pulling this through half the chain yields topological boundary represented by H_{im} . Perfect particle transmission of a σ_x perturbation is illustrated in Fig 4. Moreover, as the particle passes the boundary, it can be observed that the impurity spin flips. This is a consequence of of invisible string attachment. Note that the topological string is invisible for this model as it is absorbed by the symmetric ground state.

Conclusion and Outlook. In this paper, we demonstrated how perfect transmission arises from the presence of duality defects. By applying a unitary MPO that represents the duality transformation or symmetry to a subsystem of the Hamiltonian, we derive a lattice impurity model with perfect transmission. The impurity degrees of freedom emerge from the virtual legs of the MPO, revealing hidden insights into the duality or symmetry of the MPO. These impurities are topological and attach a topological string to the scattered particle. Conversely, we conjecture that any interface between two theories

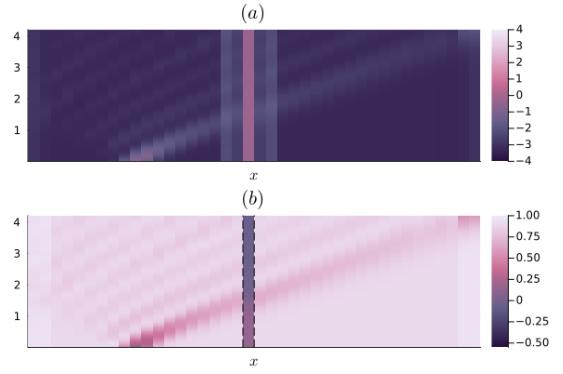


FIG. 4. The time evolution for the $\text{Rep}(\mathcal{S}_3)$ model with $(g_L, g_R) = (4.0, 4.0)$ of (a) the local energy and (b) the local magnetization $\langle Z_x \rangle$. Total transmittance is visible, as well as the spin flip on the impurity. The topological string is invisible due to the symmetry respecting ground state.

which allows for perfect transmission implies the existence of a duality between the theories at both sides.

Our model serves as a simple lattice example of the physics underlying the monopole paradox. In relativistic fermionic models with U(1) symmetry, a symmetric MPO, if it exists, should exhibit U(1) irreducible representation virtual degrees of freedom at low energies. Indeed, the impurity manifests as a bosonic rotor within the fermion-rotor model. Looking ahead, we conjecture a duality between charge 3 and 4 Weyl fermions and charge 5 and 0 Weyl fermions. However, studying chiral fermions on a lattice remains challenging. We aim to explore this using recent advancements in tensor-network-based chiral fermion formalisms [49–51] in the near future.

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¹ An extra on-site Hadamard gate is needed if we use U_{KW} defined in the previous section

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SUPPLEMENTAL MATERIAL**Videos**

FIG. 5. The video version of the scattering process for the Ising model, Haldane chain, and Rep(S_3) model. The PDF animation is best viewed in Acrobat/Okular.