Coordinate Vectors

Last Class

We saw that every vector space has a basis
- turns out that the # of vectors in a basis for a given VS is fixed
  - Call this # the ______ of V

ex) $\dim(\mathbb{R}^n) =$

$\dim(\mathbb{R}_n[x]) =$

$\dim(M_{m \times n}(\mathbb{R})) =$

Section 5.4 – Coordinate vectors / Change of Basis

Recall: In checking if the vector

i) $(2+3x-5x^2)$ in span $(1+x-2x^2, 2+x-3x^2)$

$$\Updownarrow$$

Checking if the 3-vector $\begin{pmatrix} \\ \\ \end{pmatrix}$ in span $\left(\begin{pmatrix} \\ \\ \end{pmatrix}, \begin{pmatrix} \\ \\ \end{pmatrix}\right)$

ii) Checking if $\begin{pmatrix} 4 & 1 \\ -2 & -3 \end{pmatrix}$ in span $\left(\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}\right)$

$$\Updownarrow$$

Checking if 4-vector $\begin{pmatrix} \\ \\ \end{pmatrix}$ in span $\left(\begin{pmatrix} \\ \\ \end{pmatrix}, \begin{pmatrix} \\ \\ \end{pmatrix} \right)$

Why care about Basis in first place?

Let $\mathcal{B} = (v_1, \dots, v_n)$ be a basis for V

- then every vector w in V can be expressed uniquely as the sum

$$w = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$$

Def: The coordinate vector for w with respect to a basis $\mathcal{B}$

$$[w]_{\mathcal{B}} = \begin{pmatrix} \\ \\ \\ \end{pmatrix}$$

ex) $V = \mathbb{R}_2[x]$, $\mathcal{B} = (1, x, x^2)$

- then $3 - x + 5x^2 =$

so $[3 - x + 5x^2]_{\mathcal{B}} = \begin{pmatrix} \\ \\ \end{pmatrix}$

- $\mathcal{B}' = (2+x,\ 3+x^2,\ x - x^2)$ is another basis for V

- then $3 - x + 5x^2 = c_1(2+x) + c_2(3+x^2) + c_3(x - x^2)$

$3 - x + 5x^2 = 9(2+x) - 5(3+x^2) - 5(x - x^2)$ ← turns out

$\Rightarrow [3 - x + 5x^2]_{\mathcal{B}'} = \begin{pmatrix} \\ \\ \end{pmatrix}$

Thrm: Let V vs and $\mathcal{B} = (v_1, \dots, v_n)$ a basis.

Then a vector w in span $(u_1, u_2, \dots, u_m)$ if and only if

in span $\Big(\qquad\qquad\qquad\qquad\qquad\qquad \Big)$

We already saw this!

ex) $V = M_{2\times 2}(\mathbb{R})$, $\mathcal{B} = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$

Then $\begin{pmatrix} 4 & 1 \\ -2 & -3 \end{pmatrix}$ in span$\left(\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} \right)$ if and only if in span $\left([\quad]_{\mathcal{B}}, [\quad]_{\mathcal{B}} \right)$

$$\begin{pmatrix} 4 \\ 1 \\ -2 \\ -3 \end{pmatrix} \text{ in span } \left(\begin{pmatrix} \\ \\ \\ \end{pmatrix}, \begin{pmatrix} \\ \\ \\ \end{pmatrix} \right) ?$$

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ii) $V = \mathbb{R}_2[x]$, $\mathcal{B} = (1, x, x^2)$

Checking if $2+3x-5x^2$ in span $(1+x-2x^2, 2+x-3x^2)$?

$[\quad]_{\mathcal{B}}$ in span $\left([\quad]_{\mathcal{B}}, [\quad]_{\mathcal{B}} \right)$?

Check if $\begin{pmatrix} \\ \end{pmatrix}$ in span $\left(\begin{pmatrix} \\ \end{pmatrix}, \begin{pmatrix} \\ \end{pmatrix} \right)$?

What this thm tells us is questions about spanning in general can be reduced to questions about spanning in $\mathbb{R}^n$

$\Downarrow$

<u>Thm</u>: V n-dim VS. Let $(w_1, \dots, w_k)$ be vectors in V

1) If $(w_1, \dots, w_k)$ span V then
2) If $(w_1, \dots, w_k)$ is LI then

Also have "half is good enough" statements (see HW) ex) $V = \mathbb{R}_2[x]$ $\Big(\qquad \qquad \Big)$ - have 4 vectors in 3-dim space

but do not span.

ex) $V = M_{2\times 2}(\mathbb{R})$ $\Big(\Big(\quad \Big), \Big(\quad \Big), \Big(\quad \Big) \Big)$

- have 3 vectors in 4-dim space, but they are not LI

Recall: We saw that $[w]_{\mathcal{B}}$ changes depending on our basis.

$V = \mathbb{R}_2[x]$

- $\mathcal{B} = (1, x, x^2)$
- $\mathcal{B}' = (2+x, 3+x^2, x-x^2)$

$\Longrightarrow$ $[3-x+5x^2]_{\mathcal{B}} = \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix}$

$[3-x+5x^2]_{\mathcal{B}'} = \begin{pmatrix} 9 \\ -5 \\ -5 \end{pmatrix}$

Are there any relationship between $[w]_{\mathcal{B}}$ and $[w]_{\mathcal{B}'}$

Yes! - The relationship is given as follows:

Def: Let V n-dim vector space. Let

$\mathcal{B} = (v_1, \dots, v_n)$ and $\mathcal{B}' = (w_1, \dots, w_n)$ be 2 basis.

The matrix

$$P_{\mathcal{B} \to \mathcal{B}'} = \begin{pmatrix} \\ \\ \end{pmatrix}$$

is called the ______ matrix from $\mathcal{B} \to \mathcal{B}'$

Thm: V n-dim VS, $\mathcal{B}, \mathcal{B}'$ 2 diff basis.

Then $[w]_{\mathcal{B}'} =$

______

• Moreover $\left(P_{\mathcal{B} \to \mathcal{B}'}\right)^{-1} =$

______

ex) $V = \mathbb{R}_2[x]$ and $\mathcal{B} = (1, x, x^2)$

$\mathcal{B}' = (1, 2x + 4x^2, x^2)$

Let $f = 2 - 6x + 3x^2$

a) Find $[f]_{\mathcal{B}}$

b) Find $P_{\mathcal{B} \to \mathcal{B}'}$

$P_{\mathcal{B} \to \mathcal{B}'} = \begin{pmatrix} \quad \end{pmatrix}$

c) Find $[f]_{\mathcal{B}'}$

Thm tells us $[f]_{\mathcal{B}'} =$

Check:

d) Check that $\left(P_{\mathcal{B}\to\mathcal{B}'}\right)^{-1} = P_{\mathcal{B}'\to\mathcal{B}}$

$1 =$ $\Rightarrow$ $[1]_{\mathcal{B}} =$

$2x+4x^2 =$ $\Rightarrow$ $[2x+4x^2]_{\mathcal{B}} =$

$x^2 =$ $\Rightarrow$ $[x^2]_{\mathcal{B}} =$

$$P_{B' \to B} = \begin{pmatrix} \quad \end{pmatrix}$$

$$\begin{pmatrix} \quad \end{pmatrix}\begin{pmatrix} \quad \end{pmatrix} = \begin{pmatrix} \quad \end{pmatrix}$$

$$\begin{pmatrix} \quad \end{pmatrix}\begin{pmatrix} \quad \end{pmatrix} = \begin{pmatrix} \quad \end{pmatrix}$$