Seriously, What is It?

Recall: We say two sets A, B have the same ________ if $\exists$ a ________ function ________________

and we write ______ = ______

Today we will show that

$|\mathbb{N}|$ $|\mathbb{Z}|$ $|\mathbb{Q}|$ $|\mathbb{R}|$ $|\mathbb{C}|$

Def: A set A is said to be ________ if

$|A| =$ ____

⇝ this means we can "enemurate" ie, ______ the elements of A by the ________

⇝ we'll write $A = \{ \quad \ldots \}$

- A is said to be ________ if $|A| <$ __ or if $|A| =$ __
- A is said to be ________ otherwise

Thrm: 1) An infinite subset of a ______ set is ______

Pf)

(Hilberts Hotel!!!)

ex) The set $2\mathbb{N} \subseteq \mathbb{N}$ of even positive integers is ______

[Note this means $|2\mathbb{N}| =$ ____ even though $2\mathbb{N} \subsetneq$ ____

- In this case one can explicitly construct a ______

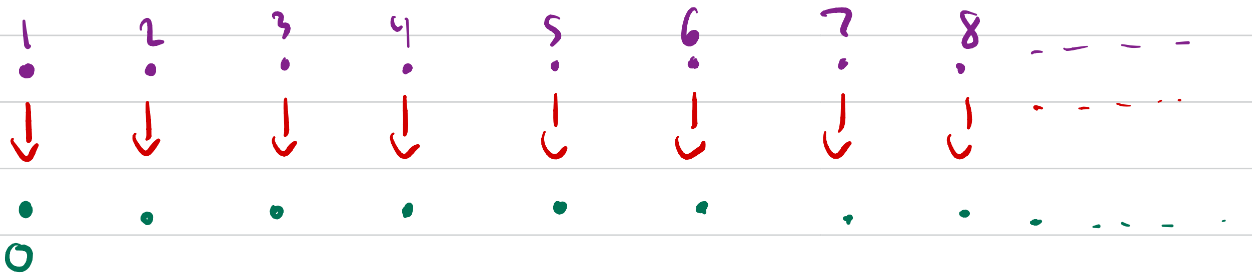
$f: \mathbb{N} \to 2\mathbb{N}$ by $f(\quad) =$ ____]

<u>RmK</u>: (1) tells us that for an ______ set A, it is enough to just have an ______ function $f: A \quad X$ where X ______, to conclude that A is ______

Thm: $|\mathbb{Z}| = |\mathbb{N}|$

Pf) The strategy is as follows: Send ______ #

to ______ #'s and ______ to ______



$\rightsquigarrow$ Can explicitly define $f : \mathbb{N} \to \mathbb{Z}$ as

$$f(n) = \begin{cases} & n \text{ even} \\ & n \text{ odd} \end{cases}$$

and check this is a ______

Thm: A cartesian product of denumerable sets is denumerable

Pf) We will sketch the proof by giving a picture

$A: \{ \qquad\qquad\qquad \}$

$B: \{ \qquad\qquad\qquad \}$

Thm: $\mathbb{Q}$ is ________

Pf) We will construct an injective function $f: \mathbb{Q} \hookrightarrow$ some countable set,

Uncountable Sets

Thm: The open interval is ______________

Pf) (Cantor's Diagonal argument)

Thm: If $A \subseteq B$ and A is uncountable then B is too

Pf) Suppose FTSOC that _ _ _.

Cor: $\mathbb{R}$ is ________

Pf)

<u>Thm</u>: $|(0,1)| = |\mathbb{R}|$

"Pf" (Idea ~ "stretch out" the interval lol)

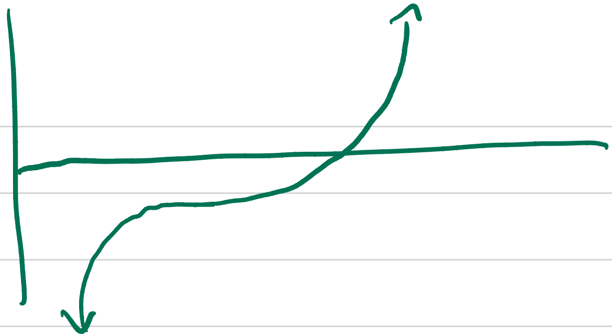
Define $f: (0,1) \longrightarrow \mathbb{R}$ by

$$f(x) = -\frac{1}{x} + \frac{1}{1-x}$$

$\rightsquigarrow$ Can check __________ no problem

$\rightsquigarrow$ __________ relies on some limit arguments but

if you graph it you get



so not hard to believe this is ________

So far we have

$$|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}| < |(0,1)| = |\mathbb{R}|$$

these are all called $\aleph_0$ ("aleph null")

first letter in Hebrew Alphabet

Continum Hypothesis

$\nexists$ set S such that

$$|\mathbb{N}| < |S| < |\mathbb{R}|$$

(1931 - Gödel)

$\rightsquigarrow$ Not able to prove using axioms of Set theory

(1963 - Cohen)

$\rightsquigarrow$ Not able to <u>disprove</u> using axioms of Set theory

However: We CAN prove that $|\mathbb{R}|$ is NOT the end of the <u>road</u>

<u>Thrm</u>: If A is any set $|A| < |P(A)|$

Pf) For A finite, obvious.

Assume A not finite and FSOC assume $\exists$ bijection $f: A \to P(A)$

Cor: There is NO "largest" set (or NO "biggest ∞")

Pf) Always have

$$|A| < |P(A)| < \underline{\qquad\qquad} < \underline{\qquad\qquad} < \cdots$$

Thrm: $|P(\mathbb{N})| = |\mathbb{R}|$

$\leadsto$ We know $|\mathbb{N}| < |P(\mathbb{N})|$

$|\mathbb{N}| < |\mathbb{R}|$

$\rightsquigarrow$ But how to show $\exists$ bijection $\mathcal{P}(\mathbb{N}) \simeq \mathbb{R}$?

Thrm: (Schröder-Bernstein Thrm)

If $\exists$ ______ $f: A \quad B$ AND

______ $g: B \quad A$

then $|A| \quad |B|$

$\rightsquigarrow$ Use this to show $|\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$)

$\rightsquigarrow$ Construct ______ $\mathcal{P}(\mathbb{N})$ $\quad (0,1))$

$(0,1) \quad \mathcal{P}(\mathbb{N})$

Fin