Yiddish of the Day

"Zolsti vaksn vi a
tsibele mitn kop
in der erd" =

זאָלסטו וואַקסן ווי אַ
ציבעלע מיטן קאָפּ
אין דער ערד

HW 2 Due today (late due date Sunday)

Proof - Portfolio Posted - START IT
ASAP

Quantified Statements

Let $P(x)$ be an open sentance over some domain S

→ We will produce special kinds of statements from this open sentance called quantified statements

ex1) for all $x \in S$, $P(x)$ is true

The phrase "for - all" is referred to as the universal qualifier and is denoted by the symboll $\forall$ (\forall)

(this is also refferred to as for each, for every or for any)

~> Symbollically, we write $\forall x \in S, P(x)$

ex 2) there exists $x \in S$ such that $P(x)$ is true

The phrase "there exists" is referred to as the existential quantifier and is denoted by the symboll $\exists$ (\exists)

~> Symbolically, we write $\exists x \in S, P(x)$

ex) Consider the open sentence

$P(n)$: n^2+n is even with domain $\mathbb{Z}$

→ • for all $n \in \mathbb{Z}$, n^2+n is even

$(\underline{\forall}\, n \in \mathbb{Z}, P(n))$ ~ <u>true</u> !

• there exists an $n \in \mathbb{Z}$, n^2+n is even

$(\underline{\exists}\, n \in \mathbb{Z}, P(n))$ ~ <u>true</u> !

Negating Quantified Statements

- $\neg(\underline{\forall}\, x \in S, P(x))$ = it is __not__ the case that __for all__ $x \in S$ $P(x)$ is true

  = __there exists__ an $x \in S$ such that $P(x)$ is __false__

  = $\underline{\exists x \in S, \neg P(x)}$

- $\neg(\underline{\exists}\, x \in S, P(x))$ = it is __not__ the case that there __exists__ an $x \in S$ such that $P(x)$ is true

  = __for all__ $x \in S$, $P(x)$ is __false__

  = $\underline{\forall x \in S, \neg P(x)}$

$\rightsquigarrow$ In summary, under negation, we have

$$\wedge \longleftrightarrow \vee$$

$$\forall \longleftrightarrow \exists$$

$$P(x) \longleftrightarrow \neg P(x)$$

Let $x \in S$ and $y \in T$ be variables. Consider

$$\underline{\ \forall\ }\ x \in S,\ \underline{\ \forall\ }\ y \in T,\ P(x,y)$$

(= <u>for</u> all $x \in S$, and <u>for all</u> $y \in T$, $P(x,y)$ is <u>true</u>)

$\xrightarrow{\text{negate it}} \neg\left(\underline{\forall}\, x\in S,\ \underline{\forall}\, y\in T,\ P(x,y)\right) \equiv \underline{\exists}\, x\in S,\ \neg\left(\underline{\forall}\, y\in T,\ P(x,y)\right)$

$\equiv \underline{\exists}\, x\in S,\ \underline{\exists}\, y\in T,\ \neg P(x,y)$

ex) Consider the statement:

for all $x\in\mathbb{R}$, $y\in\mathbb{R}$, $x^2+y^2>0$

$\leadsto$ translate into symbols: $\forall x, y\in\mathbb{R},\ x^2+y^2>0$

$\leadsto$ negate it: $\exists x, y\in\mathbb{R},\ x^2+y^2\leq 0$

(Which is true from above? <u>the second</u>)

Now consider the stument

$$\forall\, x \in S,\ \underline{\exists}\ y \in T,\ P(x,y)$$

negate it $\rightsquigarrow$ $\neg(\forall x \in S, \underline{\exists}\, y \in T, P(x,y)) \equiv \underline{\exists}\, x \in S, \neg(\exists\, y \in T, P(x,y))$

$\equiv \underline{\exists}\, x \in S, \underline{\forall}\, y \in T, \neg P(x,y)$

in words $\rightsquigarrow$ $\neg$(for all x in S, <u>there exists</u> a y in T such that $P(x,y)$ is true)

$\equiv$ there exists a x in S, such that for all y in T, $P(x,y)$ is false

However,

$\neg(\underline{\exists}\, x \in S, \forall y \in T, P(x,y)) \equiv \underline{\forall}\, x \in S, \neg(\forall y \in T, P(x,y))$

$\equiv \underline{\forall}\, x \in S, \underline{\exists}\, y \in T, \neg P(x,y)$

negation : for every x in S, there exists a y in T such that $P(x,y)$ is false ex) Lets negate the following statement:

for all integers x, y, if their product is even, then x is even or y is even

1) Locate the component statements

$P(x,y)$: the product xy is even

$Q(x)$: x is even $\quad$ ($\neg Q(x)$: x is odd)

2) Domain := $\mathbb{Z}$

3) Write in symbolic notation

$$\forall x, y \in \mathbb{Z},\ P(x,y) \Rightarrow \big(Q(x) \vee Q(y) \big)$$

4) Negate it

- First, recall how to find $\neg(P \Rightarrow Q)$

$$\rightsquigarrow \neg(P \Rightarrow Q) \equiv \neg(\neg P \vee Q) \quad \text{(previous thm)}$$

$$\equiv \neg(\neg P) \wedge (\neg Q) \quad \text{(De-Morgan's Laws)}$$

$$\equiv P \wedge (\neg Q) \quad \text{(Double negation law)}$$

$$\rightsquigarrow \neg\left(\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, \; P(x,y) \Rightarrow \left(Q(x) \vee Q(y)\right)\right)$$

$$\equiv \exists x \in \mathbb{Z}, \exists y \in \mathbb{Z}, \neg\left(P(x,y) \Rightarrow \left(Q(x) \vee Q(y)\right)\right)$$

$$\equiv \exists x \in \mathbb{Z}, \exists y \in \mathbb{Z}, P(x,y) \land \neg(Q(x) \lor Q(y))$$

$$\equiv \exists x \in \mathbb{Z}, \exists y \in \mathbb{Z}, P(x,y) \land \neg Q(x) \land \neg Q(y)$$

in words $\rightsquigarrow$

"there exists integers x, y such that their product is even and x is odd and y is odd"

Proof Techniques (Yay!!!)

Trivial and Vacuous Proofs

ex) Let $x \in \mathbb{R}$. Show that if $0 < x < 1$ then $x^2 - 2x + 2 > 0$

- $P(x)$: $0 < x < 1$
- $Q(x)$: $x^2 - 2x + 2 > 0$

$\rightsquigarrow$ Goal: $\forall x \in \mathbb{R},\ P(x) \Rightarrow Q(x)$

Note: $x^2 - 2x + 2 = (x-1)^2 + 1 > 0$

$\Rightarrow Q(x)$ is always true!

P	Q	$P \Rightarrow Q$
T	T	T
F	T	T

$\Rightarrow$ Our statement $\underline{\forall}\, x \in \mathbb{R},\ P(x) \underline{\Rightarrow} Q(x)$ is <u>true</u> for every $x \in \mathbb{R}$

This type of proof is called a <u>trivial</u> proof, one where the conclusion is always true

ex 2) Let $x \in \mathbb{R}$. Show that if $x^2 - 2x + 2 \le 0$ then $x^3 \ge 0$

into logic $\rightsquigarrow$ $P(x):\ x^2 - 2x + 2 \le 0$

$Q(x):\ x^3 \ge 0$

$\rightsquigarrow \underline{\forall}\, x \in \mathbb{R},\ P(x) \underline{\Rightarrow} Q(x)$

However: Remember that $P(x)$ is always false

P	Q	$P \Rightarrow Q$
F	T	T
F	F	T

So our statement is always true!!!

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	T	T

Remember truth tables!

This type of proof is called a Vacuous proof, one where the assumption is always false.

More interesting: Direct Proofs

Let $P(x)$ and $Q(x)$ be open sentences over a domain S.

Goal: Show that $P(x) \Rightarrow Q(x)$ is true for all $x \in S$.

We saw that 1) if $P(x)$ is F then this is always true

2) if $Q(x)$ is always true then this is also always true

$\rightsquigarrow$ interested in neither of these cases.

In a direct proof for $\forall x \in S, P(x) \Rightarrow Q(x)$ we consider an arbitrary element $x \in S$, for which $P(x)$ is true and then show that $Q(x)$ is also true

ex 1) For every odd integer n, show that $3n+7$ is even

into symbols $\rightsquigarrow$ $\forall n \in \mathbb{Z}, \underbrace{n \text{ odd}}_{P(n)} \Rightarrow \underbrace{3n+7 \text{ even}}_{Q(n)}$

[For any question involving "parity" (ie, even or odd)

- an integer x is even iff $x = 2K$ for some $K \in \mathbb{Z}$

- an integer x is odd iff

  $x = 2\ell + 1$ for some $\ell \in \mathbb{Z}$

Pf) Let n be an odd integer. Then we can write

$n = \underline{2\ell+1}$ for some $\ell \in \mathbb{Z}$.

We now compute

$$3n+7 = 3(2\ell+1) + 7 = 6\ell + 10$$
$$= 2(3\ell+5)$$

Since $\underline{3\ell+5} \in \mathbb{Z}$, we have that $3n+7 = 2(\text{integer})$ so $3n+7$ is even

$\square$

Recall: For a positive integer m. We say that k is congruent to l modulo m, denoted $k \equiv l \mod m$

- iff $k - l$ is divisible by m.
- iff k and l leave the same remainder when divided by m

ex) For an integer n, show that $n^2 \equiv 0 \mod 4$ or $n^2 \equiv 1 \mod 4$.

Scratch work:

n	0	1	2	3	4	5	6	7	8	9	10	11	12
n^2	0	1	4	9	16	25	36	49	64	81	100	121	144
$n^2 \bmod 4$	0	1	0	1	0	1	0	1	0	1	0	1	0

$\rightsquigarrow$ Conjecture: If n is even then $n^2 \equiv \underline{0} \mod 4$

If n is odd then $n^2 \equiv \underline{1} \mod 4$

Pf) Let n be an integer. Then note that n is either even or odd. Let us first assume that n is even.
Then we can write $n = 2\ell$ for some $\ell \in \mathbb{Z}$.
So we compute that $n^2 = 4\ell^2$. Since 4 divides $4\ell^2$, we have that n^2 has remainder 0, when divided by 4, so $n^2 \equiv 0 \mod 4$
Now assume that n is odd. Then $n = 2k+1$ for $k \in \mathbb{Z}$.
So we compute

$$n^2 = (2k+1)^2 = 4k^2 + 4k + 1$$

This has remainder 1 when divided by 4, so $n^2 \equiv 1 \mod 4$ $\square$

Lemma: For an integer n, the number n^2+n is even.

Pf) First we note that $n^2+n = \underline{n(n+1)}$

Now we again consider $\underline{2}$ cases; when n is $\underline{\text{even}}$ and when n is $\underline{\text{odd}}$.

So assume n is even. Then we can write $n=2\ell$ for $\ell \in \mathbb{Z}$.

So now we compute

$$\begin{aligned} n^2+n &= n(n+1) \\ &= 2\ell(2\ell+1) \\ &= 2(\ell(2\ell+1)) \end{aligned}$$ where $\ell(2\ell+1) \in \mathbb{Z}$

and is hence even. Now we assume n is odd. So we write $n=2k+1$ for some $k \in \mathbb{Z}$. We then compute

$n^2+n = (2k+1)(2k+2) = 2\Big(\underbrace{(2k+1)(k+1)}_{\in \mathbb{Z}}\Big)$, and is hence even

Let's revisit the previous theorem to make another conjecture

n	0	1	2	3	4	5	6	7	8	9
n^2	0	1	4	9	16	25	36	49	64	81
$n^2 \bmod 8$	0	1	4	1	0	1	4	1	0	1

~ We already know that when n is odd, $n^2 \equiv 1 \bmod 4$.

However, it seems we also have that n odd $\Rightarrow n^2 \equiv \underline{1 \bmod 8}$

Prop: If n is an odd integer then $n^2 \equiv$ $\underline{1 \bmod 8}$

Pf) Let n be an odd integer. Then we can write $n = 2k+1$ for some $k \in \mathbb{Z}$. Then we compute

$$n^2 = (2k+1)^2$$
$$= 4k^2 + 4k + 1$$
$$= 4[k(k+1)] + 1$$

By the previous lemma, $k(k+1)$ is even. So we can write $k(k+1) = 2\ell$ for some $\ell \in \mathbb{Z}$.

Hence we get

$$n^2 = 4(k(k+1)) + 1$$
$$= 4(2\ell) + 1$$
$$= 8\ell + 1$$

So the remainder of n^2 when divided by 8 is 1 as required. Hence $n^2 \equiv 1 \mod 8$