More on functions

<u>Last time</u>: $f: A \to B$ function, $X \subseteq A$, $Y \subseteq B$

- $\underline{\qquad\qquad} = \{ \qquad\qquad \mid x \in X \} \subseteq$

- $\underline{\qquad\qquad} = \{ \qquad \in A \mid \qquad \in Y \} \subseteq$

Def: A, B sets. Define

$$\underline{\hspace{2cm}} := \{ \qquad \mid \qquad \}$$

Lemma: For finite sets A, B have

$$| \quad | = | \quad |^{| \quad |}$$

Pf)

<u>Def</u>: Let $f: A \to B$ be a function.

1) f is ________ (or __-to-__) if

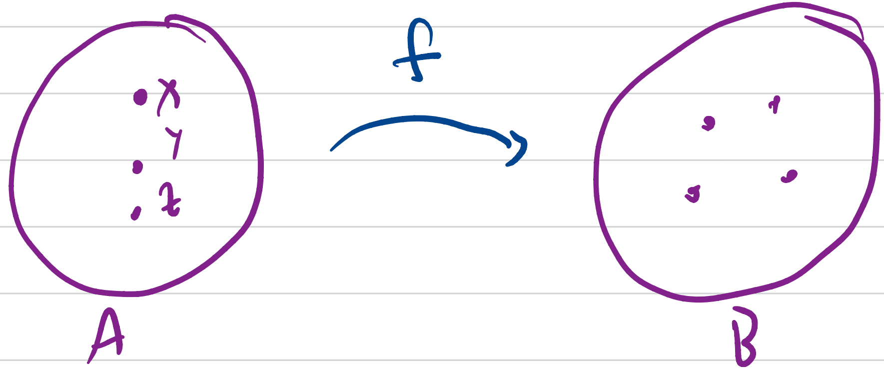
f maps ________ elements to ______ elements.

$\rightsquigarrow$ i.e., if __ $\neq$ __ then __ $\neq$ __

contrapositive

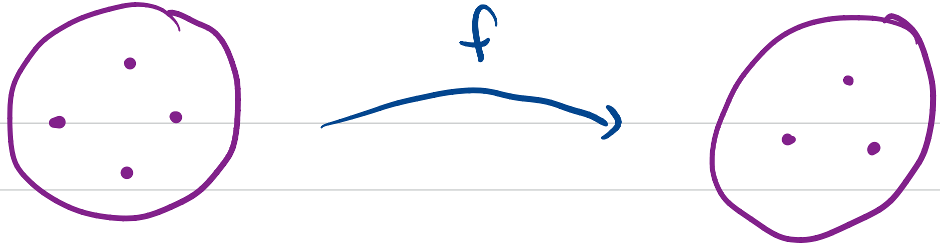
$\iff$ if ____ = ____ then ____ = ____

Pictural ex



2) f is ________ (or ______) if ____ = ____

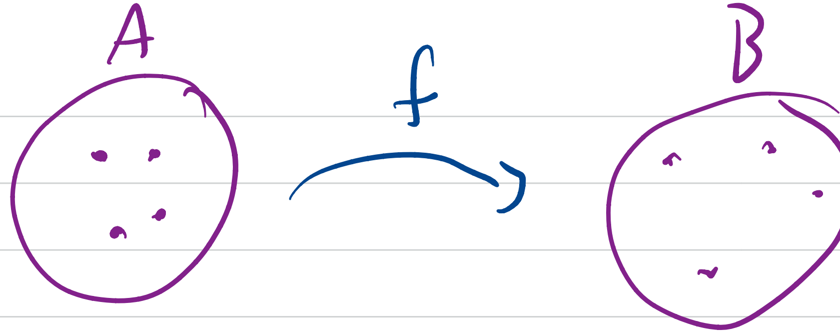
A B



$\rightsquigarrow$ ie, every ______ of ______ is mapped to by ______ element of ______

$\Longleftrightarrow$ $\forall$ ___ $\in$ ___ , $\exists$ ___ $\in$ ___ such that ___ = ___

3) f is ______ if f is both ______ and ______



<u>examples</u>:

1) X any set. Then ___ : $X \to X$ is ______

2) $A \subseteq X$. Then ___ : $A \to X$ is ____ but not ____ in general

3) $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$

- ________ ?

- ________ ?

4) $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = 3x + 2$

- ________

. ______ ?

Another way to characterize ______ and

______

Let $f: A \to B$ be a function. For any $b \in B$

consider

$$\_\_\_\,(\{b\}) = \{\,\_\_\_ \in \_\_\_ \mid \_\_\_ = b\,\}$$

$\longrightarrow$ 1) f ______ iff $|\qquad\qquad| \leq |\quad|$ $\forall b \in B$

2) f ______ iff $|\qquad\qquad| \geq |\quad|$ $\forall b \in B$

$\Rightarrow$ 3) f ______ iff $|\qquad\qquad| = |\quad|$ $\forall b \in B$

Def: Composition of functions

Let $f: A \to B$ and $g: B \to C$ be functions. Then we can define the new function

$$g \circ f : A \to C$$

by $(g \circ f)(a) =$ ________

Prop: Let $f: A \to B$ and $g: B \to C$ functions

1) If ______ f and g are ______ then the composite ______ is also ______

2) If ______ f and g are ______ then the composite ______ is also ______

Pf) Left to you ☺ Good way to check if you know the definitions.

Prop: Let $f: A \to B$ and $g: B \to C$ functions.

1) If ______ is ______ then f is ______

⌜Q: If ____ is ____ is g also?⌟

2) If ______ is ______ then g is ______

⌜Q: If ____ is ____ is f also?⌟

Pf) (1) Suppose ____ is ____.

For the Q

(2) Suppose ______ is ______ .

⌜Q left to you :) ⌟

- Sometimes there is an easier way to show a function f is ________

<u>Thm</u>: Let $f: A \to B$ and $g: B \to A$ be functions.

Suppose 1) ____ = ____

2) ____ = ____

Then f and g are both ______ and are each others ____ relation

Pf) Since $g \circ f$ = ____ which is ____ we know that ____ is ____ and ____ is ____

Similarly, since $f \circ g =$ ___ which is ___

we have that ___ is ___ and ___ is ___

<u>Moreover</u>

<u>Thm</u>: Suppose $f: A \to B$ bijective. Then the ______ $: B \to A$ is a function and $f \circ$ __ $=$ __ and __ $\circ f =$ ___

Pf) For $b \in B$, have ___ $(b) := a$ where a is the unique element such that ___ $= b$

(Do you see why this is well defined?)

Then $f(\underline{\qquad}(b)) = \underline{\qquad}$

and $\underline{\qquad}(f(\quad)) =$

ex) Let $f : \mathbb{R} \setminus \{2\} \longrightarrow \mathbb{R} \setminus \{3\}$ given by

$$f(x) = 3 + \frac{6}{x-2}$$

Show f is bijective

[Scratch work: 1) write $\underline{\qquad} = f(x)$

$\rightsquigarrow \underline{\qquad} = \underline{\qquad\qquad\qquad\qquad}$

2) ______ the values ___ and ___

$\rightsquigarrow$ ___ $=$ ______________

3) Solve for ___

4) The resulting $\hat{s}$ is the predicted choice for ______ function

Pf) Define $g: \mathbb{R}\setminus\{3\} \to \mathbb{R}\setminus\{2\}$ by

$g(x) =$

Some more notation

- We commonly use the word _____ to mean function
  - if $b = f(a)$ we say a _____ to b under f ($a \mapsto b$ \mapsto)
- Given $f : A \to B$ function, If f is
  1) _____ we write $f : A \hookrightarrow B$ (\hookrightarrow)
  2) _____ we write $f : A \twoheadrightarrow B$ (\twoheadrightarrow)
  3) _____ we write $f : A \xrightarrow{\sim} B$ (\xrightarrow{\sim})

Cardinality

Goal: Count the # of elements in ANY set

⤳ When finitely many elements ☺ ☺ ☺

⤳ But MANY sets with ______ many elements

⌜ $\mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R} \subsetneq \mathbb{C}$ ⌟

Q: How to distinguish between "how big" sets like these are?

Lemma: Let A, B be ________ sets. Then

_____ $=$ ______ iff $\exists$ ____________________

Pf)

Def: Let A, B be sets. Then A and B are said to be ____________________ (or have the same ________________), denoted by

________ = ________ if $\exists$ ____________________

- If A is ____ with n-elements we write ____ $= n$

Thm: ____________ is an ____ amongst sets