Equivalence Relations

(ER)

We call a relation R on A an ______ relation if R is

1) ______

2) ______

3 ______

- in this case, if ______ we say that

__________________ and we will sometimes write ______ (or ______)

As mentioned last time, this is a generalization of ______

- $R =$ ___ is the "usual" notion of ______

  and we saw last class this is indeed an ______ ex) Consider the relation on $A = \mathbb{R}^2 \setminus \{0,0\}$ given by

$(x,y) \, R \, (a,b)$ iff $\exists \, \lambda \in \mathbb{R}_{\neq 0}$ such that

$(a,b) = (\lambda x, \lambda y)$.

Show this is an ER

1)

2)

3)

<u>Rmk</u>: This example is <u>very important</u>! It is called the "<u>Real projective plane</u>", denoted $\mathbb{R}P^2$. We'll return to it shortly.

Def: Let R be an ______ on A.

For $a \in A$ define the ______ represented by a to be the set

$$[a] = \{ \quad | \quad \}$$

$$= \{ \quad | \quad \in R \}$$

$= \{$ all elements ______ to a $\}$

· Note: Since R is ______ we know that

$[a] \neq \phi$. ($\underline{\text{Why}}$? ________)

Lets find the EC's for the following ER's

a) $R = \{(1,1), (2,2), (3,3), (4,4), (5,5), (1,3), (1,5), (5,1), (5,3), (3,1), (3,5), (2,4), (4,2)\}$ ($A = \{1,2,3,4,5\}$)

$[1]$: {all elements ________ to 1 under R}

$= \{x \in A \mid$ ________ $\in R\}$

$= \{ \qquad \}$

$[2] =$

$[3] =$

$[4] =$

$[5] =$

$\leadsto$ The EC's for R are

___ and ___

Since $[\ \ ] = [\ \ \ \ ] = [\ \ \ \ ]$

and $[\ \ ] = [\ \ \ \ ]$

Note: For $a, b \in A$. If a ___ b then

$[a] \cap [b] =$ ______

and if a ____ b then $[a]$ _ $[b]$

b) Real projective plane example.

$(x,y) R (a,b)$ if $\exists\, 0 \neq \lambda \in \mathbb{R}$ st $(a,b) = (\lambda x, \lambda y)$

Lets find the equiv class of $(3,5)$

$[(3,5)] = \{$ ____________ $\mid (3,5) R \quad \}$

$$= \left\{ \qquad\qquad \middle| \; \_\_\_ = (3\lambda, 5\lambda) \text{ for some } 0 \neq \lambda \in \mathbb{R} \right\}$$

$$= \left\{ \qquad\qquad \middle| \; \_\_\_ = 3\lambda \text{ and } \_\_\_ = 5\lambda \text{ for some } 0 \neq \lambda \in \mathbb{R} \right\}$$

$$= \left\{ \qquad\qquad \middle| \; \lambda \in \mathbb{R}_{\neq 0} \right\}$$

⌜Graphically this is the ______ defined by

$3y = 5x$ minus the origin

⌟

c) Congruence. Let $n \in \mathbb{N}$ and $A = \mathbb{Z}$. Then have the ER

aRb iff $a \equiv b \mod n$

Lets find all the ECs for $n = 5$

$[0] = \{ \qquad \}$

$= \{$ integers with ____ 0 when ____ by 5 $\}$

$[1] : \{ \qquad \}$

$= \{$ integers with ____ 1 when ____ by 5 $\}$

$[2] = \{ \qquad \}$

$= \{$ integers with ______ 2 when ______ by 5 $\}$

$[3] = \{ \qquad \}$

$= \{$ integers with ______ 3 when ______ by 5 $\}$

$[4] = \{ \qquad \}$

$= \{$ integers with ______ 4 when ______ by 5 $\}$

~~Anymore ?~~

$\rightsquigarrow$ Note again

1) If a ___ b then $[a] \cap [b] =$ __

2) If a ___ b then $[a]$ $[b]$

Properties of EC's

Thm: Let R be an ER on A. Then for $a, b \in A$

$[a]$ __ $[b]$ iff a __ b

Pf)

Thm 2: Let R be ER on A. Then for $a, b \in A$ if

$[a] \cap [b] = \_\_\_\_$ then $\_\_\_\_\_\_\_\_$

Pf)

Recall:

A ______________ of a set A is a collection of ______________ $X_\alpha \subseteq A$ where $\alpha \in I$ is an indexing set, such that

1) $X_\alpha \cap X_\beta =$ ______ whenever ______

2) $\bigcup_{\alpha \in J} X_\alpha =$ ______

Thm 3: Let R be an ER on A. Then the set of ______ for R forms a ______ of A

We denote the collection of all EC's by A/R

and read this as "A mod R" - This is called the "quotient set" of A by R ⌟

Pf)

Important examples

a) When $A = \mathbb{R}^2 \setminus \{0,0\}$, with ER R given by

$$(x,y) \, R \, (a,b) \quad \text{iff} \quad \exists \, 0 \neq \lambda \text{ st } (x,y) = (a,b)$$

$$\rightsquigarrow A/R \overset{\text{denoted}}{:=} \mathbb{RP}^1$$

$$= \left\{ \begin{array}{l} \text{all} \quad \underline{\hspace{6cm}} \\ \text{through} \quad \underline{\hspace{6cm}} \end{array} \right\}$$

b) Fix $n \in \mathbb{N}_{\geq 2}$ and $A = \mathbb{Z}$, with R given by

aRb iff $a \equiv b \mod n$

$\rightsquigarrow A/R := \underline{\hspace{6cm}}$

= the integers "mod n"

$= \{ [\quad], \ldots, [\quad] \}$

$\rightsquigarrow$ We'll return to this example next class