Back to Sets

We have the following __________ for proving statements involving sets

1)

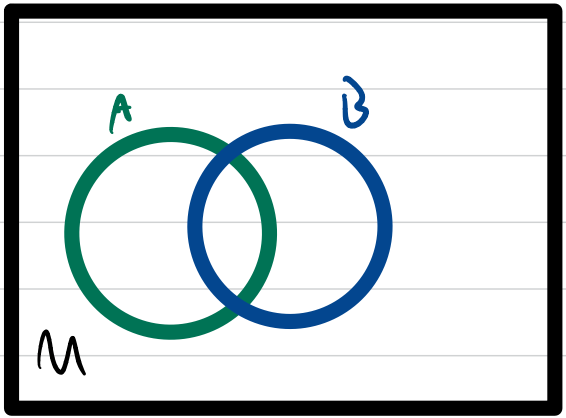
2)

3)

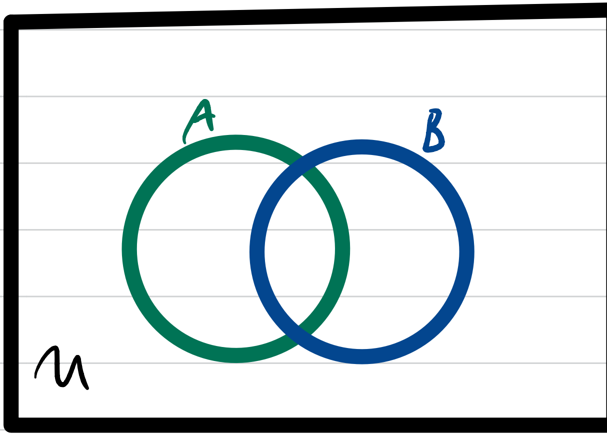
ex) If A B and B C then
A C

ex) Show that $A \setminus B = A \cap B^c$

Pf) We'll give a proof by Venn-diagram first



$A \setminus B$



$A \cap B^c$

ex) Show that $A \cup B = A$ iff $B \subseteq A$

First: this is an "iff" statement, So we have to prove 2 things

a) If ______ then ______ and

b) If ______ then ______

<u>Also</u>: Remember, showing two sets are equal $X = Y$ requires showing ______ and ______

Pf) Let us first assume that $A \cup B = A$. Then we want to show ____________ .

Prop: (Set-Operation Laws)

Let A, B, C be sets, then

- **Distributive laws:** a) $A \cup (B \cap C) =$

  b) $A \cap (B \cup C) =$

- **De-Morgan's Laws:** a) $(A \cup B)^c =$

  b) $(A \cap B)^c =$

Pf) Most of these will be left as an excercise to you! Lets prove that

$$(A \cup B)^c =$$

ex) Show that $(A \setminus B) \cap (A \setminus C) = A \setminus (B \cup C)$

Pf)

Counterexamples

Consider a __________ quantified statement

$\_\_\_\, x \in S,\ R(x)$

⇝ Will be either _____ or _____

⇝ If _____, prove it!!!

⇝ If _____, then its _____, $\_\_\_\, x \in S,\ \neg P(x)$ is _____. So to prove original statement is _____ we have to find (at least one) element $x_0 \in S$ such that $R(x_0)$ is _____

⇝ this element x_0 is called a __________ ex) Consider the statement: If $n \equiv 0, 1, 2 \mod 4$ then n is a sum of two squares $(n = a^2 + b^2)$

Pf) Lets expirement

n	0	1	2	~~3~~	4	5	6	~~7~~
Sum of squares								

ex) Prove or disprove the following. Let $a, b \in \mathbb{R}$ st $a, b \neq 0$
For all $x, y \in \mathbb{R}_{>0}$ $\quad \frac{a^2}{2b^2}x^2 + \frac{b^2}{2a^2}y^2 > xy$

[We will try and prove it. If the proof works ☺ ☺ ☺
If not, the reason it didn't may help us find a ______]

$$LHS - RHS = \frac{a^2}{2b^2}x^2 + \frac{b^2}{2a^2}y^2 - xy$$

$$= \frac{1}{2}\left[\left(\frac{ax}{b}\right)^2 + \left(\frac{by}{a}\right)^2 - \_\_\_\_\right]$$

$$= \frac{1}{2}\left[\left(\frac{ax}{b}\right)^2 + \left(\frac{by}{a}\right)^2 - (\quad)(\quad)\right]$$

$$= \frac{1}{2}\left(\_\_\_ - \_\_\_ \right)^2$$

$\rightsquigarrow$ LHS − RHS $\geq$ _____

$\rightsquigarrow$ but what about if it = _____

We would have ____ = ____

$\rightsquigarrow$ A ______ can be given by

$x =$ and $y =$

$\Rightarrow$ The statement is ________

Proof by Contradiction

Let R be a statement. A ________________ follows the structure

- to show that R is ______ we show that the ____________ of R leads to a ________________

In logical symbols this is $R \equiv (\neg R \Rightarrow \bot)$

Indeed we have the truth table

R	$\neg R$	$\bot$	$\neg R \Rightarrow \bot$

T		F	
F		F	

- In order to show ________ is ________ we have $\neg R$ must be ______ (ie R is ______)

Special/Important Case

If the statement R is of the form ____________
then to prove ________ we can prove

$$\underbrace{\rule{6cm}{0.4pt}}_{\substack{\| \text{ recall} \\ \neg(P \Rightarrow Q)}} \Rightarrow \bot$$

ex) If $q \in \mathbb{Q}$ and r is an irrational # then $q+r$ is irrational.

Pf)

ex) Prove that if $n \nmid ab$ then $n \nmid a$ and $n \nmid b$.

Rmk: this will be usefull for one of your proof portfolio problems

Pf)

ex) Let $m \in \mathbb{Z}$ st $2 \mid m$ but $4 \nmid m$. Show that there are no integer solutions to the equation $x^2+3y^2=m$ (changed)

Rmk: For non-existance statements, we often will use contradiction.

- Lets do some scratch work before the proof.
  - If $2 \mid m$ but $4 \nmid m$ then

    $m \equiv \_\_\_\_ \bmod 4$

$\rightsquigarrow$ Goal: If $m \equiv \_\_\_\_ \bmod 4$ then $x^2+2y^2=m$ has no integer solutions

Pf) For the sake of contradiction, let us assume that

$m \equiv$ ___ mod 4 and that $\exists x, y \in \mathbb{Z}$ st $x^2 + 3y^2 = m$