Yiddish of the Day

"Bisti in bod, darfsti shvitsn" = ביסטו אין באָד, דאַרפֿסטו שוויצן

Reminder: HW5 Due Today

HW6 Due Friday

Proof Portfolio 1 Due Friday

"What is this Proving" Due Friday

Principles of Induction

- **Notation**: $\mathbb{Z}_{>0} = \{1, 2, 3, 4, \dots\}$

- The principle of mathematical induction is a method of proof for statements like $\forall n \in \mathbb{Z}_{>0}, P(n)$

- To prove these statements, we do the following

1) (Base Step): $P(1)$ true

2a) (Inductive hypothesis): Assume $P(k)$ true for some $k \geq 1$

2b) (Inductive leap) : Show that $P(K)$ true $\Rightarrow P(K+1)$ also true

Rmks: • An intuition for this can be gotten from comparing to pushing over dominos

• The logical foundation is that

$$P \wedge (P \Rightarrow Q) \Rightarrow \underline{Q} \text{ is a tautology}$$

$\rightsquigarrow$ if $P(K)$ and $P(K) \Rightarrow P(K+1)$ true then so is $P(K+1)$

ex) Show that for $n \geq 1$

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

Pf) Let us first identify the open sentance $P(n)$.

$$P(n): \; 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

We will prove this statement using induction.
We first show the base case holds.
The base case is claiming that $1 = \frac{1(1+1)}{2} = \frac{2}{2}$
which is true. So the base case is satisfied.
So now we assume that $P(k)$ is true for some

$k \geq 1$. Ie, that $\exists\, k \geq 1$ such that

$$1 + \cdots + k = \frac{k(k+1)}{2}$$

Goal: Show that $\uparrow$ implies $P(k+1)$ is true.

$\rightsquigarrow$ Want to show

$$1 + \cdots + k + (k+1) = \frac{(k+1)(k+2)}{2}$$

We then compute that

$$1 + \cdots + k + (k+1) = \frac{k(k+1)}{2} + (k+1)$$

$$= \frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$$

Hence we see that

$$1 + \cdots + k + (k+1) = \frac{(k+1)(k+2)}{2}$$

so $P(k+1)$ is also true.

Therefore, by mathematical induction, $P(n)$ true $\forall n \in \mathbb{Z}_{>0}$

ex) Let $n \geq 1$. Prove that $\frac{d^n}{dx^n}\left(e^{x^2}\right) = P_n(x)e^{x^2}$ where $P_n(x)$ is a degree n polynomial.

Pf) We will first show that the base case holds. We will show that $\frac{d}{dx}\left(e^{x^2}\right) = P_1(x)\,e^{x^2}$

Indeed $\frac{d}{dx}\left(e^{x^2}\right) = 2x\,e^{x^2}$ so base case holds.

Now assume that $\frac{d}{dx^k}\left(e^{x^2}\right) = P_k(x)\,e^{x^2}$ for $k \geq 1$.

Goal: Show that $\frac{d}{dx^{k+1}}\left(e^{x^2}\right) = P_{k+1}(x)\,e^{x^2}$

We will compute

$$\frac{d}{dx^{k+1}}\left(e^{x^2}\right) = \frac{d}{dx}\left(\frac{d}{dx^k}e^{x^2}\right)$$

$$= \frac{d}{dx}\left(P_k(x)e^{x^2}\right)$$

$$= \frac{d}{dx}\left(P_k(x)\right)e^{x^2} + P_k(x)\frac{d}{dx}\left(e^{x^2}\right)$$

$$= P_{k-1}(x)e^{x^2} + P_k(x)\left(2x\,e^{x^2}\right)$$

$$= \left(P_{k-1}(x) + 2xP_k(x)\right)e^{x^2}$$

$$= \left(P_{k+1}(x)\right)e^{x^2}$$

It can happen that our base case, that is, our first instance where the statement is true does not hold at $n=1$, but instead holds for some $m>1$

→ In these cases we will still induction by "starting at $n=m>1$"

One then proves

1) Base Case: $P(m)$ is true for some $m>1$

2a) Inductive hypothesis : Assume true for $P(k)$, $k \geq m$

2b) Inductive leap : Show $P(k) \Rightarrow P(k+1)$ true

This is referred to as the 2nd (or general) principle of induction ex) Recall ~~last class~~ earlier today we defined, for $n \geq 2$ integer.

$$P_n = \left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\cdots\left(1-\frac{1}{n^2}\right)$$

We conjectured it is also of form

$$P_n = \frac{n+1}{2n}$$

Lets prove it.

Pf) We will use the general principle of induction to prove this. Our base case is when $n=2$. Indeed we have that $P_2 = \frac{3}{4} = \frac{2+1}{2(2)}$ ✓

Now we assume that $P_k = \left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{k^2}\right) = \frac{k+1}{2k}$

Goal: Show that $\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{k^2}\right)\left(1-\frac{1}{(k+1)^2}\right) = \frac{(k+1)+1}{2(k+1)}$

Therefore we have $\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{k^2}\right)\left(1-\frac{1}{(k+1)^2}\right) = \frac{k+1}{2k}\left(1-\frac{1}{(k+1)^2}\right)$

$= \underline{(\text{do algebra})}$

$\vdots$

$= \frac{k+2}{2(k+1)}$ ☺

ex) If $A_1 \ldots A_n$ are sets, for $n \geq 2$ then

$$(A_1 \cup \cdots \cup A_n)^c = A_1^c \cap A_2^c \cap \cdots \cap A_n^c$$

Pf) We will prove this using general induction.

For the base case we have to show

$$(A_1 \cup A_2)^c = A_1^c \cap A_2^c$$

This is true by De-Morgan's law, which we proved earlier.

Now we will assume that $\exists k \geq 2$ such that

$$(A_1 \cup \cdots \cup A_k)^c = A_1^c \cap A_2^c \cap \cdots \cap A_k^c$$

Goal: $(A_1 \cup \cdots A_k \cup A_{k+1})^c = A_1^c \cap \cdots \cap A_k^c \cap A_{k+1}^c$

Let $B = A_1 \cup \cdots \cup A_k$. Then

$$\begin{aligned}
(A_1 \cup \cdots \cup A_k \cup A_{k+1})^c &= (B \cup A_{k+1})^c \\
&= B^c \cap A_{k+1}^c \quad \text{(base case)} \\
&= (A_1 \cup \cdots A_k)^c \cap A_{k+1}^c \quad \text{(by def)} \\
&= A_1^c \cap A_2^c \cap \cdots \cap A_k^c \cap A_{k+1}^c
\end{aligned}$$

(induction hyp)

ex) Let $n \geq 2$, and let $f_1 \ldots f_n$ be differentiable functions. Prove that

$$\frac{(f_1 \cdots f_n)'}{f_1 \cdots f_n} = \frac{f_1'}{f_1} + \frac{f_2'}{f_2} + \cdots + \frac{f_n'}{f_n}$$

pf) Left as exercise

Strong Principle of Induction

The strong principle of mathematical induction is a variant of the first principle of induction that we saw

$\leadsto$ this will be usefull for recursive problems

Still trying to prove statements like

$\forall n \geq m, P(n)$ for some fixed $m \in \mathbb{Z}_{\geq 0}$

This time though, our steps will be

1) Base Case : $P(m)$ true

2a) Inductive hypothesis: Assume true for $m \leq i \leq k$ for some k.

$\ulcorner P(m) \wedge P(m+1) \wedge \cdots \wedge P(k) \urcorner$

2b) Inductive leap

$P(m) \wedge P(m+1) \wedge \cdots \wedge P(k) \implies P(k+1)$ true.

<u>Thm: Prime Factorization of the integers</u>

Every positive integer $n \geq 2$ is a product of prime #'s.

Pf) We will use strong induction to prove this. Our base is $n=2$, which is true because 2 is itself prime. Now let $k \geq 2$ and assume that every integer l, $2 \leq l \leq k$ can be expressed as a product of prime #'s.

[Goal: $K+1$ also a product of prime #'s]

Note $K+1$ is either prime or not prime.
If $K+1$ is prime then the statement is true.
Therefore we only have to consider when $K+1$ is not prime. Therefore we have

$$K+1 = ab \quad \text{where} \quad \begin{matrix} 1 < a < K+1 \\ 1 < b < K+1 \end{matrix} \longleftrightarrow \begin{matrix} 2 \le a \le K \\ 2 \le b \le K \end{matrix}$$

Since $2 \le a \le K$ and $2 \le b \le K$ our induction hypothesis tells us that we can write a and b as a product of prime #'s.
Hence so is $K+1$ ☺

Corollary: Every integer $n \geq 2$ has at least one prime # dividing it.

Corollary: There are ∞ many prime #'s.

Pf) Let's assume FSOC that there are only finitly many primes. Lets call them $p_1 \cdots p_k$

Consider $n = p_1 \cdots p_k + 1$

Note that none of the primes divide n $\rightarrow\leftarrow$

Recall: ~~Last class~~ earlier we defined the sequence

$a_1 = 1$

$a_2 = 4$

$a_n = 2a_{n-1} - a_{n-2} + 2$ for $n \geq 3$.

$\rightsquigarrow$ Conjectured that $a_n = n^2$. Let's prove it!

Pf) We will prove this using strong form of mathematical ind.

The conjecture is true for $n = 1, 2$. We earlier also saw that it is true for $n = 3$.

So the base case holds. So now assume

The conjecture is true for $3 \le i \le k$ for some k.

Goal: $a_{k+1} = (k+1)^2$

We know that

$$a_{k+1} = 2a_k - a_{k-1} + 2$$

$$= 2(k)^2 - (k-1)^2 + 2 \quad \text{(by induction hyp)}$$

$$= 2k^2 - (k^2 - 2k + 1) + 2$$

$$= 2k^2 - k^2 + 2k - 1 + 2$$

$$= k^2 + 2k + 1$$

$$= (k+1)^2$$