Coordinate Vectors

- **Recall**: We say $\mathcal{B} = (v_1, \ldots, v_n)$ is a __________ for V if

  1)

  2)

$\iff$ every vector $w \in V$

This gives us a way to "name" vectors easilly

Def: Let $\mathcal{B} = (v_1 \ldots v_n)$ be ________ for V

Let $w \in V$ be arbitrary and write $w =$ ________

Then we call the vector

$$[\quad]_{\mathcal{B}} = \begin{pmatrix} \\ \\ \\ \end{pmatrix} \in \mathbb{F}^n$$

the ________ of w ________

this allows us to pass questions about "abstract vectors"
to ______________

Exc: Let $\mathcal{B} = (v_1, \ldots, v_n)$ be basis for V.

1) A vector $w \in \operatorname{span}(z_1, \ldots, z_k)$

$\iff [w]_{\mathcal{B}} \in \operatorname{span}([z_1]_{\mathcal{B}}, \ldots, [z_k]_{\mathcal{B}})$

2) A list of vectors $w_1, \ldots, w_k$ are LI in V

$\iff [w_1]_{\mathcal{B}}, \ldots, [w_k]_{\mathcal{B}}$ are LI in $\mathbb{F}^n$

Potential issue

- these "______________" are dependent on $\mathcal{B}$

- **Q?** What if we chose a different basis $\mathcal{B}'$

That is, how are

$$[\quad]_{\mathcal{B}} \qquad\qquad [\quad]_{\mathcal{B}'}$$

related?

- To answer this, we will turn to ______

Linear transformations

- What should be the "proper" def of a function between 2 $\mathbb{F}$-vs V, W?

    - we can ________ in V, W
    - we can ________ by a $\# \in \mathbb{F}$

  $\rightsquigarrow$ the functions should preserve this.

[... whispers .... category theory]

Def: A function $T: V \to W$ between 2 $\mathbb{F}$-vs is

a ______________ if

1)

2)

ex) i) $V = \mathbb{F}^n$, $W = \mathbb{F}^m$ and $A \in M_{m \times n}(\mathbb{F})$

Then $T(\vec{x}) :=$ ______________

ii) Let $S \xrightarrow{f} S'$ be a function of sets.

This gives a linear map

$$\mathrm{Fct}(\underline{\quad}, \mathbb{F}) \xrightarrow{f^*} \mathrm{Fct}(\underline{\quad}, \mathbb{F})$$

$$g \longmapsto \underline{\qquad}$$

Pf)

Properties of linear maps

Def: Let $T: V \to W$ be linear. Then

1) the __________ of T is the subset

$\subseteq V$

____

2) The __________ of T is the dimension of

____

3) The __________ (or ________) of T

is ______ $\subseteq W$

4) The ______ of T is the dimension of ______

Warm up exc: 1) Show ______ $\subseteq V$ is a subspace

2) Show ______ $\subseteq W$ is a subspace.

pf)

Recall: We say a function $f: X \to Y$ is ______________ if

____

Prop: $T: V \to W$ linear. Then T is ______________ iff

____

Pf)

Recall: We say a function $f: X \to Y$ is ________ if
$\exists$ ________ $: Y \to X$ st ________ and
____

Prop: Let $T: X \to Y$ be a ________ linear map. Then
________ is also linear.

Pf)

<u>Def</u>: We call a ______________ linear map $T: V \to W$ an ______________

and we write ______________

Ways to think about basis through lense of linear maps

Prop: Let V be fd with basis $\mathcal{B} = (v_1, \dots, v_n)$
For any other vector space W and vectors $y_1 \dots y_n \in W$
$\exists!$ linear map $T: V \to W$ st

$$T(v_1) = \_\_\_, \quad \dots, \quad T(v_n) = \_\_\_$$

(ie, we can send basis anywhere we want, and that uniquely defines the map)

Pf)

Prop: V fd with basis $\mathcal{B} = (v_1, \ldots, v_n)$, and let

$T: V \to W$ be a linear map. Then

1) T is injective $\Longleftrightarrow$

HW 2) T surjective $\Longleftrightarrow$

3) T isomorphism $\Longleftrightarrow$

Pf) 1)

Cor: 1) Let $\dim(V) = n$. Then $V \simeq \mathbb{F}^n$

2) 2 vectors spaces are isomorphic iff they have the same dimension

Pf)

__Rmk__: The proof of (1) tells us the following: A choice of basis amounts to choosing an isomorphism

$$\varphi_B : \mathbb{F}^n \xrightarrow{\sim} V$$

• Under this recognition

$$[\ v\ ]_B =$$

- So, if we chose two bases then we have

$$\mathbb{F}^n \xrightarrow[\sim]{\varphi_B} V \xleftarrow[\sim]{\varphi_{B'}} \mathbb{F}^n$$

Then since $\varphi_B, \varphi_{B'}$ are bijective

$$\varphi_B(\qquad) = v = \varphi_{B'}(\qquad)$$

$$\begin{tikzcd} \mathbb{F}^n \arrow[r, "\varphi_B"] & V \\ & \mathbb{F}^n \arrow[u, "\varphi_{B'}"'] \arrow[ul, "\_\_\_"'] \end{tikzcd}$$

$$\Rightarrow [\qquad] =$$

$$\Rightarrow [\qquad] = \_\_\_\,[\quad]$$ for some bijective map $\_\_\_ : \mathbb{F}^n \to \mathbb{F}^n$

Connection to Matrices

- Recall we started today by discussing ______________

and we wanted to know how they changed if we

" ______________ " from $\mathcal{B} \to \mathcal{B}'$

We are now in a spot to answer this

Thm: Let $V \xrightarrow{T} W$ be linear and let

$\mathcal{B} = (v_1 \ldots v_n)$ $\quad \mathcal{C} = (w_1 \ldots w_m)$ be basis for V, W.

Then $\exists!$ matrix $A \in M_{m\times n}(\mathbb{F})$ such that

$$= A \qquad \forall v \in V$$

(We often write $A := [T]_{\mathcal{B}}^{e}$ the matrix of T wrt $\mathcal{B}, e$)

Pf)

Cor: Let V be a vector space and let $\mathcal{B}, \mathcal{B}'$ be two basis for V.

Consider the map

$$(V, \mathcal{B}) \longrightarrow (V, \mathcal{B}')$$

(that is $\longrightarrow$)

Then $[\quad]_{\mathcal{B}}^{\mathcal{B}'} := P_{\mathcal{B}}^{\mathcal{B}'}$ is called the

and $\forall v \in V$

$[\quad]_{\mathcal{B}'} = \qquad [\quad]_{\mathcal{B}}$

$\rightsquigarrow$ Justifies the name " "

Cor 2: a) If V is n-dim, W m-dim then there is an isomorphism

$$\underline{\qquad\qquad} \xrightarrow{\sim} \underline{\qquad\qquad}$$

b) $\dim(\qquad\qquad) =$

More generally: V W vector spaces B_V, B_V' and B_W, B_W' basis for V, W respectively.

If $T: V \to W$ linear then how are

$[\qquad\qquad]$ and $[\qquad\qquad]$ related?

⌜ • Consider the composable maps

$$\underbrace{V \xrightarrow{T} W \xrightarrow{S} Z}_{S \circ T}$$

1) $[T]$

2) $[S]$

3) $[S \circ T] =$ ______________ !!!

(see HW for example)

⌟

Back to the question above: Write $(V, \mathcal{B}_V)$ to emphasize what basis using; Write $P = P_{\mathcal{B}_V}^{\mathcal{B}_V'}$ and $Q = P_{\mathcal{B}_W}^{\mathcal{B}_W'}$

$(V, \mathcal{B}_V)$ $(W, \mathcal{B}_W)$

$(V, \mathcal{B}_V')$ $(W, \mathcal{B}_W')$