10/12/21 Section

- From HW 3 on you must type your solutions.
- They will also be submitted on Gradescope

Coordinate vectors

- Let V be fd vector space with basis $\mathcal{B} = (v_1, \dots, v_n)$.
  - Let $w \in V$
  - Then $w = c_1 v_1 + c_2 v_2 + c_3 v_3 + \dots + c_n v_n$

$$\Rightarrow [w]_{\mathcal{B}} :\overset{\text{def}}{=} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \in \mathbb{F}^n$$

ex) $V = \mathbb{R}_2[x]$ $\quad \mathcal{B} = (1, x, x^2)$

Find $[2 - 10x + 20x^2]_{\mathcal{B}} = \begin{pmatrix} 2 \\ -10 \\ 20 \end{pmatrix}$

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$2(1) - 10(x) + 20(x^2)$

ex) $V = \mathbb{R}_2[x]$ $\quad \mathcal{B}' = (2+x,\ 3+x^2,\ x-x^2)$

Find $[2 - 10x + 20x^2]_{\mathcal{B}'} = \begin{pmatrix} 28 \\ 18 \\ -38 \end{pmatrix}$

Goal: Express

$$2 - 10x + 20x^2 = c_1(2+x) + c_2(3+x^2) + c_3(x-x^2)$$

$$2 - 10x + 20x^2 = 2c_1 + 3c_2 + (c_1 + c_3)x + (c_2 - c_3)x^2$$

$$\Rightarrow \begin{aligned} 2c_1 + 3c_2 \quad\quad &= 2 \\ c_1 \quad\quad + c_3 &= -10 \\ c_2 - c_3 &= 20 \end{aligned} \longrightarrow \left(\begin{array}{ccc|c} 2 & 3 & 0 & 2 \\ 1 & 0 & 1 & -10 \\ 0 & 1 & -1 & 20 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 3/2 & 0 & 1 \\ 0 & -3/2 & 1 & -11 \\ 0 & 1 & -1 & 20 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 3/2 & 0 & 1 \\ 0 & 3 & -2 & 22 \\ 0 & 1 & -1 & 20 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 3/2 & 0 & 1 \\ 0 & 1 & -2/3 & 22/3 \\ 0 & 0 & -1/3 & 38/3 \end{array}\right) \Rightarrow \boxed{\begin{aligned} c_1 &= 28 \\ c_2 &= -18 \\ c_3 &= -38 \end{aligned}}$$

Q: How are

$$[2-10x+20x^2]_{B} \quad \text{and} \quad [2-10x+20x^2]_{B'}$$

related?

A: By the "Change of Basis Matrix"
$P_{B \to B'}$

That is $[2-10x+20x^2]_{B'} = P_{B \to B'} \, [2-10x+20x^2]_{B}$

$\mathcal{B} = (1, x, x^2)$

$\mathcal{B}' = (2+x,\ 3+x^2,\ x-x^2)$

Check: Find $P_{\mathcal{B}\to\mathcal{B}'}$

$$P_{\mathcal{B}\to\mathcal{B}'} = \begin{pmatrix} [1]_{\mathcal{B}'} & [x]_{\mathcal{B}'} & [x^2]_{\mathcal{B}'} \\ \downarrow & \downarrow & \downarrow \end{pmatrix}$$

- $1 = c_1(2+x) + c_2(3+x^2) + c_3(x-x^2)$

$1 = 2c_1 + 3c_2 + (c_1 + c_3)x + (c_2 - c_3)x^2$

$2c_1 + 3c_2 = 1$

$\Rightarrow c_2 = c_3 \qquad \Rightarrow -c_1 = 1 \qquad c_1 = -1$

$c_1 = -c_3 \qquad\qquad\qquad\qquad c_2 = c_3 = 1$

$$[1]_{\mathcal{B}'} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

Q: Why is $P_{\mathcal{B} \to \mathcal{B}'}$ defined this way?
Why does it "work"?

To answer this, need to talk about linear transformations.

Recall: A linear transf $T: V \longrightarrow W$ is a function

st

1) $T(v_1+v_2) = T(v_1) + T(v_2)$

2) $T(cv) = cT(v)$

To compute things, we chose basis for V, W

- doing so gives us a Matrix

Let $\mathcal{B}_V = (v_1, \dots, v_n)$ be basis for V

$\mathcal{B}_W = (w_1, \dots, w_m)$ W

- Then note for any vector $v \in V$

$$T(v) = c_1 w_1 + c_2 w_2 + \cdots + c_m w_m$$ for unique $c_1 \ldots c_n$

- In particular

$$T(v_1) = c_1 w_1 + \cdots + c_m w_m \qquad [T(v_1)]_{\mathcal{B}_W}$$

$$\vdots \qquad \Rightarrow \qquad \vdots$$

$$T(v_n) = \tilde{c}_1 w_1 + \cdots + \tilde{c}_m w_m \qquad [T(v_n)]_{\mathcal{B}_W}$$

Def: $[T]_{B_V}^{B_W} = \begin{pmatrix} [T(v_1)]_{B_W} & [T(v_2)]_{B_W} & \cdots & [T(v_n)]_{B_W} \\ \downarrow & \downarrow & & \downarrow \end{pmatrix}_{m \times n}$

Fact: $[T(v)]_{B_W} = \underline{[T]_{B_V}^{B_W}} \, [v]_{B_V}$

Back to Change of Basis matrix

Let $\mathcal{B}_1$, $\mathcal{B}_2$ be 2 basis for V

Consider the identity transformation

$$\mathbb{1} : (V, \mathcal{B}_1) \longrightarrow (V, \mathcal{B}_2)$$

from V with basis $\mathcal{B}_1$ to V with basis $\mathcal{B}_2$.

(that is $\mathbb{1}(v) = v$)

Q: What is $[\mathbb{1}]_{B_1}^{B_2}$?

$B_1 = (v_1 \ldots v_n)$

$B_2 = (w_1 \ldots w_n)$

$$[\mathbb{1}]_{B_1}^{B_2} = \begin{pmatrix} [\mathbb{1}(v_1)]_{B_2} & \cdots & [\mathbb{1}(v_n)]_{B_2} \\ \downarrow & & \downarrow \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} [v_1]_{B_2} & \cdots & [v_n]_{B_2} \\ \downarrow & & \downarrow \end{pmatrix}}$$

$$= P_{B_1 \to B_2}$$

ex) $V = \mathbb{R}_2[x]$ $\quad \mathcal{B}_V = (1, x, x^2)$

$W = M_{2\times 2}(\mathbb{R})$ $\quad \mathcal{B}_W = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$

$T : V \longrightarrow W$ by

$$T(a_0 + a_1 x + a_2 x^2) = \begin{pmatrix} a_0 & a_1 - a_2 \\ 0 & a_1 + a_2 \end{pmatrix}$$

Find $[T]_{\mathcal{B}_V}^{\mathcal{B}_W}$

$$\bullet\ T(1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow [T(1)]_{B_W} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\bullet\ T(x) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \Rightarrow [T(x)]_{B_W} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\bullet\ T(x^2) = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix} \Rightarrow [T(x^2)]_{B_W} = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow [T]_{B_V}^{B_W} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

- $v = 2 + 3x + 4x^2$

$$T(v) = \begin{pmatrix} 2 & 3-4 \\ 0 & 3+4 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 0 & 7 \end{pmatrix}$$

$$\Rightarrow \left[T(v) \right]_{B_W} = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 7 \end{pmatrix}$$

<u>Check</u>: $\left[2 + 3x + 4x^2 \right]_{B_V} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$

<u>Claim</u>: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \overset{?}{=} \begin{pmatrix} 2 \\ -1 \\ 0 \\ 7 \end{pmatrix}$

$\parallel$

$$2\begin{pmatrix}1\\0\\0\\0\end{pmatrix}+3\begin{pmatrix}0\\1\\0\\1\end{pmatrix}+4\begin{pmatrix}0\\-1\\0\\1\end{pmatrix}$$

$$=\begin{pmatrix}2\\0\\0\\0\end{pmatrix}+\begin{pmatrix}0\\3\\0\\3\end{pmatrix}+\begin{pmatrix}0\\-4\\0\\4\end{pmatrix}=\begin{pmatrix}2\\-1\\0\\7\end{pmatrix}\checkmark$$