Yiddish of the Day

to drag = shlepn = שלעפּן

to sweat = shvitzn = שוויצן

dirt = shmutz = שמוץ
(shmitz)

person = mentch = מענטש

Last time

- <u>quantified statements</u>
  - <u>$\forall$</u> = for all
  - <u>$\exists$</u> = there exists

  + how to negate them

- <u>Proofs</u>
  - <u>trivial</u> proofs and <u>vacuous</u> proofs
  - <u>Direct</u> proofs (how to prove that $P \Rightarrow Q$)

Contrapositive

Thm: Let P and Q be statements. Then

$$P \Rightarrow Q \equiv \neg Q \Rightarrow \neg P$$

Pf) $\neg Q \Rightarrow \neg P \equiv \neg(\neg Q) \vee (\neg P)$

$\equiv Q \vee (\neg P)$ (double negation law)

$\equiv \neg P \vee Q$ (commutative law)

$\equiv P \Rightarrow Q$

$\square$

A proof by contrapositive of $P \Rightarrow Q$ is giving a direct proof of $\underline{\neg Q \Rightarrow \neg P}$

ex 1) Let $x \in \mathbb{Z}$. Show that if $5x-7$ is even, then x is odd.

Pf) Rather than proving directly, which could be done, but is much harder (see pg 24 on typed notes), we will use the method of proof by contrapositive.
That is we will prove that if x is even then $5x-7$ is odd. We then assume that x is even.
Therefore we can write $x = 2k$ for $k \in \mathbb{Z}$.

Now we compute $5x-7 = 5(2k)-7$

$= 10k - 7$

$= 2(5k-4)+1$

Re-write the following as a contrapositive statement.

ex1) Suppose $x, y \in \mathbb{R}$. If $y^3 + yx^2 \leq x^3 + xy^2$ then $y \leq x$.

- We will use proof by contrapositive. That is we want to show that if $y > x$ then $y^3 + yx^2 > x^3 + xy^2$

So let us assume that $y > x$.

ex2) Suppose $x, y \in \mathbb{Z}$. If xy is not divisible by 5 then x is not divisible by 5 and y is not divisible by 5.

- Lets say $P(x,y)$:
  $Q(x,y)$:

→ So a proof by contraposition would be showing:

$$\neg Q(x,y) \Rightarrow \neg P(x,y)$$

- $\neg Q(x,y)$ = x is divisible by 5 <u>or</u> y is divisible by 5
- $\neg P(x,y)$ = xy <u>is</u> divisible by 5

$\rightsquigarrow$ So we want to show: If x or y is divisible by 5 then xy is divisible by 5.

WARNING!!!

- Proving <u>$P \Rightarrow Q$</u> is equivalent to proving the <u>contrapositive</u> <u>$\neg Q \Rightarrow \neg P$</u>.
- This is **NOT** the same as proving <u>$\neg P \Rightarrow \neg Q$</u>

Biconditional Statements

- **Recall**: We have

$$\underline{P \Longleftrightarrow Q} \equiv (P \Rightarrow Q) \land (Q \Rightarrow P)$$

and we say P <u>iff</u> Q

$\rightsquigarrow$ To prove $\underline{P \Longleftrightarrow Q}$ need to prove two things

1) $P \Rightarrow Q$

2) $Q \Rightarrow P$

Describe what needs to be proved for the following

ex1) Let $a \in \mathbb{Z}$. Then $a^3 + a^2 + a$ is even iff a is even

(contrapos) • If a^3+a^2+a is even then a is even $(P \Rightarrow Q)$

(direct) • If a is even then a^3+a^2+a even $(Q \Rightarrow P)$

ex2) Suppose $x, y \in \mathbb{R}$. Then $(x+y)^2 = x^2+y^2$ iff $x=0$ or $y=0$

• If $(x+y)^2 = x^2+y^2$ then $x=0$ or $y=0$

• If $x=0$ or $y=0$ then $(x+y)^2 = x^2+y^2$

Now lets prove something

ex) An integer n is even iff n^2 is even.

Pf) We will first prove that if n is even then n^2 is even.
So assume n is even. That is, $n = 2k$ for $k \in \mathbb{Z}$.
Then we compute that

$$\begin{aligned} n^2 &= (2k)^2 \\ &= 4k^2 \\ &= 2(2k^2) \quad \text{where } 2k^2 \in \mathbb{Z} \end{aligned}$$

Hence n^2 is also even.
Now we will prove that if n^2 is even then n is even.
We will prove the contrapositive; that is we will prove that if n is odd then n^2 is odd.

Assume n is odd, that is $n = 2\ell + 1$ $\ell \in \mathbb{Z}$.

Now we compute

$$\begin{aligned} n^2 &= (2\ell+1)^2 \\ &= 4\ell^2 + 4\ell + 1 \\ &= 2(\ell^2 + 2\ell) + 1 \quad \text{where } \ell^2 + 2\ell \in \mathbb{Z} \end{aligned}$$

hence n^2 is odd as desired

We have proven that n odd implies n^2 odd, which by contrapositive was equivalent to the statement n^2 even implies n even $\square$

Proof by Cases

(i.e., are both even or both odd)

ex) Let $x, y \in \mathbb{Z}$. Show that x and y have the same parity iff x+y is even.

- $P(x,y) =$
- $Q(x,y) =$

$\rightsquigarrow$ $P \Rightarrow Q$: If x and y have the same parity then x+y is even

$Q \Rightarrow P$: If x+y even then x and y have the same parity

Pf) First we will prove that if x and y have the same parity then $x+y$ is even. We can consider 2 cases.

- Case 1: We assume x, y are both even. Then we can write $x = 2k$ and $y = 2l$ for some $k, l \in \mathbb{Z}$.

  We then compute $x+y = 2k + 2l = 2(k+l)$ for $k+l \in \mathbb{Z}$ and hence is also even

- Case 2: We assume that x and y are both odd
  Then we can write

$$x = 2m+1 \qquad \text{for } m,n \in \mathbb{Z}$$
$$y = 2n+1$$

Again we compute

$$x+y = 2m+1 + 2n+1 = 2(m+n+1) \text{ where } m+n+1 \in \mathbb{Z}$$

which is again even

Now we prove that if $x+y$ is even then x and y have the same parity.

Scratch work: Proving $Q(x,y) \Rightarrow P(x,y)$

$\leadsto$ Contrapositive is $\neg P(x,y) \Rightarrow \neg Q(x,y)$

What is $\neg P(x,y)$: x and y have diff parity

$\neg Q(x,y)$: $x+y$ is odd

We will prove the contrapositive : that is we will prove that if x and y have a different parity, then $x+y$ is odd

Scratch work : Could consider two cases

1) x even y odd

2) x odd y even

however, the situation is symmetric so to prove both cases can really just prove one case

Without loss of generality, assume that x is even and y is odd. That is $x = 2k$, $k \in \mathbb{Z}$ and $y = 2\ell + 1$ for $\ell \in \mathbb{Z}$. Now we compute

$$x + y = 2k + 2\ell + 1$$
$$= 2(k+\ell) + 1 \quad \text{with } k+\ell \in \mathbb{Z}$$

So $x+y$ is odd.

Since the other case (x odd, y even) is symmetric we have proven that case as well. Hence we have proven that if x, y have diff parity then $x+y$ is odd, which by contrapositive is equivalent to the claim if $x+y$ even then x, y have same purity ☺