Math 117

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OH: T 4-5 PM

Th 11-12

Other resources:

- Linear Alg done Right - Axler
- Linear Alg done wrong - Treil
- Abstract Algebra - Dummit, Foote

Spanning, Linear Ind, Basis

Q: Let S be spanning set for V

L be LI set for V

B be basis for V

1) $|S| \geq |L|$

2) $|S| \geq |B|$

ex) $\mathbb{R}^2$ has $|B|=2$

$W = \text{span}\left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right)$

3) $|L| \leq |B|$

- Is there a linear ind set L with $|L| = 10$ in $\mathbb{F}_3^3$?

$\dim \mathbb{F}_3^3 = 3$ $\qquad$ $B = \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$

ex) Let $X = \overset{x_1}{\bullet}$ and $V = \mathbb{F}_5^2$

Consider $Fun(X, V)$. Find $|Fun(X, V)|$

How many functions are there from $X \to V$.

- A function $f: X \to V$ will take x_1 as input and spit out a vector in V.
- When x_1 goes determines a new function,
- x_1 can be mapped to $5^2 = 25$ diff vectors.

ex) X: $\overset{x_1}{\bullet}$ $\overset{x_2}{\bullet}$ $\qquad V$: $\mathbb{F}_3^4$

Again find $|\mathrm{Fun}(X, V)|$

ex) $f(x_1) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ $\qquad f(x_1) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$f(x_2) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $\qquad f(x_2) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

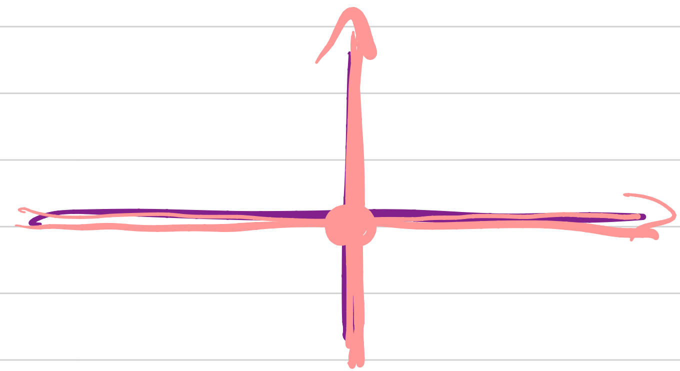
$$\underline{3^4} \cdot \underline{3^4} = 3^8$$

Subspaces and Direct Sum

ex) 1) $V = \mathbb{R}^2$

$W_1 = \text{span}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$W_2 = \text{span}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

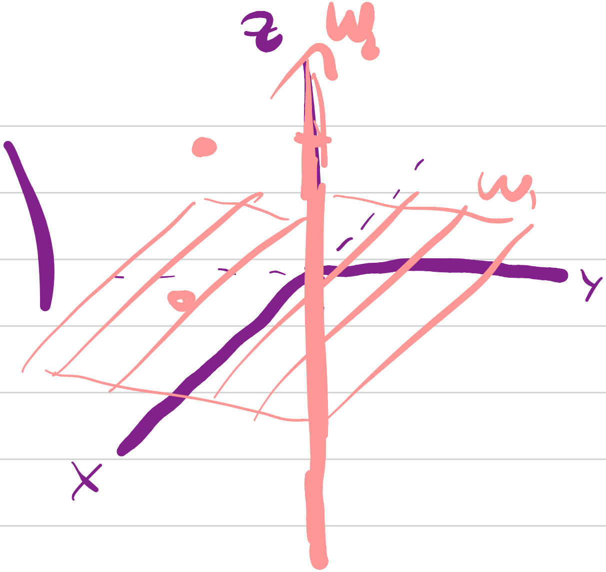


$\Rightarrow V = W_1 + W_2$

$\Rightarrow V = W_1 \oplus W_2$

2) $V = \mathbb{R}^3$

- $W_1 = \text{span}\left(\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right)$
- $W_2 = \text{span}\begin{pmatrix}0\\0\\1\end{pmatrix}$

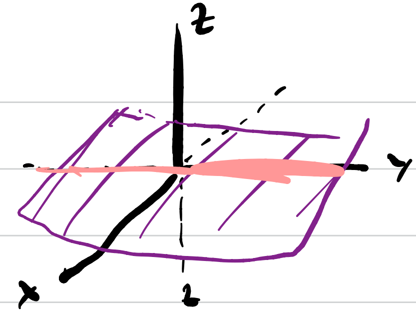


$V = W_1 + W_2$

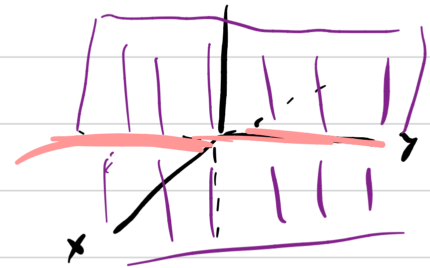
$V = W_1 \oplus W_2$

3) $V = \mathbb{R}^3$

xy-plane $\cdot\ W_1 = \left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} : x, y \in \mathbb{R} \right\}$



yz-plane $\cdot\ W_2 = \left\{ \begin{pmatrix} 0 \\ x \\ y \end{pmatrix} : x, y \in \mathbb{R} \right\}$



$V = W_1 + W_2$ but not direct sum

Q: $|V| = 3$ $\quad |W_1| = 2$ $\quad |W_2| = 2$

$|W_1 \cap W_2| = 1$

Note: $|V| = |W_1 + W_2| = |W_1| + |W_2| - |W_1 \cap W_2|$

This formula holds in general.

Q: $V = \mathbb{R}^4$

- $W_1 = \operatorname{span}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $\qquad W_2 = \left\{ \begin{pmatrix} x \\ 0 \\ y \\ 0 \end{pmatrix} ; x, y \in \mathbb{R} \right\}$

- $W_3 = \operatorname{span}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

Find

- $\dim W_1 = 1$
- $\dim W_2 = 2$
- $\dim W_3 = 1$

Q: Is it possible that

$$|W_1 + W_2 + W_3| = |W_1| + |W_2| + |W_3| \ ?$$

- $\dim(W_1 + W_2 + W_3) = \dim(W_2) = 2$
- $\dim(W_1 \cap W_2) = 1$
- $\dim(W_1 \cap W_3) = 0$
- $\dim(W_2 \cap W_3) = 1$
- $\dim(W_1 \cap W_2 \cap W_3) = 0$

Note: $\underbrace{\dim(W_1+W_2+W_3)}_{\shortparallel \; 2}$

$$2 = \dim(W_1) + \dim(W_2) + \dim(W_3)$$
$$- \dim W_1 \cap W_2 - \dim W_1 \cap W_3 - \dim W_2 \cap W_3$$
$$+ \dim(W_1 \cap W_2 \cap W_3)$$

Pigeonhole Principle

ex) $\mathbb{F}_5$

Q: How many nonzero elements are in $\mathbb{F}_5$

A = 4

Q2: Let $a \neq 0$ in $\mathbb{F}_5$.

Is $a^n = 0$ in $\mathbb{F}_5$?

Suppose $a^n = 0$ in $\mathbb{F}_5$

Then $5 \mid a^n = a \cdot a \cdots a$

Recall: p prime iff whenever $p \mid ab$, $p \mid a$ or $p \mid b$

(Q2 a) Is this true in $\mathbb{Z}/4\mathbb{Z}$? No!)

Consider the list

$a^0 \quad a^1 \quad a^2 \quad a^3 \quad a^4$

This is a list of 5 nonzero #'s.

$\Rightarrow a^i = a^j$ for some some $j < i \leq p-1$

$\Rightarrow a^{i-j} = 1$ in $\mathbb{F}_S$

$\Rightarrow a^{i-j} \equiv 1 \mod S$ with $0 < i-j \leq p-1$