"Free"-vector Space

- Recall: A vector space V is a "set with extra steps"
  - its a set, with all the axioms

Goal: Go "the other way"

- Given an arbitrary set X can we find a vector space, which we will denote, $\text{Free}(X)$ that has X as its basis?

$\longrightarrow$ Why might this be usefull?

- Recall, a linear transformation

$$T: V \longrightarrow W$$

is uniquely determined by where it sends the basis

- So, suppose $Fre(X)$ does exist

  - then any function $f: X \longrightarrow W$

  into a vector space W

  can be uniquely extended to a

linear function $\text{Free}(f) : \text{Free}(X) \to W$

Put more succinctly, there is a bijection of sets

$$\mathcal{L}(\text{Free}(X), W) \simeq \text{Fcts}(X, W)$$

between linear maps from $\text{Free}(X) \to W$ and arbitrary functions $X \to W$

- This will show up in the construction of the $\otimes$-product

OK, so the upshot is

Thm: X any set. Then $\text{Free}(X)$ exists

(in words, for any set X, there is a vector space with X as its basis)

Pf) Recall the vector space $\mathbb{F}^X := \text{Fcts}(X, \mathbb{F})$

We define a subset of this as

$$F^{(X)} := \left\{ f : X \to \mathbb{F} \;\middle|\; \begin{array}{l} f(x)=0 \text{ for all but finitely} \\ \text{many } x \in X \end{array} \right\}$$

$\ulcorner$ if X is finite $F^{(X)} = F^X$ but in general they differ $\lrcorner$

Then define the "indicator function"

$$e_x : X \to \mathbb{F} \text{ by}$$

$$e_x(y) = \begin{cases} 1 & y = x \\ 0 & \text{else} \end{cases}$$

$\boxed{\underline{\text{Fact}}: (e_x)_{x \in X} \text{ is a basis for } F^{(X)}}$

Showing LI is easy. Spanning is not much harder.

If $f \in F^{(X)}$ then there's a finite set

$(x_1 \dots x_n)$ such that f is non-zero on.

Then check that $f \in \operatorname{span}(e_{x_1}, \dots, e_{x_n})$

We can now identify $x \longleftrightarrow e_x$

Thus we have a vector space $F^{(X)}$ with a basis

that's in bijection with X

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