Complexification

<u>Idea</u>: $\mathbb{R} =$

$\mathbb{C} =$

$\rightsquigarrow$ turn _____-vs into _____-vs

<u>Recall</u>: $\mathbb{C}$ is a _____-vector space (of dim _____)

Def: Let V be $\mathbb{R}$-vs. Then the ________ of V

denoted ____ is ________

Prop: V a $\mathbb{R}$-vs with basis $\mathcal{B} = (v_1, \ldots, v_n)$. Then

1) ____ is a $\mathbb{C}$-vs (with scalar mult $\alpha \cdot (\quad) = \quad$)

2) A $\mathbb{C}$-basis of V is $\mathcal{B}_{\mathbb{C}} =$

(ie $\dim_{\mathbb{R}} V =$ ________)

P4)

ex) i) $V = \mathbb{R}^n \rightsquigarrow V_{\mathbb{C}} = \mathbb{C} \otimes \mathbb{R}^n \xrightarrow{\sim}$ ______

ii) $V = M_{m \times n}(\mathbb{R}) \rightsquigarrow V_{\mathbb{C}} = \mathbb{C} \otimes M_{m \times n}(\mathbb{R}) \xrightarrow{\sim}$

iii) $V = \mathbb{R}[t]_{\leq n} \rightsquigarrow V_{\mathbb{C}} = \mathbb{C} \otimes \mathbb{R}[t]_{\leq n} \xrightarrow{\sim}$

<u>Warning</u>: If we view $V = \mathbb{C}$ as a $\mathbb{R}$-vs then

$V_{\mathbb{C}} \neq$ ______

(HW)

The complexification is more than just a $\mathbb{C}$ vs constructed out of an $\mathbb{R}$-vs. It is "the best" (universal) one

Thm: V be an $\mathbb{R}$-vs and W a $\mathbb{C}$-vs.

Suppose $f: V \to W$ is an $\mathbb{R}$-linear map.

Then $\exists!$ $\mathbb{C}$-linear map $\tilde{f}: \_\_\_\_ \to W$

such that

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "="] & W \\ \arrow[ur, "\exists!\,\tilde{f}"'] & \end{tikzcd}$$

Now let $g: V \to V'$ be linear map of $\mathbb{R}$-vs.

Then by HW, have linear map

$$\_\_\_ : \_\_\_ \otimes V \longrightarrow \_\_\_ \otimes V'$$

$$\begin{array}{ccc} \_\_\_ \otimes V & & \_\_\_ \otimes V' \\ \| & & \| \\ \_\_\_ & & \_\_\_ \end{array}$$

(Denote $\_\_\_ := \_\_\_$)

Thm: Keeping the notation above, let

$\mathcal{B}_V = (v_1 \dots v_n)$ basis for V, V' $\rightsquigarrow$ $\mathcal{B}_{V_{\mathbb{C}}}$ complexified basis

$\mathcal{B}_{V'} = (w_1 \dots w_m)$ $\mathcal{B}_{V'_{\mathbb{C}}}$

Then $[\varphi]_{\mathcal{B}_V}^{\mathcal{B}_{V'}} =$ ______

Pf)

Cor: $\varphi : V \to V'$ linear of $\mathbb{R}$-vs. Then

i) $(\operatorname{Ker}\varphi)_{\mathbb{C}} =$

ii) φ injective $\iff$

iii) $(\operatorname{im}\varphi)_{\mathbb{C}} =$

iv) φ surjective $\iff$

Conjugation:

Recall the function $\overline{(\ )} : \mathbb{C} \to \mathbb{C}$

$$z \longrightarrow \_\_\_\_$$

- then note that, for $r \in$ ____

$$(\overline{r}) = \_\_\_\_$$

On the other hand if $(\overline{z}) =$ ____

then ____

$\rightsquigarrow$ that is we can recover ____ as the

____

Def: V be $\mathbb{R}$-vs. Define the map

$$\tau : \underline{\qquad\qquad} \longrightarrow \underline{\qquad\qquad}$$

by $\tau(\qquad) = \underline{\qquad\qquad\qquad}$

We call this map the standard ____________ on $V_{\mathbb{C}}$

HW) Show the following commutes

$$\begin{tikzcd}
\mathbb{C} \otimes \mathbb{R}^n \arrow[r, "\sim"] \arrow[d, "\tau"] & \mathbb{C}^n \arrow[d, "\overline{(\ )}"] \\
\mathbb{C} \otimes \mathbb{R}^n \arrow[r, "\sim"] & \mathbb{C}^n
\end{tikzcd}$$

(that is, this "recovers" the standard ____ on $\mathbb{C}^n$)

Prop: In the set-up above, the fixed points of

$$\tau : \underline{\qquad\qquad} \longrightarrow \underline{\qquad\qquad}$$

is the subspace $\underline{\qquad\qquad\qquad \cong \qquad\qquad\qquad}$

Pf)