Homework 1 Answers
David Rubinstein - Math 117 - Fall 2021

1) Find the inverse of the real matrix A=$\begin{pmatrix} 6 & -3 \\ 2 & 6 \end{pmatrix}$

Solution: Recall that for a 2x2 matrix, the inverse can be found via a convenient formula: divide by the determinant, swap the diagonal entries and negate the off diagonal terms. Thus we get $A^{-1} = \frac{1}{42}\begin{pmatrix} 6 & 3 \\ -2 & 6 \end{pmatrix}$

2) Consider the field $\mathbb{F} = \mathbb{F}_p$ and the matrix A= $\begin{pmatrix} \overline{6} & \overline{-3} \\ \overline{2} & \overline{6} \end{pmatrix}$.

(a) Compute the inverse of A when p=5

(b) Compute the inverse of A when p=17

(c) Show that A has no inverse when p=7

Solution: a) First note that $det(A) = 42 \equiv 2 \bmod 5$. Therefore if we want to use the formula given above, we will need to multiple all the entries by the inverse of 2 (mod 5). Since $2 \times 3 = 6 \equiv 1 \bmod 5$ we have $\overline{3} = \overline{2}^{-1}$. Then we can do the same formula as above, making sure to reduce the elements mod 5. Thus we get $A^{-1} = \begin{pmatrix} \overline{3} \times \overline{1} & \overline{3} \times \overline{3} \\ \overline{3} \times \overline{3} & \overline{3} \times \overline{1} \end{pmatrix} = \begin{pmatrix} \overline{3} & \overline{4} \\ \overline{4} & \overline{3} \end{pmatrix}$
b) Same principle applies for this one, however in this case we are reducing mod 17 so $det(A) = 42 \equiv 8 \bmod 17$. Hence we will multiply all the elements by $\overline{8}^{-1} = \overline{15}$. Therefore we get $A^{-1} = \begin{pmatrix} \overline{15} \times \overline{6} & \overline{15} \times \overline{3} \\ \overline{15} \times \overline{15} & \overline{15} \times \overline{6} \end{pmatrix} = \begin{pmatrix} \overline{5} & \overline{11} \\ \overline{4} & \overline{5} \end{pmatrix}$
c) When p=7 we have $det(A) = 42 \equiv 0 \bmod 7$ so A is not invertible.

3) Let p be any prime number:

(a) Show that for any nonzero $a \in \mathbb{F}_p$ there exists a positive integer $n \leq p-1$ such that $a^n = \overline{1}$ in $\mathbb{F}_p$

(b) Deduce that $a^{(p-1)!} = \overline{1}$ in $\mathbb{F}_p$ for all nonzero $a \in \mathbb{F}_p$.

(c) Using (b) construct a nonzero polynomial $f \in \mathbb{F}_p[T]$ that has no roots.

Solution: a) We will use the pigeonhole principle as hinted at. Not that the number of nonzero elements in $\mathbb{F}_p = p - 1$. Then let $0 \neq a \in \mathbb{F}_p$. Consider the list

$$a^0, a^1, a^2, \ldots, a^{p-1}$$

We claim that all of the elements in this list are nonzero. Indeed if one of them, say $a^i = 0$ in $\mathbb{F}_p$ then we know that p divides a^i. But p is a prime number so if p divides a^i then p must also divide a, hence $a = \overline{0}$, a contradiction. Therefore we have a list of p nonzero elements so by the pigenhole principle two of them must be equal. Thus we have $a^i = a^j \in \mathbb{F}_p$ for some $i > j$. Hence we have $a^{i-j} = \overline{1} \in \mathbb{F}_p$.
b). Note that since the n we found in part (a) is less than or equal to p-1, we have that n divides (p-1)!. Therefore we can write (p-1)!=nk for some positive integer k. Hence

$$a^{(p-1)!} = a^{nk} = (a^n)^k = \overline{1}^k = \overline{1}$$

c) We consider two cases.

(1) First suppose $p \neq 2$ and consider the polynomial $f(T) = T^{p-1)!} + 1$ Then we have $f(a) = \overline{2}$ for all $0 \neq a \in \mathbb{F}_p$. So since $p \neq 2$ f never equals 0 and hence never has a root.

(2) Now consider the case p=2. Then we claim the polynomial $f(T) = T^2 + T + 1$. Indeed $f(\overline{0}) = \overline{1}$ and $f(\overline{1}) = \overline{1} + \overline{1} + \overline{1} = \overline{1}$.

4) Consider the field $F = \mathbb{F}_{29}$ and the subset of the vector space $V = F^3$ given by

$$S = \left(\begin{pmatrix} \overline{3} \\ \overline{10} \\ \overline{5} \end{pmatrix}, \begin{pmatrix} \overline{2} \\ \overline{7} \\ \overline{4} \end{pmatrix}, \begin{pmatrix} \overline{11} \\ \overline{7} \\ \overline{17} \end{pmatrix} \right)$$

Determine whether the set is Linear independent.

Solution: For this we need to determine the solution set to the system of equations given by

$$a_1 \begin{pmatrix} \overline{3} \\ \overline{10} \\ \overline{5} \end{pmatrix} + a_2 \begin{pmatrix} \overline{2} \\ \overline{7} \\ \overline{4} \end{pmatrix} + a_3 \begin{pmatrix} \overline{11} \\ \overline{7} \\ \overline{17} \end{pmatrix} = \begin{pmatrix} \overline{0} \\ \overline{0} \\ \overline{0} \end{pmatrix}$$

with $a_1, a_2, a_3 \in \mathbb{F}_{29}$. Writing this system of equations as a matrix gives us

$$\left(\begin{array}{ccc|c} \overline{3} & \overline{2} & \overline{11} & \overline{0} \\ \overline{10} & \overline{7} & \overline{7} & \overline{0} \\ \overline{5} & \overline{4} & \overline{17} & \overline{0} \end{array} \right)$$

Now we can solve this by putting it in Echelon Form- we must only be careful that we are working in $\mathbb{F}_{29}$ the whole time. For example we want to make the top left entry $\overline{1}$ so we need to multiply the whole row by $\overline{10}$ (as opposed to "dividing" by 3). By applying the Gauss Jordan elimination method to this matrix above, we will eventually get the matrix

$$\left(\begin{array}{ccc|c} \overline{1} & \overline{20} & \overline{20} & \overline{0} \\ \overline{0} & \overline{1} & \overline{27} & \overline{0} \\ \overline{0} & \overline{0} & \overline{0} & \overline{0} \end{array}\right)$$

Hence the vectors are linearly dependent because we have a free variable.

5) Show that the three functions $cos(t), cos(2t), cos(3t)$ are Linearly Independent in the real vector space $\mathcal{C}([-\pi,\pi],\mathbb{R}) = \{f : [-\pi,\pi] \to \mathbb{R} : f(t)$ is continuous for every $t \in [-\pi,\pi]\}$

Solution: To simplify notation let $V := \mathcal{C}([-\pi,\pi],\mathbb{R})$. Note that the 0 vector in V is the "0 function". That is, the function defined by 0(t)=0 for every t. So again, we want to determine the solution set to the equation $a_1 cos(t) + a_2 cos(2t) + a_3 cos(3t) = 0_V$ where $a_1, a_2, a_3 \in \mathbb{R}$ and 0_V is the 0 vector in V. There are many ways of doing this, the point is that it the sum must always be 0. You can plug in three t values, and then get a three by three system of equations that makes the coefficients all be 0 for example.

6) For any field F the set $M_{m\times n}(F) := \{A = (a_{ij} \in F)_{1\le i\le m, 1\le j\le n}\}$ of $m \times n$ matrices forms a vector space. Consider the case m=n and the subset Σ of Symetric Matrices.

(a) Show that Σ is a subspace and find a basis

(b) For the case $F = \mathbb{F}_3$ and n=3, how many elements are in Σ

Solution: a) Note that for any matrices A,B we have $(A+B)^T = A^T + B^T$. So if $A, B \in \Sigma$ then $(A+B)^T = A^T + B^T = A + B \in \Sigma$. Similarly we have that for any matrix A, and scalar x we have $(xA)^T = x(A)^T$. So for $A \in \Sigma, x \in F$ we have $(xA)^T = x(A)^T = xA \in \Sigma$. Thus, Σ is an F-subspace of $M_{n\times n}(F)$.
A basis for Σ can be given by taking all the diagonal matrices and then matching the off diaganol entries to be symmetric. For example (because I don't want to type the general case lol) here is the basis for n=3:

$$\mathcal{B} = \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right)$$

b) By part a we know that any matrix $A \in \Sigma$ can be written as a unique combination

$$A = a_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + a_3 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + a_4 \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + a_5 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + a_6 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

where $a_1, a_2, a_3, a_4, a_5, a_6 \in \mathbb{F}_3$. Different choices of the constants give rise to different matrices, and we have 3 choices for each constant- so we have a total of 3^6 different matrices. Hence $|\Sigma| = 3^6$.