Yiddish of the Day

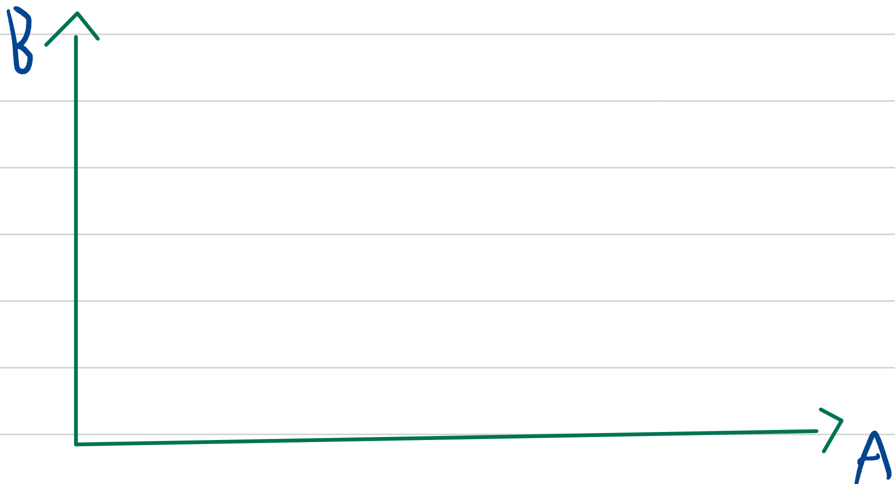
"Er kikt mit di oygn,
hert mit di oyern,
in farshtayt vi di vant" =

ער קוקט מיט די אויגן,
הערט מיט די אויערן,
און פֿאַרשטייט ווי די וואַנט"

Relations

Let A, B be sets

- For visual intuition, lets depict $A \times B$ as



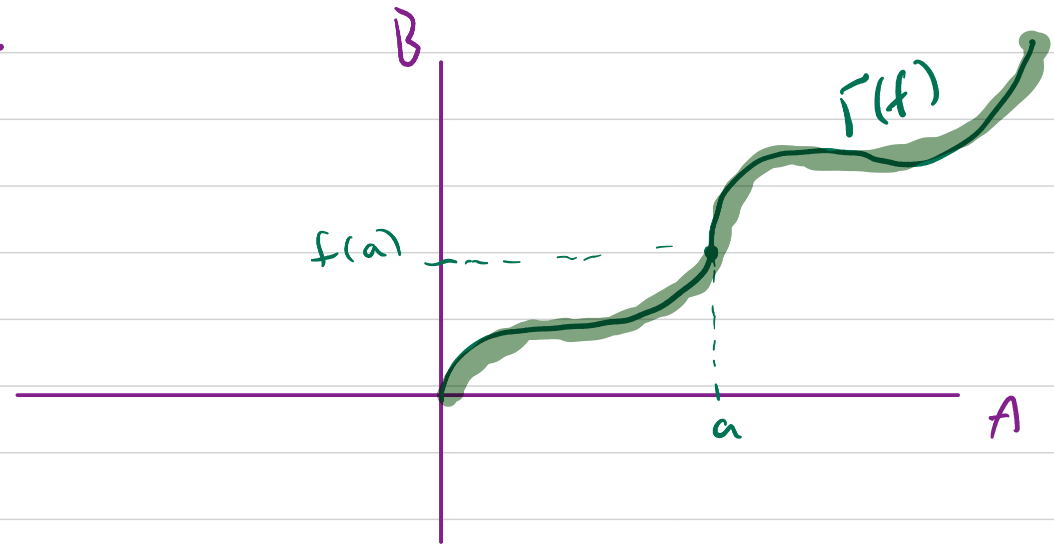
Def: A relation from A to B is a subset $R \subseteq A \times B$

- For a relation $R \subseteq A \times B$, given an element $(a, b) \in R$, we will write aRb and read it as "a is R-related to b"
- If $(a, b) \notin R$ we will write $a \not R b$ and say that a is not related to b

Examples

a) Let $f: A \longrightarrow B$ be a function. Define the graph of f

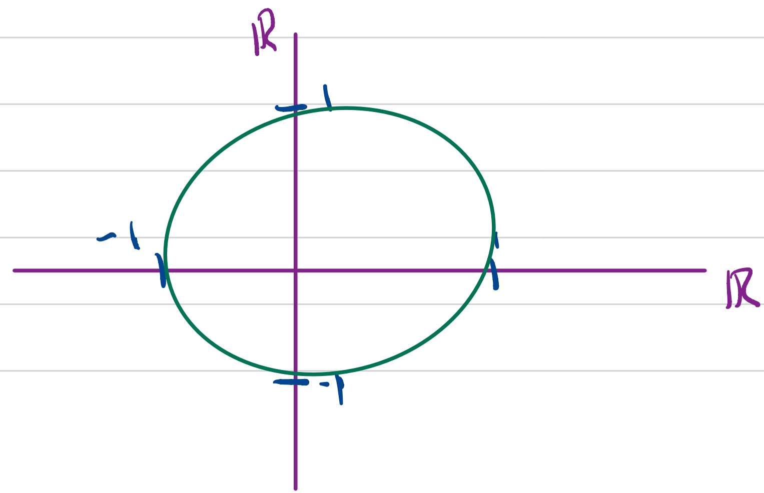
$$\Gamma(f) = \{ (a, f(a)) \mid a \in A \} \subseteq A \times B$$



this is a relation from A to B

$\longrightarrow$ In this way a function is just a special case of a relation!

b) Let $S = \{ (x, y) \in \mathbb{R}^2 \mid \underline{x^2 + y^2 = 1} \} \subseteq \mathbb{R} \times \mathbb{R}$



- Note that we can solve for x:

$$x^2+y^2=1 \longrightarrow y=\pm\sqrt{1-x^2}$$

this is not a function, but it is still a relation

Def: Let $R \subseteq A \times B$ be a relation from A to B.

Then the

i) Domain of R is the set

$$\mathrm{dom}(R) = \{ a \in A \mid aRb \text{ for some } b \in B \}$$

$\rightsquigarrow$ this is the set of all first coordinates that occur in elements of R.

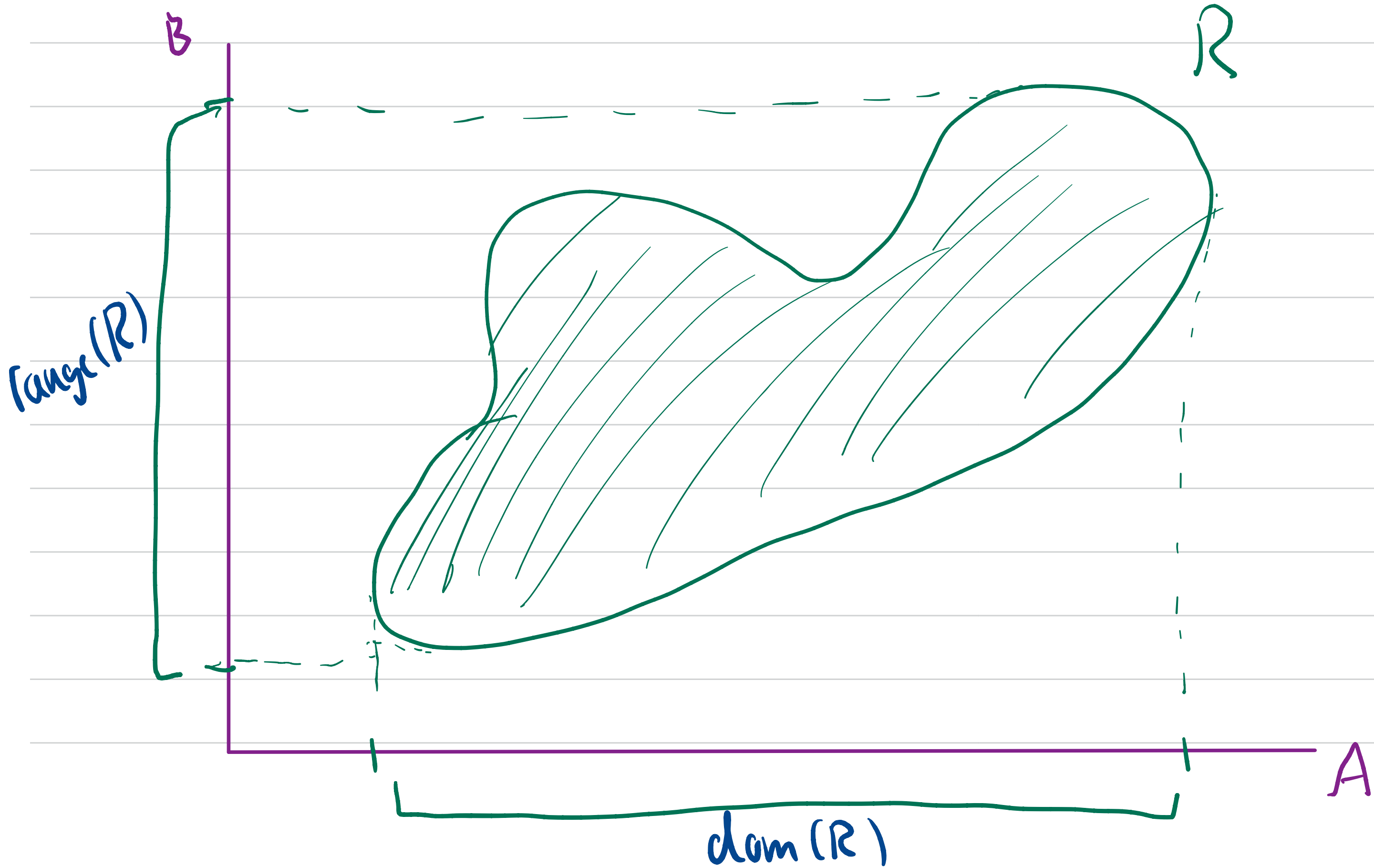
ii) The range of R (or the image of R)

is the set

$$\text{range}(R) = \{ b \in B \mid aRb \text{ for some } a \in A \}$$

→ this is the set of all second components that occur as elements of R

⌜Visually we can think of it this way



⌟

ex) Again consider the relation

$$S = \{ (x,y) \in \mathbb{R}^2 \mid \underline{x^2+y^2=1} \} \subseteq \mathbb{R}\times\mathbb{R}$$

Then $\underline{\text{dom}(S)} = \{ \underline{x\in\mathbb{R}} \mid \underline{x^2+y^2=1} \text{ for some } y \in \mathbb{R} \}$

$= [-1, 1]$ (from picture)

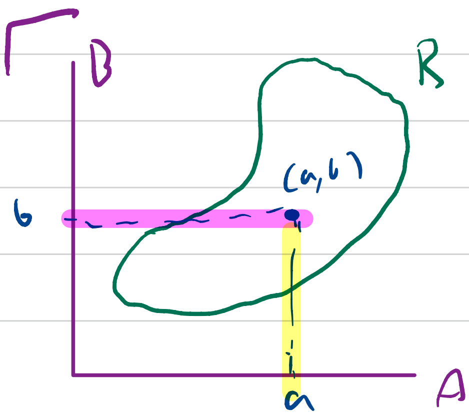
$\text{range}(S) = \{ \underline{y\in\mathbb{R}} \mid \underline{x^2+y^2=1} \text{ for some } x \in \mathbb{R} \}$

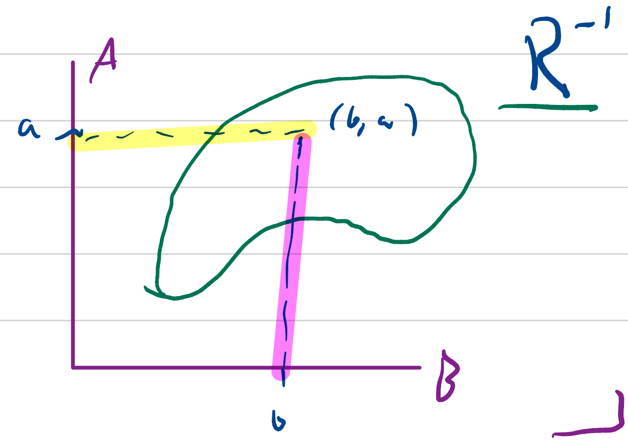
$= [-1, 1]$ (from picture)

<u>Def</u>: Let $R \subseteq A \times B$ be a relation from A to B. Then the <u>inverse relation</u>, denoted $\underline{R^{-1}}$ is

$$\underline{R^{-1}} = \{(b,a) \in B \times A \mid \underline{aRb} \ (\text{ie } (a,b) \in R)\} \subseteq \underline{B \times A}$$

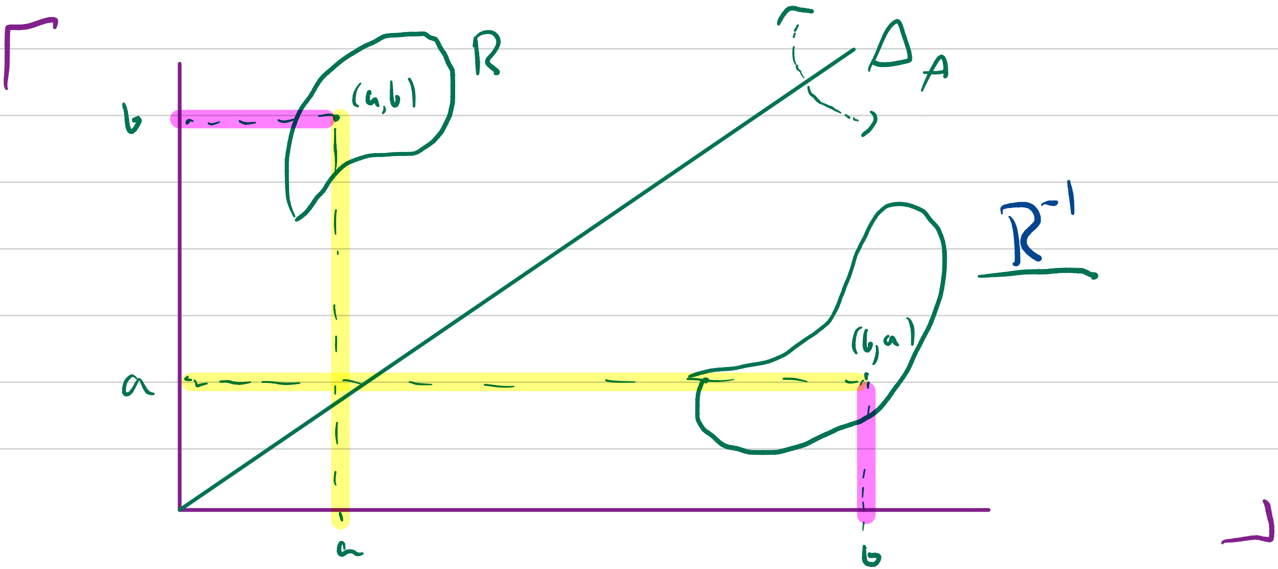
is a relation from B to A





- If $A = B$ then we obtain R^{-1} by reflecting along the "diagnol"
- this is the subset defined as

$$\Delta_A = \{ (a,a) \mid a \in A \} \subseteq A \times A$$



ex) Consider the relation $R \subseteq \mathbb{R} \times \mathbb{R}$.

defined by $R = \{ (a,b) \mid a \leq b \}$

• What is $\underline{R^{-1}}$:

$$\to R^{-1} = \{ (x,y) \mid yRx \}$$

$$= \{ (x,y) \in \mathbb{R}^2 \mid y \leq x \}$$

$$= \{ (x,y) \in \mathbb{R}^2 \mid x \geq y \}$$

ex2) Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{u, v, w, x, y, z\}$.

Consider the relation from A to B given by

$R = \{(2,x), (1,z), (2,v), (4,x), (4,u), (5,w)\}$

$\rightsquigarrow$ • $\text{dom}(R) = \{2, 1, 4, 5\}$

• $\text{range}(R) = \{u, v, w, x, z\}$

$R^{-1} = \{(b,a) \in B \times A \mid (a,b) \in R\}$

$\rightsquigarrow R^{-1} = \{(x,2), (z,1), (v,2), (x,4), (u,4), (w,5)\} \subseteq B \times A$

Special terminology when $A = B$

- From now on, assume $A = B$. Then a relation from A to A is just called a relation on A

ex) Equality

- Recall the relation on A

$$R = \Delta_A = \{ (a,a) \mid a \in A \} \subseteq A \times A$$

$\rightsquigarrow$ have aRb iff $a = b$

$\rightsquigarrow$ this relation is just "equality"

Later Goal: Generalize this concept of equality to "equivalence - relation"

- For this, lets note the fundamental properties of equality

1) for every $a \in A$ $a = a$

2) If $a = b$ then $b = a$

3) If $a = b$ and $b = c$ then $a = c$

Def: Let R be a relation on A.

1) R is called reflixive if aRa $\forall a \in A$

⌈ This is equiv to saying $\Delta_A \subseteq R$ ⌋

2) R is called symmetric if, whenever aRb then bRa

⌈ This is equiv to saying $R = R^{-1}$ ⌋

3) R is called transitive if whenever aRb and bRc then aRc

Previously seen examples

a) Let $n \in \mathbb{N}$. Then define R relation on $\mathbb{Z}$ by

aRb iff $a \equiv b \bmod n$.

• We saw that

($\equiv$ reflexive) 1) $\forall a \in \mathbb{Z}$ $a \equiv a \bmod n$ $(\forall a \in \mathbb{Z}, aRa)$

($\equiv$ symmetric) 2) If $a \equiv b \bmod n$ then $b \equiv a \bmod n$
(aRb then bRa)

($\equiv$ transitive) 3) If $a \equiv b \bmod n$ and $b \equiv c \bmod n$ then
$a \equiv c \bmod n$ (aRb and bRc then aRc)

b) R on $\mathbb{Z}$ defined by aRb iff $a \leq b$

What does R satisfy?

1) Reflexive? Yes, because $a \leq a$ for any $a \in \mathbb{Z}$

2) Symmetric? No: ex: $2 \leq 7$ so $2R7$ but $7 \not R 2$ because $7 \nleq 2$

3) Transitive? Need to check: If aRb and bRc is aRc?

Assume $a \leq b$ and $b \leq c$ then indeed $a \leq c$

Can also give "basic" examples too

Let $A = \{a, b, c\}$. Define the relation on A by

$R = \{(a,a), (b,b), (c,c), (a,b), (b,c), (a,c)\}$

- Is R reflexive? Need to check: Is aRa bRb cRc?

  $(a,a) \in R$ so aRa and $(c,c) \in R$
  $(b,b) \in R$ so bRb so cRc

- Is R symmetric? Need to check: If xRy then yRx

  $\rightsquigarrow$ No! aRb but $b \not R a$

- Is R transitive?

aRa

aRb $\quad bRb$

aRc $\quad bRc$ $\quad cRc$

Yes! aRb and bRc and also aRc ✓

Geometric Example

Let $A = \mathbb{R}$ and consider the relation R on $\mathbb{R}$.

$$R = \{ (x,y) \in \mathbb{R}^2 \mid |x-y| \leq 1 \}$$

- So if xRy then $|x-y| \leq 1 \longleftrightarrow \underline{-1} \leq x-y \leq \underline{1}$
  - $\rightsquigarrow$ So R is bounded by lines
    - $y = x+1$
    - $y = x-1$

- R <u>reflexive</u>? (Note, $\underline{\Delta_{\mathbb{R}}} \subseteq R$ so should be yes!)
  - Let $x \in \mathbb{R}$, Then $|x-x| = 0 \leq 1$

  So yes <u>reflexive</u>

- R symmetric
  - Let $x, y \in \mathbb{R}$ and suppose xRy

    Then $|x-y| \leq 1$

    Q: Is yRx? Ie is $|y-x| \leq 1$

    - $y-x = -1(x-y)$
    - $|y-x| = |x-y| \leq 1$ ✓

    So yes!

• R **transitive**

Goal: Given $a, b, c \in \mathbb{R}$ with aRb and bRc determine whether or not aRc

$\rightarrow$ ie if $|a-b| \leq 1$ and $|b-c| \leq 1$

is $|a-c| \leq 1$

Let $x = 1.5$, $y = 1$ and $z = 0$

Then $|x-y| = |1.5-1| = .5 \leq 1$ so xRy

$|y-z| = |1-0| = 1 \leq 1$ so yRz

but $|x-z| = |1.5-0| = 1.5 > 1$ so $x \not{R} z$

so R is **NOT transitive!**