

10/19/21 Section

Today

- Recap on Linear Transf \longleftrightarrow Matrices
- How does change of basis change Transformations?
- Using Matrix to find Ker/Image of T

Transformations \longleftrightarrow Matrices

Recall: Let $T: V \rightarrow W$ be linear, and let

$$B_V = (v_1, \dots, v_n)$$
$$B_W = (w_1, \dots, w_m)$$

be 2-basis. Then

$$[T]_{B_W}^{B_V} := \begin{pmatrix} [T(v_1)]_{B_W} & [T(v_2)]_{B_W} & \cdots & [T(v_n)]_{B_W} \\ \downarrow & \downarrow & \cdots & \downarrow \\ m \times n \end{pmatrix}$$

\Rightarrow have the property that, $\forall v \in V$

$$[T(v)]_{B_W} : \underline{[T]_{B_W}^{B_V} [v]_{B_V}}$$

Q: Let (V, \mathcal{B}_V) , (W, \mathcal{B}_W) , (Z, \mathcal{B}_Z) be 3 vector spaces.

Consider:

$$V \xrightarrow{T} W \xrightarrow{S} Z$$

Then recall we have new transf

$S \circ T : V \longrightarrow Z$ defined by

$$(S \circ T)(v) = S(T(v))$$

(check): This function is a linear transf.

So we have 3 matrices

$$T \longrightarrow [T]$$

$$S \longrightarrow [S]$$

$$S \circ T \longrightarrow [S \circ T]$$

How are they related? Fact: $[S \circ T] = [S][T]$

Special case

$$\begin{array}{ccc} V & \leadsto & B_v, B_v' \\ W & \leadsto & B_w, B_w' \end{array}$$

$$\begin{array}{ccc} (V, B_v) & \xrightarrow{T} & (W, B_w) \\ f = \text{id} \uparrow & \text{"} & \uparrow g = \text{id} \\ (V, B_v') & \xrightarrow{T} & (W, B_w') \end{array}$$

Let P : change basis from
 $B_v \rightarrow B_v'$

Q : change basis matrix
from $B_w \rightarrow B_w'$

That is Tof = $g \circ T$

→ now take their matrices

$$\Rightarrow [T_{of}]_{B_j}^{B_w} = [g_o T]_{B_j}^{B_w}$$

$$\Rightarrow [T]_{B_j}^{B_w} P = Q [T]_{D_j}^{B_w}$$

$$\Rightarrow \boxed{[T]_{B_j}^{B_w} = Q^{-1} [T]_{B_j}^{B_w} P}$$

ex) $V = \mathbb{R}_2[x]$

$$B_J = (1, x, x^2) \quad B_{J'} = (2+x, 3+x, x-x^2)$$

$W = \mathbb{R}^3$

$$B_W = (e_1, e_2, e_3), \quad B_{W'} = \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right)$$
$$= \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right)$$
$$\xrightarrow[e_1]{} \quad \xrightarrow[e_2]{}$$

$T: V \rightarrow W$ by

$$T(a_0 + a_1 x + a_2 x^2) = \begin{pmatrix} a_0 \\ a_1 - a_2 \\ a_0 + a_1 \end{pmatrix}$$

Q: Find $[T]_{B_J}^{B_W}$ and $[T]_{B_{J'}}^{B_W}$

$\xrightarrow[A]$ $\xrightarrow[A']'$

Verify the formula

$$A' = \underline{G^{-1} A P}$$

$$\bullet T(1+0x+0x^2) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 1e_1 + 0e_2 + 1e_3$$

$$\bullet T(0+1x+0x^2) = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 0e_1 + 1e_2 + 1e_3 \Rightarrow A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$\bullet T(0+0x+1x^2) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\bullet T(2+x) = \begin{pmatrix} 2 \\ -1 \\ 2+1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{---} \quad \text{---} \quad \text{---}$$

$$\bullet T(3+x^2) = \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 4\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow A' = \begin{pmatrix} 2 & 3 & 0 \\ -2 & -4 & 1 \\ 3 & 3 & 1 \end{pmatrix}$$

$$\bullet T(x - x^*) \in \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Now find

$$B_J = (1, x, x^2)$$

$$B_J' = (2+x, 3x^2, x-x^2)$$

$\cdot P$ = change basis matrix from $B_J \rightarrow B_J'$

$$= \left(\begin{matrix} [2+x]_{B_J} & [3+x^2]_{B_J} & [x-x^2]_{B_J} \\ \downarrow & \downarrow & \downarrow \end{matrix} \right)$$

$$= \begin{pmatrix} 2 & 3 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

$$B_w = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right)$$

$$B_{w'} = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$

$$\underline{Q}^{-1} = (\text{change of basis matrix from } B_w \rightarrow B_{w'})^{-1}$$

= change of basis matrix from $B_{w'} \rightarrow B_w$

$$\therefore \left(\begin{bmatrix} [1]_{B_{w'}} \\ [0]_{B_{w'}} \\ [0]_{B_{w'}} \end{bmatrix}_{B_w}, \begin{bmatrix} [0]_{B_{w'}} \\ [1]_{B_{w'}} \\ [0]_{B_{w'}} \end{bmatrix}_{B_w}, \begin{bmatrix} [0]_{B_{w'}} \\ [0]_{B_{w'}} \\ [1]_{B_{w'}} \end{bmatrix}_{B_w} \right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$Q^{-1} A P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 3 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 3 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 3 & 0 \\ -2 & -4 & 1 \\ 3 & 3 & 1 \end{pmatrix} = A^4 \quad \checkmark$$

How do we use Matrices to compute things

$T: V \rightarrow W$ linear

$B_V = (v_1, \dots, v_n)$ basis for V .

Recall $\text{im}(T) := \left\{ w \in W \mid \exists v \in V \text{ with } T(v) = w \right\}$

$$\text{ker}(T) := \left\{ v \in V \mid T(v) = 0_W \right\} \subseteq V$$

Claim: $\text{im}(T) = \text{span}(\{T(v_1), \dots, T(v_n)\})$. (Note \geq for free)

Pf) Let $w \in \text{im}(T)$. Then $w = T(v)$ for some v .

Now $v = c_1 v_1 + \dots + c_n v_n$ for c_1, \dots, c_n .

$$w = T(v) = T(c_1 v_1 + c_2 v_2 + \dots + c_n v_n)$$

$$= T(c_1 v_1) + T(c_2 v_2) + \dots + T(c_n v_n) \quad T \text{ linear}$$

$$w = c_1 T(v_1) + \dots + c_n T(v_n) \in \text{span}\{T(v_1), \dots, T(v_n)\}$$

Algorithm:

1) Write $[T]_{B_0}^{B_\infty} := A$

2) Put A in EF

- the columns with leading variables correspond to the LI vectors in list $T(v_1), \dots, T(v_n)$
Those are your basis for $\text{im}(T)$
- the solution set to homogeneous system given by A are the coordinate vectors of basis for Ker.

$$ex) V = \mathbb{R}_v[x]$$

$$B_V = (1, x, x^2)$$

$$W = \mathbb{R}^3$$

$$B_W = (e_1, e_2, e_3)$$

$$T(a_0 + a_1x + a_2x^2) = \begin{pmatrix} a_0 \\ a_1 - a_2 \\ a_0 + a_1 \end{pmatrix}$$