Yiddish word of the day

Yiddish expression of the day

Matrix Multiplication

1st - Matrix Addition

Let A, B be $m \times n$ matrices.

$$A = (a_{ij})_{\substack{i=1,n \\ j=1,m}} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}, \quad B = (b_{ij})$$

Then $A+B = (\quad\quad) = \begin{pmatrix} \quad \end{pmatrix}_{m \times n}$

note: If we think of vector $\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ in $\mathbb{R}^n$

we can think of this as $n \times 1$ matrix

So this addition we defined for matrices just generalizes the addition we defined for vectors.

ex) $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 2 \\ 6 & 8 & 1 \end{pmatrix} = \begin{pmatrix} & & \\ & & \end{pmatrix}$

There's another reason to define matrix add this way

Recall: $T: \mathbb{R}^n \to \mathbb{R}^m \xrightarrow{\text{we get}}$

Now, let $T, S: \mathbb{R}^n \to \mathbb{R}^m$ be two linear transf.
we get a new transformation

$$(\quad) : \mathbb{R}^n \to \mathbb{R}^m \quad \text{defined by}$$

$$(\quad)(\vec{v}) =$$

$T \rightsquigarrow$ $\qquad (T+S) \rightsquigarrow$

$S \rightsquigarrow$

Then the way we've defined matrix addition makes

ex) $T : \mathbb{R}^2 \to \mathbb{R}^3$ by $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \\ y \end{pmatrix}$

$S : \mathbb{R}^2 \to \mathbb{R}^3$ by $S\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ x \\ y \end{pmatrix}$

Then $(T+S)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \\ \\ \end{pmatrix} + \begin{pmatrix} \\ \\ \end{pmatrix} = \begin{pmatrix} \\ \\ \end{pmatrix}$

Find A_{T+S}, A_S, A_T check $A_{T+S} = A_S + A_T$

$A_T = \begin{pmatrix} & & \end{pmatrix}, \quad A_S = \begin{pmatrix} & & \end{pmatrix}$

$A_{T+S} = \begin{pmatrix} & & \end{pmatrix} =$

ex) $T : \mathbb{R}^3 \to \mathbb{R}^5 \qquad T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \\ x_2 \\ 0 \\ x_3 \end{pmatrix}$

$S : \mathbb{R}^3 \to \mathbb{R}^5 \qquad S\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_1 \\ 2x_2 \\ 0 \\ x_3 \end{pmatrix}$

$(T+S): \mathbb{R} \to \mathbb{R}$, $(T+S)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \stackrel{def}{=} \begin{pmatrix} \\ \\ \end{pmatrix} + \begin{pmatrix} \\ \\ \end{pmatrix}$

$(T+S)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \\ \\ \end{pmatrix}$

$A_T = \begin{pmatrix} \\ \\ \end{pmatrix}$ $A_S = \begin{pmatrix} \\ \\ \end{pmatrix}$

$A_T + A_S = \begin{pmatrix} \\ \\ \end{pmatrix}$

Ans = $\left(\qquad \right) = \underline{\qquad} + \underline{\qquad}$

Recall- Function Composition

$f: A \to B$, $g: B \to C$ we get a new function

$g \circ f:$ ___ $\to$ ___ defined by $(g \circ f)(a) =$ ______

A ______ B ______ C

ex) $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$

$g: \mathbb{R} \to \mathbb{R}$ by $g(x) = x-2$

Then $g \circ f: \mathbb{R} \to \mathbb{R}$, $(g \circ f)(x) =$

ii) $f : \mathbb{R} \longrightarrow \mathbb{R}^2$ by $f(x) = \begin{pmatrix} 2x \\ x^2 \end{pmatrix}$

$g : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ by $g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$

$g \circ f : \mathbb{R} \longrightarrow \mathbb{R}$

$(g \circ f)(x) =$

Note: In example 2 can't do

In example 1, **we** can do ______ as well,

but $\neq g \circ f$

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<u>Now</u>: Suppose
$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "T"] \arrow[rr, bend right, "S\circ T"'] & \mathbb{R}^m \arrow[r, "S"] & \mathbb{R}^q \end{tikzcd}$$
that $T: \mathbb{R}^n \to \mathbb{R}^m$, $S: \mathbb{R}^m \to \mathbb{R}^q$ are linear transf.

Then the function $S\circ T : \mathbb{R}^{\_\_} \to \mathbb{R}^{\_\_}$ is still a linear transformation (you can check this)

$T: \mathbb{R}^n \to \mathbb{R}^m \rightsquigarrow$ ____ (____ matrix)

$S: \mathbb{R}^m \to \mathbb{R}^q \rightsquigarrow$ ____ (____ matrix)

$S\circ T: \mathbb{R}^{\_\_} \to \mathbb{R}^{\_\_} \rightsquigarrow$ ____ (____ matrix)

Q: How are these 3 ______ related?

- Recall that given $T: \mathbb{R}^n \to \mathbb{R}^m$ we have the corresponding matrix A_T and the property that $T(\vec{v}) =$ ______

- So the matrix associated to $S \circ T$ should satisfy a similar property as above
  - That is, we want $(S \circ T)(\vec{v}) =$ ______

We will define matrix mult in such a way, that this relationship holds

Let A $m \times n$ matrix $\qquad$ B $n \times q$ matrix

$\downarrow$ $\qquad\qquad\qquad\qquad$ $\downarrow$

$T: \mathbb{R}^n \to \mathbb{R}^m$ $\qquad\qquad\qquad$ $S: \mathbb{R}^q \to \mathbb{R}^n$

AB will be a $m \times q$ matrix whose $(i, j)^{th}$ component will be

- "multiply" row i of matrix A to "column j" of matrix B

Examples

$A = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$ $\qquad$ $B = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix}$

$\Rightarrow$ AB should be a ______ matrix

(1,1) : __________

(1,2) : __________

(2,1) : __________

(2,2) : __________

$\Rightarrow AB = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$

$B = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix}_{2\times 2}$, $A = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}_{2\times 2}$

(1,1) : ______

(1,2) : ______

(2,1) : ______

(2,2) : ______

$\Rightarrow BA = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$ ______

Check: That this def of mult gives us what we want.

ex) $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ y \end{pmatrix}$

$S: \mathbb{R}^2 \to \mathbb{R}^2$ by $S\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$

Then $T \circ S: \mathbb{R}^2 \to \mathbb{R}^2$ by $(T \circ S)\begin{pmatrix} x \\ y \end{pmatrix} =$

That is $(T \circ S)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \quad \end{pmatrix}$

$T \longrightarrow A_T = \begin{pmatrix} \quad \end{pmatrix}$

$S \longrightarrow A_S = \begin{pmatrix} \quad \end{pmatrix}$

$T_{OS} \longrightarrow A_{T_{OS}} := \left(\qquad \right)$

Claim is: $A_{T_{OS}} =$ ______

Reverse way

$(S \circ T) : \mathbb{R}^2 \to \mathbb{R}^2$ by $(S \circ T)\begin{pmatrix} x \\ y \end{pmatrix} =$

$S \circ T \longrightarrow A_{S \circ T} = \begin{pmatrix} & \end{pmatrix}$

Now check that $\underline{A_{S \circ T} = \qquad\qquad}$

Ways matrix mult is different than "regular" multiplication.

1) We've seen it's not commutative, ie ___ $\neq$ ___

for many reasons!

ex) $A = \begin{pmatrix} 1 & 1 & 2 \\ 3 & 0 & 1 \end{pmatrix}_{2\times 3}$ $\quad B = \begin{pmatrix} 1 & 6 \\ 2 & 0 \\ 3 & 0 \end{pmatrix}_{3\times 2}$

$$AB = \begin{pmatrix} \quad \end{pmatrix} = \begin{pmatrix} \quad \end{pmatrix}_{2\times 2}$$

$$BA = \begin{pmatrix} \quad \end{pmatrix} = \begin{pmatrix} \quad \end{pmatrix}_{3\times 3}$$

ex2: $A = \begin{pmatrix} 1 & 1 & 2 \\ 3 & 0 & 1 \end{pmatrix}_{2\times3}$ $B = \begin{pmatrix} 1 & 6 & 0 \\ 2 & 0 & 0 \\ 3 & 0 & 1 \end{pmatrix}_{3\times3}$

Note: Cannot do ______, not defined!

______ $= \begin{pmatrix} \quad \end{pmatrix} = \begin{pmatrix} \quad \end{pmatrix}_{2\times3}$

2) It's possible for 2 non-zero matrices to ______

(shows that matrices aren't a so called integral domain) ex) $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$AB = \begin{pmatrix} & \end{pmatrix}$

3) Can't "cancel" out matrices.

ex) $A = \begin{pmatrix} 2 & -3 \\ -4 & 6 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ $C = \begin{pmatrix} 4 & -2 \\ 3 & -1 \end{pmatrix}$

$AB = \begin{pmatrix} & \end{pmatrix} \Rightarrow$

$AC = \begin{pmatrix} & \end{pmatrix}$