Yiddish of Day

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Recap

To prove $\underline{P \Rightarrow Q}$ we have 3 options

1) Direct proof $\quad (\underline{P \Rightarrow Q})$

2) Contrapositive $\quad (\neg \underline{Q} \; \underline{\Rightarrow} \; \neg \underline{P})$

3) Contradiction $\quad (\underline{P} \; \underline{\wedge} \; \neg \underline{Q} \; \underline{\Rightarrow} \; \underline{\perp})$

ex) Let $x \in \mathbb{R} \setminus \{0\}$. Show that if $x + \frac{1}{x} < 2$ then $x < 0$

Pf 1) Let us assume that $x + \frac{1}{x} < 2$.
Then we have

$$x + \frac{1}{x} - 2 < 0 \quad (\star)$$

Multiplying by $\left(\frac{x}{x}\right)$ we get that $\star$ is equal to

$$\frac{x^2 - 2x + 1}{x} < 0$$

But this is equal to

$$\frac{(x-1)^2}{x} < 0$$

Since $(x-1)^2 \geq 0$ we must have $x < 0$ for $\frac{(x-1)^2}{x}$ to be less than 0.

Pf2) We will prove this using contrapositive. That is we will prove that if $x > 0$ then $x + \frac{1}{x} \geq 2$.

We have by the $AM \geq GM$ theorem that

$$\frac{x + \frac{1}{x}}{2} \geq \sqrt{x \cdot \frac{1}{x}}$$

[$u, v \in \mathbb{R}_{>0}$ then

$$\frac{u+v}{2} \geq \sqrt{u \cdot v}$$]

$$\Rightarrow \frac{x+\frac{1}{x}}{2} \geq 1$$

$$\Rightarrow x + \frac{1}{x} \geq 2$$

Pt3) For the sake of contradiction we assume that $x + \frac{1}{x} < 2$ and $x > 0$.

However, by the AM $\geq$ GM theorem we have that

$$\frac{x+\frac{1}{x}}{2} \geq \sqrt{x \cdot \frac{1}{x}}$$

$$\Rightarrow \frac{x+\frac{1}{x}}{2} \geq 1 \Rightarrow x + \frac{1}{x} \geq 2$$

This contradicts $x + \frac{1}{x} < 2$ so $x < 0$

Existence Proofs

Goal: Prove statements of the form

$\exists x \in S, R(x)$

$\rightsquigarrow$ Show that there exists an $x \in S$ making statement true

Methods

1) Explicitly find the $x \in S$ making $R(x)$ true

2) Use other existance theorems

3) Use proof by contradiction

ex) Show that $\exists\ a, b \in \mathbb{Q}$ such that a^b is irrational.

- $a = 2$, $b = 1/2$ $\rightsquigarrow$ $\sqrt{2}$ irrational

(HW 5 or 6)

ex2) Show $\exists\, a \in \mathbb{Q}$, b irrational such that

1) $a^b \in \mathbb{Q}$

2) a^b irrational

(1) $a = 2$ $\quad b = \log_2(3)$

$$a^b = 2^{\log_2 3} = 3$$

(2) $a = 2$ $\quad b = \log_2(\sqrt{2})$

$$a^b = 2^{\log_2(\sqrt{2})} = \sqrt{2}$$

ex 3) Show $\exists\, a, b$ irrational such that
$a^b \in \mathbb{Q}$

Consider the following : $a = \sqrt{2}$ $b = \sqrt{2}$

$\rightsquigarrow a^b = (\sqrt{2})^{\sqrt{2}}$

Two options: If $\sqrt{2}^{\sqrt{2}} \in \mathbb{Q}$ then this is our example,

If irrational : Then consider $\left((\sqrt{2})^{\sqrt{2}}\right)^{\sqrt{2}}$

$= (\sqrt{2})^2 = 2 \in \mathbb{Q}$

These were all direct - How about Method $\underline{2}$

ex) Show that $x^2+2x-5=0$ has a solution on $[1,2]$

Pf) We will use the intermediate value thrm. (IVT)

Recall the IVT says that if

$f:[a,b] \to \mathbb{R}$ is continuous then

if $f(a)<0$ (or $f(a)>0$) and $f(b)>0$ (or $f(b)<0$) then $\exists\, c\in(a,b)$ such that $f(c)=0$

We have that x^2+2x-5 is continuous. Note that

- $(1)^2+(2)(1)-5<0$
- $(2)^2+(2)(2)-5>0$

$\Rightarrow$ by the IVT $\exists\, c\in(1,2)$ such that $c^2+2c-5=0$

Now a method 3 example

Thrm: (Pigeonhole Principle)

Suppose n objects are placed in m boxes. If $n > m$ then there exists a box containing at least 2 objects.

Pf) For the sake of contradiction, assume that $n > m$ but that each box contains at most one object. Therefore there is at most m objects. Hence $m \geq n$ a contradiction $\square$

ex) Let S be a set of 3 integers. For $\phi \neq A \subseteq S$ let

$\sigma_A =$ sum of elements in A

Prove there exists two distinct non-empty subsets $B \neq C \subseteq S$ such that

$$\sigma_B \equiv \sigma_C \mod 6$$

Scratch work: $S = \{2, 5, 7\}$

A	$\{2\}$	$\{5\}$	$\{7\}$	$\{2,5\}$	$\{2,7\}$	$\{5,7\}$	$\{2,5,7\}$
σ_A	2	5	7	7	9	12	14

$\rightsquigarrow$ For $B = \{7\}$ and $C = \{2, 5\}$

have $\sigma_B \equiv \sigma_C \mod 6$ $\lrcorner$

Pf) Let S be a subset of the integers containing 3 integers.

Then the # of non-empty subsets of S is $|P(S)| - 1 = 2^3 - 1 = 7$

Thus we have 7 options for our σ_A

But there are only 6 distinct #'s mod 6.

By the pigeonhole principle $\exists$ B, C subsets such that $\sigma_B \equiv \sigma_C \mod 6$ $\square$

Uniqueness

Rather than just asking for an existance of element we can ask for a unique element such that the statement is true.

$\rightsquigarrow$ We write $\exists! x \in S$, $R(x)$ ($\exists$ unique)

In this case, there are 2 steps to proving such a statement

1) Show an element exists

2) Show it is the only element

→ typically goes like this:

- Suppose $x, y \in S$ such that $R(x)$, $R(y)$ true
  then show $x = y$

or

- Suppose $x \neq y \in S$ with $R(x)$, $R(y)$ true
  → we get a contradiction

ex) Show $x^5 + 2x - 5 = 0$ has unique solution in $[1, 2]$.

Pf) We will apply the IVT to $f(x) = x^2 + 2x - 5$, since it is continuous. We again see that $f(1) < 0$ and $f(2) > 0$ so IVT $\Rightarrow$ $\exists c \in [1,2]$ such that $f(c) = 0$

Note that we have

$$f'(x) = 5x^4 + 2 > 0 \quad \text{on} \quad [1,2]$$

So if $\exists c_1 \neq c_2$ on $[1,2]$ such that $f(c_1) = 0$ and $f(c_2) = 0$ we would get a contradiction. Indeed, we can assume $c_1 > c_2$. Then $f(c_1) > f(c_2)$ since f is increasing. But this contradicts that $f(c_1) = f(c_2)$ $\square$

Disproving Existance Statements

To disprove, $\exists x \in S, R(x)$ we will prove it's negation is true

$\rightsquigarrow$ Remember: $\neg(\exists x \in S, R(x)) \equiv \forall x \in S, \neg R(x)$

ex) Disprove that there exists integers $a \geq 2$ and $n \geq 1$ such that $a^2 + 1 = 2^n$

Goal: Prove for all $a \geq 2$, $n \geq 1$ we have $a^2 + 1 \neq 2^n$

Pf) For the sake of contradiction, let us assume $\exists\, a \geq 2, n \geq 1$ such that

$$a^2+1=2^n$$

Note that 2^n is even so a^2+1 is even. Hence a^2 is odd, and so therefore a is odd.

Let us write $a=2k+1$ for $k \in \mathbb{Z}$.

Then we have that

$$(2k+1)^2+1=2^n$$

$$\Rightarrow 4k^2+4k+1+1=2^n$$

$$\Rightarrow 2(2k^2+2k+1)=2^n$$

Hence we get $2k^2+2k+1=2^{n-1}$

Rewriting we get that $2^{n-1}-1=2k(k+1)$

Note that $k \geq 1$ so $2k(k+1) \geq 4$

So $2^{n-1} \geq 5$ so $\underline{n \geq 4}$

So 2^{n-1} is even but then $2^{n-1}-1$ is odd

yet it equals $2k(k+1)$ an even #.

Hence we have a contradiction $\square$

Prove or Disprove

General tip for universally quantified statements

- if you think they are true, PROVE THEM!
- if you think they are false, find a counterexample

ex) Prove or disprove: If ab, bc, ac are all even then a, b, c are even.

False: $a = 7$
$b = 4$
$c = 2$

ex) Prove or disprove: Suppose A is a set such that

$A \cap B = \emptyset$ for every set B. then

$A = \emptyset$. (such that $B \neq A$)

Pf) Bradley/Arthi: Assume FSOC, that $A \neq \emptyset$. Then $\exists x \in A$

However, $\{x\} \cap A = \{x\} \neq \emptyset$ $\rightarrow\leftarrow$

David: $A \cap A = A$ but by assumption $A \cap A = \emptyset$

so $A = \emptyset$

ex) Prove or disprove: If A, B, C are sets such that

$A \times C = B \times C$ then $A = B$

Strategy: Elements of $A \times C$ are ordered pairs (a, c) with $a \in A$, $c \in C$.
If $A \times C = B \times C$ then given an $(a, c) \in A \times C$ it is also in $B \times C$ so $a \in B$. This makes it seem plausible. HOWEVER. Is there a hidden assumption we made?

$A = \{2\}$ $B = \{3\}$ $C = \emptyset$

⟶ Modify the claim because $A \times C = B \times C = \emptyset$

ex) Let A, B, C sets with $C \neq \emptyset$.
If $A \times C = B \times C$ then $A = B$

This is true: "Flush out" the strategy above to write a full proof.

↓↓↓ Left for next class

Modified new claim from last class

Let A be a set. Suppose for every set B such that

$$A \nsubseteq B \text{ and } B \nsubseteq A$$

that $A \cap B = \emptyset$. Prove or disprove that $A = \emptyset$.

Forming Conjectures (basically educated guesses)

ex) Consider the following Product, for $n \geq 2$

$$P_n = \left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{9}\right) \cdots \left(1 - \frac{1}{n^2}\right)$$

Let's compute a few terms and make a conjecture about the value

$$P_2 = \left(1 - \frac{1}{4}\right) = 3/4$$

$$P_3 = \left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{9}\right) = (3/4)(8/9) = 2/3 = 4/6$$

$$P_4 = \left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{9}\right)\left(1 - \frac{1}{16}\right) = \left(\frac{3}{4}\right)\left(\frac{8}{9}\right)\left(\frac{15}{16}\right) = \left(\frac{2}{3}\right)\left(\frac{15}{16}\right) = \frac{5}{8}$$

$$P_5 = \left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\left(1-\frac{1}{16}\right)\left(1-\frac{1}{25}\right) = \left(\frac{5}{8}\right)\left(\frac{24}{25}\right) = \frac{3}{5} = \frac{6}{10}$$

$\rightsquigarrow$ Conjecture: For $n \geq 2$ $\boxed{P_n = \frac{n+1}{2n}}$

- We will see how to prove this ~~next class.~~ later today

Similar example

Sequence defined by $a_1 = 1$, $a_2 = 4$ and

$a_n = 2a_{n-1} - a_{n-2} + 2$ for $n \geq 3$.

$\rightsquigarrow$ Lets conjecture a formula.

$$a_3 = 2a_2 - a_1 + 2 = 2(4) - 1 + 2 = 9$$

$$a_4 = 2a_3 - a_2 + 2 = 2(9) - 4 + 2 = 16$$

$$a_5 = 2a_4 - a_3 + 2 = 2(16) - 9 + 2 = 25$$

$\rightsquigarrow$ $a_n = \underline{\quad n^2 \quad}$

We'll prove ~~next class~~ in just a bit