Tensor-Products Cont

<u>Last time</u>: For vector spaces $V_1 \ldots V_m$, W over $\mathbb{F}$, we saw that the "$\otimes$" operation gave us the bijection

$$\text{Mult}(\qquad\qquad , W) \simeq \mathcal{L}(\underline{\qquad\qquad} , W)$$

Moreover, we mentioned

- how we can find a basis for it
- how it "behaves" with linear maps

Let us now compute some examples. Here is a quick comment to note

- The thm about the basis of V⊗W can be used to show certain examples are ______ just by counting ______

However! Sometimes the form of the isomorphism is more important (ie, what is the actual map)

ex) Prove $\mathbb{F} \otimes \mathbb{F} \cong$

Pf 1) Dimension

Pf 2) Universal property

(**Rmk**: Compare the above to $\mathbb{F} \oplus \mathbb{F}$)

ii) Prove that $\exists$ iso $V \otimes W \cong$ ______ that sends $v \otimes w \mapsto$ ______

Pf)

iii) Let V_1, V_2, W be vector spaces and $\mathbb{F}$. Prove $\exists$ iso

$$(V_1 \oplus V_2) \otimes W \cong$$

that sends the simple tensor $(v_1, v_2) \otimes w \longrightarrow$

ex) $\mathbb{R}^3 \otimes \mathbb{R}^2 \simeq$

iv) For V_1, V_2, W $\exists!$ isomorphism

$$(V_1 \otimes V_2) \otimes W \longrightarrow \underline{\hspace{4cm}}$$

that sends the simple tensor $(v_1 \otimes v_2) \otimes w \longmapsto$

(ie, the $\otimes$-product is $\underline{\hspace{4cm}}$)

v) There is a well-defined linear isomorphism $\mathbb{R}^n \otimes \mathbb{R}^n \to M_n(\mathbb{R})$ sending a simple tensor

$$\vec{v} \otimes \vec{w} \longmapsto$$

$P(x)$

Rmk: There is a sense in which row vectors $(x_1 \ldots x_n) \in (\mathbb{R}^n)^*$

Indeed choose a basis $\mathcal{B} = (v_1 \ldots v_n)$ for V and let

$\varphi \in V^*$. Then $[\ \ ]_{\mathcal{B}}$ is just a ___ $\times$ ___ matrix (i.e., ...)

So column vectors = "elements of $\mathbb{R}^n$"

row vectors = "elements of $(\mathbb{R}^n)^*$"

With this, and the last prop in mind, consider the following

Prop: $V \otimes V^* \cong \mathcal{L}(V, V)$

(Rmk: This is how tensors often show up in physics!)

We denote this map we get from the bijection

$(V \otimes g)(v') =$

HW

iv) Generalize the above to show

$$V \otimes W^* \simeq \mathcal{L}(V,W)$$

v) Use (iv) to prove

$$\mathcal{L}(V \otimes W, \mathbb{F}) \simeq \mathcal{L}(V, \mathcal{L}(W,\mathbb{F}))$$

(i.e. $(V \otimes W)^* \simeq \mathcal{L}(V, W^*)$)

"Unimportant" Rmk: Such an isomorphism as in (v) true more generally. Challenge problem!
Prove that for V, W, Z vector spaces

$$\mathcal{L}(V \otimes W, Z) \simeq \mathcal{L}(V, \mathcal{L}(W,Z))$$

(the famous "$\otimes$-hom adjunction")

ex) $V = \mathbb{R}^2$ with standard basis (e_1, e_2).

Let's see what map $\mathbb{R}^2 \to \mathbb{R}^2$ (ie a 2×2 matrix) corresponds to the tensors described below

Let $g\begin{pmatrix} x \\ y \end{pmatrix} = x$

1) $e_1 \otimes g$

2) $e_2 \otimes g$

For $g\begin{pmatrix} x \\ y \end{pmatrix} = x+y$

1) $3e_1 \otimes g$

2) $-2e_2 \otimes g$