Rank-Nullity Thrm

Recall: A $m \times n$ matrix

- $\text{null}(A) = \{ \vec{x} \text{ in } \mathbb{R}^n : \qquad \}$
  $=$ solution set to the $\qquad$ system

- We saw that this was a subspace
  - so in particular null(A) has $\qquad$

Def: The dimension of null(A) is called $\qquad$ of A

$= \dim(\text{null}(A))$

Recall: the # ______ is the # vectors in the basis for null(A)

$\Rightarrow$ nullity(A) = ______

- col(A) = span( )

= the vectors $\vec{b}$ in $\mathbb{R}^m$ such ______

______

Def: The ______ of matrix A is

= dim(col(A))

- Since col(A) = span(v_1 . . . v_n)

The ______________ in this list $V_1 - - V_n$
will be the # of vectors in the basis.

- This implies rank $(A) =$
$=$

Note: A $m \times n$ matrix

- $n =$

$\Rightarrow n =$

ex) Let A be a 6×5 matrix and the null space of A is

$$\text{null}(A) = \text{span}\left(\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}, \begin{pmatrix} 9 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right)$$

Q: What is rank of A?

ex) A 3×8 matrix with $\text{nullity}(A) = 6$.

Do the columns of A span $\mathbb{R}^3$?

Chapter 6 - Linear Transformations

Def: V, W be vector spaces. Then a function

$T: V \to W$ is a linear transformation if

1) for u, v in V

2) for c in $\mathbb{R}$, u in V

ex) $T: M_{2\times2}(\mathbb{R}) \to \mathbb{R}$

$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a+d$ = (trace of matrix)

ii) $D : \mathbb{R}_n[x] \longrightarrow \mathbb{R}_{n-1}[x]$

$$D(a_0 + a_1 x + \cdots + a_n x^n) = a_1 + 2a_2 x + \cdots + n a_n x^{n-1}$$

iii) $D(f) : C^\infty(\mathbb{R}^n; \mathbb{R}^m) \longrightarrow M_{m\times n}(\mathbb{R})$

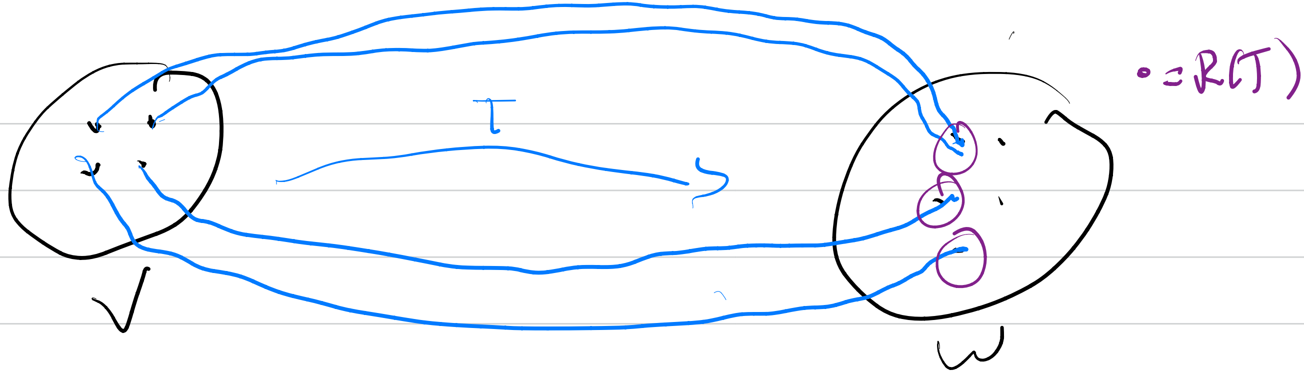
$\cup$

$f : \mathbb{R}^n \to \mathbb{R}^m \rightsquigarrow Df$ is a $m \times n$ matrix

Def: $T : V \to W$ be linear transformation.

1) $Ker(T) = \{ v \text{ in } V : \quad \}$ (Kernal of T)

2) Range of T, $R(T) = \{ w \text{ in } W : \quad \}$



Recall: If $T:\mathbb{R}^n \to \mathbb{R}^m$ linear we saw that T was uniquely defined on the standard basis

- Same is true for $T: V \to W$

In particular: Let $\mathcal{B}_V = (v_1 \dots v_n)$ be a basis for V

then $R(T) = \text{span}(\qquad\qquad\qquad)$

· So in particular, a basis for the range is just

ex) $T: \mathbb{R}_2[x] \to M_{2\times 2}(\mathbb{R})$

$$T(a+bx+cx^2) = \begin{pmatrix} a & a+b+c \\ c & -b \end{pmatrix}$$

$=$

$=$

$=$

So to find basis for $R(T)$, check which of the matrices

, , are LI

$$\Longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

So basis for range is $\mathcal{B} = (\qquad)$

More on Kernel

ex) $D: \mathbb{R}_3[x] \to \mathbb{R}_2[x]$ the derivative.

- What is $\ker(D)$?

ex) $T(a+bx+cx^2) = \begin{pmatrix} a & a+b+c \\ c & -b \end{pmatrix}$

$\ker(T) =$

Prop: T linear $\quad T(0_V) =$ (show!)

Def: $T: V \to W$ linear transf

1) nullity of T is

2) rank of T is

Recall: $T: \mathbb{R}^n \to \mathbb{R}^m$ linear $\rightsquigarrow$ A_T $m \times n$ matrix

such that $T(\vec{x}) = A_T \vec{x}$

$T: \mathbb{R}^n \to \mathbb{R}^m$	A_T $m \times n$ matrix
	$\text{null}(A_T)$
	$\text{nullity}(A_T)$
	$\text{Col}(A_T)$
	$\text{rank}(A_T)$

In the case of $T: \mathbb{R}^n \to \mathbb{R}^m$ we saw that

$\text{rank}(T) + \text{nullity}(T) = n$

Is this true in general?

Let V n-dim VS, W m-dimensional VS.

$T: V \to W$ linear transf.

Then $n = \dim(V) =$ ______________________

______________ Linear transformations!

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d] & H \\ G/\ker f \arrow[ur, dashed] & \end{tikzcd}$$

1st Isomorphism Thrm

• the cover of our canvas page!!

Def: $f: V \to W$ any function

1) We say f is surjective if
   (onto)

2) We say f is injective if whenever
   (1-1)

Thm: $T: V \to W$ be linear transf.

Then T is injective if and only if

ex) $T(a+bx+cx^2) = \begin{pmatrix} a & a+b+c \\ c & -b \end{pmatrix}$

- We saw $\ker(T) = \{0\}$ so T is ____

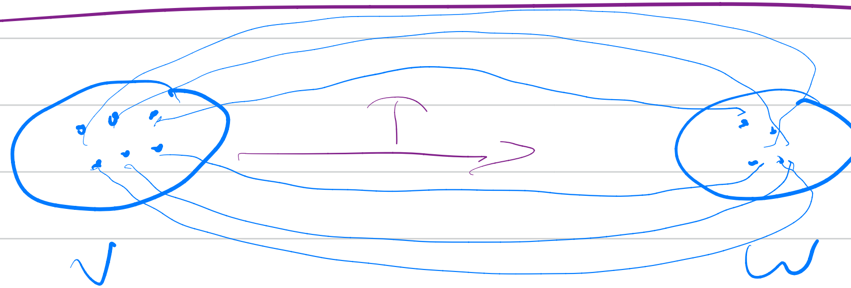
  but T is not ______

  ($R(T)$ has dim = __, but $M_{2\times 2}(\mathbb{R})$ has dim = __)

We will adress how to check for surjective/injective in a second, but first we have some applications of rank-nullity.

Thm: V n-dim, W m-dim $T: V \to W$ linear

1) If $\dim V > \dim W$ then



$\dim V = 6$ $\qquad\qquad$ $\dim W = 4$

$$\begin{aligned} n &= \operatorname{rank}(T) + \operatorname{nullity}(T) \\ &\leq m + \operatorname{nullity}(T) \\ 0 < n - m &\leq \operatorname{nullity}(T) \end{aligned} \implies \operatorname{nullity}(T) > 0$$

2) If $\dim V < \dim W$ then

Pf Uses Rank-Nullity

3) <u>Def</u>: We say T is an isomorphism if T is both injective and surjective.

- If $T: V \to W$ is an isomorphism we say V is isomorphic to W (write $V \cong W$)

Note if T is injective, need $\dim V \leq \dim W$
if T is surjective, need $\dim V \geq \dim W$

So if T is an isomorphism we have $\dim V = \dim W$

Thrm: Two vectors are isomorphic if and only if

ex) $M_{m \times n}(\mathbb{R}) \cong \qquad \cong$

$M_{3 \times 2}(\mathbb{R}) \cong \qquad \cong$