Yiddish of the day

"S'iz varem vi ... — ס'איז וואַרעם ווי ...

- in an oyer = — אין אַן אויער
- fayer volt gefaln fun himl — פֿײַער וואָלט געפֿאַלן פֿון הימל
- S'iz keyn trop luft nishto — ס'איז קיין טראָפּן לופֿט נישטאָ

Equivalence Relations (ER)

We call a relation R on A an equivalence relation if R is

1) Reflexive
2) Symmetric
3) transitive

- in this case, if aRb we say that

a is equivalent to b and we will

sometimes write $a \sim b$ (or $b \sim a$)

As mentioned last time, this is a generalization of equality

- $R = \Delta_A$ is the "usual" notion of equality

  and we saw last class this is indeed an ER ex) Consider the relation on $A = \mathbb{R}^2 \setminus \{0,0\}$ given by

$(x,y) \, R \, (a,b)$ iff $\exists \, \lambda \in \mathbb{R}_{\neq 0}$ such that

$(a,b) = (\lambda x, \lambda y)$.

Show this is an ER

1) Reflexive: Goal: Let $(a,b) \in \mathbb{R}^2 \setminus \{0,0\}$. Is $(a,b) \, R \, (a,b)$?

Yes: $\lambda = 1$ we have $(a,b) = (1a, 1b)$ ✓

2) Symmetric: Lets assume that $(a,b) R (x,y)$. Then $\exists \lambda \neq 0$ such that

$$(a,b) = (\lambda x, \lambda y)$$

$\xRightarrow{\text{since } \lambda \neq 0}$ $\frac{1}{\lambda}(a,b) = (x,y)$ $\rightsquigarrow$ $(x,y) = \left(\frac{1}{\lambda} a, \frac{1}{\lambda} b\right)$

$\Rightarrow$ $(x,y) R (a,b)$ ✓

3) Transitive: Lets assume $(a,b) R (c,d)$ and $(c,d) R (e,f)$

$\rightsquigarrow$ $\exists \lambda_1, \lambda_2 \in \mathbb{R}_{\neq 0}$ such that

$$(a,b) = \lambda_1 (c,d)$$

$$(c,d) = \lambda_2 (e,f)$$

$\rightsquigarrow$ plug in (c,d) to get

$$(a,b) = \lambda_1(\lambda_2(e,f)) = \underbrace{\lambda_1\lambda_2}_{\in\mathbb{R}_{\neq 0}}(e,f)$$

so $(a,b)\,R\,(e,f)$

Rmk: This example is very important! It leads to the "Real projective plane", denoted $\mathbb{RP}^2$. We'll return to it shortly

Def: Let R be an equivalence relation on A.

For $a \in A$ define the equivalence class represented by a to be the set

$$[a] = \{x \in A \mid xRa\}$$

$$= \{x \in A \mid (x,a) \in R\}$$

$$= \{\text{all elements in } A \text{ related to } a\}$$

- Note: Since R is reflexive we know that

$[a] \neq \phi$. (Why? $a \in [a]$)

Lets find the EC's for the following ER's

a) $R = \{(1,1), (2,2), (3,3), (4,4), (5,5), (1,3), (1,5), (5,1), (5,3), (3,1), (3,5), (2,4), (4,2)\}$ $(A = \{1,2,3,4,5\})$

$[1] = \{$all elements in A related to 1 under R$\}$

$= \{x \in A \mid (x,1) \in R\}$

$= \{1, 3, 5\}$

$[2] = \{2, 4\}$

$[3] = \{3, 1, 5\}$

$[4] = \{4, 2\}$

$[5] = \{5, 1, 3\}$

$\rightsquigarrow$ The EC's for R are

$\underline{[1]}$ and $\underline{[2]}$

Since $[1] = [3] = [5]$

and $[2] = [4]$

Note: For $a, b \in A$. If $a \cancel{R} b$ then

$[a] \cap [b] = \underline{\phi}$

and if $a \underline{R} b$ then $[a] \underline{=} [b]$

b) Real projective plane example.

$(x,y) R (a,b)$ if $\exists\, 0 \neq \lambda \in \mathbb{R}$ st $(a,b) = (\lambda x, \lambda y)$

Lets find the equiv class of $(3,5)$

$[(3,5)] = \{ \underline{(x,y) \in \mathbb{R}^2} \setminus \{0,0\} \mid (3,5) R (x,y) \}$

$$= \left\{ (x,y) \in \mathbb{R}^2 \setminus \{0,0\} \;\middle|\; \begin{array}{l} \underline{(x,y)} = (3\lambda, 5\lambda) \\ \text{for some } 0 \neq \lambda \in \mathbb{R} \end{array} \right\}$$

$$= \left\{ (x,y) \in \mathbb{R}^2 \setminus \{0,0\} \;\middle|\; \begin{array}{l} \underline{x} = 3\lambda \text{ and } y = 5\lambda \\ \text{for some } 0 \neq \lambda \in \mathbb{R} \end{array} \right\}$$

$$= \left\{ (3\lambda, 5\lambda) \;\middle|\; \lambda \in \mathbb{R}_{\neq 0} \right\}$$

Graphically this is the <u>line</u> defined by
$3y = 5x$ minus the origin

c) Congruence. Let $n \in \mathbb{N}$ and $A = \mathbb{Z}$. Then have the ER

aRb iff $a \equiv b \mod n$

Lets find all the ECs for $n = 5$

$[0] = \{x \in \mathbb{Z} \mid x \equiv 0 \mod 5\}$

$= \{$integers with remainder 0 when divided by 5$\}$

$[1] = \{x \in \mathbb{Z} \mid x \equiv 1 \mod 5\}$

$= \{$integers with remainder 1 when divided by 5$\}$

$[2] = \{x \in \mathbb{Z} \mid x \equiv 2 \bmod 5\}$

$= \{$integers with <u>remainder</u> 2 when <u>divided</u> by 5$\}$

$[3] = \{x \in \mathbb{Z} \mid x \equiv 3 \bmod 5\}$

$= \{$integers with <u>remainder</u> 3 when <u>divided</u> by 5$\}$

$[4] = \{x \in \mathbb{Z} \mid x \equiv 4 \bmod 5\}$

$= \{$integers with <u>remainder</u> 4 when <u>divided</u> by 5$\}$

Are there anymore? No!

$\rightsquigarrow$ Note again

1) If $a \not\mathrel{\underline{R}} b$ then $[a] \cap [b] = \emptyset$

2) If $a \mathrel{\underline{R}} b$ then $[a] = [b]$

Properties of EC's

<u>Thm</u>: Let R be an ER on A. Then for $a, b \in A$

$[a] \underline{=} [b]$ <u>iff</u> $a \underline{R} b$

Pf) First we will prove that if $[a] = [b]$ then aRb.
Since $[a] = [b]$ we have that $b \in [b] = [a]$ is in $[a]$.
Hence aRb

Now we assume that aRb. Let us assume that $x \in [a]$. Hence xRa and aRb by assumption. Since R is transitive, xRb so $x \in [b]$

Now let $y \in [b]$, then yRb and bRa by assumption (and the fact that R is symmetric)

Hence yRa because R is transitive so $y \in [a]$

Hence $[a] = [b]$ $\square$

Thm 2: Let R be ER on A. Then for $a, b \in A$ if

$[a] \cap [b] \neq \emptyset$ then aRb

(ie, $[a] = [b]$)

Pf) Suppose $[a] \cap [b] \neq \emptyset$. Then $\exists x \in [a] \cap [b]$.

Then xRa and xRb

Since R is symmetric we know aRx

So by transitivity of R, aRx and $xRb \Rightarrow aRb$ $\square$

Recall:

A <u>partition</u> of a set A is a collection of <u>subsets</u> $X_\alpha \subseteq A$ where $\alpha \in I$ is an indexing set, such that

1) $X_\alpha \cap X_\beta = \underline{\phi}$ whenever $\underline{\alpha \neq \beta}$

2) $\bigcup_{\alpha \in J} X_\alpha = \underline{A}$

Thm 3: Let R be an ER on A. Then the set of <u>equivalence classes</u> for R forms a <u>partition</u> of A

We denote the collection of all EC's by A/R

and read this as "A mod R" — This is called the "quotient set" of A by R

Pf) We already know that $[a] \cap [b] = \emptyset$ if they aren't the same set

Just need to show that $A = \bigcup_{x \in A} [x]$

Indeed, since $\forall x \in A$ $[x] \subseteq A$ so $\bigcup_{x \in A} [x] \subseteq A$

Take $y \in A$. Then $y \in [y]$ so $y \in \bigcup_{x \in A} [x]$

Important examples

a) When $A = \mathbb{R}^2 \setminus \{0,0\}$, with ER R given by
$(x,y) \, R \, (a,b)$ iff $\exists \, 0 \neq \lambda$ st $(x,y) = \lambda(a,b)$

$$\rightsquigarrow A/R \overset{\text{denoted}}{:=} \mathbb{RP}^1$$

$$= \left\{ \text{all } \underline{\text{lines}} \text{ through } \underline{\text{origin}} \right\}$$

b) Fix $n \in \mathbb{N}_{\geq 2}$ and $A = \mathbb{Z}$, with R given by

aRb iff $a \equiv b \mod n$

$\rightsquigarrow A/R := \underline{\mathbb{Z}/n\mathbb{Z}}$

$=$ the integers "mod n"

$= \{ [0], [1], \ldots, [n-1] \}$

$\rightsquigarrow$ We'll return to this example next class