Symmetric / Alternating Products

Def: Suppose V, W are $\mathbb{F}$-vs. Then a multilinear function $f: \underbrace{V \times \cdots \times V}_{k\text{-times}} \to W$ is

1) ______________ if

2) ______________ if

ex) Suppose $f: V \times V \to W$ is ________. Then

$f(v_1, v_2) =$ (this is called ________)

Pf) (maybe HW?)

(Rmk: True more generally than 2 vector spaces)

We saw before that a ________ map

$$f : V \times V \longrightarrow W$$

is the "same thing" as a ________ map

$$\tilde{f} : \_\_\_\_\_\_\_\_ \longrightarrow W$$

Now, what if this f was ________ ?

- if $m_1 = m_2$ we have

$$0 = \_\_\_\_\_\_\_\_ =$$

- That is the subspace

$N := \langle \qquad \mid \qquad \rangle \subseteq$

will be contained in the ______ of $\tilde{f}$

- We consider the ______ and denote it by $\Lambda^2(V) :=$

(call this the (second) exterior power of V)

- the coset of $v_1 \otimes v_2$ in $\Lambda^2(V)$ is denoted

______

Why? Universal Prop of quotient! We saw above that if

$$f : V \times V \longrightarrow W$$

is ______

then the induced map

$\tilde{f}$: ______ $\longrightarrow W$

Contains ______ in ______

$\rightsquigarrow$ We get a well defined map

$\tilde{f}$: ______ $\longrightarrow W$

Thm: The composite

$$V \times V \longrightarrow \underline{\qquad\qquad} \longrightarrow \underline{\qquad\qquad}$$

is alternating.

Moreover, given any alternating map

$$f : V \times V \longrightarrow W$$

there's a unique ______ map

$$\tilde{f} : \underline{\qquad} \longrightarrow W$$

such that

$$\begin{tikzcd} V \times V \arrow[r, "f"] \arrow[d] & W \\ {} \arrow[ur, dashed, "\exists ! \tilde{f}"'] \arrow[u, phantom, "\circlearrowleft"] & \end{tikzcd}$$

(ie $\tilde{f}(\quad) = f(\quad)$)

(that is there's a bijection

$$\mathrm{Alt}(V \times V, W) \cong \mathcal{L}(\quad , W)$$
)

So to define a map out of ______

one just defines a map $V \times V \to W$ and verifies

it is alternating

ex) Prove there is a unique linear map

$$\Lambda^2(V) \longrightarrow \underline{V \otimes V}$$

that sends $v_1 \wedge v_2 \longmapsto$

Pf)

Questions

1) What does it mean that $v_1 \wedge v_2 = 0$?

2) What does it mean that $\Lambda^2(V) = 0$?

3) What does it mean to say
$$v_1 \wedge v_2 = v_1' \wedge v_2' \,?$$

Rmk: All this works for ________ maps
$$f : \overbrace{V \times \cdots \times V}^{k\text{-times}} \longrightarrow W$$

$\rightsquigarrow$ get the space $\Lambda^k(V)$ for all k

ex) Suppose V is n-dim. Show in HW that if $k > n$

HW then any alternating map

$$\underbrace{f: V \times \cdots \times V}_{k\text{-times}} \longrightarrow W \text{ is } 0$$

$\Longrightarrow$ if $k > \dim V$ then $\Lambda^k(V) = 0$

Q: What if $k = \dim V$?

ex) Suppose V is 2-dim with basis $\mathcal{B}_V = (v_1, v_2)$.

Lets compute the "elementary wedge"

$$(av_1 + bv_2) \wedge (cv_1 + dv_2)$$

$=$

Wedge Products (cont)

- We saw last time that, if $k > \dim V$ then

$$\Lambda^k(V) =$$

(compare to $\dim(\underbrace{V \otimes \cdots \otimes V}_{k\text{-times}}) =$ ______)

- We saw that, for a 2-dim vs with basis (v_1, v_2)

wedge $(av_1 + bv_2) \wedge (cv_1 + dv_2)$

$=$

Thm: (Notes) Let $\dim V = n$. Then

$$\dim \Lambda^k(V) = \underline{\qquad\qquad}$$

If $\mathcal{B} = (v_1, \ldots, v_n)$ is a basis for V then

$$\mathcal{C} = (v_{i_1} \wedge \cdots \wedge v_{i_k} \mid 1 \le i_1 \le \cdots \le i_k \le n)$$

is a basis for $\Lambda^k(V)$

Cor: 1) If $\mathcal{B} = (v_1 \ldots v_n)$ basis for V then

$\dim(\Lambda^2(V)) =$ ______

with basis

$\mathcal{C} =$

ex) $\mathcal{B} = (v_1, v_2, v_3)$ then

$\mathcal{C} =$

Note that $\dim(\Lambda^2(V)) =$

So $\Lambda^2(V) \cong$

Here's a particular isomorphism in the case $V = \mathbb{R}^3$

ex) $\mathcal{B} = (e_1, e_2, e_3)$ be standard basis for $\mathbb{R}^3$

Define the isomorphism $\Lambda^2(\mathbb{R}^3) \xrightarrow{\sim} \mathbb{R}^3$ by

- $e_1 \wedge e_2 \longrightarrow e_3$

(☆) - $e_1 \wedge e_3 \longrightarrow -e_2$

- $e_2 \wedge e_3 \longrightarrow e_1$

(why is this an iso?)

Now let $v, w \in \Lambda^2(\mathbb{R}^3)$. Then compute

$v \wedge w$ and check what is is in $\mathbb{R}^3$ under ($\star$)

2) $\dim(\Lambda^n(V)) =$ ___

with basis

$\mathcal{C} =$

ex) $\mathcal{B} = (v_1, v_2)$

$\mathcal{C} =$