Math 117

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OH : T 4-5 PM

Th 11-12

Other resources :

- Linear Alg done Right - Axler
- Linear Alg done wrong - Treil
- Abstract Algebra - Dummit, Foote

Spanning, Linear Ind, Basis

Q: Let S be spanning set for V

L be LI set for V

B be basis for V

1) $|S| \geq |L|$ ex) $\mathbb{R}^2$ $B = \left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right)$

span $(\vec{v}_1)$ = line

2) $|S| \geq |B|$

3) $|L| \leq |B|$

- Is there a Linear ind set L with $|L| = 10$ in $\mathbb{F}_3^3$?

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Q2: How many elements are in $\mathbb{F}_3^3$?

$$|\mathbb{F}_3^3| = \underline{3} \cdot \underline{3} \cdot \underline{3} = 27$$

$$|\mathbb{F}_p^n| = p^n$$

ex) Let $X = x_1$ and $V = \mathbb{F}_5^2$

Consider $\mathrm{Fun}(X, V)$. Find $|\mathrm{Fun}(X, V)|$

Note: Functions determined by what they "do" to inputs
• in this case only have 1 input.

$$x_1 \longmapsto \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$x_1 \longrightarrow \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

$$x_1 \longrightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\vdots$$

$$|\mathrm{Fun}(X, V)| = 25$$

ex) X: $\overset{x_1}{\bullet}$ $\overset{x_2}{\bullet}$ V: $\mathbb{F}_3^4$

Again find $|\mathrm{Fun}(X, V)|$

Note: A function $f: X \to V$

$x_1 \mapsto \vec{v}_1$

$x_2 \to \vec{v}_2$

- Have 3^4 choices for where x_1 maps
- Have 3^4 choices for where x_2 maps

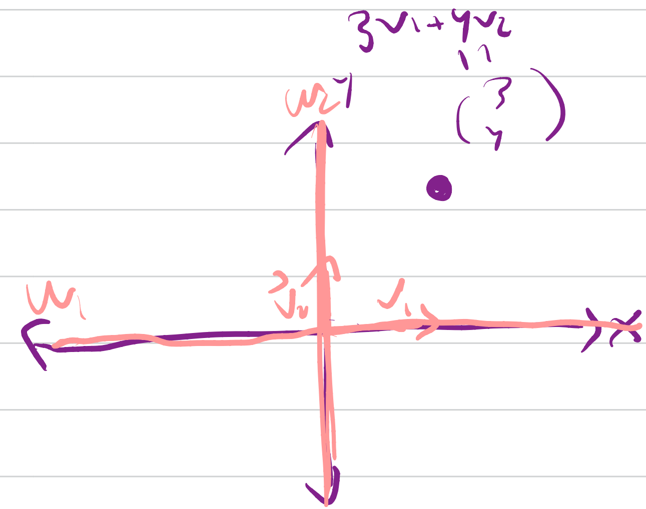
$\Rightarrow$ Have $3^4 \cdot 3^4$ choices of functions

Subspaces and Direct Sum

ex) 1) $V = \mathbb{R}^2$

$W_1 = \text{span}\overset{v_1}{\begin{pmatrix}1\\0\end{pmatrix}}$

$W_2 = \text{span}\overset{v_2}{\begin{pmatrix}0\\1\end{pmatrix}}$



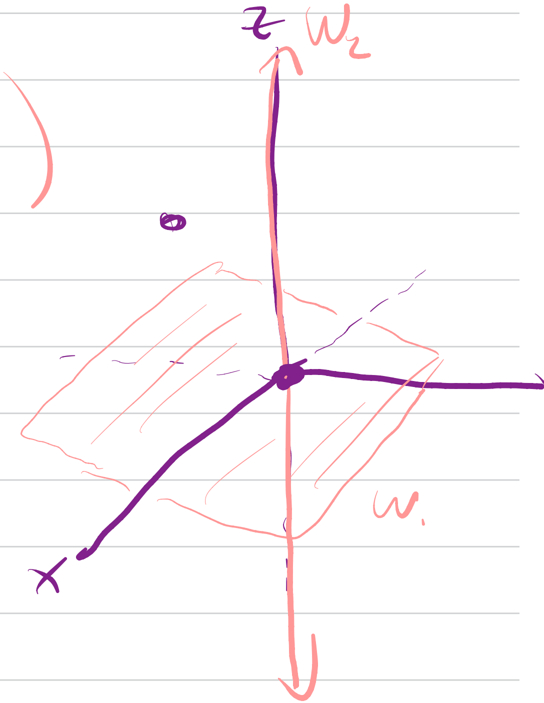
$\Rightarrow V = \underline{W_1 + W_2}$

$\Rightarrow V = W_1 \oplus W_2$ direct sum

2) $V = \mathbb{R}^3$

- $W_1 = \text{Span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right)$

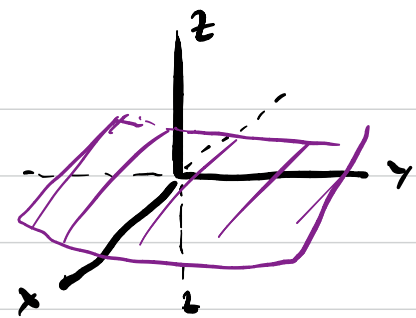
- $W_2 = \text{Span} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$



$V = \underline{W_1 + W_2}$

$V = W_1 \oplus W_2$

3) $V = \mathbb{R}^3$



- $W_1 = \left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} : x, y \in \mathbb{R} \right\}$

- $W_2 = \left\{ \begin{pmatrix} 0 \\ x \\ y \end{pmatrix} : x, y \in \mathbb{R} \right\}$

$V = \underline{W_1 + W_2}$ but $\underline{\text{not}}$ direct sum

$\underline{Q}$: $\dim V = 3$ $\dim W_1 = 2$ $\dim W_2 = 2$

$$\dim(W_1 \cap W_2) = 1$$

$$\dim(W_1 + W_2) = \dim V = 3 = \dim W_1 + \dim W_2 - \dim W_1 \cap W_2$$

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim W_1 \cap W_2$$

Q: $V = \mathbb{R}^4$

- $W_1 = \text{span}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $W_2 = \left\{ \begin{pmatrix} x \\ 0 \\ y \\ 0 \end{pmatrix} ; x, y \in \mathbb{R} \right\}$

- $W_3 = \text{span}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

Find

- $\dim W_1 = 1$
- $\dim W_2 = 2$
- $\dim W_3 = 1$

Q: $W_1 + W_2 + W_3$
$\parallel$
W_2

- $\dim(W_1 + W_2 + W_3) = \dim W_2 = 2$
- $\dim(W_1 \cap W_2) = \dim(W_1) = 1$
- $\dim(W_1 \cap W_3) = 0$
- $\dim(W_2 \cap W_3) = \dim(W_3) = 1$
- $\dim(W_1 \cap W_2 \cap W_3) = 0$

$$2 = \dim(W_1 + W_2 + W_3) = \dim(W_1) + \dim(W_2) + \dim(W_3) - \dim(W_1 \cap W_2) - \dim(W_1 \cap W_3) - \dim(W_2 \cap W_3) + \dim(W_1 \cap W_2 \cap W_3)$$

$$= 1 + 2 + 1 - 1 + 0 - 1 + 0$$

$$= 4 - 2 = 2 \quad \checkmark$$

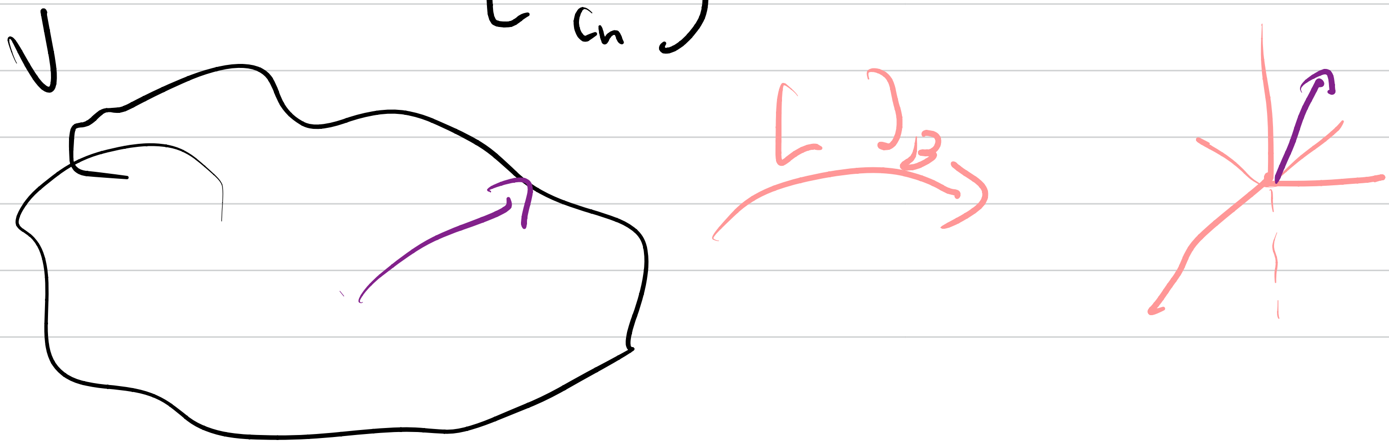
Q: What is $W_1 \cap (W_2 + W_3)$ for any subspaces W_1, W_2, W_3?

Def: V vector spaces $\mathcal{B} = (v_1, v_2, \dots, v_n)$ is basis.

Then for any $w \in V$ we can write w uniquely like

$$w = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$$

$$[w]_{\mathcal{B}} \overset{\text{def}}{=} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n$$



If $\mathcal{B}' = (w_1, \ldots, w_n)$ is a different basis

$w = c_1 v_1 + \cdots + c_n v_n$

$= d_1 w_1 + \cdots + d_n w_n$

Q: How are the vectors

$$[w]_{\mathcal{B}} = \underbrace{P_{\mathcal{B} \to \mathcal{B}'}}_{\text{change of basis matrix}} [w]_{\mathcal{B}'}$$

related