SETs REMINDER

~> Right now only 13/29 taken (44%)

~> Only 1 more Day (Tuesday it ends)

Yiddish family page

משפחה – ווי גאָט טוט, אַבי אין איינעם

"Mish**po**khe – Whatever happens, as long as we're together.

elter

great-... ...-עלטער

di bobeh

די באָבע (ס)

grandmother

Der zayde

דער זיידע (ס)

grandfather

די באָבע-זיידע

grandparents

di mimeh

די מומע (ס)

די טאַנטע (ס)

aunt

di tanteh

Der feter

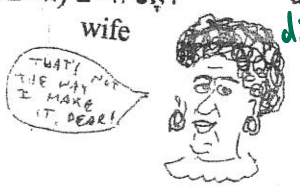
דער פֿעטער (ס)

uncle

dos vayb

דאָס ווײַב (ווײַבער)

wife



di mameh

די מאַמע (ס)

mother

Der tateh

דער טאַטע (ס)

father

דער מאַן (מענער)

husband, man

Der man

די טאַטע-מאַמע (ס)

די עלטערן

parents

di eltern

di tokhter

די טאָכטער (טעכטער)

די שוועסטער (-)

daughter, sister

דער זון (די זין)

Der zin

דער ברודער (ברידער)

Der brider

son, brother

דער פּלימעניק (ן)

די פּלימעניצע (ס)

pli**me**nik – nephew

pli**me**nitze – niece

di Kuzine

די קוזינע (ס)

cousin (f)

Der Kuzine

דער קוזין (ע)

דאָס שוועסטערקינד (שוועסטערקינדער)

cousin (m.)

Dos shvesterkind

די מחותּנתטע (ס)

Di shviger

די שוויגער (ס)

דער שווער (ן)/דער מחותּן

Der shver

די שווער-און-שוויגער

די מחותּנים

mekhu**te**neste – mother-in-law

me**khu**tn – father-in-law

mekhutonim – the in-laws

di shvigerin

די שוועגערין (ס)

sister-in-law

דער צד (צדדים)

side (tzad, tze**do**dim)

Der shvoger

דער שוואָגער (ס)

brother-in-law

די שנור (ן)

daughter-in-law

di shnor

דער איידעם (ס)

son-in-law

der aydem

More on functions

__Last time:__ $f: A \to B$ function, $X \subseteq A$, $Y \subseteq B$

- $\underline{f(X)} = \{ f(x) \mid x \in X \} \subseteq B$
- $\underline{f^{-1}(Y)} = \{ x \in A \mid f(x) \in Y \} \subseteq A$

$\rightsquigarrow f^{-1}(f(X)) = \{ z \in A \mid f(z) \in f(X) \}$

$$f(f^{-1}(Y)) = \{ f(x) \mid x \in f^{-1}(Y) \}$$

<u>examples</u>

1) $f : \mathbb{Z} \to \mathbb{Z}$ by $f(x) = x^2$

$$X = \{-1, 1, 2\}$$

$\rightsquigarrow$ 1) $f(X) = \{1, 4\}$

2) $f^{-1}(f(X)) = f^{-1}(\{1, 4\})$

$$= \{1, -1, 2, -2\}$$

2) $f: \mathbb{Z} \to \mathbb{Z}$ by $f(x) = x^3$

$X = \{-1, 1, 3, 4\}$

1) $f(X) = \{-1, 1, 27, 64\}$

2) $f^{-1}(f(X)) = f^{-1}(\{-1, 1, 27, 64\})$

$= \{a \in \mathbb{Z} \mid a^3 \in \{-1, 1, 27, 64\}\}$

$= \{-1, 1, 3, 4\}$

$= X$

<u>Def</u>: Let $f: A \to B$ be a function.

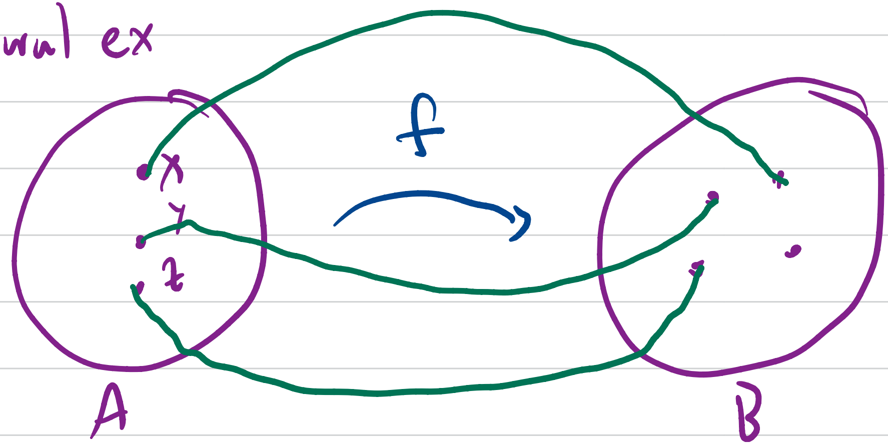
1) f is <u>injective</u> (or <u>one</u>-to-<u>one</u>) if f maps <u>distinct</u> elements to <u>distinct</u> elements.

$\rightsquigarrow$ i.e., if $\underline{x} \neq \underline{y}$ then $\underline{f(x)} \neq \underline{f(y)}$

contrapositive

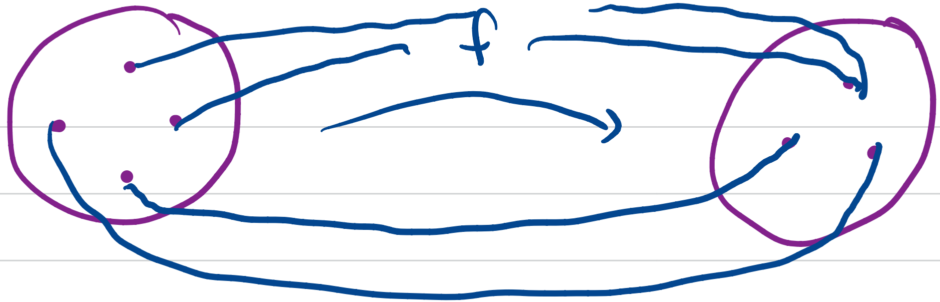
$\Longleftrightarrow$ if $\underline{f(x)} = \underline{f(y)}$ then $\underline{x} = \underline{y}$

Pictural ex



2) f is <u>surjective</u> (or <u>onto</u>) if <u>range(A)</u> = <u>B</u>

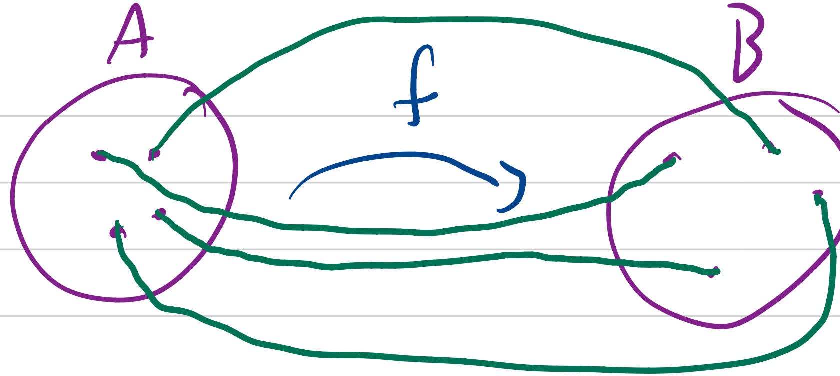
A   B



⇝ ie, every element of B is mapped to by an element of A

$\iff \forall\, b \in B,\ \exists\, a \in A$ such that $f(a) = b$

3) f is bijective if f is both injective and surjective



examples:

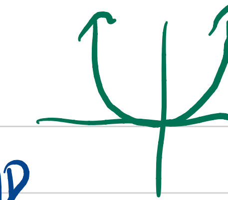
1) X any set. Then $\mathrm{id}_X : X \to X$ is bijective

2) $A \subseteq X$. Then $\iota_A : A \to X$ is injective but not surjective in general

$X = \{1,2,3\}$ $\quad A = \{1,2\}$ $\quad \iota_A : A \to X$ $\quad \iota(1) = 1$

$\iota(2) = 2$

3) $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^4$



- Injective? No! $f(x) = f(-x)$ $\forall x \in \mathbb{R}$

- Surjective? No! $-1 \notin \text{range}(f)$

However! $f: \mathbb{R} \to [0, \infty)$ by $f(x) = x^4$ is surjective!

4) $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = 3x + 2$

- Injective: Let $x, y \in \mathbb{R}$ and assume $f(x) = f(y)$.

Then $3x + 2 = 3y + 2$. Thus $x = y$ as required

• Surjective? Let $b \in \mathbb{R}$.

⌜Goal: Find $x \in \mathbb{R}$ such that $f(x) = b$⌟

$\rightsquigarrow$ Want $x \in \mathbb{R}$ such that $3x+2 = b$. That is $3x = b-2$
and $x = \dfrac{b-2}{3}$

$\rightsquigarrow$ $x = \frac{b-2}{3}$ then $f(x) = 3\left(\frac{b-2}{3}\right) + 2 = b$ as desired $\square$

Another way to characterize injective and surjective

Let $f : A \to B$ be a function. For any $b \in B$

consider

$$f^{-1}(\{b\}) = \{ a \in A \mid f(a) = b \}$$

$\rightarrow$ 1) f injective iff $|f^{-1}(\{b\})| \leq 1 \quad \forall b \in B$

2) f surjective iff $|f^{-1}(\{b\})| \geq 1 \quad \forall b \in B$

$\Rightarrow$ 3) f bijective iff $|f^{-1}(\{b\})| = 1 \quad \forall b \in B$

Def: Composition of functions

$$A \xrightarrow{f} B \xrightarrow{g} C$$

Let $f: A \to B$ and $g: B \to C$ be functions. Then we can define the new function

$$g \circ f : A \longrightarrow C$$

by $(g \circ f)(a) = \underline{g(f(a))}$

ex) $f: \mathbb{Z} \to \mathbb{Z}$ by $f(x) = x^2$

$g: \mathbb{Z} \to \mathbb{Z}$ by $g(x) = x - 2$

$\rightsquigarrow$ $(f \circ g)(x) = f(x-2) = (x-2)^2$

$(g \circ f)(x) = g(x^2) = x^2 - 2$

Prop: Let $f: A \to B$ and $g: B \to C$ functions

1) If both f and g are injective then the composite $g \circ f$ is also injective

2) If both f and g are surjective then the composite $g \circ f$ is also surjective

Pf) Left to you :) Good way to check if you know the definitions.

Prop: Let $f: A \to B$ and $g: B \to C$ functions.

1) If $g \circ f$ is injective then f is injective

⌜Q: If $g \circ f$ is injective is g also?⌟

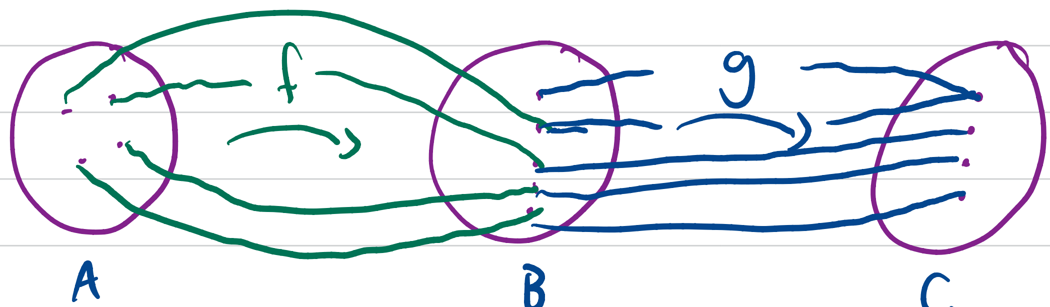
2) If $g \circ f$ is surjective then g is surjective

⌜Q: If $g \circ f$ is surjective is f also?⌟

Pf) (1) Suppose $g \circ f$ is injective. Let $x, y \in A$ such that $f(x) = f(y)$. Then $g(f(x)) = g(f(y))$

Hence $(g \circ f)(x) = (g \circ f)(y)$, so since $g \circ f$ injective we have $x = y$ as desired $\square$

For the Q



(2) Suppose $g \circ f$ is surjective. Let $c \in C$ be arbitrary.
Since $g \circ f : A \to C$ is surjective, $\exists a \in A$ such that

$g \circ f(a) = c$. That is $g(f(a)) = c$. Hence $f(a) \in B$, $\exists$ an element in B such that g of that element $= c$ $\boxed{}$

⌜ Q left to you :) ⌟

- Sometimes there is an easier way to show a function f is <u>bijective</u>

Thm: Let $f: A \to B$ and $g: B \to A$ be functions.

Suppose 1) $\underline{g \circ f} = \underline{id_A}$

2) $\underline{f \circ g} = \underline{id_B}$

Then f and g are both <u>bijective</u> and are each others <u>inverse</u> relation

Pf) Since $g \circ f = \underline{id_A}$ which is <u>bijective</u> we know that <u>f</u> is <u>injective</u> and <u>g</u> is <u>surjective</u>

Similarly, since $f \circ g = \mathrm{id}_B$ which is bijective we have that g is injective and f is surjective

Moreover

Thm: Suppose $f: A \to B$ bijective. Then the preimage $f^{-1}: B \to A$ is a function and $f \circ f^{-1} = \mathrm{id}_B$ and $f^{-1} \circ f = \mathrm{id}_A$

Pf) For $b \in B$, have $f^{-1}(b) := a$ where a is the unique element such that $f(a) = b$ (Do you see why this is well defined?)

Then $f(f^{-1}(b)) = f(a)$ where $a \in A$ such that $f(a) = b$

$= b$

and $f^{-1}(f(a)) = a$

ex) Let $f: \mathbb{R}\setminus\{2\} \to \mathbb{R}\setminus\{3\}$ given by

$f(x) = 3 + \dfrac{6}{x-2}$

Show f is bijective

[Scratch work: 1) write $y = f(x)$

$\rightsquigarrow y = 3 + \dfrac{6}{x-2}$

2) Swap the values x and y

$\rightsquigarrow \quad x = 3 + \frac{6}{y-2}$

3) Solve for y

$x - 3 = \frac{6}{y-2} \quad \rightsquigarrow \quad y - 2 = \frac{6}{x-3}$

$\rightsquigarrow \quad y = 2 + \frac{6}{x-3}$

4) The resulting y is the predicted choice for inverse function

Pf) Define $g: \mathbb{R} \setminus \{3\} \longrightarrow \mathbb{R} \setminus \{2\}$ by

$g(x) = 2 + \dfrac{6}{x-3}$ is our inverse to f.

$$(f\circ g)(x) = f\left(2 + \frac{6}{x-3}\right) = 3 + \frac{6}{2+\frac{6}{x-3}-2}$$

$$= 3 + \frac{6}{\frac{6}{x-3}} = 3 + x - 3 = x \checkmark$$

$$(g\circ f)(x) = g\left(3 + \frac{6}{x-2}\right) = 2 + \frac{6}{3+\frac{6}{x-2}-3}$$

$$= 2 + \frac{6}{\frac{6}{x-2}} = 2 + x - 2 = x \checkmark$$

Some more notation

- We commonly use the word **map** to mean function
  - if $b = f(a)$ we say a **maps** to b under f ($a \longmapsto b$ \mapsto)
- Given $f: A \to B$ function, If f is
  1) **f injective** we write $f: A \hookrightarrow B$ (\hookrightarrow)
  2) **f surjective** we write $f: A \twoheadrightarrow B$ (\twoheadrightarrow)
  3) **f bijective** we write $f: A \xrightarrow{\sim} B$ (\xrightarrow{\sim})

Cardinality

Goal: Count the # of elements in ANY set

⤳ When finitly many elements :) :) :)

⤳ But MANY sets with not finitly many elements

[$\mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R} \subsetneq \mathbb{C}$]

Q: How to distinguish between "how big" sets like these are?

Lemma: Let A, B be finite sets. Then

$|A| = |B|$ iff $\exists$ bijection $f: A \to B$

Pf) "$\Rightarrow$" Assume that $|A| = |B| = n$

Then $A = \{a_1, a_2, \dots, a_n\}$ $\longrightarrow$ Define $f: A \to B$ by

$B = \{b_1, b_2, \dots, b_n\}$ $\qquad f(a_i) = b_i$

Check this is bijective ✓

"$\Leftarrow$" Assume $\exists$ bijection $f: A \xrightarrow{\sim} B$. Let $A = \{a_1, \ldots, a_n\}$

Consider the set $\{f(a_1), f(a_2), \ldots, f(a_n)\} = \text{range}(f) = B$

That is $B = \{f(a_1), f(a_2), \ldots, f(a_n)\}$ so $|A| = |B| = n$ $\square$

Def: Let A, B be sets. Then A and B are said to be numerically equivalent (or have the same cardinality), denoted by "$|A| = |B|$" if $\exists$ bijection $f: A \to B$

- If A is finite with n-elements we write

$|A| = n$

Thm: Numerical equivalence is an ER amongst sets