Yiddish of the Day

"Er halt az er iz der pipik fin der velt" = ער האַלט אַז ער איז דער פּופּיק פֿון דער וועלט

=

~> **HW 1 Due today**

Logic

Logic is the grammer of math, i.e. the rules of the language

Def: A statement is a declarative sentance which can be objectively determined to be true or false

ex) (and non examples)

a) The integer 11 is divisible by 4 — false :(

b) The integer 11 is an odd # — true :)

c) Is $10^{10} \in \mathbb{Z}$ - Not a statement

d) The integer 10^{10} is bigggg ~ not a statement

Def: Given a <u>statement</u> P, the <u>negation</u> of P is the statement <u>not P</u> denoted <u>$\neg P$</u> (or <u>$\sim$</u> P) (\neg)

⇝ It is characterized by the following <u>truth table</u>

P	$\neg P$
T	F
F	T

ex) a) P: The # 11 is odd

$\neg P$ = The # 11 is not odd

= the # 11 is even

b) P: The day of the week is Wednesday

$\neg P$: The day of the week is not Wednesday

Def: Given statements P and Q, the disjunction of P and Q is the statement

P or Q denoted $P \vee Q$ (\lor)

- It is defined via the following truth table

P	Q	P∨Q
T	T	T
T	F	T
F	T	T
F	F	F

⇝ ie, P∨Q is true if at least one of P or Q is true

ex) a) P: 5 is odd T
Q: 10 is prime F

$\Rightarrow$ P∨Q is T

b) Let P be any statement. Then $P \vee \neg P$ is always true.

P	$\neg P$	$P \vee \neg P$
T	F	T
F	T	T

⇝ we call such statements **tautologies** (ie always true)

Def: Given statements P and Q the conjunction of P and Q is the statement
P and Q denoted $P \wedge Q$ (\land)

It is defined via the truth table

P	Q	$P \wedge Q$
T	T	T

→ ie only true when both

T	F	F
F	T	F
F	F	F

P and Q is true

Def: Given statements P and Q the implication is the statement if P then Q, denoted $P \Rightarrow Q$ (P implies Q)

- We call P the hypothesis and Q the conclusion.

Its truth table is defined as

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Note that when the <u>hypoth</u> P is false the <u>conclusion</u> Q may be T or F, but <u>$P \Rightarrow Q$</u> is always <u>true</u>

ex) P: It is raining
Q: I will stay at home

$\rightsquigarrow$ <u>$P \Rightarrow Q$</u> : If it is raining, then I will stay home.

If it doesn't rain and I go out, did I lie?

$\underbrace{\text{doesn't rain}}_{P \text{ false}}$ $\underbrace{\text{go out}}_{Q \text{ false}}$

$\longrightarrow P \Rightarrow Q = \underline{\text{true}}$

Some terminology: $\underline{P \Rightarrow Q}$ is read in diff ways

- if $\underline{P}$ then $\underline{Q}$
- $\underline{P}$ implies $\underline{Q}$
- $\underline{Q}$ if $\underline{P}$
- $\underline{P}$ only if $\underline{Q}$
- P is <u>sufficient</u> for Q
- Q is <u>necessary</u> for P

all these mean $\underline{P \Rightarrow Q}$

We will be considering the truth value of statements that depend on a variable.

Def: An open sentence is a declarative sentence which contains variables, where each variable can assume any value in a given set, called the domain of the variable, which becomes a statement if the variables are replaced with specific values.

ex) For $x \in \mathbb{R}$ consider the statement

$$P(x): \ |x| = 3$$

$x = 1$: $P(1)$: $|1| = 3$ false

$x=-2$: $P(-2)$: $\underline{|-2|=3}$ false

$x=-3$: $P(-3)$: $\underline{|-3|=3}$ true

- $P(x)$: x is odd for $\underbrace{x \in \mathbb{Z}}$

this is the **domain**

We can combine open sentences using $\neg, \vee, \wedge, \Rightarrow$ to form new open sentences

ex) Domain : $x \in \mathbb{R}$ $\quad$ $P(x)$: $|x|=3$

$Q(x)$: $x=-3$

$\rightsquigarrow \neg P(x)$: $|x| \neq 3$ $\qquad$ $\neg P(1)$: $|1| \neq 3$ T

$P(x) \wedge Q(x)$: $|x| = 3$ and $x = -3$ $\qquad$ $P(3) \wedge Q(3)$: False

always true $(Q(x) \Rightarrow P(x))$: If $x = -3$ then $|x| = 3$

$P(x) \vee Q(x)$: $|x| = 3$ or $x = -3$

Def: The implication $\underline{Q \Rightarrow P}$ is called the <u>converse</u> of $P \Rightarrow Q$

- Given statements P and Q the <u>biconditional</u> of P and Q is the statement

$$\underline{(P \Rightarrow Q)} \wedge \underline{(Q \Rightarrow P)}$$

denoted $\underline{P \Leftrightarrow Q}$ (\iff)

- We read $P \iff Q$ as
  - P if and only if Q
  - P is equivalent to Q
  - P is necessary and sufficient for Q
    ($Q \Rightarrow P$) ($P \Rightarrow Q$)

The truth table is given as

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$P \Rightarrow Q \wedge Q \Rightarrow P$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F

F	F	T	T	T

$\rightsquigarrow$ $P \iff Q$ is true when both P and Q are either true or false

Compound Statements

Def: A compound statement is a statement consisting of least one statement involving at least one connective connective $(\neg, \wedge, \vee, \Rightarrow, \Leftarrow)$

- Each statement in a compound statement is called a component statement ex) $P \Leftrightarrow Q$ is a compound statement with component statements

- $P \Rightarrow Q$
- $Q \Rightarrow P$

Def) 1) A compound statement is called a <u>tautology</u> if it is <u>true</u> for all possible <u>values</u> of truth values for its <u>component</u> statements

2) A compound statement is called a <u>contradiction</u> if it is <u>false</u> for all possible <u>combinations</u> of truth values for its <u>component</u> statements

ex) i) $P \vee (\neg P)$ is a tautology

ii) $P \wedge (\neg P)$ is a contradiction

P	$\neg P$	$P \wedge (\neg P)$
T	F	F
F	T	F

iii) $(P \wedge (P \Rightarrow Q)) \Rightarrow Q$ is a tautology

P	Q	$P \Rightarrow Q$	$P \wedge (P \Rightarrow Q)$	$(P \wedge (P \Rightarrow Q)) \Rightarrow Q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

iv) $((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$ is a <u>tautology</u>

notice how there's more options now

P	Q	R	$P \Rightarrow Q$	$Q \Rightarrow R$	$(P \Rightarrow Q) \wedge (Q \Rightarrow R)$	$P \Rightarrow R$	$((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Logical Equivalences

Def: Two compound statements R and S are logically equivallent if they have the same truth values for all possible combinations of truth values for its component statements

⇝ We denote this as $R \equiv S$ (\equiv)

Thrm: Let P and Q be statements. Then

$$P \Rightarrow Q \equiv ((\neg P) \vee Q))$$

Pf) We will build the truth table

P	Q	$\neg P$	$P \Rightarrow Q$	$(\neg P) \vee Q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Thm: Let P, Q, R be statements. Let $\underline{\;T\;}$ and $\underline{\;\perp\;}$ be a tautology and contradiction. Then

1) Identity laws: $P \vee \perp \equiv P$
$P \wedge T \equiv P$

2) Domination laws: $P \vee T \equiv T$
$P \wedge \perp \equiv \perp$

3) Double negation: $\neg(\neg P) \equiv P$

4) Commutative laws: $P \vee Q \equiv Q \vee P$
$P \wedge Q \equiv Q \wedge P$

5) Associative laws: $(P \vee Q) \vee R \equiv P \vee (Q \vee R)$
$(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$

★ 6) Distributive laws : $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$

$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$

★ ★ 7) De-Morgan's Laws : $\neg(P \vee Q) \equiv (\neg P) \wedge (\neg Q)$

$\neg(P \wedge Q) \equiv (\neg P) \vee (\neg Q)$

Pf) We will prove De-Morgan's Laws (the first part)

P	Q	$\neg P$	$\neg Q$	$P \vee Q$	$\neg(P \vee Q)$	$(\neg P) \wedge (\neg Q)$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Application: Prove that

1) $\neg(P \Rightarrow Q) \equiv P \wedge (\neg Q)$

Pf) $\neg(P \Rightarrow Q) \equiv \neg\big((\neg P) \vee Q\big)$ $\quad (P \Rightarrow Q \equiv \neg P \vee Q)$

$\equiv \neg(\neg P) \wedge (\neg Q)$ $\quad$ De Morgan's Law

$\equiv P \wedge \neg Q$ $\quad$ Double-negation law

2) $((\neg Q) \Rightarrow (P \land \neg P)) \equiv Q$

Pf) $\neg Q \Rightarrow (P \land \neg P) \equiv \neg(\neg Q) \lor (P \land \neg P)$

$P \Rightarrow Q \equiv \neg P \lor Q$

$\equiv Q \lor (P \land \neg P)$ double negation

$\equiv Q \lor \bot$

$\equiv Q$ identity law

3) $((P \land \neg Q) \Rightarrow (R \land \neg R)) \equiv P \Rightarrow Q$

Pf) $P \land \neg Q \Rightarrow (R \land \neg R) \equiv \neg(P \land \neg Q) \lor (R \land \neg R)$

$\equiv \neg(P \land \neg Q) \lor \bot$

$\equiv \neg(P \land \neg Q)$ identity law

$\equiv \neg P \lor \neg(\neg Q)$ De-Morgan's law

$\equiv \neg P \lor Q$ Double-negation

$\equiv P \Rightarrow Q$