"After a few months, though, I realized something: I hadn't gotten any better at understanding tensor-products, but I was getting used to *not* understanding them. It was pretty amazing; I no longer felt anguish when tensor products came up; I was instead amused by their cunning ways"

— Cathy O'Neil

"It is the things you can prove that tell you how to think about tensor products. Ie, you let lemmas and examples shape your intuition of the mathematical objects in question. There's nothing else, no magical intuition will appear to help you understand it"

— Johan de Jong

Multilinear Algebra

- **Def:** Let V_1, V_2, W be vector spaces, Then a __________ is a function

$$f: \_\_\_\_\_\_ \longrightarrow \_\_\_\_\_\_$$

such that

ii) More generally if $V_1 \dots V_n$, W are vector spaces, then a
______________ is a function

$$f: \underline{\hspace{4cm}} \longrightarrow \underline{\hspace{2cm}}$$

st

Rmk: When $n=1$ ⟶

$n=2$ ⟶

iii) When $W=IF$ and ____ are same

$n=1$:

$n=2$

$n>2$:

Prop: The set ______ of ______ is a vector space under pointwise addition/scaling

Pf) HW (n=2 case)

(no kernel, image not subspace, etc...)

Goal: 1) ______ maps = ☹ booooo!

______ maps = ☺ wooooh!

$\rightarrow$ Want to convert ________ $\rightarrow$ ________ maps

2) Want some way to "________" vectors

3) We know that a vector space V over $\mathbb{R}$, isn't always a vector space over $\mathbb{C}$. But $\mathbb{C}$ is better than $\mathbb{R}$, can we

________________________________ ?

Deep Breath

Goal: Want a bijection

$$\text{Bilinear}(\quad , w) \simeq \mathcal{L}(?, w)$$

- What should this ? be

General construction

Let V_1, V_2 be vector spaces.

1) Consider the ________ vector space $F(\quad)$

(see notes)

rmk: Given a function $f: \_\_\_ \longrightarrow W$

we can uniquely extend this to a linear map

$$F(\quad) \longrightarrow W$$

(why?)

2) Consider this weird subspace

$N : \left\{ \vphantom{\begin{matrix} a \\ b \\ c \\ d \end{matrix}} \right.$

Why?: If $f:$ ____ $\to W$ is ____

then have induced **linear** map

$\mathrm{Free}(f) : F(\quad) \to W$ as before.

Note: $N \subseteq$ ____

3) Consider $F^{(\qquad)} / N$

Why: Given ______ f: ______ $\to W$

we saw we had an induced map

Free(f): $F^{(\qquad)} \to W$

with $N \subseteq$ ______

$\Rightarrow$ Universal property of ______

tells us that this map can

then be extended to a unique linear map

$$\underline{\qquad\qquad\qquad} \longrightarrow W$$

<u>Define</u>: $\underline{\qquad\qquad} := F(\qquad)/N$

<u>Step back</u>: What have we acheived?

- Start with a $\underline{\qquad\qquad}$ map $f : \underline{\quad} \longrightarrow W$

Then we have the following diagram

$$
\begin{tikzcd}
\underline{\qquad} \arrow[r, "f"] \arrow[d, hook] & W \\
F(\quad) \arrow[ur, "\exists!\, \mathrm{Fru}(f)"] \arrow[d, "\pi_N"] & \\
\underline{\qquad} =: F(\quad)/N \arrow[uur, dashed, "\exists!\, \tilde{f}"'] &
\end{tikzcd}
$$

Thm: The composite $\underline{\qquad} \hookrightarrow F(\quad) \longrightarrow \underline{\qquad}$ is bilinear and it induces a (natural) isomorphism

$\mathrm{Bil}(\underline{\quad\quad}, W) \simeq \mathcal{L}(\quad\quad, W)$

(this is the so called "universal property of the ________)

- We write $\pi : \underline{\quad\quad} \longrightarrow \underline{\quad\quad}$ to be this ________ map and denote ______ $:= \pi(\quad,\quad)$

- <u>Rmks</u>: Every vector in ________ is a finite sum of these "______"

- Because ______ is really $\prod($ $)$

we have things are "linear" in each slot

ex) i)

ii)

Same exact construction holds for $V_1, \ldots, V_n, W$

and ______ maps

Thm: The composite $\underline{\qquad\qquad} \hookrightarrow F(\quad) \longrightarrow F(\quad)/N =$

is $\underline{\qquad\qquad}$ and induces a bijection

$$\mathrm{Mult}(\qquad\qquad, W) \cong \mathcal{L}(\qquad\qquad, W)$$

$$\begin{tikzcd}
\underline{\qquad\qquad\qquad} \arrow[r, "f"] \arrow[d] & W \\
\underline{\qquad\qquad} \arrow[ur, dashed, "\exists!\,\tilde{f}"'] &
\end{tikzcd}$$

"

OK: The machinery developed above is very general, and therefore very powerful.

However, in the world of vector spaces, we have

__________ !

This simplifies things!!!

Thm: Let V, W be vector spaces with basis

$B_V = (v_1 \ldots v_n) \quad B_W = (w_1 \ldots w_m)$.

Then __________ has basis

$$B_{\_\_\_\_} = \left(\qquad\qquad \middle| \begin{array}{l} 1 \le i \le n \\ 1 \le j \le m \end{array} \right)$$

<u>Cor</u>: $\dim(\qquad) = \_\_\_\_\_\_\_\_$

(Compare again to __________)

- We will use this to compute matrix of tensor product of linear maps.

Suppose $T_1 : V_1 \to W_1$ and $T_2 : V_2 \to W_2$ are linear maps between $\mathbb{F}$-vector spaces V_1, V_2, W_1, W_2

Prop: There exists a unique linear map

$$\underline{\qquad\quad} : \underline{\qquad\quad} \longrightarrow \underline{\qquad\quad}$$

such that $\underline{\qquad\quad}(\qquad) = \underline{\qquad} \quad \underline{\qquad}$

Pf) HW :) (Hint: Universal property of $\otimes$:))

Special Cases

Consider a map $V \xrightarrow{T} W$. Then for vector space V' get ! map

$$V' \otimes V \xrightarrow{1 \otimes T} V' \otimes W$$

a) Let $V = W$ and $f = id$. Then what is this map

$$V' \otimes V \xrightarrow{1 \otimes 1} V' \otimes V$$

(ie when both T_1, T_2 are the identity maps)

b) Now let $V \xrightarrow{f} W \xrightarrow{g} Z$ be linear maps

have the two maps

i) $V' \otimes V \xrightarrow{1 \otimes f} V' \otimes W \xrightarrow{1 \otimes g} V' \otimes Z$

ii) $V' \otimes V \xrightarrow{1 \otimes \underline{\quad}} V' \otimes Z$

$\Rightarrow$ The prop said there's a ! such map so $\underline{\quad} = \underline{\quad} \circ \underline{\quad}$

Rmk: This tells us $\underline{\qquad\qquad}$ is a $\underline{\qquad\qquad}$!