10/21/21 Section

Today

- Recap on Linear Transf $\longleftrightarrow$ Matrices
- How does change of basis change Transformations?
- Using Matrices to find Ker/image of T

Transformations $\longleftrightarrow$ Matrices

<u>Recall</u>: Let $T: V \to W$ be linear, and let $\mathcal{B}_V = (v_1, \dots, v_n)$, $\mathcal{B}_W = (w_1, \dots, w_m)$ be a basis. Then

$$[T]_{\mathcal{B}_V}^{\mathcal{B}_W} := \begin{pmatrix} [T(v_1)]_{\mathcal{B}_W} & [T(v_2)]_{\mathcal{B}_W} & \cdots & [T(v_n)]_{\mathcal{B}_W} \\ \downarrow & \downarrow & & \downarrow \end{pmatrix}_{m \times n}$$

$\Rightarrow$ have the property that, $\forall v \in V$

$$[T(v)]_{\mathcal{B}_W} = \underline{[T]_{\mathcal{B}_V}^{\mathcal{B}_W} [v]_{\mathcal{B}_V}}$$

Q: Let $(V, \mathcal{B}_V)$, $(W, \mathcal{B}_W)$, $(Z, \mathcal{B}_Z)$ be 3 vector spaces.

Consider:

$$V \xrightarrow{T} W \xrightarrow{S} Z$$

Then recall we have new transf

$$S \circ T : V \longrightarrow Z \text{ defined by}$$

$$(S \circ T)(v) = S(T(v))$$

Check: This function is a linear transf. ✓✓

So we have 3 matrices

$$T \longrightarrow [T]$$

$$S \longrightarrow [S]$$

$$S \circ T \longrightarrow [S \circ T]$$

How are they related? **Fact:** $[S \circ T] = [S][T]$

Special Case

$V \rightsquigarrow B_v, B_v'$

$W \rightsquigarrow B_w, B_w'$

$$\begin{tikzcd}
(V, B_v) \arrow[r, "T"] & (W, B_w) \\
(V, B_v') \arrow[r, "T"] \arrow[u, "f = id"] & (W, B_w') \arrow[u, "g = id"']
\end{tikzcd}$$

Let P = change basis from $B_v \to B_v'$

Q = change basis matrix from $B_w \to B_w'$

That is $\underline{T \circ f} = \underline{g \circ T}$

$\longrightarrow$ now take their matrices

$$\Rightarrow [T \circ f]_{B_i}^{B_w} = [g \circ T]_{B_i'}^{B_w'}$$

$$\Rightarrow [T]_{B_i}^{B_w} [f]_{B_i}^{B_i} = [g]_{B_w'}^{B_w'} [T]_{B_i'}^{B_w'}$$

$$\Rightarrow [T]_{B_i}^{B_w} P = Q [T]_{B_i'}^{B_w'}$$

$$\Rightarrow \boxed{Q^{-1} [T]_{B_i}^{B_w} P = [T]_{B_i'}^{B_w'}}$$

ex) $V = \mathbb{R}_2[x]$ $\qquad B_V = (1, x, x^2)$ $\qquad B_V' = (2+x,\ 3+x^2,\ x-x^2)$

$W = \mathbb{R}^3$ $\qquad B_W = (e_1, e_2, e_3) = \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right)$, $\qquad B_W' = \left(\underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{e_1}, \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{e_2}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right)$

$T : V \to W$ be

$$T(a_0 + a_1 x + a_2 x^2) = \begin{pmatrix} a_0 \\ a_1 - a_2 \\ a_0 + a_1 \end{pmatrix}$$

Q: Find $\underset{A}{\overset{\shortparallel}{[T]_{B_V}^{B_W}}}$ and $\underset{A'}{\overset{\shortparallel}{[T]_{B_V'}^{B_W'}}}$

Verify the formula $\quad \underline{A' = Q^{-1}AP}$

- $T(1) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 1e_1 + 0e_2 + 1e_3$

- $T(x) = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} =$

- $T(x^2) = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$

$$\Rightarrow A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{pmatrix}$$

- $T(2+x) = \begin{pmatrix} 2 \\ 1 \\ 2+1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

- $T(3+x^2) = \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 4\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \; 1$

$$\Rightarrow A' = \begin{pmatrix} 2 & 3 & 0 \\ -2 & -4 & 1 \\ 3 & 3 & 1 \end{pmatrix}$$

$$\bullet\ T(x-x^2) = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = 0\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Now find

$B_V = (1, x, x^2)$

$B_V' = (2+x,\ 3+x^2,\ x-x^2)$

$\bullet\ P$ = change basis matrix from $B_V \to B_V'$

$$= \begin{pmatrix} [2+x]_{B_V} & [3+x^2]_{B_V} & [x-x^2]_{B_V} \\ \downarrow & \downarrow & \downarrow \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 3 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

$$B_W = \left(\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right)$$

$$B_W' = \left(\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix} \right)$$

$\underline{Q^{-1}} = \left(\text{change of basis matrix from } B_W \to B_W'\right)^{-1}$

$= \text{change of basis matrix from } B_W' \to B_W$

$$= \begin{pmatrix} \left[\begin{pmatrix}1\\0\\0\end{pmatrix}\right]_{B_W'} & \left[\begin{pmatrix}0\\1\\0\end{pmatrix}\right]_{B_W'} & \left[\begin{pmatrix}0\\0\\1\end{pmatrix}\right]_{B_W'} \\ \downarrow & \downarrow & \downarrow \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$G^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 3 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 3 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 3 & 0 \\ -2 & -4 & 1 \\ 3 & 3 & 1 \end{pmatrix} = A' \checkmark$$

How to use Matrices to compute things

$T : V \longrightarrow W$ linear $\qquad \mathcal{B}_V = (v_1, \ldots, v_n)$ basis for V.

Recall $\quad \operatorname{im}(T) := \left\{ w \in W \;\middle|\; \begin{array}{l} \exists\, v \in V \text{ st} \\ T(v) = w \end{array} \right\} \subseteq W$

$$\ker(T) = \{ v \in V \mid T(v) = 0_W \} \subseteq V$$

Claim: $\text{im}(T) = \text{span}(T(v_1), \ldots, T(v_n))$. (Note $\supseteq$ for free)

Pf) Need to show $\text{im}(T) \subseteq \text{Span}(T(v_1) \ldots T(v_n))$

Let $w \in \text{im}(T)$. That is $w = T(v)$ for some $v \in V$.

Note $v = c_1 v_1 + \cdots + c_n v_n$ for $!$ $c_1 \ldots c_n$

$$\Rightarrow w = T(v) = T(c_1 v_1 + \cdots + c_n v_n)$$

$$= T(c_1 v_1) + T(c_2 v_2) + \cdots + T(c_n v_n) \quad T \text{ linear}$$

$$w = c_1 T(v_1) + c_2 T(v_2) + \cdots + c_n T(v_n) \quad T \text{ linear}$$

$$\Rightarrow w \in \text{span}(T(v_1) \ldots T(v_n)) \quad \square$$

Algorithm:

1) Write $[T]_{B_V}^{B_W} := A$

2) Put A in EF

- the columns with leading variables correspond to the LI vectors in list $T(v_1), \dots, T(v_n)$. Those are your basis for $\mathrm{im}(T)$

- the solution set to homogeneous system given by A are the coordinate vectors of basis for Ker.

ex) $V = \mathbb{R}_2[x]$ $\qquad B_V = (1, x, x^2)$

$W = \mathbb{R}^3$ $\qquad B_W = (e_1, e_2, e_3)$

$$T(a_0 + a_1 x + a_2 x^2) = \begin{pmatrix} a_0 \\ a_1 - a_2 \\ a_0 + a_1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

$\Rightarrow \left(T(v_1), T(v_2), T(v_3)\right)$ LI list

Basis for image $= \left(\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \right)$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightsquigarrow T(v_1), T(v_2)$$

are basis.