Dual Vector Spaces

Every vector space gives rise to another

$\longrightarrow$ so called " ________ "

We will see that many familiar stuff arise as this way

Def: i) Let V be a vector space over $\mathbb{F}$. Then a ________
is a map

____

ii) The vector space of ______ $= \mathcal{L}(V, \mathbb{F}) :=$
is called the ________ of V

ex) i) $V = M_n(\mathbb{F})$ have

ii) $V = \mathbb{F}[t]$ have

iii) $V = C^\infty(\mathbb{F})$ have

(or, more generally fix $g \in C^\infty(\mathbb{F})$ and define

______________)

<u>Rmk</u>: When we study ______________ we will see "morally" where these come from

Important example

Def: Let V be vector space $0 \neq v \in V$. Define the ______ vector, denoted ______ $: V \to \mathbb{F}$ by

$$\_\_(\quad) = \begin{cases} \\ \\ \end{cases}$$

- In particular, suppose now V is fd and let $B = (v_1, \ldots, v_n)$ be a basis for V.

Then have list of vectors (______) in ______

Thm: In the setup above the vectors (______)
are a basis for ______
(we call this the ______)

(Rmk: We know dim( ) =
So we just have to check these vectors
either ______ or are ______.)

Pf)

ex) i) $V = \mathbb{R}^3$ $\quad \mathcal{B} = (e_1, e_2, e_3)$

Compute e_1^*

e_2^*

e_3^*

ii) $V = \mathbb{F}_{\leq 2}[t]$ $\quad \mathcal{B} = (\underset{f_0}{1}, \underset{f_1}{t}, \underset{f_2}{t^2})$

Compute f_0^*

f_1^*

f_2^*

iii) $V = M_{2\times 2}(\mathbb{R})$ $\quad \mathcal{B} = \left(\overset{m_1}{\overset{\shortparallel}{\begin{pmatrix}1&0\\0&0\end{pmatrix}}}, \overset{m_2}{\overset{\shortparallel}{\begin{pmatrix}0&1\\0&0\end{pmatrix}}}, \overset{m_3}{\overset{\shortparallel}{\begin{pmatrix}0&0\\1&0\end{pmatrix}}}, \overset{m_4}{\overset{\shortparallel}{\begin{pmatrix}0&0\\0&1\end{pmatrix}}} \right)$

Compute

m_1^*

m_2^*

m_3^*

m_4^*

The "double ______"

Def: V vector space. Then the double ______ is $(\quad)$

Rmk: We know that $V \cong$ ______ $\cong$ ______

so we already know that $V \cong$ ______

However! There's a "more natural" isomorphism.

(see hw about this)

Def: Let $v \in V$. Then define ______ $\in$ ______ ("evaluation at v")

by

______ $(f) :=$ ______ for $f \in$ ______

This defines a function $V \xrightarrow{\Phi}$ ______

$v \longmapsto$ ______

Thm: If V is fd the map $\Phi : V \to$ ___ is an isomorphism

Def: Let $T: V \to W$ be a linear map
and let _____ $\in$ _____

Then we get a new linear map _____ : _____ $\to$ _____
called the _____ of T defined by

_____ =

(Rmk: If $\dim V = n$, $\dim W = m$ we can identify $T \longleftrightarrow [T] \in M_{\_\_}(\mathbb{F})$

Then this new map _____ $\longleftrightarrow [\quad] \in M_{\_\_}(\mathbb{F})$

hmmm ....)

ex) $V = \mathbb{R}_{\leq 2}[x]$ $\mathcal{B}_V = (f_0, f_1, f_2)$

$W = M_{2\times 2}(\mathbb{R})$ $\mathcal{B}_W = (m_1, m_2, m_3, m_4)$

$T: V \to W$ by $T(a_0 + a_1 x + a_2 x^2) = \begin{pmatrix} a_0 & a_1 - a_2 \\ a_2 & 0 \end{pmatrix}$

i) Compute $[T]_{\mathcal{B}_V}^{\mathcal{B}_W}$

ii) Compute $\longrightarrow$

iii) What is $[\quad]^{B_V^*}_{B_W^*}$

iv) Anything you notice ??

Thm: V, W finite dim with basis $\mathcal{B}_V = (v_1 \dots v_n)$
$\mathcal{B}_W = (w_1 \dots w_m)$

Then $\left[\quad \right]_{\mathcal{B}_W^*}^{\mathcal{B}_V^*} = \left(\left[\quad \right]_{\mathcal{B}_V}^{\mathcal{B}_W} \right)^{-}$

Pf)

Applications

Def: Let $W \subseteq V$ subspace. Define $W^0 = \{ \_\_\_\_\_\_\_\_\_\_\_\_ \}$ and call it the ____________ of W

HW) i) If V is fd show $\dim W^0 = \dim V - \dim W$

(hint, dual basis)

ii) Use universal property of quotient to show that

$(V/W)^* \simeq W^0$

Prop: Let V, W fd and $T: V \to W$ linear. Then

$$\dim(\qquad) = \dim(\qquad)$$

Cor: For a matrix A ________ = ________