10/14/21 Section

- From HW 3 on you must type your solutions.
- They will also be submitted on Gradescope

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Coordinate vectors

- Let V be fd vector space with basis $\mathcal{B} = (v_1, \dots, v_n)$.
  - Let $w \in V$

  Then $w = c_1 v_1 + c_2 v_2 + c_3 v_3 + \dots + c_n v_n$
  for $!$ $c_1, \dots, c_n \in \mathbb{F}$

$$\Rightarrow [w]_{\mathcal{B}} :\overset{\text{def}}{=} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \in \mathbb{F}^n$$

ex) $V = \mathbb{R}_2[x]$ $\quad \mathcal{B} = (1, x, x^2)$

Find $[3 - x + 5x^2]_{\mathcal{B}} = \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix} \in \mathbb{R}^3$

$3(1) - 1(x) + 5(x^2)$

ex) $V = \mathbb{R}_2[x]$ $\quad \mathcal{B}' = (2+x,\ 3+x^2,\ x-x^2)$

Find $[3 - x + 5x^2]_{\mathcal{B}'}$

Goal: Express

$$3 - x + 5x^2 = c_1(2+x) + c_2(3+x^2) + c_3(x-x^2)$$

$$3 - x + 5x^2 = 2c_1 + 3c_2 + (c_1 + c_3)x + (c_2 - c_3)x^2$$

$$\Rightarrow \begin{aligned} 2c_1 + 3c_2 + 0c_3 &= 3 \\ c_1 + 0c_2 + c_3 &= -1 \\ 0c_1 + c_2 - c_3 &= 5 \end{aligned} \xrightarrow[\text{matrix}]{\text{coefficient}} \left(\begin{array}{ccc|c} 2 & 3 & 0 & 3 \\ 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 5 \end{array}\right)$$

put matrix in EF $\Rightarrow$

$$\boxed{\begin{aligned} c_1 &= 9 \\ c_2 &= -5 \\ c_3 &= -5 \end{aligned}} \Rightarrow [3 - x + 5x^2]_{B'} = \begin{pmatrix} 9 \\ -5 \\ -5 \end{pmatrix}$$

Q: How are

$$[3-x+5x^2]_{\mathcal{B}} \quad \text{and} \quad [3-x+5x^2]_{\mathcal{B}'}$$

related?

A: By the "Change of Basis Matrix"

$P_{\mathcal{B}\to\mathcal{B}'}$

That is $[3-x+5x^2]_{\mathcal{B}'} = P_{\mathcal{B}\to\mathcal{B}'}\,[3-x+5x^2]_{\mathcal{B}}$

Check: Find $P_{\mathcal{B}\to\mathcal{B}'}$

$\mathcal{B} = (1, x, x^2)$

$\mathcal{B}' = (2+x, 3+x^2, x-x^2)$

$$P_{\mathcal{B}\to\mathcal{B}'} = \begin{pmatrix} [1]_{\mathcal{B}'} & [x]_{\mathcal{B}'} & [x^2]_{\mathcal{B}'} \\ \downarrow & \downarrow & \downarrow \end{pmatrix}$$

$$1 = c_1(2+x) + c_2(3+x^2) + c_3(x-x^2)$$

$$x = d_1(2+x) + d_2(3+x^2) + d_3(x-x^2)$$

$$x^2 = e_1(2+x) + e_2(3+x^2) + e_3(x-x^2)$$

Note: $$P_{\mathcal{B}'\to\mathcal{B}} = \begin{pmatrix} 2 & 3 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} = \left(P_{\mathcal{B}\to\mathcal{B}'}\right)^{-1}$$

$\underline{Q}$: Why is $P_{B \to B'}$ defined this way?
Why does it "work"?

To answer this, need to talk about linear transformations.

Recall: A linear transf $T: V \longrightarrow W$ is a function st

1) $T(v_1+v_2) = T(v_1) + T(v_2)$

2) $T(cv_1) = cT(v_1)$

To compute things, we chose basis for V, W

- doing so gives us a Matrix

Let $B_V = (v_1, \ldots, v_n)$ be basis for V

$B_W = (w_1, \ldots, w_m)$ be basis for W

- Then note for any vector $v \in V$

$$T(v) = T(c_1 v_1 + c_2 v_2 + \cdots + c_n v_n) = c_1 T(v_1) + c_2 T(v_2) + \cdots + c_n T(v_n)$$

- We also have

$$W \ni T(v_1) = d_1 w_1 + d_2 w_2 + \cdots + d_m w_m \qquad [T(v_1)]_{B_W}$$

$$\vdots \qquad \Rightarrow$$

$$W \ni T(v_n) = \tilde{d}_1 w_1 + \tilde{d}_2 w_2 + \cdots + \tilde{d}_m w_m \qquad [T(v_n)]_{B_W}$$

Def: $[T]_{\mathcal{B}_V}^{\mathcal{B}_W} = \begin{pmatrix} [T(v_1)]_{\mathcal{B}_W} & \cdots & [T(v_n)]_{\mathcal{B}_W} \\ \downarrow & & \downarrow \end{pmatrix}_{m \times n}$

Fact: $[T(v)]_{\mathcal{B}_W} = \underline{[T]_{\mathcal{B}_W}^{\mathcal{B}_W}} \, [v]_{\mathcal{B}_V}$

Back to Change of Basis matrix

Let $\mathcal{B}_1, \mathcal{B}_2$ be 2 basis for V

Consider the identity transformation

$$\mathbb{1} : (V, \mathcal{B}_1) \longrightarrow (V, \mathcal{B}_2)$$

from V with basis $\mathcal{B}_1$ to V with basis $\mathcal{B}_2$.

(that is $\mathbb{1}(v) = v$)

$\mathcal{B}_1 = (v_1 \dots v_n)$

$\mathcal{B}_2 = (w_1 \dots w_n)$

Q: What is $[\mathbb{1}]_{\mathcal{B}_1}^{\mathcal{B}_2}$?

$$[\mathbb{1}]_{\mathcal{B}_1}^{\mathcal{B}_2} \overset{\text{def}}{=} \begin{pmatrix} [\mathbb{1}(v_1)]_{\mathcal{B}_2} & \dots & [\mathbb{1}(v_n)]_{\mathcal{B}_2} \\ \downarrow & & \downarrow \end{pmatrix}$$

$$= \begin{pmatrix} [v_1]_{\mathcal{B}_2} & [v_2]_{\mathcal{B}_2} & \dots & [v_n]_{\mathcal{B}_2} \\ \downarrow & \downarrow & & \downarrow \end{pmatrix}$$

$$= P_{\mathcal{B} \to \mathcal{B}'}$$

ex) $V = \mathbb{R}_2[x]$ $\quad B_V = (1, x, x^2)$

$W = M_{2\times 2}(\mathbb{R})$ $\quad B_W = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$

$T : V \longrightarrow W$ by

$$T(a_0 + a_1 x + a_2 x^2) = \begin{pmatrix} a_0 & a_1 - a_2 \\ 0 & a_1 + a_2 \end{pmatrix}$$

Find $[T]_{B_V}^{B_W}$

$\bullet\ T(1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow [T(1)]_{B_W} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$\bullet\ T(x) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \Rightarrow [T(x)]_{B_W} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$

$\bullet\ T(x^2) = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix} \Rightarrow [T(x^2)]_{B_W} = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$

$\Rightarrow [T]_{B_V}^{B_W} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

- $v = 1 + 2x + x^2$

$$T(v) = \begin{pmatrix} 1 & 2-1 \\ 0 & 2+1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}$$

$$\Rightarrow \left[T(v) \right]_{B_W} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 3 \end{pmatrix}$$

Check: $\left[1 + 2x + x^2 \right]_{B_V} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

Claim: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \overset{?}{=} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 3 \end{pmatrix}$ ✓

check

$$= 1\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + 2\begin{pmatrix}0\\1\\0\\1\end{pmatrix} + 1\begin{pmatrix}0\\-1\\0\\1\end{pmatrix}$$

$$= \begin{pmatrix}1\\0\\0\\0\end{pmatrix} + \begin{pmatrix}0\\2\\0\\2\end{pmatrix} + \begin{pmatrix}0\\-1\\0\\1\end{pmatrix}$$

$$= \begin{pmatrix}1\\1\\0\\3\end{pmatrix}$$