Quantified Statements

Let $P(x)$ be an ____________ over some ________ S

→ We will produce __________ kinds of statements from this ____________ called ____________ statements

ex1) for ______ $x \in S$, $P(x)$ is ______

The phrase "______" is referred to as the ____________ qualifier and is denoted by the symboll ____ (\forall)

(this is also reffered to as ______, ______ or

____)

~> Symbollically, we write ____________

ex 2) there ______ such that ______ is true

The phrase "______" is referred to as the __________ quantifier and is denoted by the symboll ______ (\exists)

~> Symbolically, we write ____________

ex) Consider the open sentence

$P(n)$: n^2+n is even with domain $\mathbb{Z}$

$\rightsquigarrow$ • for all $n \in \mathbb{Z}$, n^2+n is even

$(\_\_\_ n \in \mathbb{Z}, P(n)) \sim \_\_\_\_\_$!

• there exists an $n \in \mathbb{Z}$, n^2+n is even

$(\_\_\_ n \in \mathbb{Z}, P(n)) \sim \_\_\_\_\_$!

Negating Quantified Statements

- $\neg(\_\_\_\_ x \in S, P(x))$ = it is ______ the case that ______ $x \in S$ $P(x)$ is true

  = ________ an $x \in S$ such that $P(x)$ is ______

  = ________________

- $\neg(\_\_\_\_ x \in S, P(x))$ = it is ______ the case that there ______ an $x \in S$ such that $P(x)$ is true

  = ______ $x \in S$, $P(x)$ is ______

  = ________________

$\rightsquigarrow$ In summary, under negation, we have

$$\wedge \longleftrightarrow$$

$$\forall \longleftrightarrow$$

$$P(x) \longleftrightarrow$$

Let $x \in S$ and $y \in T$ be variables. Consider

$$\_\_\_\_ \, x \in S, \; \_\_\_\_ \, y \in T, \; P(x,y)$$

(= ____ all $x \in S$, and ____ $\in T$, $P(x,y)$ is ____) negate it $\rightsquigarrow \neg(\_\_\, x \in S, \_\_\, y \in T, P(x,y)) \equiv \_\_\, x \in S, \neg(\_\, y \in T, P(x,y))$

$\equiv \_\_\, x \in S, \_\_\, y \in T, \neg P(x,y)$

ex) Consider the statement:

for all $x \in \mathbb{R}$, $y \in \mathbb{R}$, $x^2 + y^2 > 0$

$\rightsquigarrow$ translate into symbols:

$\rightsquigarrow$ negate it:

(Which is true from above? ________)

Now consider the stument

$$\forall x \in S, \_\_\_ y \in T, P(x,y)$$

negate it $\rightsquigarrow$ $\neg(\forall x \in S, \_\_\_ y \in T, P(x,y)) \equiv \_\_\_ x \in S, \neg(\_\_ y \in T, P(x,y))$

$\equiv \_\_ x \in S, \_\_\_ y \in T, \neg P(x,y)$

in words $\rightsquigarrow$ ¬(for all x in S, ______ a y in T such that P(x,y) is true)

$\equiv$

However,

$$\neg(\_\_\_ x \in S, \forall y \in T, P(x,y)) \equiv \_\_\_ x \in S, \neg(\forall y \in T, P(x,y))$$

$$\equiv \_\_\_ x \in S, \_\_ y \in T, \neg P(x,y)$$

ex) Lets negate the following Statement:

for all integers x, y, if their product is even, then x is even or y is even

1) Locate the component statements

$P(x,y)$: the product xy is ______

$Q(x)$: x is ______ ($\neg Q(x)$: x is ____)

2) Domain : = ______

3) <u>Write in symbolic notation</u>

4) Negate it

- First, recall how to find $\neg(P \Rightarrow Q)$

$\rightsquigarrow \neg(P \Rightarrow Q) \equiv \neg(\neg P \vee Q)$ (previous thrm)

$\equiv$ (De-Morgan's Laws)

$\equiv$ (Double negation law)

$$\rightsquigarrow \neg\left(\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, \; P(x,y) \Rightarrow \left(Q(x) \vee Q(y)\right)\right)$$

$$\equiv \_\_ x \in \mathbb{Z}, \_\_ y \in \mathbb{Z}, \neg\left(P(x,y) \Rightarrow (Q(x) \vee Q(y))\right)$$

$$\exists \_\_ x \in \mathbb{Z}, \_\_ y \in \mathbb{Z}, P(x,y) \_\_\_\_ (\qquad\qquad\qquad)$$

$$\exists \_\_ x \in \mathbb{Z}, \_\_ y \in \mathbb{Z}, P(x,y) \_\_\_\_ \_\_\_\_ \_\_ \_\_\_\_$$

in words
$\rightsquigarrow$

"there ____ integers x, y such that their product is ____
____ x is __ and y is ____ "

Proof Techniques (Yay!!!)

Trivial and Vacuous Proofs

ex) Let $x \in \mathbb{R}$. Show that if $0 < x < 1$ then $x^2 - 2x + 2 > 0$

- $P(x)$:
- $Q(x)$:

$\rightsquigarrow$ Goal: ___ $x \in \mathbb{R}$, $P(x)$ ___ $Q(x)$

Note: $x^2 - 2x + 2 = (x-1)^2 + 1$ ___ 0

$\Rightarrow Q(x)$ is __________ !

$\Rightarrow$ Our statement ____ $x \in \mathbb{R}$, $P(x)$ ____ $Q(x)$ is ____ for every $x \in \mathbb{R}$

This type of proof is called a __________ proof, one where the conclusion is always true

ex 2) Let $x \in \mathbb{R}$. Show that if $x^2 - 2x + 2 \leq 0$ then $x^3 \geq 0$

into logic $\rightsquigarrow$ $P(x)$:

$Q(x)$:

$\rightsquigarrow$ __ $x \in \mathbb{R}$, $P(x)$ ____ $Q(x)$

However: Remember that $P(x)$ is always ______

So our statement is ______ true!!!

P	Q	P=>Q
T	T	
T	F	
F	T	
F	T	

Remember truth tables!

This type of proof is called a ______ proof, one where the assumption is always false.

More interesting: Direct Proofs

Let $P(x)$ and $Q(x)$ be open sentences over a domain S.

Goal: Show that $P(x)$ ____ $Q(x)$ is true for all $x \in S$.

We saw that 1) if $P(x)$ is ____ then this is ______ true

2) if $Q(x)$ is always ____ then this is also always ____

$\rightsquigarrow$ interested in neither of these cases.

In a ______ for ______ we consider an ______ element $x \in S$, for which $P(x)$ is ______ and then show that ______

ex1) For every odd integer n, show that $3n+7$ is even

into symbols $\rightarrow$ $\forall n \in \mathbb{Z}$, $\underbrace{\rule{4cm}{0.4pt}}_{P(n)}$, then $\underbrace{\rule{4cm}{0.4pt}}_{Q(n)}$

For any question involving "parity" (ie, even or odd)

- an integer x is even iff $x = 2K$ for some $K \in \mathbb{Z}$

- an integer x is odd iff

  $x = 2\ell + 1$ for some $\ell \in \mathbb{Z}$

Pf) Let n be an odd integer. Then we can write

$n =$ __________ for some $\ell \in \mathbb{Z}$.

We now compute

$3n + 7 =$

$=$

Since ______ $\in \mathbb{Z}$

$\boxed{:)}$

Recall: For a positive integer m. We say that ____________ modulo m, denoted ____________ iff ____________ is ____________ by m.

iff ____________ leave the same remainder when ____________ by m

ex) For an integer n, show that $n^2 \equiv 0 \mod 4$ or $n^2 \equiv 1 \mod 4$.

Scratch work:

n	0	1	2	3	4	5	6	7	8	9	10	11	12
n^2	0	1	4	9	16	25	36	49	64	81	100	121	144
$n^2 \mod 4$													

$\rightsquigarrow$ Conjecture: If n is ____ then $n^2 \equiv$ ___ mod 4

If n is ____ then $n^2 \equiv$ ___ mod 4

Pf) Let n be an integer. Then note that n is either _____ or odd. Let us first assume that n is _____.
Then we can write $n =$ ________ for some $l \in \mathbb{Z}$.

⋮

Lemma: For an integer n, the number n^2+n is even.

Pf) First we note that $n^2+n =$ __________

Now we again consider ______ cases; when n is ____ and when n is _____.

Let's revist the previous theorem to make another conjecture

n	0	1	2	3	4	5	6	7	8	9
n^2	0	1	4	9	16	25	36	44	64	81

~ We already know that when n is odd, $n^2 \equiv 1 \bmod 4$.

However, it seems we also have that n odd $\Rightarrow n^2 \equiv$ ______

Prop: If n is an odd integer then $n^2 \equiv$ ______