Last Yiddish of Day

"Gay kakn afn yam" = גײ קאַקן אױפֿן ים

"Ikh shep nakhes fin aykh" = איך שעפּ נחת פֿון אײַך

Seriously, What is It?

Recall: We say two sets A, B have the same cardinality if $\exists$ a bijective function $f: A \to B$

and we write $|A| = |B|$

Today we will show that

$$|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}| \text{ "} < \text{"} |\mathbb{R}| = |\mathbb{C}|$$

Def: A set A is said to be <u>denumerable</u> if $|A| = \underline{|\mathbb{N}|}$

$\rightsquigarrow$ this means we can "enumerate" i.e, <u>list</u> (order) the elements of A by the <u>natural #s</u>

$\rightsquigarrow$ we'll write $A = \{a_1, a_2, a_3, a_4, \ldots\ldots\}$

- A is said to be <u>countable</u> if $|A| < \infty$ or if $|A| = |\mathbb{N}|$
- A is said to be <u>uncountable</u> otherwise

<u>Thm</u>: 1) An infinite subset of a <u>countable</u> set is <u>countable</u>

Pf) Let A be countable and $S \subseteq A$ be an infinite subset,

We know $\exists$ bijection $f: \mathbb{N} \xrightarrow{\sim} A$

Consider $f^{-1}(S) \subseteq \mathbb{N}$

We can order $f^{-1}(S) = \{s_1, s_2, s_3, \dots \dots\}$

(with $s_i < s_j$ for $i < j$)

Define $g: \mathbb{N} \xrightarrow{\sim} f^{-1}(S)$

$i \longrightarrow s_i$

Now since $f: \mathbb{N} \to A$ is a bijection

the restricted $f|_{f^{-1}(S)} : f^{-1}(S) \xrightarrow{\sim} S$

$\Rightarrow \exists$ bijection $\mathbb{N} \xrightarrow{\sim} S$ $\square$

(Hilberts Hotel!!!)

ex) The set $2\mathbb{N} \subsetneq \mathbb{N}$ of even positive integers is countable

⌈Note this means $|2\mathbb{N}| = |\mathbb{N}|$ even though $2\mathbb{N} \subsetneq \mathbb{N}$

- In this case one can explicitly construct a bijection

$f: \mathbb{N} \to 2\mathbb{N}$ by $f(i) = 2i$ ⌋

Rmk: (1) tells us that for an infinite set A, it is enough to just have an injective function

$f: A \hookrightarrow X$ where X countable, to conclude that

A is countable

Thm: $|\mathbb{Z}| = |\mathbb{N}|$

Pf) The strategy is as follows: Send <u>odd #</u> to <u>non-neg #'s</u> and <u>even #</u> to <u>neg</u>

$$
\begin{array}{ccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \cdots \\
0 & -1 & 1 & -2 & 2 & -3 & 3 & -4 & \cdots
\end{array}
$$

$\rightsquigarrow$ Can explicitly define $f : \mathbb{N} \to \mathbb{Z}$ as

$$
f(n) = \begin{cases} n/2 & n \text{ even} \\ \dfrac{1-n}{2} & n \text{ odd} \end{cases}
$$

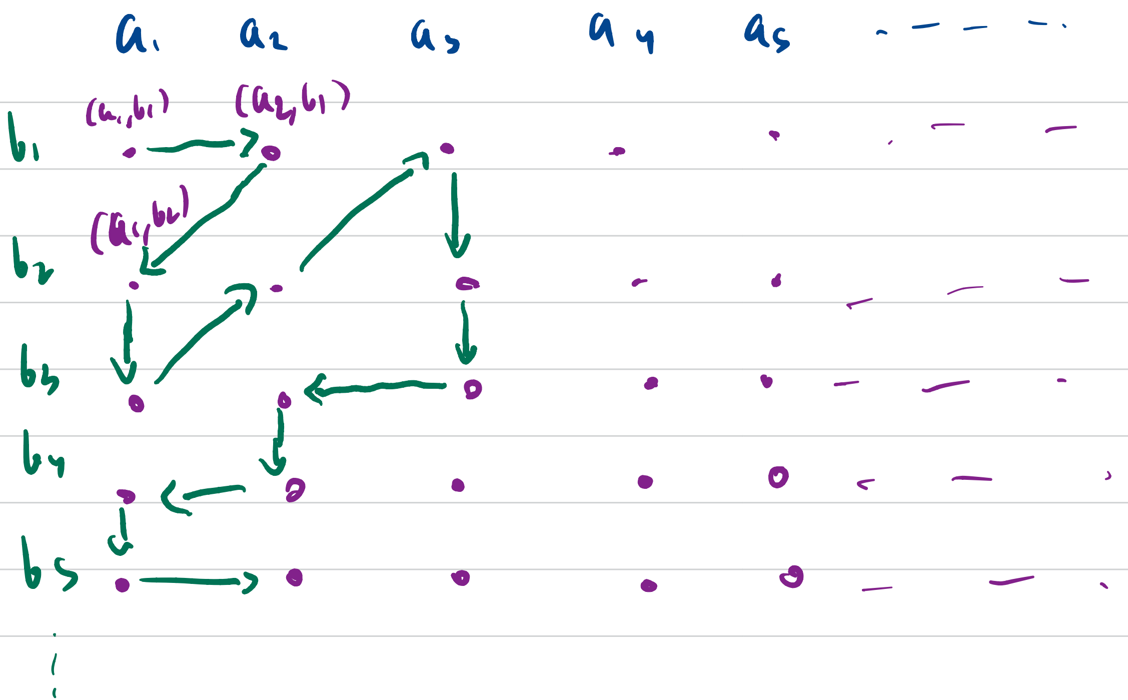
and check this is a <u>bijection</u>

<u>Thm</u>: A cartesian product of denumerable sets is denumerable

Pf) We will sketch the proof by giving a picture

$A: \{a_1, a_2, a_3, \ldots\ \ \}$

$B: \{b_1, b_2, b_3, \ldots\ \ \}$



Construct function f that does this

$f: \mathbb{N} \xrightarrow{\sim} A \times B$

Thm: $\mathbb{Q}$ is countable (i.e. $|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}|$)

Pf) We will construct an injective function $f: \mathbb{Q} \hookrightarrow$ some countable set.

Any rational # $r = \frac{a}{b}$ with $a \in \mathbb{Z}$, $b \in \mathbb{N}$ in reduced form. Then define the function

$$f: \mathbb{Q} \hookrightarrow \mathbb{Z} \times \mathbb{N}$$

$$r = \frac{a}{b} \longmapsto (a, b)$$

This is injective! If $f(r_1) = f(r_2)$ then $(a_1, b_1) = (a_2, b_2)$ then $a_1 = a_2$, $b_1 = b_2$ so $r_1 = r_2$ ☺

Def / Notation

For finite sets: A, B finite then $|A| < |B|$ iff $\exists$ injection but no bijection from $A \to B$

Def: A, B sets. Write $|A| < |B|$ if $\exists$ injection $A \hookrightarrow B$ but no bijection.

Uncountable Sets

Thm: The open interval $(0,1)$ is uncountable (1920's)

Pf) (Cantor's Diagonal argument)

Lets assume FTSOC $\exists$ bijection $f: \mathbb{N} \xrightarrow{\simeq} (0,1)$

We can then list every # in $(0,1)$ as follows

$f(1) = a_1 = 0.a_{11}a_{12}a_{13}a_{14} \cdots$

$f(2) = a_2 = 0.a_{21}a_{22}a_{23}a_{24} \cdots$

$f(3) = a_3 = 0.a_{31}a_{32}a_{33}a_{34} \cdots$

$f(4) = a_4 = 0.a_{41}a_{42}a_{43}a_{44} \cdots$

$\vdots$

where $0 \le a_{ij} \le 9$

Now define the sequence $(b_k)_{k\in\mathbb{N}}$

$$b_k = \begin{cases} a_{kk}+1 & a_{kk}\neq 9 \\ 0 & a_{kk}=9 \end{cases}$$

Now consider the number

$$b = 0.b_1b_2b_3b_4b_5\ldots\ldots$$

whose decimal expansion are the #'s in this sequence

This number cannot be in the list above, ie b is not in the image of f, but this contradicts that f is a bijection

Thm: If $A \subseteq B$ and A is uncountable then B is too

Pf) Suppose FTSOC that B is countable. But then $A \subseteq B$ would also be countable (thm 1) $\rightarrow\leftarrow$

Cor: $\mathbb{R}$ is uncountable

Pf) $(0,1) \subset \mathbb{R}$

Thm: $|(0,1)| = |\mathbb{R}|$

"Pf" (Idea ~ "stretch out" the interval lol)

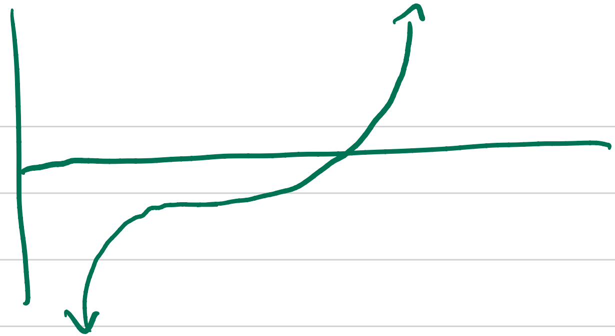
Define $f: (0,1) \to \mathbb{R}$ by

$$f(x) = -\frac{1}{x} + \frac{1}{1-x}$$

$\leadsto$ Can check <u>injective</u> no problem

$\leadsto$ <u>surjective</u> relies on some limit arguments but

if you graph it you get



so not hard to believe this is surjective

So far we have

$$|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}| < |(0,1)| = |\mathbb{R}|$$

these are all called $\aleph_0$ ("aleph null")

first letter in Hebrew Alphabet

Continum Hypothesis

$\nexists$ set S such that

$$|\mathbb{N}| < |S| < |\mathbb{R}|$$

(1931 - Gödel)
$\rightsquigarrow$ Not able to prove using axioms of Set theory

(1963 - Cohen)
$\rightsquigarrow$ Not able to <u>disprove</u> using axioms of Set theory

However! We CAN prove that $|\mathbb{R}|$ is NOT the end of the road

Thrm: If A is any set $|A| < |P(A)|$

Pf) For A finite, obvious.. An injection is given by $f: A \hookrightarrow P(A)$ $x \to \{x\}$

Assume A not finite and FSOC assume $\exists$ bijection $f: A \to P(A)$. For every $x \in A$, denote $A_x \subseteq A$ $(f(x) = A_x)$

Define $B = \{y \in A \mid y \notin A_x\}$. Because f is bijection $\exists y \in A$ such that $B = A_y$. Get a contradiction

Cor: There is NO "largest" set (or NO "biggest ∞")

Pf) Always have

$$|A| < |P(A)| < |P(P(A))| < |P(P(P(A)))| < \cdots$$

Thrm: $|P(\mathbb{N})| = |\mathbb{R}|$

$\rightsquigarrow$ We know $|\mathbb{N}| < |P(\mathbb{N})|$

$|\mathbb{N}| < |\mathbb{R}|$

$\rightsquigarrow$ But how to show $\exists$ bijection $\mathcal{P}(\mathbb{N}) \simeq \mathbb{R}$?

Thrm: (Schröder-Bernstein Thrm)

If $\exists$ injection $f: A \hookrightarrow B$ AND

injection $g: B \hookrightarrow A$

then $|A| = |B|$

$\rightsquigarrow$ Use this to show $|\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$

$\rightsquigarrow$ Construct <u>injection</u> $\mathcal{P}(\mathbb{N}) \hookrightarrow (0,1)$

injection $(0,1) \hookrightarrow \mathcal{P}(\mathbb{N})$

<u>Fin</u>