UNIVERSITY OF CALIFORNIA, SANTA CRUZ

MATH 21– FINAL EXAM

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Name: ______________________________
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This exam contains 12 pages (including this cover page) and 10 questions. Total of points is 100.
Good luck, it's been a pleasure being your instructor this term!

Distribution of Marks

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total:	100	

1. (a) (10 points) Let $T : \mathbb{R}_3[x] \to M_{2\times 2}(\mathbb{R})$ be a linear transformation, whose matrix with respect to the standard basis of $\mathbb{R}_3[x]$ and $M_{2\times 2}(\mathbb{R})$ is

$$A_T = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 3 & 1 & 3 \\ 1 & 2 & 1 & 2 \\ 2 & 3 & 1 & 2 \end{pmatrix}$$

Find $T(1 - 2x + 2x^2 + 4x^3)$

2. (a) (5 points) Show that $M_{2\times 2}(\mathbb{R})$ is a vector space. That is, show the way we defined matrix addition, and scalar multiplication satisfy the necessary axioms.

   (b) (5 points) Give an example of a set that is NOT a vector space. Explain why it is not one.

3. Let $\mathcal{B} = (1, x, x^2)$ be the standard basis for $\mathbb{R}_2[x]$, and suppose

$$T : \mathbb{R}_2[x] \to \mathbb{R}_2[x]$$

is a linear transformation whose matrix with respect to $\mathcal{B}$ is

$$A_{T,\mathcal{B}} = \begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix}$$

This matrix has the following eigenvectors with associated eigenvalues;

$$v_1 = \begin{pmatrix} -1 \\ 6 \\ 13 \end{pmatrix} \text{ with } \lambda_1 = 0$$
$$v_2 = \begin{pmatrix} -2 \\ -3 \\ 2 \end{pmatrix} \text{ with } \lambda_2 = 3$$
$$v_3 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \text{ with } \lambda_3 = -4$$

(a) (2 points) Show that $\mathcal{C} = (v_1, v_2, v_3)$ is a basis for $\mathbb{R}^3$.

(b) (4 points) Let $\mathcal{S} = (e_1, e_2, e_3)$ be the standard basis for $\mathbb{R}^3$. Find

$$\mathcal{P}_{\mathcal{S}\to\mathcal{C}} \tag{1}$$
$$\mathcal{P}_{\mathcal{C}\to\mathcal{S}} \tag{2}$$

.

(c) (2 points) Find the matrix multiplication

$$D = (\mathcal{P}_{\mathcal{S}\to\mathcal{C}})(A_{T,\mathcal{B}})(\mathcal{P}_{\mathcal{C}\to\mathcal{S}})$$

(d) (2 points) What are the eigenvectors for T?

4. State whether the following are true or false. In each case, explain why it is true or false, or give a counter example if it is false.

   (a) (2 points) Suppose A is an $n \times n$ matrix, and λ is an eigenvalue for A. Then the columns of $(A - \lambda I_n)$ are a basis for $\mathbb{R}^n$

   (b) (2 points) Suppose A is an $n \times n$ matrix with det(A)=12, and suppose B is another $n \times n$ matrix such that $B = P^{-1}AP$ for some invertible matrix P. Then B is an invertible matrix.

   (c) (2 points) Suppose $T : V \to W$ is an isomorphism of n-dimensional vector spaces, and let A_T be a matrix of T (with respect to any basis for V and W). Then for any vector $\vec{b} \in \mathbb{R}^n$ the linear system $[A_T|\vec{b}]$ (matrix A_T augmented by $\vec{b}$) has a unique solution.

   (d) (2 points) Suppose W is a vector space that is isomorphic to $\mathbb{R}^4$. There exists a sequence of vectors $(w_1, w_2, \ldots, w_6)$ that are linearly independent in W.

   (e) (2 points) Consider the vector spaces $V = \mathbb{R}_4[x], W = \mathbb{R}^3$. Then for every linear transformation $T : V \to W$ the columns of the associated matrix A_T (with respect to the standard basis of V,W) are linearly dependent.

5. Consider the determinant function, $\det : M_{n\times n}(\mathbb{R}) \to \mathbb{R}$

   (a) (10 points) Is det a linear transformation? If so prove it, if not explain why.

6. Consider the following two bases for $M_{2\times 2}(\mathbb{R})$:

$$\mathcal{B} = \{\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\}$$
$$\mathcal{C} = \{\begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix}, \begin{pmatrix}1 & 2\\ 1 & 1\end{pmatrix}, \begin{pmatrix}1 & 1\\ 2 & 1\end{pmatrix}, \begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\}$$

(a) (2 points) Suppose m is a matrix with $[m]_{\mathcal{B}} = \begin{pmatrix}2\\ -8\\ 1\\ 12\end{pmatrix}$ Find what the matrix m is.

(b) (5 points) Find the change of basis matrix $\mathcal{P}_{\mathcal{B}\to\mathcal{C}}$

(c) (3 points) Find $[m]_{\mathcal{C}}$

7. For the following, give an example if one exists, or explain why no such example exists.
   (a) (2 points) A matrix $m \in M_{3\times 3}(\mathbb{R})$ with eigenvalues $\lambda_1 = 2, \lambda_2 = 1, \lambda_3 = 12$
   (b) (2 points) A vector space W that is isomorphic to $\mathbb{R}_3[x]$ but is not isomorphic to $M_{2\times 2}(\mathbb{R})$
   (c) (2 points) A vector space W that is isomorphic to $\mathbb{R}_{209}[x]$ and is also isomorphic to $M_{21\times 10}(\mathbb{R})$
   (d) (2 points) A matrix $A \in M_{4\times 3}(\mathbb{R})$ with rank(A)=2 and nullity(A)=1
   (e) (2 points) A matrix $A \in M_{3\times 3}(\mathbb{R})$ with det(A)=24, along with a nonzero 3×3 matrix $B \in M_{3\times 3}(\mathbb{R})$ such that $AB = 0_{3\times 3}$ (the zero-matrix).

8. Find the eigenvalues and eigenvectors for the following matrices.

   (a) (5 points) $A = \begin{pmatrix} 3 & 0 & -3 \\ 0 & 3 & 0 \\ 1 & 0 & -1 \end{pmatrix}$

   (b) (5 points) $B = \begin{pmatrix} -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -10 & 0 & 0 & 0 & 0 \\ 0 & 0 & 62 & 0 & 0 & 0 \\ 0 & 0 & 0 & 93 & 0 & 0 \\ 0 & 0 & 0 & 0 & -12 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

9. Consider the following sequence of vectors in $\mathbb{R}_2[x]$

$$\mathcal{B} = \{1 + x, x^2 - x, 1 + x + x^2\}$$

   (a) (5 points) Show that $\mathcal{B}$ is a basis for $\mathbb{R}_2[x]$
   (b) (5 points) Find $[4 - x]_{\mathcal{B}}$

10. Find the values of x for which the following matrices are not invertible. Explain your answer.

  (a) (5 points)

$$A = \begin{pmatrix} x & 1-x \\ 1 & 2x+1 \end{pmatrix}$$

  (b) (5 points)

$$B = \begin{pmatrix} x+3 & 2 & 3 \\ 0 & x-12 & 1 \\ 0 & 0 & 2x-8 \end{pmatrix}$$

This page is intentionally left blank to accommodate work that wouldn't fit elsewhere and/or scratch work.