6.3 - Matrices of Linear Transf.

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Canvas

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!!!!

<u>Recall</u>: $T: \mathbb{R}^n \to \mathbb{R}^m$ we got the matrix A_T whose columns are $T(e_i)$

$\Rightarrow T(\vec{v}) = A_T \vec{v}$

<u>Now</u>: V, W with basis $\mathcal{B}_V = (v_1 \ldots v_n)$
$\mathcal{B}_W = (w_1, \ldots, w_m)$

- $T: V \to W$ then $T(v) =$
  - in particular $T(v_i) =$
  - So can form $[\quad]_{\mathcal{B}_W} = \begin{pmatrix} \\ \\ \end{pmatrix}$

We define the matrix of T with respect to $\mathcal{B}_V, \mathcal{B}_W$ as

$$A_{T,\mathcal{B}_V,\mathcal{B}_W} = \begin{pmatrix} [\quad]_{\mathcal{B}_W} & [\quad]_{\mathcal{B}_W} & \quad & [\quad]_{\mathcal{B}_W} \\ & & & \end{pmatrix}_{m\times n}$$

$\Rightarrow$ gives us $[\quad]_{\mathcal{B}_W} = \quad [\ ]_{\mathcal{B}_V}$

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ex) $V = \mathbb{R}_2[x]$ $\quad \mathcal{B}_V = (1, x, x^2)$

$W = M_{2\times 2}(\mathbb{R})$ $\quad \mathcal{B}_W = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$

$T: V \to W$ as

$$T(a_0 + a_1 x + a_2 x^2) = \begin{pmatrix} 2a_0 & a_1 - a_2 \\ a_0 & 4a_1 \end{pmatrix}$$

Find A_{T,B_v,B_w}:

- $\Rightarrow [\quad]_{B_w} =$
- $\Rightarrow [\quad]_{B_w} =$
- $\Rightarrow [\quad]_{B_w} =$

$$A_{T,B_v,B_w} = \begin{pmatrix} \\ \\ \end{pmatrix}$$

<u>Check</u>: Consider the vector $v = 2+x-x^2$ in V

- $T(v) =$ $\Rightarrow$ $[T(v)]_{\mathcal{B}_W} =$

- $[v]_{\mathcal{B}_V} =$ $\Rightarrow$ $\begin{pmatrix} \quad \end{pmatrix}\begin{pmatrix} \quad \end{pmatrix} =$

Again let V, W vector spaces, $\mathcal{B}_V$, $\mathcal{B}_W$ bases.

$T: V \to W$	$A_{T,\mathcal{B}_V,\mathcal{B}_W} := A$ ($m \times n$ matrix)
$\ker(T)$	$\text{null}(A)$
$\text{nullity}(T)$	$\text{nullity}(A)$
$\text{Range}(T)$	$\text{col}(A)$
$\text{rank}(T)$	$\text{rank}(A)$
	columns of A LI
	column space spans $\mathbb{R}^m$ (pivot in every row)
	A is invertible

ex) $T: \mathbb{R}_2[x] \to M_{2\times 2}(\mathbb{R})$ with $\mathcal{B}_V = (1, x, x^2)$

$\mathbb{R}_2[x] = V$, $M_{2\times 2}(\mathbb{R}) = W$

$\mathcal{B}_W: \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$

$$T(a_0 + a_1 x + a_2 x^2) = \begin{pmatrix} 2a_0 & a_1 - a_2 \\ a_0 & 4a_1 \end{pmatrix}$$

Find $\ker(T)$

Use above connection, we'll find

$$\Rightarrow \begin{pmatrix} & & \\ & & \\ & & \\ & & \end{pmatrix} \xrightarrow{\text{put into echelon form}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

Pivot in every column $\Rightarrow \operatorname{null}(A_{T,\mathcal{B}_V,\mathcal{B}_W}) = \{0\} \Rightarrow \ker T = \{0\}$

ex2) $V = \mathbb{R}_3[x]$ $\quad \mathcal{B}_V = (1, x, x^2, x^3)$

$W = \mathbb{R}_2[x]$ $\quad \mathcal{B}_W = (1, x, x^2)$

$D: V \to W$ derivative $\quad (D(a_0 + a_1 x + a_2 x^2 + a_3 x^3)$

$= a_1 + 2a_2 x + 3a_3 x^2$

Find A_{D,B_V,B_W} and find $\ker(D)$

$\Rightarrow A_{D,B_V,B_W} = \begin{pmatrix} \end{pmatrix}$

$\Rightarrow \text{null}(A_{D,B_V,B_W}) =$

$\begin{pmatrix} \end{pmatrix} = [v]_{B_V} \Rightarrow v =$

$\ker(D) =$ =

ex) $V = M_{2\times 2}(\mathbb{R})$ $\qquad W = \mathbb{R}^4$

$B_V = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$ $\qquad B_W = (e_1, e_2, e_3, e_4)$

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+b \\ c+d \\ a+c \\ d \end{pmatrix}$$

Find A_{T,B_V,B_W}

$=$ $=$

$=$ $=$ $\left[\quad \right]_{B_W}$

$= \Big[\qquad \Big]_{\mathcal{B}_W}$

$= \Big[\qquad \Big]_{\mathcal{B}_W}$

$\Rightarrow A_{T, \mathcal{B}_V, \mathcal{B}_W} = \Big(\qquad \Big)$

ex) <u>Change of basis matrix</u>

V vector space with two bases $\mathcal{B}_1$, $\mathcal{B}_2$

$\mathcal{B}_1 = (v_1 \ldots v_n)$, $\mathcal{B}_2 = (w_1 \ldots w_n)$

Then consider the identity transformation

$$\mathbb{1} : (V, \mathcal{B}_1) \longrightarrow (V, \mathcal{B}_2) \qquad \mathbb{1}(v) = v$$

$$A_{\mathbb{1}, \mathcal{B}_1, \mathcal{B}_2} = \begin{pmatrix} \quad \end{pmatrix}$$

$$=$$

$$\left([\quad]_{\mathcal{B}_2} = [\quad]_{\mathcal{B}_1} \right)$$

Determinants for Transformations

We want to apply the concept of determinants to

$T: V \to V$ (linear operator)

$\quad \mathcal{B} \qquad \mathcal{B}$

We just showed how to get a matrix $A_{T,\mathcal{B},\mathcal{B}}$ from T

Can we define ____________ ?

- A problem is, ____________________

____

• If B' is another basis for V, we need to know the relation between

and

turns out

$\Rightarrow$ =

=

$\Rightarrow \det(\quad) = \det(\quad)$

$=$

$=$

$=$

So

Chapter 7 - Eigenvalues / vectors

- We saw that we get a diff matrix for $T: V \to V$ for each basis.
  - Is there a "nice" basis that makes the matrix of T as simple as possible.

ex) For example, is there a basis for V such that the matrix of T with respect to this basis is diagonal.

$$A_{T,\mathcal{B}} = \begin{pmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_n \end{pmatrix}$$

- If $\mathcal{B} = (v_1 \dots v_n)$ is a basis where the matrix of T is as above then $T(v_1) =$ ______, $T(v_2) =$ ______, ..., $T(v_n) =$

Def: $T: V \to V$, and $0 \neq v$ a vector in V.

Then if ______ for some ______, we call v ______

and ______

i) A $n \times n$ matrix. A nonzero vector $\vec{o} \neq \vec{v}$ in $\mathbb{R}^n$ is ______ of A if ______ (this ______ is called ______.)

How to find eigenvectors/values for a given matrix:

- Suppose $\vec{v}$ is an eigenvector for A, so $A\vec{v} = \lambda\vec{v}$

$\Rightarrow$

$\Rightarrow$

- So we set ______ $= 0$ and solve for ______

Then we find the ______

ex) $A = \begin{pmatrix} 5 & 4 \\ -2 & -1 \end{pmatrix}$ Find eigenvalues/vectors of A.

$\lambda_1 =$:

So $\vec{v}_1 = \begin{pmatrix} \\ \end{pmatrix}$ is an eigenvector with eigenvalue ____

<u>Check</u>: $A\vec{v}_1 = \begin{pmatrix} & \\ & \end{pmatrix}\begin{pmatrix} \\ \end{pmatrix} =$

<u>$\lambda_2 =$</u>

So $\vec{v}_2 = \begin{pmatrix} \\ \end{pmatrix}$ is an eigenvector with eigenvalue ____

<u>Check</u>: $A\vec{v}_2 = \begin{pmatrix} & \\ & \end{pmatrix}\begin{pmatrix} \\ \end{pmatrix} =$