Lecture 1

Yiddish of the Day

"A tsig hot ohket a bord, in iz dokh nor a tsig"

= אַ ציג האָט אָנקעט אַ באָרד,
און איז דאָך נאָר אַ ציג

=

- **OH** : T, Th 10-11 AM
- **TA** : Brian Ma W, F 12:30-1:30

	M	T	W	Th	F	
6/26			PS 1 Due		PS 2 Due	6/30
7/3			PS 3 Due		• PS 4 due • Glossary (1)	7/7
7/10			PS 5 Due		• Proof Part (1) due • PS 6 Due	7/14
7/17					PS 7 Due	7/21
7/24			PS 8 Due		• Proof Part (2) Due • Glossary (2)	7/28

Sets!

- **Def**: A set is a collection of elements

⇝ **Notation**:
- upper-case letters (A, B, C, X, Y, Z) will denote sets typically
- lower-case letters (a, b, c, x, y, z) will denote elements of sets
- the symbol $\in$ denotes that an element belongs to a set

(Latex: \in)

We read this as

- $a \in A$: "a is an element of A"

- $a \notin A$ = "a is <u>not</u> an element of A"

(Latex: \notin)

How to describe a set?

1) We know all the elements of the set and we write them all out within { } \{ \}

ex) $A = \{-3, -2, 2, 3\} = \{2, -3, 3, -2\}$

(ie, ordering doesn't matter)

$A = \{\text{blue, green, purple, peach}\}$

2) We have a <u>rule</u> that describes a <u>general element</u> in the set

- We call this <u>set-builder</u> notation, and it takes the form

$$\left\{ \underline{\text{general element of set}} \;\middle|\; \text{properties of this element} \right\}$$

We read this as
"the set of all elements such that this property holds"

First some important sets

$\mathbb{N}$ = the set of natural #'s = $\{0, 1, 2, 3, 4, \ldots\}$ (\mathbb{N})

$\mathbb{Z}$ = set of integers = $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ (\mathbb{Z})

$\mathbb{Q}$ = set of rational #'s = $\left\{ \frac{a}{b} \;\middle|\; a, b \in \mathbb{Z}, b \neq 0 \right\}$ (\mathbb{Q})

$\mathbb{R}$ = set of real #'s (\mathbb{R})

ex) ii) $A = \{n \in \mathbb{Z} \mid n \text{ even and } |n| \leq 4\}$

$= \{n \in \mathbb{Z} \mid n = 2k, \text{ for } k \in \mathbb{Z} \text{ and } |k| \leq 2\}$

$= \{-4, -2, 0, 2, 4\}$

= "the set of even integers of absolute value less than or equal to 4"

A weird set

Def: The set with <u>no elements</u> is called the <u>empty set</u> and is denoted $\underline{\phi}$ (\empty)

- There are different ways of describing this set

ex) $\{x \in \mathbb{Z} \mid x^2 < 0\}$ or

{days of the week that end in r}

WARNING !!

- the empty set has no elements, but the set

$X = \{ \phi \}$ is NOT the empty set

It has an element! $\phi \in X$

Def: The cardinality of a set is the # of elements in the set. For a set X we denote the cardinality by $|X|$

ex) $A = \{ x \in \mathbb{Z} \mid x \text{ even and } |x| \leq 4 \}$

$\rightsquigarrow$ $|A| = 5$

Rmk: This notion of cardinality is straightforward when the set is finite

For infinite sets it is tricky (but super interesting)

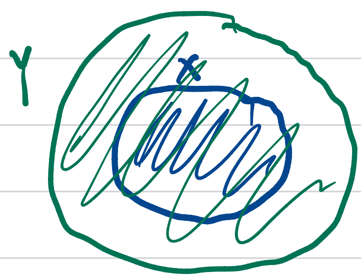
ex) We will see that the following all have the same cardinality

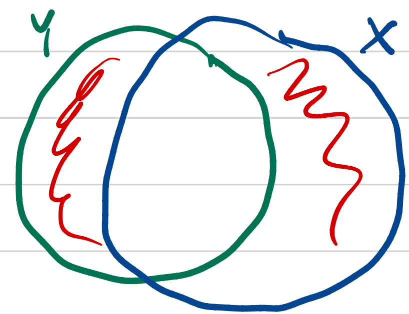
($\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$)

but that $\mathbb{R}$ has a larger cardinality

Subsets

Def: Given 2 sets X, Y we say that X is a **subset** of Y (denoted $X \subseteq Y$ \subseteq) if every element of X is also an element of Y

ex) 

Non ex: 

Q: Does a set always have at least one subset?

A Yes! $\emptyset \subseteq X$ vacuously true

$X \subseteq X$

ex) Let $B = \{\emptyset, \{\emptyset\}, 1, 2, \{1, 2\}\}$

• note $\emptyset \in B$, $1 \in B$ $\Rightarrow$ $\{\emptyset\} \subseteq B$, $\{1\} \subseteq B$, $\{2\} \subseteq B$

$\{\emptyset\} \in B$ $\Rightarrow$ $\{\{\emptyset\}\} \subseteq B$

$1, 2 \in B$ $\Rightarrow$ $\{1, 2\} \subseteq B$

$\{1, 2\} \in B$ $\Rightarrow$ $\{\{1, 2\}\} \subseteq B$

And always $\emptyset \subseteq B$

$B \subseteq B$

- Is $\{\{2\}\}$ subset of B ? No ! because $\{2\} \notin B$

$\{1\} \in B$? No !

Def: We say two sets A, B are equal if both

1) $A \subseteq B$
2) $B \subseteq A$

We will write this as $A = B$

- A set X is said to be a proper subset of a set Y

if 1) $X \subseteq Y$

2) $X \neq Y$

We will write this as $X \subset Y$ (\subset)

(or sometimes $X \subsetneq Y$) (\subsetneq)

Def: The power-set of a set X is the set of all subsets of X, including $\varnothing$ and X.

- We denote this set $\mathcal{P}(X)$

ex) For the following, find $|X|$, $P(X)$, and $|P(X)|$

a) $X = \{1, 2, 3\} \rightsquigarrow |X| = 3$

- $\varnothing \subset X \rightsquigarrow \varnothing \in \mathcal{P}(X)$
- $1 \in X \rightsquigarrow \{1\} \subseteq X \rightsquigarrow \{1\} \in P(X)$

$\{2\} \in P(x)$, $\{3\} \in P(x)$

$\{1,3\} \subseteq X \quad \{2,3\} \subseteq X$

- $1,2 \in X \rightsquigarrow \{1,2\} \subseteq X \rightsquigarrow \{1,2\} \in \mathcal{P}(X)$
- $1,2,3 \in X \rightsquigarrow \{1,2,3\} \subseteq X \rightsquigarrow \{1,2,3\} \in \mathcal{P}(X)$

$\vdots$

$\mathcal{P}(X) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

$\rightsquigarrow \quad |\mathcal{P}(X)| = 8 = \underline{2^{|X|}}$

b) $X = \{1, \{2\}\} \longrightarrow |X| = 2$

$\mathcal{P}(X) = \{\emptyset, \{1\}, \{\{2\}\}, X\}$

$|\mathcal{P}(X)| = 4 = \underline{2^{|X|}}$

c) $X = \emptyset \rightsquigarrow |X| = 0$

$$\mathcal{P}(X) = \{\ \emptyset\ \}$$

$$|\mathcal{P}(X)| = \underline{1} = \underline{2^{|X|}}$$

Rmk: We shall indeed prove later that $|\mathcal{P}(X)| = \underline{2^{|X|}}$

For now, we will accept it as fact

Set Operations

For this section, we are going to assume all sets are subsets of some set $\mathcal{U}$ which we call the universal set

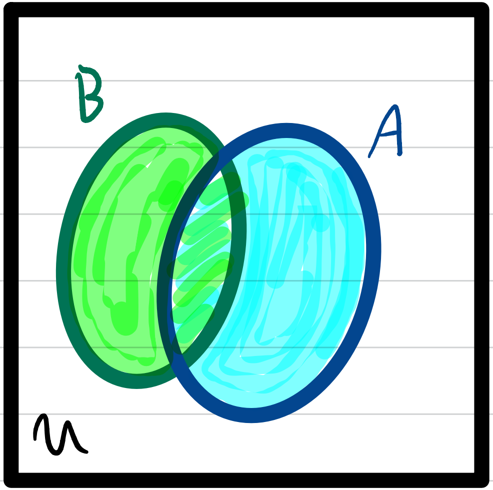
Def: Let $A, B \subseteq \mathcal{U}$. Then

1) The union of A, B, denoted $A \cup B$ (\cup) is the set

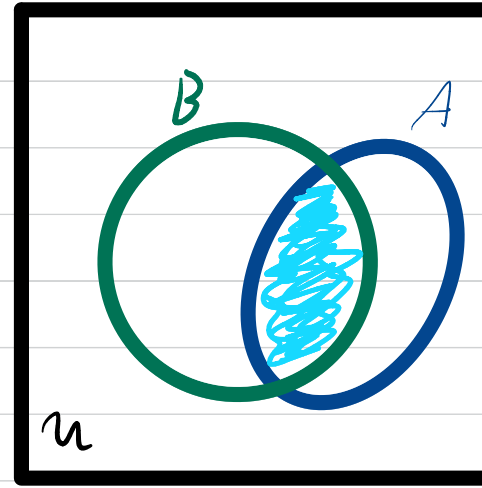
$$A \cup B = \{ x \in \mathcal{U} \mid x \in A \text{ or } x \in B \}$$

2) The intersection of A, B, denoted $A \cap B$ (\cap) is the set

$$A \cap B = \{ x \in \mathcal{U} \mid x \in A \text{ and } x \in B \}$$



$A \cup B$



$A \cap B$

ex) $A = \{1, 2, 3, \star, \square\}$ $B = \{\star, 3, b, 1\}$

$\rightsquigarrow A \cup B = \{\star, 1, 3, \square, 2, b\}$

$$A \cap B = \{3, \star, 1\}$$

[Note that we did not <u>repeat</u> elements in <u>union</u> that were in both <u>A</u>, <u>B</u>. We only write them once]

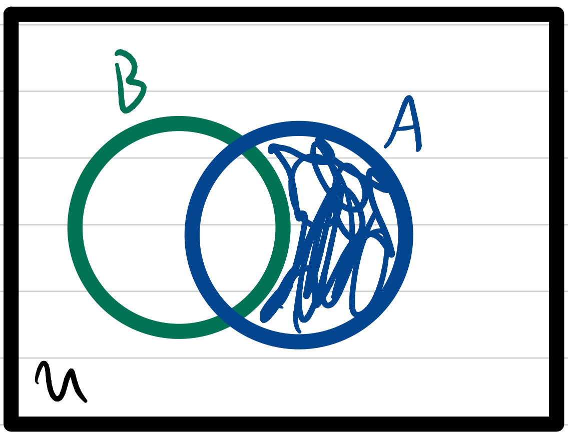
3) The <u>difference</u> of A and B is the set

$$\underline{A \setminus B} = \underline{A - B} = \{x \in \mathcal{U} \mid x \in A \text{ and } x \notin B\}$$

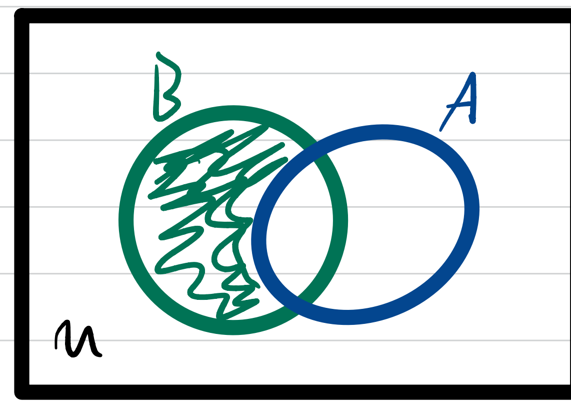
ie: We remove the part of A that is in B

⤳ also have the set

$$\underline{B - A} = \{ x \in \mathcal{U} \mid x \in B \text{ and } x \notin A \}$$



$A \setminus B$

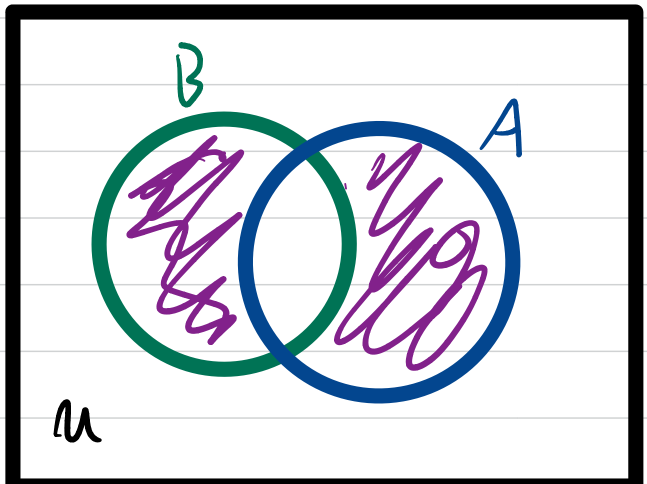


$B \setminus A$

⇝ related to this is the so called symmetric - difference of A, B

$$\underline{A \,\Delta\, B} = A \setminus B \cup B \setminus A$$

(\triangle)



Rmk: $A \,\Delta\, B$

$$\shortparallel$$

$$B \,\Delta\, A$$

$A \,\Delta\, B$

ex) $A = \{1, 2, 3, A, \square\}$ $B = \{*, 3, b, 1\}$

$\rightsquigarrow A \setminus B = \{2, \square \}$

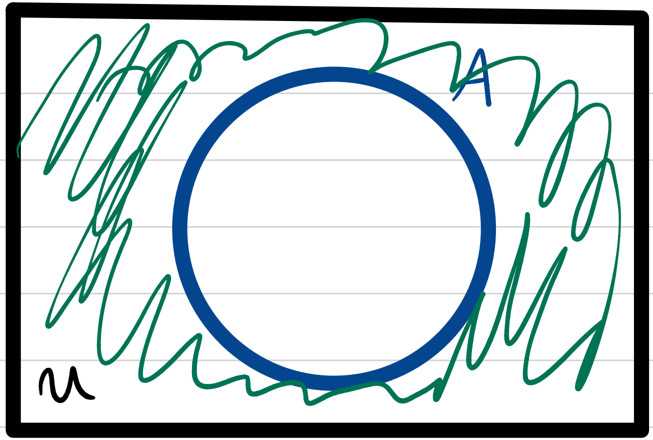
$B \setminus A = \{ b \}$

$A \Delta B = \{2, \square, b \}$

4) The compliment of A in $\mathcal{U}$ is the set

$$A^c = \bar{A} = \{x \in \mathcal{U} \mid x \notin A\} = \mathcal{U} \setminus A$$

= "the set of elements of $\mathcal{U}$ that are not in A"



ex) $\mathcal{U} = \mathbb{Z}$ and A = the set of <u>even</u> #'s.

$\rightsquigarrow$ then $\underline{A^c}$ = the set of <u>odd</u> #'s

$$\boxed{\rightsquigarrow (A^c)^c = A}$$ think about this

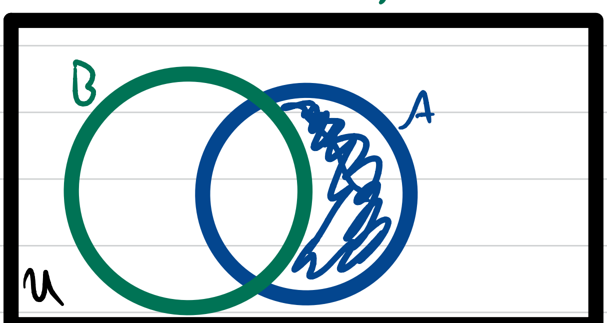
Prop: We have the following identities.

1) $A \setminus B = A \cap B^c$

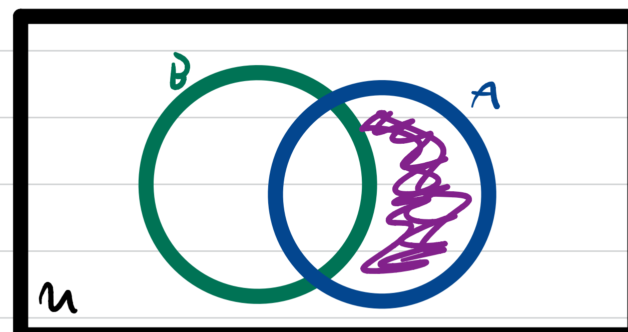
2) $(A \cap B)^c = A^c \cup B^c$

3) $(A \cup B)^c = A^c \cap B^c$

We will prove (1) by a picture. We will learn how to prove such statements without pictures soon.



$A \setminus B$



$A \cap B^c$

Indexed Sets

$\rightsquigarrow$ Want a way to concisely express elements of sets

ex) $\{a_1, a_2, a_3, \ldots, a_n\} = \{a_i\}_{i=1}^{n}$

$= \{a_i\}_{i \in I}$

where $I = \{1, 2, 3, 4, \ldots, n\}$

ex) $A_1 \cap A_2 \cap \cdots \cap A_n = \bigcap_{i=1}^{n} A_i = \bigcap_{i \in I} A_i$ (\bigcap)

with $I = \{1, 2, 3, \ldots, n\}$

$A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{i=1}^{n} A_i = \bigcup_{i \in I} A_i$ (\bigcup)

for $I = \{1, 2, \ldots, n\}$

Why? - We will be interested in collections of sets that aren't enumerable (can't list them all out)

$\longrightarrow$ this motivates us to consider arbitrary collection of sets, which is indexed by some indexing set I
(can very well be infinite)

$\rightsquigarrow \{a_\alpha, a_\beta, a_\gamma, \ldots\} = \{a_i\}_{i \in I}$

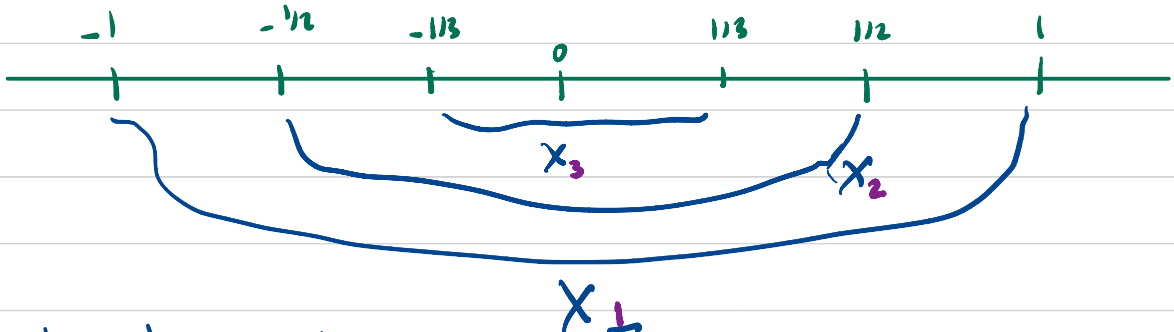
$$X_\alpha \cap X_\beta \cap \cdots = \bigcap_{i \in I} X_i \qquad I = \{\alpha, \beta, \gamma, \ldots\}$$

$$X_\alpha \cup X_\beta \cup \cdots = \bigcup_{i \in I} X_i$$

ex) Let $I = [1, \infty)$

→ for each $r \in I$ consider the set

$X_r = \left(-\frac{1}{r}, \frac{1}{r}\right)$



→ also have sets like $X_\pi = \left(-\frac{1}{\pi}, \frac{1}{\pi}\right)$

→ One can show that

1) $\bigcap_{r \in I} X_r = \{0\}$

2) $\bigcup_{r \in I} X_r = \underline{X_1} = (-1, 1)$

Partitions of Sets

Let A be a set, and let $\mathcal{S} = \{X_\alpha\}_{\alpha \in I}$ be a collection of non-empty subsets of A for some indexing set I

[ie, $\mathcal{S} \subseteq \underline{\mathcal{P}(A)}$]

Def: We say that $\mathcal{S}$ is a partition of A if

1) $X_\alpha \cap X_\beta = \phi$ when $\alpha \neq \beta$

2) $\bigcup_{g \in I} X_g = \underline{A}$

ex) $A = \mathbb{Z}$. Can someone tell me a <u>partition</u> of A?

$X_1 =$ even #s

$X_2 =$ odd #'s

ex) Let's generalize this. Let $A = \mathbb{Z}$ and let $n \in \mathbb{N}$

<u>Goal</u>: Find a partition of A wrt $\underline{n}$

$\rightsquigarrow$ For any integer $0 \le r < n$ consider the subset

$X_r = [r]_n = \{k \in \mathbb{Z} \mid k$ has remainder r when divided by $n\}$

ex: When $n=2$. What is

$[1]_2 = \{k \in \mathbb{Z} \mid k$ has remainder 1 when divided by $2\}$ = odd #'s

$[0]_2 = \{k \in \mathbb{Z} \mid k$ has no remainder when divided by $2\}$ = even #s

ex: When $n=3$

$[0]_3 = \{\ldots, -9, -6, -3, 0, 3, 6, 9, \ldots\}$

$[1]_3 = \{\ldots, -7, -4, -1, 4, 7, \ldots\}$

$[2]_3 = \{\ldots, -8, -5, -2, 2, 5, 8, \ldots\}$

- Note $[0] \cap [1] = [0] \cap [2] = [1] \cap [2] = \underline{\emptyset}$

• Also we have

$$[0]_3 \cup [1]_3 \cup [2]_3 = \underline{\mathbb{Z}}$$

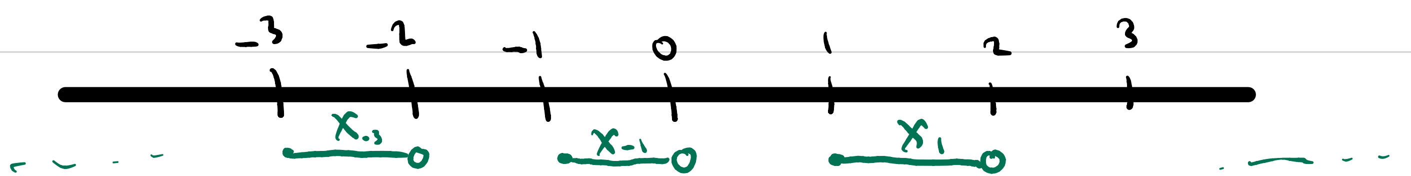
$\rightsquigarrow$ $\mathcal{S} = \{[0]_3, [1]_3, [2]_3\}$ is a partition for $\mathbb{Z}$

$\rightsquigarrow$ This turns out to be true for any n. This partition is called modular arithmetic and we will return to it often!

ex) $A = \mathbb{R}$. Let's find a partition of A.

$\rightsquigarrow$ for each $m \in \mathbb{Z}$ consider the subset

$$X_m = [m, m+1)$$



$\rightsquigarrow$ Note that $X_n \cap X_m = [n, n+1) \cap [m, m+1)$

$= \phi$ if $n \neq m$

Moreover $\bigcup_{n \in \mathbb{Z}} X_n = \mathbb{R}$

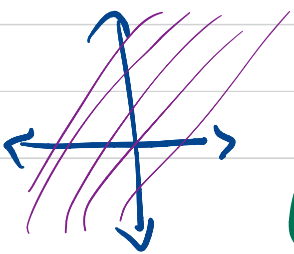
$\rightsquigarrow$ the set $\mathcal{S} = \{X_n \mid n \in \mathbb{Z}\}$ is a partition for $\mathbb{R}$.

Cartesian Products

Def: For sets A and B, we define the <u>cartesian product</u>, denoted <u>$A \times B$</u> (\times) as the set of <u>ordered-pairs</u>

$$\underline{A \times B} = \{ (a,b) \mid a \in A \text{ and } b \in B \}$$

ex 1) Let $A = B = \underline{\mathbb{R}}$. Then



$$\underline{\mathbb{R} \times \mathbb{R}} = \{ (a,b) \mid a, b \in \mathbb{R} \}$$

(this is often denoted by $\underline{\mathbb{R}^2}$)

ex2) $A = \{1, 2, 3\}$ and $B = \{dog, cat\}$

$\underline{A \times B} = \{(1, dog), (1, cat), (2, dog), (2, cat), (3, dog), (3, cat)\}$

- Note that we can also find

$\underline{B \times A} = \{(dog, 1), (dog, 2), (dog, 3), (cat, 1), (cat, 2), (cat, 3)\}$

These are <u>DIFFERENT</u> in general!

- Indeed, if $\underline{A \times B} = \underline{B \times A}$ then ie, $A = B$

1) $\underline{A} \subseteq \underline{B}$ and

2) $\underline{B} \subseteq A$

[In our case $(1, dog) \in \underline{A \times B}$ but

$(1, dog) \notin \underline{B \times A}$ so $\underline{A \times B} \neq \underline{B \times A}$]

Lemma: If A, B are finite sets then

$$|\underline{A \times B}| = |A| \cdot |B|$$

multiplication.