Lecture 11 - Inner Product

Recall

- We based our idea of a ________ on $\mathbb{R}^n$

  → but $\mathbb{R}^n$ has more stuff!

  (________ , ________ , ________)

- Really these come from the same thing!

Def: Let V be $\mathbb{R}$ v.s. Then an ________ on V is a bilinear form

$$\langle -, - \rangle : V \times V \longrightarrow \mathbb{R}$$

st 1) $\langle x, y \rangle =$

2) $\langle x, x \rangle$

(and $=$ ____ iff $x =$)

ex) $V =$ ___ with $\langle -, - \rangle =$ ______

Def 2: An ______ on a $\mathbb{C}$ vector space V is a function

$\langle -, - \rangle : V \times V \to \mathbb{C}$ st

1) $\langle -, - \rangle$ is ___ in first slot

$$\left(\langle \lambda_1 x_1 + x_2, x_3 \rangle = \right)$$

2) $\langle -, - \rangle$ is ______-______ in second slot

$(\langle x_1, \lambda_1 x_2 + x_3 \rangle =$

3) $\langle x, y \rangle =$ ______

4) $\langle x, x \rangle$ ______

Q: Why do we know $\langle x, x \rangle \in$ ______ ?

and = ______ iff $x =$ ______

What does this give us?

Def: i) Let V be either $\mathbb{R}$ or $\mathbb{C}$-vs and let

$\langle -, - \rangle$ be an ________________ on V.

Say v_1, v_2 are ____________ if

$$\langle v_1, v_2 \rangle = \_\_\_\_\_\_$$

ii) Say a set $S \subseteq V$ is ____________ if

$\forall x, y \in S$

$\langle x, y \rangle =$ $\sum$

iii) Define the ______ of $x \in V$ to be

$\|x\| =$

<u>Prop</u>: Suppose $S = (v_1, \dots, v_n)$ is an ______ set. Then S is also ______

Pf)

- So ______ are nice. They can be even nicer.

Def: Say $B = (v_1, \dots, v_n)$ are an ______ of V if

1)

2)

<u>Prop</u>: Suppose $\mathcal{B} = (v_1 \ldots v_n)$ is an ______ - ______ of V

Then for any $w \in V$ we have

$$w = \sum_{i=1}^{n} \_\_\_\_\_\_$$

Pf)

Thm: V an inner-product space, Then V has an Orthonormal - basis

Pf) Google "Grahm - Schmidt"

Something else an inner-product lets us do is remove our dependance on choosing a basis.

<u>Def</u>: $W \leq V$. Define the <u>orthogonal compliment</u> to be

$W^{\perp} =$

<u>Prop</u>: For $W \leq V$ we have

$V =$

Moreover $(W^{\perp})^{\perp} =$

Pf)

There is another way to think about this:

V be either a $\mathbb{R}$ or $\mathbb{C}$ inner product space

Then we have a map

$$\varphi : V \longrightarrow V^*$$

$$v \longmapsto \_\_\_\_$$

- When V is $\mathbb{R}$-vs this is ____

  V is $\mathbb{C}$-vs this is ____

Thm: This map $\delta: V \to V^*$ is a ________ when V is finite dimensional.

Pf) Skip lol

$\Rightarrow$ This gives us a basis free ("canonical") isomorphism $V \cong V^*$ (when V is $\mathbb{R}$-vs)

This will allow us to identify V with V^* as we did before with V and V^{**}

$\rightsquigarrow$ linear maps on ________ can now be thought of as being on ________

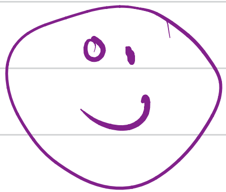
Def/Thrm: Let V be fd inner-product space and $T: V \to V$, linear. Then $\exists!$ linear map $T^*: V \to V$ st

$$\langle \quad (v), w \rangle = \langle v, \quad (w) \rangle \quad \forall v, w \in V$$

P(f) $V \xrightarrow{\delta} V^* \xrightarrow{T^*} V^* \xrightarrow{\delta^{-1}} V$

Thm: $T: V \to V$ linear and $\mathcal{B} = (v_1, \dots, v_n)$ orthonormal

Then $[T]_{\mathcal{B}} =$ ______

Pf) HW 

(**Hint:** Let $T(v_i) = \sum_{j=1}^{n} a_{ij} v_j$

Compute $T^*(v_i)$)