Direct Sums

- Common goal in mathematics
  - Decompose objects into smaller pieces.

ex) i) Every integer n =

ii) Bases: Every vector

Goal: Do this for

Q: What would "a basis of ________" be?

Def: Let V be an $\mathbb{F}$-vs and ________________

Then

1) The ________ (or, more leadingly ________)

of these ________________ is the set

Exc: Let $V_1 \ldots V_k$ be supsaces. Show

1) ______________ is a subspace

2) ______________ is smallest subspace containing all the subspaces.

(Compare to the result about span $(v_1 \ldots v_k)$)

______________ $\times$ ______________

External ______________ Sum

Def: Let $V_1 \ldots V_k$ be vector spaces over $\mathbb{F}$. Then the "external" ______________, denoted ______________ is the set

Rmk: This is a vector space!

- Addition:

- Scalar mult:

- 0 vector:

Notice: For each i, we have an identification

$$V_i \xrightarrow{\sim} \{ \qquad \} \leq$$

"Internal"

Def: Let U, W be subspaces of V. Then we say

" ______________________________________ "

if

1)

2)

Prop: U, W subspaces of V. TFAE

(internal) 1)

(uniqueness) 2)

(external) 3)

Pf)

How to get ________________ ?

Def: Let $W \subseteq V$ be subspace. Then a ____________ for W is another subspace W' st

____

Rank/HW: ________ always exist (hint, extend bases)

- Note ________ are not at all unique

ex) $V = \mathbb{R}^2$ $\quad W = \text{span}\left(\begin{pmatrix}1\\1\end{pmatrix}\right)$

Then ________________

Some way of dealing with them all

<u>Recall</u>: An __________ on a set X (denoted $\sim$) is a __________ st

1)

2)

3)

$\longrightarrow$ we denote $[\quad] = \{ \quad \}$

$X/\sim \; = \{ \quad \}$

Then there is always a surjective function

$$X \twoheadrightarrow X/\sim$$

$$\_\_\_ \longrightarrow \_\_\_\_\_$$

<u>Case we're interested in</u>

Let $W \subseteq V$ subspace. Define a relation $\sim_W$ on V as

$$x \sim_W y \iff \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

<u>Prop</u>: $\sim_W$ is a ____________________

Pf) 1)

2)
3) > HW :)

What is _____ as a set

_____ =

Def: Denote V/W =

(say "V mod W" or "V quotient W")

We call this the "quotient space"

Thm: We can make V/W into a vector space such that the canonical map

$$\pi_W : V \longrightarrow V/W$$

is a linear transformation

Moreover $\ker(\pi_W) =$ ______

Pf)

Remark / Warning / HW

- V/W is **NOT** a ______ of V.

- **However!**

HW: $W \subseteq V$ subspace and W' a complimentary

Then the composite inclusion $W' \hookrightarrow V \twoheadrightarrow V/W$

is an isomorphism $W' \cong V/W$

In particular: $\dim(V/W) =$ ______ ______

______

This construction ~~may be~~ seems ad-hoc, however it is extremly natural.

<u>Intuition</u>: $(V/W, \pi_W)$ is the "best possible" vector space that sends W to 0.

<u>Thm</u>: <u>(Universal property of the quotient)</u>

Let $T: V \to Z$ be linear and suppose ______

Then

P4)

Consequences

Thm: (1st isomorphism thm)
Let $T: V \to Z$ be linear. Then
there is isomorphism

Pf)

Cor: (Rank-Nullity thrm)

Let V be finite dim and $T: V \to Z$ linear.

Then

$$\underline{\qquad\qquad} = \underline{\qquad\qquad} \qquad \underline{\qquad\qquad}$$

Pf)