ECON 703 TA Note 1

Chia-Min Wei

August 12, 2025

1 General Information

	Name	Website	Email	Office
Lecturer	John Kennan	users.ssc.wisc.edu/~jkennan/	jkennan@ssc.wisc.edu	SSC 6434
TA	Chia-Min Wei	chiaminwei.com	cwei69@wisc.edu	SSC 7435

Dates	Lectures	TA Sections	Lecturer's OH	TA's OH
Aug 13 - 15	9:55 - 11:50 AM in SSC 6116 & 4:30 - 6:45 PM in SSC 6203	none	1:30 - 3:00 PM (Mon) in SSC 6434	2:30 - 4:00 PM in SSC 7435
Aug 18 - 22	none	9:55 - 10:45 AM & 11:00 - 11:50 AM in SSC 6116	none	2:30 - 4:00 PM
Aug 21 (Thu)	**Midterm:** 4:30 - 6:45 PM in SSC 6203	9:55 - 10:45 AM & 11:00 - 11:50 PM in SSC 6116	none	2:30 - 4:00 PM
Aug 25 - Sep 2	4:30 - 6:45 PM in SSC 6203	9:55 - 10:45 AM & 11:00 - 11:50 AM in SSC 6116	1:30 - 3:00 PM (Mon) in SSC 6434	2:30 - 4:00 PM (Fri)
Sep 4 - 16	5:30 - 6:45 PM (Tue & Thu) in SSC 6203	8:50 - 9:40 (Fri) in SSC 6113 & 9:55 - 10:45 (Fri) in SSC 6109	1:30 - 3:00 PM (Mon) in SSC 6434	2:30 - 4:00 PM (Fri)
Sep 18 (Thu)	**Final:** 5:30 - 6:45 in SSC 6203	none	none	none

Remark:

1. There are **no lectures nor discussion sections on weekends**. If no specific day is stated on the schedule, the event takes place on all weekdays.
2. There will be **no lectures nor discussion sections on Labor Day (Sep 1)**.
3. There will be **no TA discussion sections on August 28 and August 29** due to TA training.
4. All materials would be posted on canvas. My handouts would also be posted on my website.
5. The location for TA office hours next week would be announced later.

2 Preliminaries from Logic

2.1 Statements and Predicates

In classical logic, a **statement** is an expression that is either **true** or **false**, with no middle ground. For example, the following is a true statement:

John Kennan is the instructor for the math camp in 2025.

The following is a false statement:

John Kennan is the instructor for the math camp every year.

In mathematics, when we are asked to **prove** a statement, our goal is to show that it is **true**.

A **predicate** is an expression such as "is the instructor for the math camp in 2025." By itself, a predicate is not a statement—it is neither true nor false until we specify the object(s) it refers to. For example, "John Kennan is the instructor for the math camp in 2025" is obtained by applying the predicate to a particular person, producing a statement that can be evaluated as true or false.

We often use the notation $P(x)$ to represent a predicate, where x is the variable we "fill in." If

$$P(x) = \text{“}x \text{ is the instructor for the math camp in 2025,”}$$

then

$$P(\text{John Kennan}) = \text{“John Kennan is the instructor for the math camp in 2025.”}$$

A predicate can have more than one variable. For example,

$$P(x, y) = \text{“}x \text{ is } y\text{'s father.”}$$

This becomes a statement only when both x and y are specified.

2.2 Quantifiers

A quantifier must always be followed by a variable (or variables) and a predicate that involves that variable (those variables). Together, they form a complete statement. The most common quantifiers are $\forall$ (for all), $\exists$ (exists) and $\exists!$ (exists exactly one).

- $\forall x\ P(x)$ — "for all x, $P(x)$ is true"
- $\exists x\ P(x)$ — "there exists x such that $P(x)$ is true"
- $\exists! x\ P(x)$ — "there exists exactly one x such that $P(x)$ is true"
- $\nexists x\ P(x)$ — "there does not exist any x such that $P(x)$ is true"

A quantifier followed by two variables:

- $\forall xy\ P(x, y)$ — "for all x and y, $P(x, y)$ is true"

2.3 Logical Connectives

Logical connectives are symbols or words used to combine one or more statements into a more complex statement.

- $P \land Q$ — "P and Q are both true"
- $P \lor Q$ — "P or Q is true"
- $P \implies Q$ — "if P is true, then Q must be true"
- $P \iff Q$ — "P is true if and only if Q is true" (equivalence)
- $\neg P$ — "not P"

A useful result in classical logic is the theorem of contrapostive.

> **Theorem 1 (Contrapostive):** Let P and Q be two statements. Then $P \implies Q$ is logically equivalent to $\neg Q \implies \neg P$.

3 Proofs

There is no simple recipe for proofs, and there is no substitute for practice. Here, though, are some rules of thumb and strategies to keep in mind.

Work backwards from what you want. The ultimate goal is to derive the conclusion. Look at the conclusion and ask what you need to prove just before the conclusion. Then you can treat this line as if it were your goal.

Work forwards from what you have. When you are starting a proof, look at the premises; later, look at the sentences that you have derived so far. These will tell you what your options are.

Let us prove a classical theorem.

> **Theorem 2 (AM-GM Inequality):** For all $a, b \geq 0$,
>
> $$\frac{a+b}{2} \geq \sqrt{ab}.$$

Proof. Since $a, b > 0$, both the LHS and RHS are positive. Therefore, the statement

$$\left(\frac{a+b}{2}\right)^2 \geq ab$$

leads to the final statement.

$$\begin{aligned} \left(\frac{a+b}{2}\right)^2 \geq ab &\impliedby (a+b)^2 \geq 4ab \\ &\impliedby (a+b)^2 - 4ab \geq 0 \\ &\impliedby a^2 + b^2 - 2ab \geq 0 \\ &\impliedby (a-b)^2 \geq 0. \end{aligned}$$

Since the square of any real number is always non-negative, this inequality holds true, completing the proof. $\square$

3.1 Proof by Contradiction

Sometimes it is hard to prove directly that a statement is true. In such situations, it may be better to assume that the statement is **false** and show that this leads to a contradiction. This method is called **proof by contradiction**.

Theorem 3: There are infinitely many prime numbers.

Proof. Suppose there are only finitely many prime numbers, denoted $a_1, ..., a_n$. Consider the number $a = a_1 \times a_2 \times ... \times a_n + 1$. The number a is not divisible by any of the primes. We thus conclude that a is itself a prime number. A contradiction. $\square$

3.2 Proof by Induction

We can prove statements of the form "for all $n \in \mathbb{N}, n \geq a$, $P(n)$ is true," by

1. First showing that $P(a)$ holds.
2. Then, assuming $P(n)$ is true for some n, we use this assumption to prove that $P(n+1)$ is also true.

By doing this, it follows automatically that $P(n)$ is true for all $n \geq a$. In step (2), one can also assume that $P(k)$ is true for all $k = 1, ..., n-1, n$, and use this stronger assumption to show that $P(n+1)$ is true.

Theorem 4 (AM-GM Inequality for $n = 2^k$): For all $n = 2^k$, $k \in \mathbb{N}$,

$$\frac{a_1 + a_2 + ... + a_n}{n} \geq (a_1 \times a_2 \times ... \times a_n)^{1/n},$$

where $a_1, ..., a_n \geq 0$.

Proof. Prove by induction on k.

1. Base case $k = 1$: Proved in Theorem 2.
2. Inductive step: Now assume that the statement holds for k. We want to show that the statement is also true for $k+1$. Let $a_1, ..., a_{2^{k+1}} \geq 0$. Observe that

$$\begin{aligned}
\frac{a_1 + ... + a_{2^{k+1}}}{2^{k+1}} &= \frac{1}{2}\left(\frac{a_1 + ... + a_{2^k}}{2^k} + \frac{a_{2^k+1} + ... + a_{2^{k+1}}}{2^k}\right) \\
&\geq \frac{1}{2}\left((a_1 \times ... \times a_{2^k})^{1/2^k} + (a_{2^k+1} \times ... \times a_{2^{k+1}})^{1/2^k}\right) \\
&\geq \left\{(a_1 \times ... \times a_{2^k})^{1/2^k} \times (a_{2^k+1} \times ... \times a_{2^{k+1}})^{1/2^k}\right\}^{1/2} \\
&= (a_1 \times a_2 ... \times a_{2^{k+1}})^{1/2^{k+1}}.
\end{aligned}$$

We applied the induction hypothesis to obtain the first inequality, and then used the AM-GM inequality for $n = 2$ to establish the second inequality.

$\square$

Theorem 5 (Tiling with Trominoes): For integer $n \geq 1$, a $2^n \times 2^n$ chessboard with any single unit square removed can be tiled completely by $L-shaped$ trominoes (each tromino covers three unit squares in an L shape).

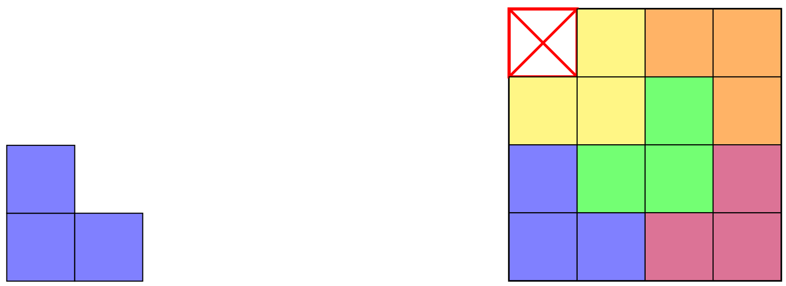
Figure 1 shows an L-shaped tromino and how a 4×4 chessboard with one square removed can be tiled using it.

Proof. Prove by induction on n.

1. Base case $n = 1$: A 2×2 chessboard with any unit square removed is exactly the shape of a L-shaped tromino.

2. Inductive step: Assume the statement holds for $n \geq 1$. We prove the statement for $n + 1$. Partition the $2^{n+1} \times 2^{n+1}$ board into four quadrants (northwest, northeast, southwest, southeast), each of size $2^n \times 2^n$. Therefore, there is exactly one quadrant with a missing square. Place one L-shaped tromino at the center so that it covers one square in each of the three quadrants that do not contain the original missing square. This creates a single missing square in each quadrant, allowing us to apply the induction hypothesis to each. Thus, the entire board can be tiled.

□



(a) A L-shaped tromino. (b) A 4×4 chessboard with one square removed tiled.

Figure 1: Tiling with Trominoes

3.3 Divide and Conquer

Sometimes it is helpful to break the problem into several simpler cases and prove the statement separately for each one.

Theorem 6: There exists a, b that are not rational such that a^b is rational.

Proof. Consider $a = b = \sqrt{2}$. We seperate two cases.

1. a^b is rational: Since $\sqrt{2}$ is not rational, the statement is true.

2. a^b is not rational: Observe that $(a^b)^{\sqrt{2}} = 2$. Since both a^b and $\sqrt{2}$ are not rational, the statement is true.

Since the statement is true in either cases, the proof is done. □