Econ 703 TA Note 6

Chia-Min Wei *

October 23, 2025

1 Differentiable Functions on $\mathbb{R}$

Definition 1.1 (Differentiable at a Point): A function $f : (a, b) \to \mathbb{R}$ is differentiable at a point c if for all $\{x_n\} \subset [a, b]$ that converges to c,

$$\lim_{n\to\infty} \frac{f(x_n) - f(c)}{x_n - c}$$

exists and equals the same value. This value is called the derivative of f at c, denoted $f'(c)$.

From the definition we can immediately see that if f is differentiable at c, then it is also continuous at c.

Definition 1.2 (Derivative of a Function): If a function $f : (a, b) \to \mathbb{R}$ is differentiable everywhere in (a, b), we say that the function is differentiable. $f'(x)$ is well-defined for all $x \in (a, b)$, and thus is also a function on (a, b). An alternative notation is $\frac{d}{dx} f(x)$.

Here are some common rules for taking derivatives:

- **Power Rule:** $\frac{d}{dx}[x^n] = nx^{n-1}, \quad n \in \mathbb{R}$.
- **Constant Multiple Rule:** $\frac{d}{dx}[c \cdot f(x)] = c \cdot f'(x)$.
- **Sum/Difference Rule:** $\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$.
- **Product Rule:** $\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$.
- **Quotient Rule:** $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}$.
- **Chain Rule:** $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$.

2 Big O, Little o

If a function is differentiable at a point c, then the linear function

$$h(x) = f(c) + f'(c)(x - c)$$

provides a "good" approximation of $f(x)$ near $x = c$. But what exactly do we mean by good? To make this precise, let us introduce the concepts of *big O* and *little o* notation.

*†

Definition 2.1 (Big O): We say that a function $f(x)$ is of $O(g(x))$ as $x \to c$, written as $f(x) = O(g(x))$ as $x \to c$, if there exists $\delta, M > 0$ such that for all $|x - c| < \delta$,

$$|f(x)| \leq M|g(x)|.$$

$f(x) = O(g(x))$ as $x \to c$ means $f(x)$ is bounded by a constant multiple of $g(x)$ near $x = c$.

Example 2.1: If f is differentiable at c, then $f(x) - f(c) = O(|x - c|)$. This is because

$$\lim_{x \to c} \left| \frac{f(x) - f(c)}{x - c} \right| = |f'(c)| < \infty.$$

Example 2.2: f being continuous at c does not necessarily mean that $f(x) - f(c) = O(|x - c|)$. For example, $f(x) = \sqrt{|x|}$ is continuous at $x = 0$, however,

$$\lim_{x \to 0} \left| \frac{\sqrt{|x|}}{|x|} \right| = \lim_{x \to 0} |x|^{-1/2} \to \infty.$$

Definition 2.2 (Little o): We say that a function $f(x)$ is of $o(g(x))$ as $x \to c$, written as $f(x) = o(g(x))$ as $x \to c$, if

$$\lim_{x \to c} \left| \frac{f(x)}{g(x)} \right| = 0.$$

$f(x) = o(g(x))$ as $x \to c$ means $f(x)$ becomes negligible compared to $g(x)$ as x approaches c.
Here are some rules regarding big O and little o.

1. $f(x) = o(g(x)) \implies f(x) = O(g(x))$.
2. $f(x) = O(g(x)) \implies \alpha f(x) = O(g(x)), \quad f(x) = o(g(x)) \implies \alpha f(x) = o(g(x))$.
3. $f(x) = O(g(x)),\ h(x) = O(g(x)) \implies f(x) + h(x) = O(g(x))$.
   $f(x) = o(g(x)),\ h(x) = o(g(x)) \implies f(x) + h(x) = o(g(x))$.
4. $f(x) = O(g(x)),\ h(x) = O(k(x)) \implies f(x)h(x) = O(g(x)k(x))$.
   $f(x) = O(g(x)),\ h(x) = o(k(x)) \implies f(x)h(x) = o(g(x)k(x))$.
5. $f(x) = O(g(x)),\ g(x) = O(h(x)) \implies f(x) = O(h(x))$.
   $f(x) = O(g(x)),\ g(x) = o(h(x)) \implies f(x) = o(h(x))$.
   $f(x) = o(g(x)),\ g(x) = O(h(x)) \implies f(x) = o(h(x))$.

Theorem 2.1: A function $f : [a, b] \to \mathbb{R}$ is differentiable at c with derivative $f'(c)$ if and only if

$$f(x) - (f(c) + f'(c)(x - c)) = o(|x - c|),$$

or alternatively,

$$f(x) = f(c) + f'(c)(x - c) + o(|x - c|).$$

Thus, saying a function is differentiable at c means that the error — the difference between $f(x)$ and its linear approximation — goes to 0 faster than the distance between x and c.

Let us use the big O and little o notation to prove the chain rule.

Theorem 2.2 (Chain Rule): Let $g : (a_1, b_1) \to \mathbb{R}$, $f : (a_2, b_2) \to \mathbb{R}$ where $(a_2, b_2) \supset g(a_1, b_1)$. Suppose g is differentiable at $c \in (a_1, b_1)$ and f is differentiable at $g(c)$, then $h(x) = f(g(x))$ is differentiable at c with derivative $f'(g(c))g'(c)$.

Proof. As x approahes c,

$$\begin{aligned}
f(g(x)) &= f(g(c)) + f'(g(c))(g(x) - g(c)) + o(|g(x) - g(c)|) \\
&= f(g(c)) + f'(g(c))g'(c)(x - c) + f'(g(c))g'(c)o(|x - c|) + o(|g(x) - g(c)|) \\
&= f(g(c)) + f'(g(c))g'(c)(x - c) + o(|x - c|).
\end{aligned}$$

Note that $|g(x) - g(c)| = O(|x - c|)$, and thus $o(|g(x) - g(c)|) = o(|x - c|)$ by rule 5, $f'(g(c))g'(c)o(|x - c|) = o(|x - c|)$ by rule 2, and the sum of them is still $o(|x - c|)$ by rule 3. □

3 First Order Condition, Rolle's Theorem, Mean Value Theorem

Theorem 3.1 (First Order Condition): Let $f : (a, b) \to \mathbb{R}$ be differentiable. If f attains a local maximum or minimum at some point $c \in (a, b)$, then $f'(c) = 0$.

Proof. We prove the case when c is a maximal point. Since f is differentiable at c,

$$f(x) = f(c) + f'(c)(x - c) + r(x),$$

where $r(x) = o(|x - c|)$ as $x \to c$. If $f'(c) > 0$, then there exists a small enough δ such that for all $|x - c| < \delta$, $|r(x)| < \left|\frac{f'(c)}{2}(x - c)\right|$. Hence,

$$\begin{aligned}
f(x) &= f(c) + f'(c)(x - c) + r(x) \\
&\geq f(c) + f'(c)(x - c) - \left|\frac{f'(c)}{2}\right| |x - c| \\
&= f(c) + \frac{f'(c)}{2}(x - c)
\end{aligned}$$

for all $c < x < c + \delta$. Therefore, for all $c < x < c + \delta$, we have $f(x) > c$. This is a contradiction. The proof that $f'(c)$ cannot be smaller than 0 is the same. □

Note that the First Order Condition is a **necessary** condition for a point to be a local maximal/minimal point. A function f can have a derivative equal to 0 at some point x, even if x is neither a maximum nor a minimum.

Example 3.1: Consider $f(x) = x^3$. $f'(0) = 0$ but $x = 0$ is neither a local maximal nor a local minimal point.

Suppose $f : [a, b] \to \mathbb{R}$ and $f(a) = f(b)$, then f must attain a global maximum or minimum in (a, b). This leads to Rolle's Theorem:

Theorem 3.2 (Rolle's Theorem): Let $f : [a, b] \to \mathbb{R}$ be differentiable on (a, b). If $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.

The generalization of Rolles's theorem is the Mean Value Theorem:

Theorem 3.3 (Mean Value Theorem): Let $f : [a, b] \to \mathbb{R}$ be differentiable on (a, b). Then there exists $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Proof. Write $\theta = \frac{f(b)-f(a)}{b-a}$. Let us consider the function:

$$h(x) \coloneqq f(x) - \theta(x - a).$$

Note that $h(a) = h(b) = f(a)$. By Rolle's Theorem, there exists $c \in (a, b)$ such that $h'(c) = 0$. But note that

$$h'(x) = f'(x) - \theta.$$

Therefore, $h'(c) = 0 \implies f'(c) = \theta$. ☐