Econ 703 TA Note 2

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1 Bellman Equation: A Motivation

1.1 Catching Fish

This is a story about catching fish: Charlie owns a lake and catches fish each day. The number of fish available tomorrow depends on how many remain after today's catch. Each day, there is a 50% chance that the remaining fish population doubles and a 50% chance that it stays the same. By consuming c fish, Charlie obtains a utility of $u(c)$, and he discounts future utility by a factor β. What is his lifetime value if he catches the optimal amount of fish every day?

Of course, his lifetime value depends on the initial number of fish in the lake. Write $V(x)$ as Charlie's lifetime value if the initial number of fish in the lake is x. V is the so-called **value function**. The right question that we should be asking is, what is Charlie's **value function** V?

Indeed, $V(x)$ has to satisfy the following:

$$V(x) = \max_{c \leq x} \left\{ u(c) + \beta \left(\frac{1}{2} V(2(x-c)) + \frac{1}{2} V(x-c) \right) \right\}. \tag{1}$$

If he catches c fish today, he enjoys utility $u(c)$ today. As for tomorrow, there is a 50% chance that the initial number of fish is $2(x-c)$ and a 50% chance that it is $x-c$. Because Charlie always chooses the optimal catch each day, his continuation value tomorrow, by the definition of V, should be either $V(x-c)$ or $V(2(x-c))$, each with probability $1/2$. Equation 1 is called a **Bellman Equation**.

1.2 Value Function as a Fixed Point of a Mapping

Assume that the fish population can be any real number from 0 to infinity, and Charlie can catch any amount $c \leq x$. Thus, V and u are functions defined on $[0, \infty)$. Suppose u is continuous and bounded. Let $\mathcal{C}$ be the set of bounded continuous functions on $[0, \infty)$. We define a mapping $T : \mathcal{C} \to \mathcal{C}$ that takes one function and returns another. For **any** $W \in \mathcal{C}$, we define $W' = T(W)$ **pointwise** by

$$W'(x) = \max_{c \leq x} \left\{ u(c) + \beta \left(\frac{1}{2} W(2(x-c)) + \frac{1}{2} W(x-c) \right) \right\}.$$

For each $x \in [0, \infty)$, $W'(x)$ is well-defined, so the function W is fully determined. And therefore, the mapping T is well-defined.

It is then clear that Charlie's value function V is a **fixed point** of T. This leads to the next question: under what conditions does T have a fixed point? Furthermore, if a fixed point exists, is it unique? These questions motivate the introduction of **Contraction Mapping Theorem**.

*This TA note is based on Prof. John Kennan's math camp lecture in 2025 at UW-Madison. All errors are mine.

2 Metric Space, Cauchy Sequences, Completeness

Before introducing the theorem, we need formal definitions of **metric spaces**, **Cauchy sequences** and **completeness**.

Definition 2.1 (Metric Space): We say that (X, d) is a metric space if $d : X \times X \to [0, \infty)$ satisfies

1. $d(x, y) = 0 \iff x = y$.
2. $d(x, y) = d(y, x)$ for all $x, y \in X$.
3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Example 2.1: Define $d : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$, for each $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n)$,

$$d(x, y) \coloneqq \left((x_1 - y_1)^2 + ... + (x_n - y_n)^2\right)^{1/2}.$$

This is the so-called **Euclidean metric**.

Example 2.2: Define $d : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ by

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

This is also a metric defined on $\mathbb{R}$.

2.1 Cauchy Sequences and Completeness

A **sequence** in a metric space is an ordered list of elements indexed by the natural numbers, typically written as $\{x_i\}_{i=1}^{\infty}$.

Definition 2.2 (Limit): We say that a sequence $\{x_n\}_{n=1}^{\infty}$ converges to a limit x if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $d(x_n, x) < \epsilon$.

This means that the points in the sequence becomes arbitrarily close to the point as we get to the tail of the sequence. In a metric space, a sequence has at most one limit.

Definition 2.3 (Cauchy Sequence): We say that a sequence $\{x_n\}_{n=1}^{\infty}$ in a metric space (X, d) is a Cauchy sequence if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n, m \geq \mathbb{N}$,

$$d(x_n, x_m) < \epsilon.$$

Intuitively, the distance between two points in the sequence becomes arbitrarily small as we move toward its tail. Does a Cauchy sequence always converge? In some metric spaces, a Cauchy sequence may not converge in the space.

Example 2.3 ($\mathbb{Q}$ is not complete): Start from $a_0 = 1, b_0 = 2$. Note that $a_0^2 < 2 < b_0^2$. Define recursively,

1. Let $m_n = \frac{a_n + b_n}{2}$.
2. If $m_n^2 > 2$, then set $a_{n+1} = a_n$ and $b_{n+1} = m_n$.
3. If $m_n^2 < 2$, then set $a_{n+1} = m_n$ and $b_{n+1} = b_n$.

We get a series of nested intervals with rational endpoints, $[a_0, b_0] \supset [a_1, b_1] \supset [a_2, b_2] \supset ...$, all of which includes $\sqrt{2}$. Moreover, the length of $[a_n, b_n]$ is 2^{-n}. It is clear that $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{Q}$. However, $\{a_n\}$ does not converge in $\mathbb{Q}$: one can show that indeed it converges to $\sqrt{2}$ in $\mathbb{R}$.

Definition 2.4 (Complete Metric Space): We say that a metric space is complete if every Cauchy sequence converges in the metric space.

3 Contraction Mapping Theorem

Now we are ready to introduce the idea of contraction mapping.

Definition 3.1 (Contraction): Let (X, d) be a metric space. We say that $T : X \to X$ is a contraction mapping if there exists $\alpha < 1$ such that for all $x, y \in X$,

$$d(T(x), T(y)) < \alpha T(x, y).$$

α is called the modulus of the contraction mapping.

There are two important points of a contraction mapping. First, it has to **map any point in the metric space back into the same metric space**. Second, there exists a **universal rate smaller than** 1 at which the distance between any two points shrinks after being mapped.

Theorem 3.1 (Contraction Mapping Theorem): Let (X, d) be a **complete** metric space and $T : X \to X$ a **contraction mapping**. Then there exists a **unique** $x^* \in X$ such that $T(x^*) = x^*$.

Figure 1 illustrates the proof of contraction mapping theorem. Starting from any point in the metric space and applying the mapping iteratively, the sequence converges to the fixed point.

Proof. Start from any point $a_0 \in X$. Define recursively $a_n = T(a_{n-1})$. We get a sequence $\{a_n\}_{n=1}^{\infty}$. We show that it is Cauchy. Write $\delta = d(a_1, a_0)$. Observe that

$$\begin{aligned} d(a_n, a_{n-1}) = d(T(a_{n-1}), a_{n-2}) &\leq \alpha d(a_{n-1}, a_{n-2}) \\ &= \alpha d(T(a_{n-2}), T(a_{n-3})) \\ &\leq \alpha^2 d(a_{n-2}, a_{n-3}) \\ &\vdots \\ &\leq \alpha^{n-1}\delta. \end{aligned}$$

Now for any $n, k \in \mathbb{N}$,

$$d(a_{n+k}, a_n) \leq \sum_{l=0}^{k-1} d(a_{n+l+1}, a_{n+l}) \leq \sum_{l=0}^{k-1} \alpha^{n+l}\delta \leq \delta \sum_{l=0}^{\infty} \alpha^{n+l} = \frac{\alpha^n}{1-\alpha}\delta.$$

For any $\epsilon > 0$, we can choose N so large so that $\frac{\alpha^N}{1-\alpha}\delta < \epsilon$. Then for any $n > m \geq N$,

$$d(a_n, a_m) \leq \frac{\alpha^m}{1-\alpha}\delta \leq \frac{\alpha^N}{1-\alpha}\delta < \epsilon.$$

We have thus proved that $\{a_n\}$ is Cauchy. Since X is complete, $\{a_n\}$ converges to a unique point a^*.

We now show that a^* is a fixed point of T. Fix $\epsilon > 0$. Since $\{a_n\}$ converges to a^*, there exists $n \in \mathbb{N}$ such that for all $k \geq n$, $d(a^*, a_k) < \epsilon$. Now observe that

$$\begin{aligned} d(f(a^*), a^*) &\leq d(f(a^*), a_{n+1}) + d(a_{n+1}, a^*) \\ &\leq d(f(a^*), f(a_n)) + d(a_{n+1}, a^*) \\ &\leq \alpha d(a^*, a_n) + d(a_{n+1}, a^*) \leq (\alpha + 1)\epsilon. \end{aligned}$$

Since ϵ is arbitrary, we conclude that $d(f(a^*), a^*) = 0$, which implies $f(a^*) = a^*$.

Finally, we show that the fixed point x^* is unique. Suppose x^* and y^* are both fixed points.

$$d(x^*, y^*) = d(f(x^*), f(y^*)) \leq \alpha d(x^*, y^*).$$

Since $\alpha < 1$, it must be $d(x^*, y^*) = 0$, which implies $x^* = y^*$. □

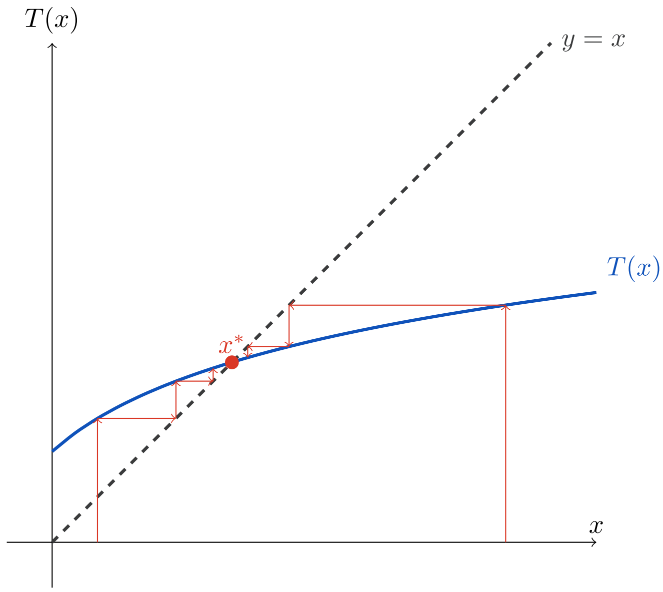

Figure 1: Contraction Mapping Theorem

Example 3.1: If we can show that the T described in subsection 1.2 is a contraction mapping on $(\mathcal{C}, d)$ with respect to some metric d (we need a way to measure the distance between two functions), then the Contraction Mapping Theorem tells us that the value function V exists and is unique. In many cases, we adopt the supnorm as the metric:

$$d(f, g) = \sup_{x \in [0,\infty)} |f(x) - g(x)|.$$

Example 3.2: A differentiable function $f : \mathbb{R} \to \mathbb{R}$ is a contraction mapping if $\sup_{x\in\mathbb{R}} |f'(x)| \leq \alpha$ for some $\alpha < 1$.

Example 3.3: If you stand in the Social Science Building holding a model of it, there is a unique point in the building that coincides with its counterpart in the model.

The fact that the fixed point is unique is a powerful result. It leads to the following corollary.

Corollary: Let (X, d) and (Y, d) be two metric spaces, and $Y \subset X$. Suppose $T : X \to X$ is a contraction mapping. If $T(Y) \subset Y$, then the fixed point of $T : X \to X$ lies in Y.

Proof. By restricting T to Y, note that $T|_Y : Y \to Y$ is a contraction mapping on Y. Therefore, it has a fix point y^* in Y. Since $y^* = T|_Y(y^*) = T(y^*)$, it follows that y^* is also the unique fix point of T. □

Example 3.4: Let $\tilde{\mathcal{C}}$ be the set of bounded, concave and continuous functions on $[0, \infty)$. If $\tilde{\mathcal{C}}$ is complete under the metric d, and $T(\tilde{\mathcal{C}}) \subset \tilde{\mathcal{C}}$, then we can conclude that the value function V is concave.