

## Lecture 15

### 1. Burnside formula

Let  $G$  be a finite gp acting on a finite set  $X$ . For  $g \in G$ , let  $X^g$  denote the

set  $\{x \in X \mid g \cdot x = x\}$ . Thus,

$X^g$  = fixed points of  $g$  under the action.

Now,

$$\begin{aligned} \sum_{g \in G} |X^g| &= \left| \left\{ (g, x) \in G \times X \mid g \cdot x = x \right\} \right| \\ &= \sum_{x \in X} |G_x| \end{aligned}$$

Let  $C_1, \dots, C_m$  be the orbits of the action.

$$\text{For } x \in C_i, |G_x| = |G| \cdot \frac{|G_x|}{|G|} = |G| \cdot \frac{1}{|C_i|}$$

Substituting

$$\sum_{g \in G} |X^g| = \sum_{i=1}^m \sum_{x \in C_i} \frac{|G|}{|C_i|} = |G| \cdot m.$$

$$\text{Hence, } m = \left[ \# \text{ of orbits} = \frac{1}{|G|} \cdot \sum_{g \in G} |X^g| \right]$$