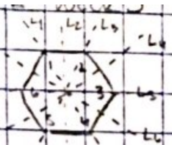
7.F.1




$R_0 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$ $R_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 1 \end{pmatrix}$ ccw 60° $R_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 1 & 2 \end{pmatrix}$ ccw 12

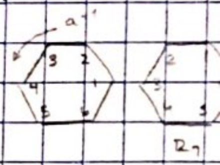
$R_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 1 & 2 & 3 \end{pmatrix}$ ccw 180° $R_4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 1 & 2 & 3 & 4 \end{pmatrix}$ ccw 240°

20/20

$R_5 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 3 & 4 & 5 \end{pmatrix}$ ccw 300° $R_6 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 5 & 4 & 3 & 2 \end{pmatrix}$ refl. $L_1$ $R_7 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 6 & 5 & 4 & 3 \end{pmatrix}$ refl. $L_2$

$R_8 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 1 & 6 & 5 & 4 \end{pmatrix}$ refl. $L_3$ $R_9 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 2 & 1 & 6 & 5 \end{pmatrix}$ refl. $L_4$ $R_{10} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 3 & 2 & 1 & 6 \end{pmatrix}$ refl. $L_5$

$R_{11} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}$ refl. $L_6$ $R_1 R_7 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 1 & 6 & 5 & 4 \end{pmatrix} = R_8$




Say $a$ = some rotation, $b$ = some reflection $\longrightarrow \langle a, b;\ a^6 = e,\ b^2 = e,\ ab = ba^m \rangle$

$b^{-1}ab = a^m$ Say $b = R_7$, $a = R_1$

$R_7^{-1} \underbrace{R_1 R_7}_{R_8} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 6 & 5 & 4 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 1 & 6 & 5 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 3 & 4 & 5 \end{pmatrix} = R_5 = R_1^5 \longrightarrow a^5$

| | $e$ | $a$ | $a^2$ | $a^3$ | $a^4$ | $a^5$ | $b$ | $ba$ | $ba^2$ | $ba^3$ | $ba^4$ | $ba^5$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $e$ | $e$ | $a$ | $a^2$ | $a^3$ | $a^4$ | $a^5$ | $b$ | $ba$ | $ba^2$ | $ba^3$ | $ba^4$ | $ba^5$ |
| $a$ | $a$ | $a^2$ | $a^3$ | $a^4$ | $a^5$ | $e$ | $ba^5$ | $b$ | $ba$ | $ba^2$ | $ba^3$ | $ba^4$ |
| $a^2$ | $a^2$ | $a^3$ | $a^4$ | $a^5$ | $e$ | $a$ | $ba^4$ | $ba^5$ | $b$ | $ba$ | $ba^2$ | $ba^3$ |
| $a^3$ | $a^3$ | $a^4$ | $a^5$ | $e$ | $a$ | $a^2$ | $ba^3$ | $ba^4$ | $ba^5$ | $b$ | $ba$ | $ba^2$ |
| $a^4$ | $a^4$ | $a^5$ | $e$ | $a$ | $a^2$ | $a^3$ | $ba^2$ | $ba^3$ | $ba^4$ | $ba^5$ | $b$ | $ba$ |
| $a^5$ | $a^5$ | $e$ | $a$ | $a^2$ | $a^3$ | $a^4$ | $ba$ | $ba^2$ | $ba^3$ | $ba^4$ | $ba^5$ | $b$ |
| $b$ | $b$ | $ba$ | $ba^2$ | $ba^3$ | $ba^4$ | $ba^5$ | $e$ | $a$ | $a^2$ | $a^3$ | $a^4$ | $a^5$ |
| $ba$ | $ba$ | $ba^2$ | $ba^3$ | $ba^4$ | $ba^5$ | $b$ | $a^5$ | $e$ | $a$ | $a^2$ | $a^3$ | $a^4$ |
| $ba^2$ | $ba^2$ | $ba^3$ | $ba^4$ | $ba^5$ | $b$ | $ba$ | $a^4$ | $a^5$ | $e$ | $a$ | $a^2$ | $a^3$ |
| $ba^3$ | $ba^3$ | $ba^4$ | $ba^5$ | $b$ | $ba$ | $ba^2$ | $a^3$ | $a^4$ | $a^5$ | $e$ | $a$ | $a^2$ |
| $ba^4$ | $ba^4$ | $ba^5$ | $b$ | $ba$ | $ba^2$ | $ba^3$ | $a^2$ | $a^3$ | $a^4$ | $a^5$ | $e$ | $a$ |
| $ba^5$ | $ba^5$ | $b$ | $ba$ | $ba^2$ | $ba^3$ | $ba^4$ | $a$ | $a^2$ | $a^3$ | $a^4$ | $a^5$ | $e$ |

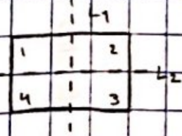
$\longrightarrow$ Where is some rotation and b is some reflection

☆ Chart redone using $R_i$ on last page

$a^2b = aba^5 = ba^5a^5 = ba^4$
$a^3b = a^2ba^5 = ba^4a^5 = ba^3$
$a^4b = a^3ba^5 = ba^3a^5 = ba^2$

$bab = bba^5 = a^5$
$ba^2b = baba^5 = bba^5a^5 = a^4$

7.F.2




$R_0 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$ $R_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$ ccw 180° $R_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$ refl. $L_1$

$R_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$ refl. $L_2$

Say $a$ is some rotation, $b$ is some reflection $\longrightarrow \langle a, b;\ a^2 = e,\ b^2 = e,\ ab = ba^m \rangle$

$b^{-1}ab = a^m$ Say $b = R_2$, $a = R_1$

$R_2^{-1} R_1 R_2 \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} = R_1 = a$

| | $e$ | $a$ | $b$ | $ba$ |
|---|---|---|---|---|
| $e$ | $e$ | $a$ | $b$ | $ba$ |
| $a$ | $a$ | $e$ | $ba$ | $b$ |
| $b$ | $b$ | $ba$ | $e$ | $a$ |
| $ba$ | $ba$ | $b$ | $a$ | $e$ |

$\longrightarrow$ Where $a$ is some rotation and $b$ is some reflection

$aba = baa = b$

$baba = bbaa = ee$

| | $e$ | $R_1$ | $R_2$ | $R_3$ |
|---|---|---|---|---|
| $e$ | $e$ | $R_1$ | $R_2$ | $R_3$ |
| $R_1$ | $R_1$ | $e$ | $R_3$ | $R_2$ |
| $R_2$ | $R_2$ | $R_3$ | $e$ | $R_1$ |
| $R_3$ | $R_3$ | $R_2$ | $R_1$ | $e$ |

8.A.1 c. $(71825)(36)(49) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 8 & 5 & 6 & 9 & 7 & 3 & 1 & 2 & 4 \end{pmatrix}$ 2/2

8.A.2 b. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 4 & 9 & 2 & 3 & 8 & 1 & 6 & 5 \end{pmatrix} = (17)(24)(395)(68)$ 2/2

8.A.3 c. $(123)(456)(1574) = (13)(12)(46)(45)(14)(17)(15)$ 2/2

8.A.4 d. $\beta^2\alpha\gamma = (123)(123)(3714)(24135) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 2 & 5 & 4 & 1 & 6 & 3 \end{pmatrix} = (1735)(2)(4)(6)$

8.C.1 c. $(12)(76)(345) = (12)(76)(35)(34) = p;\ |p| = 4 \rightarrow$ even

8.C.2 Prove the product of 2 even permutations is even 2/2

Say you have 2 permutations $p_1$ and $p_2$, and there product is $p_1p_2$. Every permutation can be written in terms of one or more cycles, and every cycle can be written in terms of one or more transpositions. So $p_1$ and $p_2$ can both be written in terms of $k_1$ and $k_2$ transpositions. More over, while $k_1$ and $k_2$ may not be unique, since $p_1$ and $p_2$ are even permutations, $k_1$ and $k_2$ MUST be even integers.

$2n_1 = k_1,\ 2n_2 = k_2 \longrightarrow p_1p_2 = (k_1 \text{ transpositions})(k_2 \text{ transpositions})$

Total # transpositions $= k_1 + k_2 = 2n_1 + 2n_2 = 2(n_1+n_2) = 2N,\ N \in \mathbb{Z} \therefore$ $p_1p_2$ is even given $p_1, p_2 \in$ even

4/4

Prove the product of 2 odd permutations is even

Use permutations $p_1, p_2$ with $k_1 = 2n_1+1,\ k_2 = 2n_2+1$ transpositions respectively. By definition of odd permutations, $k_1$ and $k_2$ MUST be odd integers.

$p_1p_2 = (k_1 \text{ transpositions})(k_2 \text{ transpositions})$

Total # transpositions $= k_1 + k_2 = 2n_1 + 1 + 2n_2 + 1 = 2n_1 + 2n_2 + 2 = 2(n_1+n_2+1) = 2N,\ N \in \mathbb{Z}$

$\therefore$ $p_1p_2$ is even given $p_1, p_2 \in$ odd

Prove the product of an even and odd permutation is odd

Use permutations $p_1, p_2$, with $k_1 = 2n_1,\ k_2 = 2n_2+1$ transpositions respectively. Here, $p_1$ is the even permutation and $p_2$ is the odd permutation, though this argument could go both ways.

$p_1p_2 = (k_1 \text{ transpotisions})(k_2 \text{ transpositions})$

Total # transpositions $= k_1 + k_2 = 2n_1 + 2n_2 + 1 = 2(n_1+n_2)+1 = 2N+1,\ N \in \mathbb{Z} \therefore$ $p_1p_2$ is odd given $p_1$ (or $p_2$) $\in$ even and $p_2$ (or $p_1$) $\in$ odd.

4/4

8.C.3 Prove a cycle of length $l$ is even if $l$ is odd

A cycle $(a_1 a_2 a_3 \cdots a_{l-1} a_l)$ can be written in terms of $l-1$ transpositions: $(a_1 a_l)(a_1 a_{l-1}) \cdots (a_1 a_2)$ So if $l$ $2n+1$, that means it can be written using $(2n+1)-1 = 2n$ transpositions. Now we have that the number of transpositions must be even. $\therefore$ cycle is even given length is odd

Prove a cycle of length $l$ is odd if $l$ is even

If $l = 2n$, that means it can be written using $2n-1$ transpositions. Now we have that the number of transpositions must be odd. $\therefore$ cycle is odd given length is even.

8.E.1 $\alpha = (a_1, \ldots, a_s)$, $\pi$ is a permutation $\in S_n \rightarrow$ symmetric group on $\{1, 2, 3, \ldots, n-1, n\}$

$\pi(i) = 1$

$\pi\alpha\pi^{-1} = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ \pi(1) & \pi(2) & \pi(3) & & \pi(n) \end{pmatrix} \cdot (a_1, \ldots, a_s) \begin{pmatrix} \pi_1 & \pi_2 & \pi_3 & \cdots \\ \pi_1^{-1} & \pi_2^{-1} & \pi_3^{-1} & \end{pmatrix} = (\pi(a_1), \ldots, \pi(a_s))$ ?

$\pi_i = 1$ from $\pi^{-1}$ will be mapped to a value in $\alpha$, which in turn will be mapped to a number on the top row. That number is then mapped to it's "bijective function 1-to-1 value" $\pi(\alpha)$. So $\pi\alpha\pi^{-1}$ visually is

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ \pi(1) & \pi(2) & \pi(3) & & \pi(n) \end{pmatrix} (a_1 \cdots a_s) \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ \pi^{-1}(1) & \pi^{-1}(2) & \pi^{-1}(3) & & \pi^{-1}(n) \end{pmatrix} \qquad 1 \longrightarrow \pi^{-1}(1) \text{ some \#} \longrightarrow a_x \longrightarrow \pi(a_x)$$

So $\pi\alpha\pi^{-1}$ is ultimately a cycle where each $a_x$ produces an output using $\pi: A \longrightarrow A$

$\therefore \pi\alpha\pi^{-1} = (\pi(a_1) \ldots \pi(s_n))$

4/4

8.E.2 Prove cycles of the same length are conjugates of each other

Say we have cycles $\alpha_1$ and $\alpha_2$ such that $|\alpha_1| = |\alpha_2|$, and we have permutation $\pi$
$\alpha_1 = (a_1\ a_2\ a_3 \cdots a_s) \qquad \alpha_2 = (b_1\ b_2\ b_3 \cdots b_s)$

$\pi\alpha_1\pi^{-1} = (\pi(b_1)\ \pi(b_2)\ \pi(b_3) \cdots \pi(b_s))$
$\pi\alpha_2\pi^{-1} = (\pi(a_1)\ \pi(a_2)\ \pi(a_3) \cdots \pi(a_s))$

~~2~~/4

$\pi$ is over $S_n$, so all values in $\alpha_1$ and $\alpha_2$ must be in $\{1, 2, 3, \ldots, n\}$
A given $\pi$ and $\pi^{-1}$ (one particular permutation) will always map to a certain value in $\{1, 2, \ldots n\}$
If both $\alpha_1$ and $\alpha_2$ contain that value, $\pi^{-1}$ and $\pi$ will be able to map to either and use the value as input to the bijective function.
$\alpha_1$ and $\alpha_2$ have all the values from $[1, n]$. When $\pi^{-1}$ maps to some $x$ in $[1, n]$, there will be some $b_i$ and some $a_i$ to map to. Since it's a permutation, you won't have to worry about repeating values, and since $|\alpha_1| = |\alpha_2|$, you know that $\alpha_1$ will always be able to map to $\alpha_2$ with a given permutation and vise versa

$\therefore$ cycles of same size are conjugates of each other

7.F.1

| | $R_0$ | $R_1$ | $R_2$ | $R_3$ | $R_4$ | $R_5$ | $R_6$ | $R_7$ | $R_8$ | $R_9$ | $R_{10}$ | $R_{11}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $R_0$ | $R_0$ | $R_1$ | $R_2$ | $R_3$ | $R_4$ | $R_5$ | $R_6$ | $R_7$ | $R_8$ | $R_9$ | $R_{10}$ | $R_{11}$ |
| $R_1$ | $R_1$ | $R_2$ | $R_3$ | $R_4$ | $R_5$ | $R_0$ | $R_7$ | $R_8$ | $R_9$ | $R_{10}$ | $R_{11}$ | $R_6$ |
| $R_2$ | $R_2$ | $R_3$ | $R_4$ | $R_5$ | $R_0$ | $R_1$ | $R_8$ | $R_9$ | $R_{10}$ | $R_{11}$ | $R_6$ | $R_7$ |
| $R_3$ | $R_3$ | $R_4$ | $R_5$ | $R_0$ | $R_1$ | $R_2$ | $R_9$ | $R_{10}$ | $R_{11}$ | $R_6$ | $R_7$ | $R_8$ |
| $R_4$ | $R_4$ | $R_5$ | $R_0$ | $R_1$ | $R_2$ | $R_3$ | $R_{10}$ | $R_{11}$ | $R_6$ | $R_7$ | $R_8$ | $R_9$ |
| $R_5$ | $R_5$ | $R_0$ | $R_1$ | $R_2$ | $R_3$ | $R_4$ | $R_{11}$ | $R_6$ | $R_7$ | $R_8$ | $R_9$ | $R_{10}$ |
| $R_6$ | $R_6$ | $R_{11}$ | $R_{10}$ | $R_9$ | $R_8$ | $R_7$ | $R_0$ | $R_5$ | $R_4$ | $R_3$ | $R_2$ | $R_1$ |
| $R_7$ | $R_7$ | $R_6$ | $R_{11}$ | $R_{10}$ | $R_9$ | $R_8$ | $R_1$ | $R_0$ | $R_5$ | $R_4$ | $R_3$ | $R_2$ |
| $R_8$ | $R_8$ | $R_7$ | $R_6$ | $R_{11}$ | $R_{10}$ | $R_9$ | $R_2$ | $R_1$ | $R_0$ | $R_3$ | $R_4$ | $R_3$ |
| $R_9$ | $R_9$ | $R_8$ | $R_7$ | $R_6$ | $R_{11}$ | $R_{10}$ | $R_3$ | $R_2$ | $R_1$ | $R_0$ | $R_5$ | $R_4$ |
| $R_{10}$ | $R_{10}$ | $R_9$ | $R_8$ | $R_7$ | $R_6$ | $R_{11}$ | $R_4$ | $R_3$ | $R_2$ | $R_1$ | $R_0$ | $R_5$ |
| $R_{11}$ | $R_{11}$ | $R_{10}$ | $R_9$ | $R_8$ | $R_7$ | $R_6$ | $R_5$ | $R_4$ | $R_3$ | $R_2$ | $R_1$ | $R_0$ |