Why abstract algebra? algebra = al jebr in Arabic = relation

- solve for roots of polynomial equations.

  competition in 14th century: Cardan, Tartaglia (cubic), Ferrari (quartic)

  1824: Niels Abel: no formula of deg ≥ 5.

- application to practical problems

  algebra of matrices, Boolean algebra

- algebraic structures

  axiomatics of algebra $\leftrightsquigarrow$ axiomatics of geometry

Galois: tied in the problem of finding the roots of equations with new discoveries on groups of permutations.

explained exactly which equations of degree 5 or higher have solutions of the traditional kind

Chap. 2

operations: An operation $*$ on a set A is a rule which assigns to each ordered pair (a,b) of elements of A exactly one element $a*b$ in A.

Options: commutative : $a*b=b*a$

associative : $(a*b)*c = a*(b*c)$

identity element : $a*e=e*a=a$ for every $a\in A$

inverse of a : $a\cdot x = x\cdot a = e$ $\quad x=a^{-1}$.

Ex: 1, $a*b=\sqrt{|ab|}$ on the set $\mathbb{Q}$ is not an operation

2, $a*b=|a-b|$ on $\mathbb{Z}_{\geq 0}$ is an operation.

commutative, not associative, identity element $=0$

$a^{-1}=a$.

3. $x*y = x+y+a$ is an operation on $\mathbb{R}$

- $y*x = y+x+a = x*y$ commutative

- $(x*y)*z = x*y+z+a = (x+y+a)+z+a$

  $x*(y*z) = x+y*z+a = x+(y+z+a)+a$ $\Rightarrow$ associative

- $x*e=x \Leftrightarrow x+e+a=x \Leftrightarrow e=-a \Rightarrow \exists$ identity

- $x*y=e=-a \Leftrightarrow x+y+a=-a \Leftrightarrow y=-2a-x$

  so for any $x\in\mathbb{R}$, x has an inverse. $\Leftrightarrow x^{-1}=-2a-x$

Q: How many operations are there on a set with n elements?

A: n^{n^2}

Q: How many commutative operations? associative operations?

Operation tables for $A=\{a,b\}$. $2^{2^2}=2^4=16$ tables.

E.g.

(a,a)	a	a	a	a	a	a	a	a	b	b	b	b	b	b	b	b
(a,b)	a	a	a	a	b	b	b	b	a	a	a	a	b	b	b	b
(b,a)	a	a	b	b	a	a	b	b	a	a	b	b	a	a	b	b
(b,b)	a	b	a	b	a	b	a	b	a	b	a	b	a	b	a	b

Ex: which are commutative, associative, having identity element? every element is invertible?

Eg: $a*b=b \neq a=b*a$ not commutative.

$$\left.\begin{array}{ll}
(a*a)*a = a*a=a & a*(a*a)=a*a=a \\
(a*a)*b = a*b=b, & a*(a*b)=a*b=b \\
(a*b)*a = a*a=a & a*(b*a)=a*a=a \\
(a*b)*b = b*b=b & a*(b*b)=a*b=b \\
(b*a)*a = a*a=a, & b*(a*a)=b*a=a \\
(b*a)*b = a*b=b, & b*(a*b)=b*b=b \\
(b*b)*a = b*a=a, & b*(b*a)=b*a=a \\
(b*b)*b = b*b=b, & b*(b*b)=b*b=b
\end{array}\right\} \Rightarrow \text{associative}$$

$a*b=b \Rightarrow b$ is not identity
$b*a=a \Rightarrow a$ is not identity
$\Rightarrow$ no identity.