Lecture 13

1. Examples of quotient gps.

__Eg 0__ Let G be a group. Then $\{e\}$, and G are normal subgps.

$G/\{e\}$ is isomorphic to G.

G/G is isomorphic to $\{e\}$.

__Eg 1__ $G = \mathbb{Z}$ $N = 4\mathbb{Z}$

G/N is isomorphic to $\mathbb{Z}_4$.

~~2~~ More examples would follow from what comes next.

2. Kernels and images of homomorphisms and the first isomorphism theorem.

__Def.__ Let $f: G \longrightarrow H$ be a gp homomorphism. The kernel of f is defined by

$$\ker(f) = \{ g \in G \mid f(g) = e_H \}.$$

The image of f (denoted $Im(f)$) is defined by

$$Im(f) = \{h \in H \mid \exists g \in G \text{ with } f(g) = h\}.$$

Thm ~~(2.2)~~ Let $f: G \longrightarrow H$ be a gp homomorphism.

Then,

(1) $\ker(f)$ is a normal subgp of G.

(2) $Im(f)$ is a subgp (not necessarily normal) of H.

Pf: Exercise.

3. First isomorphism Theorem

Thm Let $f: G \longrightarrow H$ be a gp homomorphism.

Then, $Im(f) \cong G/\ker(f)$.

"isomorphic to"

Pf: ~~Let~~ ~~$f: G$~~ We define first a map $F: G/_{\ker(f)} \longrightarrow Im(f)$

and then prove that F is an isomorphism.

1. Definition of F : Let $N = \ker(f)$

Let $F(gN) = f(g)$.

We need to check that F is well defined.

Suppose $gN = g'N$ i.e. $g' = gn$ for some $n \in N$.

(since f is a homomorphism)

Then $f(g') = f(gn) = f(g) f(n)$

$= f(g) e_H = f(g)$.

Since $n \in \ker(f)$

Hence $F(gN) = f(g) = f(g') = F(g'N)$ and so F is well defined.

2. F is a gp homomorphism

$F((g_1N) \cdot (g_2N)) = F(g_1 g_2 N)$

$= f(g_1 g_2) = f(g_1) f(g_2) = F(g_1 N) F(g_2 N)$.

$F((gN)^{-1}) = F(g^{-1}N) = f(g^{-1}) = (f(g))^{-1}$

$= (F(gN))^{-1}$.

3. F is bijective.

Clearly F is surjective.

To show that F is injective suppose

$$F(g_1 N) = F(g_2 N)$$

then $f(g_1) = f(g_2)$

$$\Rightarrow f(g_1) f(g_2)^{-1} = e_H$$

$$\Rightarrow f(g_1) f(g_2^{-1}) = e_H$$

$$\Rightarrow f(g_1 g_2^{-1}) = e_H$$

$$\Rightarrow g_1 g_2^{-1} \in N = \ker(f)$$

$$\Rightarrow g_1 \in N g_2 = g_2 N \Rightarrow g_1 N = g_2 N.$$

(since N is normal)

$\boxtimes$

<u>Cor.</u> (1) If $f: G \to H$ is a surjective gp homomorphism then $H \simeq G/\ker(f)$

(2) If $f: G \to H$ is a gp homomorphism then f is injective iff $\ker(f) = \{e\}$.