Problem 17.E.1

$$\alpha=\begin{pmatrix}a_1+b_1i & c_1+d_1i\\ -c_1+d_1i & a_1-b_1i\end{pmatrix},\beta=\begin{pmatrix}a_2+b_2i & c_2+d_2i\\ -c_2+d_2i & a_2-b_2i\end{pmatrix},\gamma=\begin{pmatrix}a_3+b_3i & c_3+d_3i\\ -c_3+d_3i & a_3-b_3i\end{pmatrix}$$

$$\mathfrak{I}=\alpha+\beta=\begin{pmatrix}(a_1+a_2)+(b_1+b_2)i & (c_1+c_2)+(d_1+d_2)i\\ -(c_1+c_2)+(d_1+d_2)i & (a_1+a_2)-(b_1+b_2)i\end{pmatrix}$$

$\alpha+\beta=\beta+\alpha\Rightarrow$ Commutative

$(\alpha+\beta)+\gamma=\alpha+(\beta+\gamma)$ by the calculation of Matrix $\Rightarrow$ Associativity

$$\alpha+\mathbf{0}=\begin{pmatrix}a_1+b_1i & c_1+d_1i\\ -c_1+d_1i & a_1-b_1i\end{pmatrix}+\begin{pmatrix}0&0\\0&0\end{pmatrix}=\begin{pmatrix}a_1+b_1i & c_1+d_1i\\ -c_1+d_1i & a_1-b_1i\end{pmatrix}\Rightarrow\mathbf{0}=\text{Addictive identity}$$

$$\alpha+(-\alpha)=\begin{pmatrix}a_1+b_1i & c_1+d_1i\\ -c_1+d_1i & a_1-b_1i\end{pmatrix}-\begin{pmatrix}a_1+b_1i & c_1+d_1i\\ -c_1+d_1i & a_1-b_1i\end{pmatrix}=\mathbf{0}\Rightarrow-\alpha=\text{Addictive inverse}$$

$$\alpha\beta=\begin{pmatrix}x+yi & z+wi\\ -z+wi & x-yi\end{pmatrix}\text{ such that }\begin{cases}x=a_1a_2-b_1b_2-c_1c_2-d_1d_2\\ y=a_1b_2+a_2b_1+c_1d_2-c_2d_1\\ z=a_1c_2-b_1d_2+a_2c_1+b_2d_1\\ w=a_1d_2+b_1c_2+a_2d_1-b_2c_1\end{cases}$$

$(\alpha\beta)\gamma=\alpha(\beta\gamma)$ by the calculation of Matrix $\Rightarrow$ Associativity

$(\alpha+\beta)\gamma=\alpha(\beta+\gamma)$ by the calculation of Matrix $\Rightarrow$ Distributivity

$$\alpha\cdot I=\begin{pmatrix}a_1+b_1i & c_1+d_1i\\ -c_1+d_1i & a_1-b_1i\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}=\begin{pmatrix}a_1+b_1i & c_1+d_1i\\ -c_1+d_1i & a_1-b_1i\end{pmatrix}\Rightarrow I=\text{Multiplicative identity}$$

However, α is not commutative under multiplication.

Suppose that $a_1=d_1=1, b_1=c_1=0, a_2=d_2=0, c_2=b_2=1$

$$\alpha\beta=\begin{pmatrix}1&i\\i&1\end{pmatrix}\begin{pmatrix}i&1\\-1&-i\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}\neq\beta\alpha=\begin{pmatrix}i&1\\-1&-i\end{pmatrix}\begin{pmatrix}1&i\\i&1\end{pmatrix}=\begin{pmatrix}2i&0\\0&-2i\end{pmatrix}$$

Problem 17.E.2

$$\alpha=\begin{pmatrix}a+bi & c+di\\ -c+di & a-bi\end{pmatrix}$$

$$1=\begin{pmatrix}1&0\\0&1\end{pmatrix},i=\begin{pmatrix}i&0\\0&-i\end{pmatrix},j=\begin{pmatrix}0&1\\-1&0\end{pmatrix},k=\begin{pmatrix}0&i\\i&0\end{pmatrix}$$

$$a\begin{pmatrix}1&0\\0&1\end{pmatrix}+b\begin{pmatrix}i&0\\0&-i\end{pmatrix}+c\begin{pmatrix}0&1\\-1&0\end{pmatrix}+d\begin{pmatrix}0&i\\i&0\end{pmatrix}=\begin{pmatrix}a+bi & c+di\\ -c+di & a-bi\end{pmatrix}=\alpha$$

Problem 17.E.3

$$(a)\begin{cases}i^2=\begin{pmatrix}i&0\\0&-i\end{pmatrix}\begin{pmatrix}i&0\\0&-i\end{pmatrix}=\begin{pmatrix}-1&0\\0&-1\end{pmatrix}=-1\\ j^2=\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}0&1\\-1&0\end{pmatrix}=\begin{pmatrix}-1&0\\0&-1\end{pmatrix}=-1\Rightarrow \boldsymbol{i^2=j^2=k^2=-1}\\ k^2=\begin{pmatrix}0&i\\i&0\end{pmatrix}\begin{pmatrix}0&i\\i&0\end{pmatrix}=\begin{pmatrix}-1&0\\0&-1\end{pmatrix}=-1\end{cases}$$

$$(\mathbf{b})\begin{cases} ij = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = k \\ -ji = -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = -\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = k \end{cases} \Rightarrow \boldsymbol{ij = -ji = k}$$ ✓

$$(\mathbf{c})\begin{cases} jk = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i \\ -kj = -\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = i \end{cases} \Rightarrow \boldsymbol{jk = -kj = i}$$ ✓

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$$(\mathbf{d})\begin{cases} ki = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = j \\ -ik = -\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = -\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = j \end{cases} \Rightarrow \boldsymbol{ki = -ik = j}$$ ✓

Problem 17.E.4

$$\alpha = \begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix} \Rightarrow \bar{\alpha} = \begin{pmatrix} a-bi & c-di \\ c-di & a+bi \end{pmatrix}$$

$$\Rightarrow \alpha\bar{\alpha} = \begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix}\begin{pmatrix} a-bi & c-di \\ c-di & a+bi \end{pmatrix} = \begin{pmatrix} a^2+b^2+c^2+d^2 & 0 \\ 0 & a^2+b^2+c^2+d^2 \end{pmatrix}$$

$$= (a^2+b^2+c^2+d^2)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = t\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} \text{ where } t = a^2+b^2+c^2+d^2$$

$$\Rightarrow \frac{1}{t}\alpha\bar{\alpha} = \alpha\left(\frac{1}{t}\bar{\alpha}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\bar{\alpha}\alpha = \begin{pmatrix} a-bi & c-di \\ c-di & a+bi \end{pmatrix}\begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix} = \begin{pmatrix} a^2+b^2+c^2+d^2 & 0 \\ 0 & a^2+b^2+c^2+d^2 \end{pmatrix} = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}$$

$$\Rightarrow \frac{1}{t}\bar{\alpha}\alpha = \left(\frac{1}{t}\bar{\alpha}\right)\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

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$$\Rightarrow \alpha\left(\frac{1}{t}\bar{\alpha}\right) = \left(\frac{1}{t}\bar{\alpha}\right)\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \text{the multiplicative inverse of } \alpha \text{ is } \left(\frac{1}{t}\right)\bar{\alpha}$$

Problem 17.E.5

From problem E. 1 we get $\alpha \cdot I = \alpha \Rightarrow I =$ Multiplicative identity $\Rightarrow \alpha$ has a unity
From problem E. 2 to E. 4 we conclude that:

If $\alpha = \begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix} \neq \mathbf{0}$, then there exists $t > 0$ such that $\left(\frac{1}{t}\right)\bar{\alpha}$ is the inverse of α.

That is, $\mathfrak{I}$ is a skew field since it is a ring with unity in which every nonzero element has a multiplicative inverse.

Problem 18.A.3

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Let $S = \{x \cdot 2^y : x, y \in \mathbb{Z}\} \subseteq \mathbb{R}$ since $x \cdot 2^y \in \mathbb{R}\ \forall\, x, y \in \mathbb{Z}$
Let $x_1, x_2 \in \mathbb{Z}, y_1, y_2 \in \mathbb{Z}$, then $x_1 \cdot 2^{y_1} \in S$ and $x_2 \cdot 2^{y_2} \in S$ ✓

$$x_2 \cdot 2^{y_2} - x_1 \cdot 2^{y_1} = \begin{cases} (x_1 \cdot 2^{y_1-y_2} - x_2) \cdot 2^{y_2} \in S, \text{if } y_1 \geq y_2 \\ (x_1 - x_2 \cdot 2^{y_2-y_1}) \cdot 2^{y_1} \in S, \text{if } y_1 \leq y_2 \end{cases} \Rightarrow x_2 \cdot 2^{y_2} - x_1 \cdot 2^{y_1} \in S\ \forall\, x_1, x_2, y_1, y_2 \in \mathbb{Z}$$

$(x_1 \cdot 2^{y_1})(x_2 \cdot 2^{y_2}) = (x_1x_2) \cdot 2^{(y_1+y_2)} \in S$ since $x_1x_2 \in \mathbb{Z}$ and $(y_1 + y_2) \in \mathbb{Z}$
$\Rightarrow$ Thus S is a subring of $\mathbb{R}$

✓

Problem 18.B.1

(a) No

Let $S = \{(n,n): n \in \mathbb{Z}\}$ and take $(1,0) \in \mathbb{Z} \times \mathbb{Z}$

Take $(1,1) = (n,m) \in S \Rightarrow (1,0) * (1,1) = (1,0) \notin S$

(b) Yes

Let $S = \{(5n,0): n \in \mathbb{Z}\}$

If $n,m \in \mathbb{Z}$ then $\begin{cases} (5n,0) + (5m,0) = (5(n+m),0) \\ -(5n,0) = (5(-n),0) \end{cases}$

$\Rightarrow S$ is a subgroup of $\mathbb{Z} \times \mathbb{Z}$

For all $a,b,n \in \mathbb{Z}$, $(a,b) \cdot (5n,0) = (5(an),0) = (5n,0) \cdot (a,b)$

Since $(5(an),0) \in S \Rightarrow (a,b) \cdot (5n,0) \in S$

$\Rightarrow S$ is an ideal of $\mathbb{Z} \times \mathbb{Z}$

(c) No

Let $S = \{(n,m): n+m \text{ is even}\}$ and take $(1,0) \in \mathbb{Z} \times \mathbb{Z}$

Take $(n,m) = (3,5)$ where $3+5 = 8 = \text{even} \in S$

$(1,0) \cdot (3,5) = (3,0)$ where $3+0 = 3 = \text{odd} \Rightarrow (3,0) \notin S$

(d) No

Let $S = \{(n,m): nm \text{ is even}\}$ and take $(1,0) \in \mathbb{Z} \times \mathbb{Z}$

Take $(3,4) = (n_1,m_1)$ and $(0,1) = (n_2,m_2)$ where $(n_1,m_1) \in S$ and $(n_2,m_2) \in S$

$(3,4) - (0,1) = (3,3)$ where $3 \times 3 = 9 = \text{odd} \Rightarrow (3,3) \notin S$

(e) Yes

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Take $a = (2n_1, 3m_1)$ and $b = (2n_2, 3m_2)$ where $(2n_1,3m_1) \in S$ and $(2n_2,3m_2) \in S$

$a - b = (2n_1,3m_1) - (2n_2,3m_2) = \big(2(n_1-n_2), 3(m_1-m_2)\big) \in S$

$ab = (2n_1,3m_1)(2n_2,3m_2) = \big(2(2n_1n_2), 3(3m_1m_2)\big) \in S$

Take $t = (t_1,t_2) \in \mathbb{Z} \times \mathbb{Z}$

$ta = (t_1,t_2)(2n_1,3m_1) = (2t_1n_1, 3t_2m_1) \in S$

$\Rightarrow S$ is an ideal of $\mathbb{Z} \times \mathbb{Z}$

Problem 18.B.5

Let $f: \mathbb{R} \to \mathbb{R}$ such that $f(x) = \begin{cases} 1, x > 0 \\ 0, x \le 0 \end{cases}$ and let $g: \mathbb{R} \to \mathbb{R}$ such that $g(x) = 2x$

Note that $f(x) \in \mathcal{F}(\mathbb{R})$ and $g(x) \in \mathcal{C}(\mathbb{R})$

$(g \circ f)(x) = g(f(x)) = \begin{cases} (2)\cdot(1), x > 0 \\ (2)\cdot(0), x \le 0 \end{cases} \Rightarrow g(f(x)) = \begin{cases} 2, x > 0 \\ 0, x \le 0 \end{cases}$

$\Rightarrow (g \circ f)(x)$ is not a continuous function

$\Rightarrow \mathcal{C}(\mathbb{R})$ is not an ideal of $\mathcal{F}(\mathbb{R})$

Problem 18.D.3

Let $\{I_i\}$ be the ideals of A and let $a,b \in I = \cap I_i$, then $a,b \in I_i \ \forall i$

Since I_i is the ideal of A thus $a - b \in I_i \Rightarrow a - b \in \cap I_i = I \Rightarrow a - b \in I$

Let $a \in I$ and $x \in A \Rightarrow a \in I_i \ \forall i$

Since I_i is the ideal of A thus $\begin{cases} ax \in I_i \ \forall i \Rightarrow ax \in \cap I_i = I \Rightarrow ax \in I \\ xa \in I_i \ \forall i \Rightarrow xa \in \cap I_i = I \Rightarrow xa \in I \end{cases}$

$\Rightarrow I$ is an ideal of A $\Rightarrow$ The intersection of any two ideals of A is an ideal of A

Question D3

Let I_1 and I_2 be ideals. Let $I = I_1 \cap I_2$. Take $x, y \in I$. $x, y \in I_1$ and $x, y \in I_2$. $x - y \in I_1$, $x - y \in I_2$. Thus $x - y \in I_1 \cap I_2$. Let $r \in R$. $rx \in I_1$ and $rx \in I_2$, $rx \in I_1 \cap I_2$.

Question D4

$\forall r \in A$. $1r \in J$ because J is an ideal. $1r = r$, hence $\forall r \in A$, $r \in J$ → $A \subseteq J$

need to prove $J \subseteq A$ $A \subseteq J$ when   2.5/3   and also $J \subseteq A$

need to write them

Question G2

Injective: if $f(a+bi) = f(c+di)$ then $\begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \begin{pmatrix} c & d \\ -d & c \end{pmatrix}$ which implies $a = c$ and $b = d$, hence $a + bi = c + di$.

Surjective: $\forall s \in \mathscr{S}$ $s = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ we always have $a + bi$ such that $f(a+bi) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$.

Homomorphic: $f(a+bi) + f(c+di) = \begin{pmatrix} a+c & b+d \\ -b-d & a+c \end{pmatrix} = f(a + bi + c + di)$

Question G5

$\mathbb{Z}$ contains an non-zero element 1 whose square equals itself, however $\nvDash\mathbb{Z}$ does not.

Suppose $2\mathbb{Z} \cong 3\mathbb{Z}$ there exists isomorphism $f : 2\mathbb{Z} \to 3\mathbb{Z}$. $f(2) = 3n$ for some $n \in \mathbb{Z}$. However, $f(4) = f(2+2) = f(2) + f(2) = 6n$ while $f(4) = f(2 \cdot 2) = f(2) \cdot f(2) = 9n^2$. $6n = 9n^2$ as $n \in \mathbb{Z}$, $n = 0$. We also know that if f is an isomorphism, $f(0) = 0$. So f is not injective. A contradiction.

Suppose $k\mathbb{Z} \cong l\mathbb{Z}$ there exists isomorphism $f : k\mathbb{Z} \to l\mathbb{Z}$. $f(k) = ln$ for some $n \in \mathbb{Z}$. $f(k^2) = f(k)f(k) = l^2n^2$. However, $f(k^2) = kf(k) = kln$, $l^2n^2 = kln$ then $k = ln$. However if $k = ln$, $f(k) = k$. $pk \in k\mathbb{Z}$ for all $p \in \mathbb{Z}$. Since f is homomorphic. $f(pk) = pf(k) = pk$. Hence $f(pk) = pk$ for all elements $pk \in k\mathbb{Z}$. Then $k = l$. A contradiction.

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Question H2

I_a is closed under subtraction. $(ax_1 + j_1 + k_1) - (ax_2 + j_2 + k_2) = a(x_1 - x_2) + (j_1 - j_2) + (k_1 - k_2)$.

I_a absorbs products. $b \in A$ then $b(ax + j + k) = abx + bj + bk$, $bx \in A$, $bj \in J$ and $bk \in K$