!!! WRITE YOUR NAME, STUDENT ID. BELOW !!!

NAME :

PURDUE ID :

$(x*y)*z = (2(x+y))*z$
$= 2(2(x+y)+z)$
$= 4x+4y+2z$

$x*(y*z) = x*(2(y+z))$
$= 2(x+2(y+z))$
$= 2x+4y+4z$

(1) Define the following operation on $\mathbb{R}$. $x * y = 2(x+y)$. Which is the following statement is true for this operation

(a) associative, commutative, there is an identity element.
(b) associative, commutative, there is no identity element.
(c) not associative, commutative, there is an identity element.
$\boxed{\text{(d)}}$ not associative, commutative, there is no identity element.

(2) Which of the following operations defines a group structure on $\mathbb{R}$:

(a) $x * y = xy + 1$.
$\boxed{\text{(b)}}$ $x * y = x + y + 100$.
(c) $x * y = -x - y$.

(3) Suppose G is a non-abelian group. Solve the following equations in G (a, b, c are different elements in G with no relations):

$$x^3 a = b, \quad x^4 = c.$$

$\boxed{\text{(a)}}$ $x = ab^{-1}c$.

$x^3 = ba^{-1} \Rightarrow x^{-3} = a \cdot b^{-1} \overset{x^4=c}{\Longrightarrow} x = ab^{-1}c$

(b) $x = a^{-1}bc$.
(c) $x = b^{-1}ac$.

(4) Suppose we have two functions $f : A \to B$ and $g : B \to C$. Which of the following statement is true?

(a) If $g \circ f$ is injective, then g is injective.

$\boxed{\text{(b)}}$ If $g \circ f$ is surjective, then g is surjective.

(c) If $g \circ f$ is bijective, then both f and g are bijective.

(5) Let S_A denote the permutation group on the set A. Which of the following permutations $f \in S_A$ is **NOT** of finite order:

(a) $A = \mathbb{Z}_{10}$, $f(x) = x + \bar{2}$.
(b) $A = \mathbb{R} - \{1\}$, $f(x) = \frac{x}{x-1}$. $f^2(x) = f \circ f(x) = \frac{\frac{x}{x-1}}{\frac{x}{x-1} - 1} = \frac{\frac{x}{x-1}}{\frac{1}{x-1}} = x = id(x) \Rightarrow ord(f) = 2$
$\boxed{\text{(c)}}$ $A = \mathbb{R} - \{0\}$, $f(x) = x^{1/3}$.

(6) What are the products of cycles: $(1234567)^4$ and $(123)(234)(456)(567)$.

(a) (1473625), (1234567).
(b) (1642753), (1234567).
$\boxed{\text{(c)}}$ (1526374), (12)(345)(67).

(7) Which of the following two groups are **NOT** isomorphic:
$\boxed{\text{(a)}}$ $\mathbb{Z}_2 \times \mathbb{Z}_4$, and $\mathbb{Z}_8$.
(b) S_3, and the symmetry group of equilateral triangle.
(c) $(\mathbb{Z}_5^*, \times)$, and $(\mathbb{Z}_4, +)$.
$\{\bar1, \bar2, \bar3, \bar4\} = \langle \bar2 \rangle = \{\bar1, \bar2, \bar4, \bar8\}$, $\bar8 = \bar3$

(8) Let G be an abelian group. Which of the follow statement is true:
by exclusion $\boxed{\text{(a)}}$ If $\text{ord}(a) = 2$ and $\text{ord}(b) = 5$, then $\text{ord}(ab) = 10$.
(b) If $\text{ord}(a) = 10$, then $\text{ord}(a^6) = 10$. $ord(a^6) = 5$
(c) If $\text{ord}(a) = 10$, then $\text{ord}(a^k) = 10$ if k is odd. $ord(a^5) = 2$

$(ab)^k = e \Leftrightarrow a^k = b^{-k} \Rightarrow \begin{cases} a^{k \cdot 5} = b^{-5k} = e \\ b^{-k \cdot 2} = a^{2 \cdot k} = e \end{cases} \Rightarrow \begin{cases} 2 \mid 5k \\ 5 \mid 2k \end{cases} \Rightarrow \begin{cases} 2 \mid k \\ 5 \mid k \end{cases} \Rightarrow 10 \mid k$

$(ab)^{10} = (a^2)^5 (b^5)^2 = e.$

(9) Which of the following is **NOT** an isomorphism of groups?
$\boxed{\text{(a)}}$ $\mathbb{Z}_{20}/\langle \bar{4} \rangle \cong \mathbb{Z}_5$ $|\langle \bar4 \rangle| = |\{\bar0, \bar4, \bar8, \overline{12}, \overline{16}\}| = 5$. $\left| \frac{\mathbb{Z}_{20}}{\langle \bar4 \rangle} \right| = \frac{20}{5} = 4$, $|\mathbb{Z}_5| = 5$.
(b) $(\mathbb{Z} \times \mathbb{Z})/\langle (1,1) \rangle \cong \mathbb{Z}$.
(c) $S_3/\langle (123) \rangle \cong \mathbb{Z}_2$.
$\boxed{\text{(d)}}$ $S_4/\langle (1234) \rangle \cong \mathbb{Z}_6$. $\langle (1234) \rangle = \{e, (1234), (13)(24), (1432)\}$ is not normal in S_4

$(12)\,(13)(24)\,(12)^{-1} = (14)(23) \notin \langle (1234) \rangle$

(b) $\mathbb{Z} \times \mathbb{Z} \xrightarrow{f} \mathbb{Z}$, $\ker(f) \cong \mathbb{Z}$
$(a, b) \longmapsto a - b$ $\quad \ker(f) = \langle (1,1) \rangle$

$\mathbb{Z} \times \mathbb{Z} / \langle (1,1) \rangle \cong \mathbb{Z}$ by (FTH).

(10) Which of the following H is **NOT** a normal subgroup of G?

(a) $G = \mathbb{Z}_2 \times \mathbb{Z}_4$, $H = \langle (1,1) \rangle$.

(b) G is the symmetry group of a square. H is the cyclic subgroup generated by a reflection.

(c) G is any group. $H = \{x; xy = yx \text{ for any } y \in G\}$.

(d) $G = S_{10}$ and $H = A_{10} = \{ \text{ all even permutations } \}$.

(11) Which of the following statements is **NOT** true (up to isomorphism):

(a) There is no non-abelian group of order 4.

(b) There is only one group of order 5.

(c) There is only one abelian group of order 4. $\mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2$

(12) In the product ring $\mathbb{Z} \times \mathbb{Z}$, which of the following subsets are ideals?

$$I_1 = \{(2m, 3m); m \in \mathbb{Z}\}$$

$(2,3)\cdot(1,0) = (2,0) \notin I_1$ does not absorb products.

$$I_2 = \{(2m, 3n); m, n \in \mathbb{Z}\}$$
$$I_3 = \{(m, 0); m \in \mathbb{Z}\}$$

(a) I_1 and I_3 are ideals.

(b) I_2 and I_3 are ideals.

(c) I_1, I_2 and I_3 are all ideals.

(13) In the ring $\mathbb{Z}$, consider the subset

$$I = \{40m + 24n; m, n \in \mathbb{Z}\}.$$

Which of the following statements is true:

(a) I is a principal ideal generated by 4.

(b) I is a principal ideal generated by 8. $\gcd(40, 24) = 8$.

(c) I is a principal ideal generated by 120.

(d) I is not a principal ideal

(14) In the ring $\mathbb{Z}$, consider the subset

$$I = \{n;\, 40|\, n \text{ and } 24|\, n\}.$$

Which of the following statements is true:

(a) I is a principal ideal generated by 4.
(b) I is a principal ideal generated by 8.
$\boxed{\text{(c)}}$ I is a principal ideal generated by 120. $\operatorname{lcm}(40, 24) = 120$
(d) I is not a principal ideal

(15) Consider the following rings

$$A_1 = \mathbb{Z}_9, \quad A_2 = \mathbb{Z}_2 \times \mathbb{Z}_3, \quad A_3 = \mathbb{Z}, \quad A_4 = \mathbb{Q}.$$

Which rings are integral domains (list all)?
(a) Only A_1 and A_4.
(b) Only A_2 and A_3.
$\boxed{\text{(c)}}$ Only A_3 and A_4.
(d) Only A_2, A_3 and A_4.

(16) Which of the following statements is **NOT** true?
(a) $\mathbb{Z}_{37}$ is an integral domain
$\boxed{\text{(b)}}$ $\mathbb{Z}_{35}$ is a field $35 = 5 \times 7$ $\bar{5} \cdot \bar{7} = \bar{0} \Rightarrow \bar{5}, \bar{7}$ not invertible
(c) Any finite field is an integral domain
(d) Any finite integral domain is a field

(17) Which of the following map $f : A \to B$ is a homomorphism of RINGS?
(a) $A = \mathbb{Z}$, $B = 2\mathbb{Z}$, $f(x) = 2x$.
(b) $A = \mathbb{R}$, $f(x) = x^2$.
$\boxed{\text{(c)}}$ $A = \mathbb{Z}_5$, $f(x) = x^5$. $(x+y)^5 = x^5 + y^5$ because 5 is prime
(d) $A = \mathbb{Z}_4$, $f(x) = x^4$. $\Leftrightarrow \begin{cases} (\bar{1}+\bar{1})^4 = \bar{2}^4 = \overline{16} = \bar{0} \\ \bar{1}^4 + \bar{1}^4 = \bar{1} + \bar{1} = \bar{2} \end{cases}$

(18) Let A be a finite integral domain with characteristic 2. Which of the following statements is **NOT** true:

(a) $8a = 0$ for every $a \in A$.

(b) $(a+b)^4 = a^4 + b^4$ for every $a, b \in A$. $(a+b)^2 = a^2+b^2$, $(a+b)^4 = (a^2+b^2)^2 = a^4+b^4$, $\cdots$

(c) There can be no nonzero element a such that $5a = 0$. $\cdots$ $\underline{(a+b)^{2^k} = a^{2^k}+b^{2^k}}$

by exclusion $\boxed{\text{(d)}}$ There is such a ring A with only 6 elements. (finite integral domain is a finite field. Any finite field has order p^k, $k \geq 1$. ($p=2$ here))

(19) What are the **complete** solutions to the following two Diophantine equation:

$$\textbf{(i)}\, 16x \equiv 32 \pmod{28}, \quad \textbf{(ii)}\, 16x \equiv 30 \pmod{28}.$$

(a) **(i)** $x \equiv 2 \pmod{14}$, **(ii)** no solutions

(b) **(i)** no solutions, **(ii)** $x \equiv 2 \pmod{14}$

$\boxed{\text{(c)}}$ **(i)** $x \equiv 2 \pmod{7}$, **(ii)** no solutions ($\gcd(16,28)=4$. $4 \nmid 30$)

(d) **(i)** $x \equiv 2 \pmod{14}$, **(ii)** $x \equiv 2 \pmod{14}$

(20) What are the **complete** solutions of the following Diophantine equation:

$$16(x+1)^2 \equiv 32 \pmod{28}.$$

$\boxed{\text{(a)}}$ $x \equiv 2 \pmod{7}$ and $x \equiv 3 \pmod{7}$.

(b) $x \equiv 2 \pmod{14}$ and $x \equiv 3 \pmod{14}$.

(c) $x \equiv 3 \pmod{7}$ and $x \equiv 4 \pmod{7}$.

(d) $x \equiv 3 \pmod{14}$ and $x \equiv 4 \pmod{14}$.

- $\gcd(16,28)=4$ $\quad 4 \mid 32$
- $\frac{16}{4}(x+1)^2 \equiv \frac{32}{4} \left(\bmod \frac{28}{4}\right) \Leftrightarrow 4(x+1)^2 \equiv 8 \pmod{7}$

$\Leftrightarrow \overline{(x+1)^2} = \bar{4}^{-1} \cdot \bar{8} = \bar{2} \cdot \bar{1} = \bar{2}$ in $\mathbb{Z}_7$

$\Leftrightarrow \overline{x+1} = \bar{x} + \bar{1} = \bar{3}$ or $\bar{4}$ in $\mathbb{Z}_7$

$\Leftrightarrow \bar{x} = \bar{2}$ or $\bar{3}$ in $\mathbb{Z}_7$

$\Leftrightarrow x \equiv 2 \pmod{7}$ or (and) $x \equiv 3 \pmod{7}$

$$\begin{array}{lccccccc} a: & \bar{0} & \bar{1} & \bar{2} & \bar{3} & \bar{4} & \bar{5} & \bar{6} \\ a^2: & \bar{0} & \bar{1} & \bar{4} & \bar{9} & \overline{16} & \overline{25} & \overline{36} \\ & & & & \shortparallel & \shortparallel & \shortparallel & \shortparallel \\ & & & & \bar{2} & \bar{2} & \bar{4} & \bar{1} \end{array}$$

(21) Let $a(x) \in \mathbb{R}[x]$ and $b(x) \in \mathbb{R}[x]$ be defined as:

$$a(x) = x^3 - 2x + 2, \quad b(x) = x - 1.$$

What is the quotient $q(x)$ and remainder $r(x)$ when $a(x)$ is divided by $b(x)$?

(a) $q(x) = x^2 + x,$ $\quad r(x) = -x + 2.$

$\boxed{\text{(b)}}$ $q(x) = x^2 + x - 1,$ $\quad r(x) = 1.$

(c) $q(x) = x^2 + x,$ $\quad r(x) = x - 2.$

(d) $q(x) = x^2 - x + 1,$ $\quad r(x) = -1.$

(22) Let $a(x) \in \mathbb{Z}_5[x]$ and $b(x) \in \mathbb{Z}_5[x]$ be defined as:

$$a(x) = x^3 + \bar{3}x + \bar{2}, \quad b(x) = x + \bar{4}.$$

What is the quotient $q(x)$ and remainder $r(x)$ when $a(x)$ is divided by $b(x)$?

$\boxed{\text{(a)}}$ $q(x) = x^2 + x + \bar{4},$ $\quad r(x) = \bar{1}.$

(b) $q(x) = x^2 + x,$ $\quad r(x) = \bar{4}x + \bar{2}.$

(c) $q(x) = x^2 + x,$ $\quad r(x) = x + \bar{3}.$

(d) $q(x) = x^2 + \bar{4}x + 1,$ $\quad r(x) = \bar{3}.$

$$
\begin{array}{r|l}
 & x^2 + x + \bar{4} \\
\hline
x+\bar{4} & x^3 + 0 \cdot x^2 + \bar{3}x + \bar{2} \\
 & x^3 + \bar{4}x^2 \\
\hline
 & \bar{1}x^2 + \bar{3}x \\
 & x^2 + \bar{4}x \\
\hline
 & \bar{4}x + \bar{2} \\
 & \bar{4}x + \overline{16} \\
\hline
 & -\overline{14} = \bar{1}
\end{array}
$$

$\Rightarrow q(x) = x^2 + x + \bar{4}, \ r(x) = \bar{1}.$