Chap 12: Partitions and equivalence relations

<u>Def</u>: A partition of a set A is a family $\{A_i;\ i\in I\}$ of nonempty subsets of A which are mutually disjoint and whose union is all of A.

More explicitly: (i) $A_i \neq \phi$, $i\in I$. $A_i \cap A_j \neq \phi \Rightarrow A_i = A_j$

(ii) $A = \bigcup_i A_i$: $\forall x\in A, \exists i\in I$ s.t. $x\in A_i$.

<u>Def</u>: A relation on a set A is any statement which is either true or false for each ordered pair (x,y) of elements of A.

<u>Ex of relation</u>: "$x=y$", "$x<y$", "x is parallel to y". "x is the offspring of y" ...

<u>Def</u>: An equivalence relation on a set A is a relation that is:

(i) Reflexive: $x\sim x\ \ \forall x\in A$

(ii) Symmetric: $x\sim y \Rightarrow y\sim x$

(iii) Transitive: $\left.\begin{matrix} x\sim y \\ y\sim z\end{matrix}\right\} \Rightarrow x\sim z$.

partition $\{A_i, i\in I\}$ $\rightsquigarrow$ equivalence relation: $x\sim y$ iff $\exists i\in I$ s.t. $x\in A_i$ and $y\in A_i$.

partition $\{[x]\}_{x\in A}$ $\leftsquigarrow$ equivalence relation

where $[x] = \{y\in A;\ y\sim x\}$.

Exercise: A.2: $n\in\mathbb{Z}$, $A_n = \{x\in\mathbb{Q}:\ n\leq x<n+1\}$.

(i) $A_n\neq\phi$. $A_n\cap A_m\neq\phi \Rightarrow x\in A_n\cap A_m$ $\begin{matrix} n\leq x<n+1 \\ m\leq x\leq m+1\end{matrix} \Rightarrow m=n$

$\Rightarrow A_m = A_n$

(ii) $\forall x\in\mathbb{Q}$, $x\in A_n$ where $n\leq x<n+1$.

B.5. $a \sim b$ iff $a - b \in \mathbb{Q}$

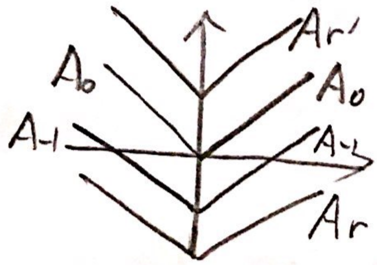
(i). $a - a = 0 \in \mathbb{Q} \Rightarrow a \sim a$

(ii) $a \sim b \Rightarrow a - b \in \mathbb{Q} \Rightarrow b - a \in \mathbb{Q} \Rightarrow b \sim a$

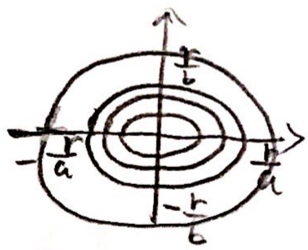
(iii) $\begin{matrix} a \sim b \\ b \sim c \end{matrix} \Rightarrow \begin{matrix} a - b \in \mathbb{Q} \\ b - c \in \mathbb{Q} \end{matrix} \Rightarrow a - c = (a-b) + (b-c) \in \mathbb{Q} \Rightarrow a \sim c.$

So $\sim$ is an equivalence relation

C. 3: $\forall r \in \mathbb{R}$, $A_r = \{(x,y) : y = |x| + r\}$



C.4: $(x,y) \sim (u,v)$ iff $ax^2 + by^2 = au^2 + bv^2$ (where $a, b > 0$).



$(x,y) \sim (u,v)$ iff (x,y) and (u,v) are on the same ellipse given by $ax^2 + by^2 = r$, $r \in \mathbb{R}$.

D.4. $a \sim b$ iff $\exists$ an integer k s.t. $a^k = b^k$.

(i) $a = a \Rightarrow a \sim a$

(ii) $a \sim b \Rightarrow \exists k$ s.t. $\begin{matrix} a^k = b^k \\ b^k = a^k \end{matrix} \Rightarrow b \sim a$

(iii) $a \sim b \Rightarrow \exists k$ s.t. $a^k = b^k$ $\Rightarrow a^{kl} = b^{kl} = (b^l)^k = (c^l)^k = c^{kl}$

$b \sim c \Rightarrow \exists l$ s.t. $b^l = c^l$

$\Downarrow$

$a \sim c.$

E.4: $\{B_i : i \in I\}$ is a partion of B. $f: A \to B$ is a function.

$\{f^{-1}(B_i) : i \in I\}$.

(i) $\exists\, x \in A$, s.t. $x \in f^{-1}(B_i) \cap f^{-1}(B_j) \Rightarrow f(x) \in B_i \cap B_j \Rightarrow B_i \cap B_j \neq \phi$

$\neg f^{-1}(B_i) \cap f^{-1}(B_j) \Leftarrow i = j$

(ii) $\forall\, x \in A,\ f(x) \in B \Rightarrow \exists\, i \in I$ s.t. $f(x) \in B_i \Rightarrow x \in f^{-1}(B_i)$