Lecture 14

1. Groups acting on sets

<u>Defn</u> Let G be a gp and X a set. An action of G on X is a gp homomorphism

$$\varphi : G \longrightarrow S_X$$

where S_X is the gp (under composition) of bijections of X to itself.

If the context is clear then we will abbreviate for $g \in G$, $x \in X$

$\varphi(g)(x)$ by $g \cdot x$.

<u>Remark</u> From the fact that φ is a gp. homomorphism it follows that the action $\varphi : G \to S_X$ has the properties

(1) $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$ for all $g_1, g_2 \in G$, $x \in X$.

(2) $e \cdot x = x$ for all $x \in X$.

(3) $g \circ (g^{-1} \circ x) = e \cdot x = x$ for all $g \in G, x \in X$

2. Examples.

Eg. (1) A group G acting on itself by

(a) Left action.

Here $X = G$ and $g \cdot x = gx$.

(b) By conjugation

Here $X = G$ and $g \cdot x = g x g^{-1}$.

(2) $G = GL(n, \mathbb{R})$ acting on $\mathbb{R}^n$ by left multiplication.

(3) $G = S_n$ acting on the set $X = \{1, \dots, n\}$ by the "permutation" action.

3. Orbits and stabilizers.

Let $\varphi : G \to S_X$ be a fixed action.

For $x \in X$, the orbit$(x) = \{ y \mid \exists g \in G,\ g \cdot x = y \}$

This is a subset of X.

The stabilizer subgroup of x, $G_x = \{ g \in G \mid g \cdot x = x \}$

This is a subgroup of G.

(4). The orbit - stabilizer formula.

Thm There is a bijection between the orbit(x) and the set of left cosets of G_x.

Pf: Let $x \in X$. Define the map:

$F :$ orbit$(x) \longrightarrow \{$left cosets of $G_x\}$ by

$F(g \cdot x) = g G_x$.

Need to prove first that this map is well defined.

Suppose $g \cdot x = g' \cdot x$. Then

$x = g^{-1} g' \cdot x \Rightarrow g^{-1} \cdot g' \in G_x$

$\Rightarrow g' \in g G_x \Rightarrow g G_x = g' G_x$.

So the map F is well defined.

Clearly ~~the~~ F is surjective.

To show F is injective, let

$F(g \cdot x) = F(g' \cdot x)$

Then $gG_x = g'G_x \Rightarrow g^{-1}g'G_x = G_x$

$\Rightarrow g^{-1}g' \in G_x$.

This implies that $g^{-1}g' . x = x$

$\Rightarrow g'.x = g.x$.

This proves that F is injective as well. $\square$

<u>Cor</u> If $[G:G_x]$ is finite then
$$|\text{orbit}(x)| = [G:G_x] \quad \ldots\ldots (1)$$

<u>5</u> <u>The class equation</u>

Suppose G is a finite group and consider the action of G on itself by conjugation.
Let C be a set of elements of G containing exactly one element from each orbit (conjugacy class). Then
$$|G| = |Z(G)| + \sum_{x \in C \setminus Z(G)} [G:G_x]$$
This follows from (1).