Chapter 2

Question B.2

Associative	*Commutative*	*Identity*	*Inverses*
Yes ☐ No ☒	Yes ☐ No ☒	Yes ☐ No ☒	Yes ☐ No ☒

(i) $1 * 2 = 1 + 4 - 2 = 3 \neq 2 = 2 + 2 - 2 = 2 * 1$

(Thus, * is not commutative)

(ii) $(x * y) * z = (x + 2y - xy) * z = (x + 2y - xy) + 2z - (x + 2y - xy)z =$
$x + 2y - xy + 2z - xz - 2yz + xyz$

$x * (y * z) = x + 2(y * z) - x(y * z) = x + 2(y + 2z - yz) - x(y + 2z - yz) =$
$x + 2y + 4z - 2yz - xy - 2xz + xyz$

(Thus, * is not associative)

(iii) Solve e for $x * e = x$, $x + 2e - xe = x$, $e = 0$. Solve e for $e * x = x$, $e + 2x - ex = x$, $e(1 - x) + 2x = x$, $e = \dfrac{-x}{1 - x} \neq 0$

(Thus, * does not have an identity element)

(iv) (Since * does not have an identity element, an inverse cannot exist.)

4/4

Question B.5

Associative	*Commutative*	*Identity*	*Inverses*
Yes ☐ No ☒	Yes ☒ No ☐	Yes ☐ No ☒	Yes ☐ No ☒

(i) $x * y = xy + 1 = yx + 1 = y * x$

(Thus, * is commutative)

(ii) $(x*y)*z = (xy+1)z+1 = xyz+z+1$, $x*(y*z) = x(yz+1)+1 = xyz+x+1$

(Thus, * is not associative)

(iii) Solve e for $x * e = x$, $xe + 1 = x$, $e = (x-1)/x$.

(e depends on x. Thus, * does not have an identity element)

(iv) (Since * does not have an identity element, an inverse cannot exist.)

Question C.2

*	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
(a,a)	a	a	a	a	a	a	a	a	b	b	b	b	b	b	b	b
(a,b)	a	a	a	a	b	b	b	b	a	a	a	a	b	b	b	b
(b,a)	a	a	b	b	a	a	b	b	a	a	b	b	a	a	b	b
(b,b)	a	b	a	b	a	b	a	b	a	b	a	b	a	b	a	b

Operation * 0, 1, 6, 7, 8, 9, E, F are commutative.

Chapter 3

Question A.3

Associative	*Commutative*	*Identity*	*Inverses*
Yes ☒ No ☐	Yes ☒ No ☐	Yes ☒ No ☐	Yes ☒ No ☐

(i) $x * y = x + y + xy = y + x + yx = y * x$

(Thus, * is commutative)

(ii) $(x * y) * z = (x + y + xy) * z = x + y + xy + z + xz + yz + xyz$

$x * (y * z) = x * (y + z + yz) = x + y + z + yz + xy + xz + xyz = (x * y) * z$

(Thus, * is associative)

(iii) Solve e for $x * e = x$, $x + e + xe = x$, $e + xe = 0$, $e(1 + x) = 0$, $e = 0$

Solve e for $e * x = x$, $e + x + ex = x$, $e + ex = 0$, $e = 0$

(Thus, * has an identity element 0)

(iv) Solve $x^{-1} * x = 0$, $x^{-1} + x + x^{-1}x = 0$, $x^{-1}(1 + x) = -x$, $x^{-1} = \dfrac{-x}{1 + x}$

(Since * is commutative, every x has an inverse)

Therefore this is an Abelian Group. ~~4/4~~

Question B.1

Associative	*Commutative*	*Identity*	*Inverses*
Yes ☒ No ☐	Yes ☒ No ☐	Yes ☒ No ☐	Yes ☒ No ☐

(i) $(a, b) * (c, d) = (ad + bc, bd)$, $(c, d) * (a, b) = (cb + da, bd) = (a, b) * (c, d)$

(Thus, * is commutative)

(ii) $((a, b) * (c, d)) * (e, f) = (ad + bc, bd) * (e, f) = (adf + bcf + bde, bdf)$ 4/4

$(a, b) * ((c, d) * (e, f)) = (a, b) * (cf + de, df) = (adf + bcf + bde, bdf)$

(Thus, * is associative)

(iii) Solve e for $x * e = x$, $(a, b) * (e_1, e_2) = (ae_2 + be_1, be_2)$, $e_1 = 0$, $e_2 = 1$

Solve e for $e * x = x$, $(e_1, e_2) * (a, b) = (e_1 b + e_2 a, e_2 b)$, $e_1 = 0$, $e_2 = 1$

(Thus, * has an identity element $(0, 1)$)

(iv) Solve $x * x^{-1} = (0, 1)$, $(a, b) * (a', b') = (ab' + ba', bb')$, $a' = \dfrac{-a}{b^2}, b' = \dfrac{1}{b}$

(Since * is commutative, every x has an inverse)

Therefore this is an Abelian Group.

Question B.4

Associative	*Commutative*	*Identity*	*Inverses*
Yes ☒ No ☐	Yes ☒ No ☐	Yes ☒ No ☐	Yes ☒ No ☐

(i) $(c, d) * (a, b) = (ca - db, cb + da) = (a, b) * (c, d)$

(Thus, * is commutative)

(ii) $((a, b) * (c, d)) * (e, f) = (ac - bd, ad + bc) * (e, f) = (ace - bde - adf - bcf, acf - bdf + ade + bce)$

$(a, b) * ((c, d) * (e, f)) = (a, b) * (ce - df, cf + de) = (ace - adf - bcf - bde, acf + ade + bce - bdf)$

(Thus, * is associative)

(iii) Solve e for $x * e = x$, $(a, b) * (e_1, e_2) = (ae_1 - be_2, ae_2 + be_1)$, $e_1 = 1$, $e_2 = 0$

Solve e for $e * x = x$, $(e_1, e_2) * (a, b) = (e_1 a - e_2 b, e_1 b + e_2 a)$, $e_1 = 1$, $e_2 = 0$

(Thus, * has an identity element $(1, 0)$)

(iv) Solve $x * x^{-1} = (1, 0)$, $(a, b) * (a', b') = (aa' - bb', ab' + a'b)$, $a' = \dfrac{a}{a^2 + b^2}$, $b' = \dfrac{-b}{a^2 + b^2}$

(Since * is commutative, every x has an inverse)

Therefore this is an Abelian Group.

C

$A+B=(A-B)\cup(B-A)$ $\quad P_D=\{A : A\subseteq D\}$

1. Prove: $A+\phi=(A-\phi)\cup(\phi-A)=A\cup\phi=A$

$\phi+A=(\phi-A)\cup(A-\phi)=\phi\cup A=A.$ $\Rightarrow \boxed{\phi}$ is identity element

✓ 2/2

2. $A+A=(A-A)\cup(A-A)=\phi\cup\phi=\phi$ for all $A\in P_D$ $\Rightarrow \boxed{A}$ is inverse of A

2/2

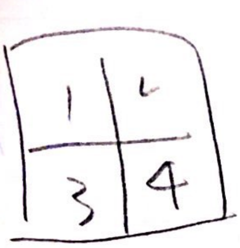
3. $D=\{a,b,c\}$ $\quad P_D=\{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\}\}$

operation table for $\langle P_D, +\rangle$ $\quad \{a,b,c\}=D.$

$+$	ϕ	$\{a\}$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{b,c\}$	$\{a,c\}$	$\{a,b,c\}$
ϕ	ϕ	$\{a\}$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{b,c\}$	$\{a,c\}$	D
$\{a\}$	$\{a\}$	ϕ	$\{a,b\}$	$\{a,c\}$	$\{b\}$	D	$\{c\}$	$\{b,c\}$
$\{b\}$	$\{b\}$	$\{a,b\}$	ϕ	$\{b,c\}$	$\{a\}$	$\{c\}$	D	$\{a,c\}$
$\{c\}$	$\{c\}$	$\{a,c\}$	$\{b,c\}$	ϕ	D	$\{b\}$	$\{a\}$	$\{a,b\}$
$\{a,b\}$	$\{a,b\}$	$\{b\}$	$\{a\}$	D	ϕ	$\{a,c\}$	$\{b,c\}$	$\{c\}$
$\{b,c\}$	$\{b,c\}$	D	$\{c\}$	$\{b\}$	$\{a,c\}$	ϕ	$\{a,b\}$	$\{a\}$
$\{a,c\}$	$\{a,c\}$	$\{c\}$	D	$\{a\}$	$\{b,c\}$	$\{a,b\}$	ϕ	$\{b\}$
$\{a,b,c\}$	D	$\{b,c\}$	$\{a,c\}$	$\{a,b\}$	$\{c\}$	$\{a\}$	$\{b\}$	ϕ

4/4

D



*	I	V	H	D
I	I	V	H	D
V	V	I	D	H
H	H	D	I	V
D	D	H	V	I

$G = \{V, H, D, I\}$

$\langle G, * \rangle$ is a group.

(1) associativity is granted $(a*b)*c = a*(b*c)$.

(2) I is the identity

(3) inverse of each element is itself.

4/4

$\Rightarrow \langle G, * \rangle$ is a group