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1. communicative: yes

$(a.b)(c.d) = (ac, ad+bc) = (ca, cb+da) = (c.d)(a.b)$

associative: yes

$((a.b)(c.d))(e.f) = (ac. ad+bc)(e.f) = (ace, acf + (ad+bc)e) = (ace, acf + ade + bce)$

$(a.b)((c.d)(e.f)) = (a.b)(ce, cf+de) = (ace, bce + a(cf+de)) = (ace, bce + acf + ade))$

identity: yes

$(x.y)(a.b) = (a.b)$

$(ax, bx+ay) = (a.b)$

$\begin{cases} ax = a \\ bx+ay = b \end{cases}$

$ax = a \quad \because a \neq 0 \quad x = 1$

$b + ay = b \quad ay = 0 \quad \because a \neq 0 \quad y = 0$

$\therefore \forall (a.b) \in G, \quad (1.0)(a.b) = (a.b)$

$\because$ it is communicative, $(a.b)(1.0) = (a.b)$

inverse: yes

$(x.y)(a.b) = (1.0)$

$(ax, bx+ay) = (1.0)$

$\begin{cases} ax = 1 \\ bx + ay = 0 \end{cases}$

$\because a \neq 0, \ x = \frac{1}{a}$

$\frac{b}{a} + ay = 0 \quad ay = -\frac{b}{a} \quad \because a \neq 0, \ y = -\frac{b}{a^2}$

thus G is an abelian group.

if a could be 0, for identity

$(x.y)(0.b) = (0.b)$

$(0x, bx+0y) = (0.b)$

$\begin{cases} 0x = 0 \\ bx + 0y = b \end{cases}$

(1.0) still works for identity

for inverse:

$(x.y)(0.b) = (1.0)$

$(0x, bx+0y) = (1.0)$

$(0, bx) = (1.0)$

where $1 = 0$, which is impossible. inverse does not work, G is not a group.

2. $x^2 a = b \quad x^5 = a$

$x^2 = ba^{-1}$

$x^5 = a$

$x^2 x^2 x = a$

$ba^{-1}ba^{-1}x = a$

$x = (ba^{-1}ba^{-1})^{-1} a$

$= ab^{-1}ab^{-1}a$

$x^2 a = b \qquad x^5 = a \qquad ab \neq ba$

$x^2 a a^{-1} = b a^{-1}$

$x^2 = b a^{-1}$

$x^2 x^2 = b a^{-1} b a^{-1} = x^4$

$x^{-4} = (b a^{-1} b a^{-1})^{-1}$

$\cancel{x^4}$

$x^5 (x^{-4}) = a(b a^{-1} b a^{-1})^{-1}$

$x = a(a b^{-1} a b^{-1})$

$= a^2 b^{-1} a b^{-1}$

2. $x^2 a = b.$ ①

$x^5 = a.$ ②

$a.b \in G.$ Solve x.

from ①. $x^2 = b a^{-1}$

$x^6 = b a^{-1} b a^{-1} b a^{-1}$

from ② $x \cdot a = b a^{-1} b a^{-1} b a^{-1}$

$x = b a^{-1} b a^{-1} b a^{-1} a^{-1}.$