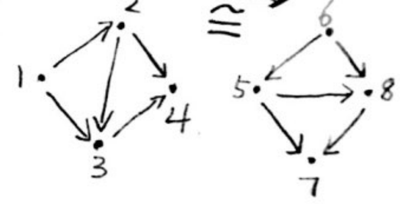
Chap 9: Isomorphism : same structure in different guises

Examples of isomorphisms : Congruent, similar (Geometry)

$1 \to 6$, $2 \to 5$, $3 \to 8$, $4 \to 7$ $\cong$



$$MADAM \xrightarrow{\cong} ROTOR$$

$M \to R$
$A \to O$
$D \to T$

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$\cong$

$0 \leftrightarrow e$
$1 \leftrightarrow a$
$2 \leftrightarrow b$

*	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

<u>Def</u>: If G_1 and G_2 are any groups, an isomorphism from G_1 to G_2 is a 1-to-1 correspondence f from G_1 to G_2 with the following property:

1-to-1 $\Updownarrow$ bijection

If $f(a) = a'$ and $f(b) = b'$, then $f(ab) = a'b' = f(a)f(b)$

If f has this property, then the table of G_1 can be transformed into the table of G_2.

In other words: G_1 and G_2 are actually the same, except that the elements of G_1 have different names from the elements of G_2.

To recognize if two groups are isomorphic:

1. make an educated guess of a function $f: G_1 \to G_2$
2. Check f is injective and surjective
3. Check f satisfies $f(ab) = f(a)f(b)$

<u>Ex</u>: $(\mathbb{R}, +) \cong (\mathbb{R}^{>0}, \cdot)$

$f: x \mapsto e^x$

To recognize when two groups are not isomorphic:

Isomorphic groups have same properties $\Longrightarrow$ if a group G_1 has a property which group G_2 does not have (or vice versa), then $G_1 \ncong G_2$.

Cayley's Theorem: Every group is isomorphic to a group of permutations.

<u>Pf</u>: $G \longrightarrow S_G$

$a \longmapsto \pi_a : G \to G$

$\pi_a(x) = ax$.

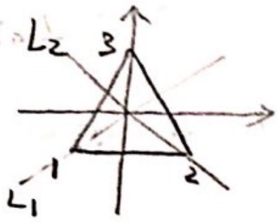
<u>Ex</u>:

B.1. $f: G_1 \to G_2$ an isomorphism. e_1 is the identity element of G_1

for any element $y \in G_2$. $f(f^{-1}(y)) = y$ and

$$y \cdot f(e_1) = f(f^{-1}(y)) f(e_1) = f(f^{-1}(y) e_1) = f(f^{-1}(y)) = y$$

So $f(e_1)$ is the identity element of G_2



C.3 $S_3 \cong D_3$

	ε = (1)	α = (12)	β = (123)	β^2 = (132)	$\alpha\beta$ = (23)	$\alpha\beta^2$ = (13)
(1)	(1)	(12)	(123)	(132)	(23)	(13)
(12)	(12)	(1)	(23)	(13)	(123)	(132)
(123)	(123)					
(132)	(132)					
(23)	(23)					
(13)	(13)					

$\alpha = (12)$: reflection by y-axis

$\beta = (123)$: rotation by $120°$ counter-clockwise

$\alpha\beta = (12)(123) = (1)(23)$: reflection by L_1

$\beta^2 = (132)$: rotation by $240°$ counter-clockwise

$\alpha\beta^2 = (12)(132) = (13)(2)$: reflection by L_2

$\varepsilon = (1)$: identity

C.1

	I	H	V	D
I	I	H	V	D
H	H	I	D	V
V	V	D	I	H
D	D	V	H	I

$\cong$

	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	i
i	i	$-i$	-1	1
$-i$	$-i$	i	1	-1

$I \leftrightarrow 1$

$H \leftrightarrow -1$

$V \leftrightarrow +i$

$D \leftrightarrow -i$.

D.3. $P_3 = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$

$A+B=(A-B)\cup(B-A)$ is commutative $\Longrightarrow P_3 \not\cong D_4$

$P_3 = \langle \underset{\alpha}{\underset{\shortparallel}{\{a\}}}, \underset{\beta}{\underset{\shortparallel}{\{b\}}}, \underset{\gamma}{\underset{\shortparallel}{\{c\}}} \mid \alpha^2=e,\ \beta^2=e,\ \gamma^2=e,\ \alpha\beta=\beta\alpha,\ \alpha\gamma=\gamma\alpha,\ \beta\gamma=\gamma\beta \rangle$

$\cong \mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$

In $\mathbb{Z}_8$, $\bar{1}^{-1}=\bar{7}\neq\bar{1}$. In $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$, $x=x^{-1}$ for any element

so $\mathbb{Z}_8\not\cong\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$

E1: $\mathbb{Z}\cong 2\mathbb{Z}=E$ $\quad a\mapsto 2a$.

E5: $\mathbb{Z}\not\cong\mathbb{Q}$. For $1\in\mathbb{Z}$, there is no element $x\in\mathbb{Z}$ s.t. $x+x=1$.

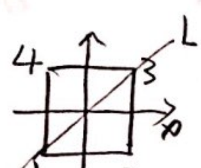
But any element $a\in\mathbb{Q}$ can be written as $a=\frac{a}{2}+\frac{a}{2}$ with $\frac{a}{2}\in\mathbb{Q}$.

F4: $G=\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$; $G'=\{e,a,b,c,ab,ac,bc,abc\}$

$=\langle a,b,c;\ a^2=b^2=c^2=e,\ (ab)^2=(bc)^2=(ac)^2=e\}$

$a\mapsto(1,0,0)$, $b\mapsto(0,1,0)$, $c\mapsto(0,0,1)$.

F1: $G=\langle(24),(1234)\rangle$; $G'=\langle a,b;\ a^2=e,\ b^4=e,\ ba=ab^3\rangle$.



reflection by L

$G\cong G'\cong D_4$

rotation by 90° counterclockwise

$(12)\mapsto a$

$(1234)\mapsto b$

F3: $G''=\langle a,b;\ \underset{\Downarrow}{\underline{a^2=b^2=e}},\ (ab)^4=e\rangle$ $\qquad D_4\cong G''$

a, b are reflections $\Rightarrow$ ab is a rotation.

$a\mapsto(24)$ $\leftarrow$ reflection by L.

$b\mapsto(24)(1234)=(14)(23)$ $\leftarrow$ reflection by x-axis

$\underline{G2}$: $(\mathbb{R}, x * y = x + y + 1)$ identity element: $x + e + 1 = x \ \forall x \in \mathbb{R}$
$\Rightarrow e = -1$.

$f: \mathbb{R} \longrightarrow \mathbb{R}$
$x \longmapsto x - 1$
bijective

$$\underbrace{f(x+y)}_{= x+y-1} = \underbrace{f(x) * f(y)}_{= (x-1)*(y-1) = x-1+y-1+1 = x+y-1}$$

$\underline{H4}$: $f: G \rightarrow H$
$x \longmapsto x^{-1}$

- $f(x \cdot y) = (x \cdot y)^{-1} = y^{-1} \cdot x^{-1} = x^{-1} * y^{-1} = f(x) * f(y)$
- f is bijective $\Rightarrow f$ is an isomorphism

$\underline{I2}$: $f_1 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 2 & 4 & 1 & 3 \end{pmatrix}$. $f_1: \mathbb{Z}_5 \rightarrow \mathbb{Z}_5$, $\bar{x} \mapsto \overline{2x}$ $\qquad \underbrace{f_1^{-1}}_{= f_2}: \mathbb{Z}_5 \rightarrow \mathbb{Z}_5$, $\bar{a} \mapsto \overline{3a}$

$f_1^{-1} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 3 & 1 & 4 & 2 \end{pmatrix} = f_2$

$f_3 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 4 & 3 & 2 & 1 \end{pmatrix}$ $\qquad f_3: \mathbb{Z}_5 \rightarrow \mathbb{Z}_5$, $\bar{x} \mapsto \overline{4x}$ $\qquad \underbrace{f_3^{-1}}_{= f_3}: \mathbb{Z}_5 \rightarrow \mathbb{Z}_5$, $\bar{x} \mapsto \overline{4x}$

$\underbrace{f_3}_{= f_3^{-1}} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 4 & 3 & 2 & 1 \end{pmatrix}$

$f_3^{-1} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 4 & 3 & 2 & 1 \end{pmatrix}$

$\underline{J3}$: $G =$

	I (=1)	A (=2)	B (=3)	C (=4 = AB)	D (=5 = B^2)	K (=6 = AB^2)
$1 = I$	I	A	B	C	D	K
$2 = A$	A	I	C	B	K	D
$3 = B$	B	K	D	A	I	C
$4 = C$	C	D	K	I	A	B
$5 = D$	D	C	I	K	B	A
$6 = K$	K	B	A	D	C	I

$= \langle A, B \mid A^2 = I, B^3 = I, AB = B^2 A \rangle$

$\pi_A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 3 & 6 & 5 \end{pmatrix} = (12)(34)(56)$

$\pi_B = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 5 & 2 & 1 & 4 \end{pmatrix} = (135)(264)$

$\pi_A \pi_B = (12)(34)(56)(135)(264) = (14)(25)(36)$

$\pi_B^2 \pi_A = [(135)(264)(135)(264)][(12)(34)(56)]$
$= (153)(246)(12)(34)(56)$
$= (14)(25)(36)$

$\rho_A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 6 & 5 & 4 & 3 \end{pmatrix} = (12)(36)(45)$

$\rho_B = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 1 & 2 \end{pmatrix} = (135)(246)$

$\rho_A \circ \rho_B = \rho_{BA} = \rho_{AB^2} = \rho_{B^2} \circ \rho_A = \rho_B^2 \circ \rho_A$

$BA = B^4 A = AB^2$