MATH 453 LECTURE 2 Aug 23, 2017

1. PROPERTY OF INJECTIVE MAPS (LEFT CANCELLATION)

Given two maps $g, h : X \longrightarrow Y$ and a map $f : Y \longrightarrow Z$, if f in injective and $f \circ g = f \circ h$ then $g = h$.

How to "prove" such a statement.
Notice first the conclusion that we need to prove "$g = h$". To prove two maps equal we need to show that $g(x) = h(x)$ for every $x \in X$.
So the "proof" starts:

Proof: Let $x \in X$. . . . . .

By assumption

$$f \circ g(x) = f \circ h(x)$$

But $f \circ g(x) = f(g(x))$
$f \circ h(x) = f(h(x))$

Since f is injective

$f(g(x)) = f(h(x))$ implies that

$g(x) = h(x)$.

Thus we have shown that for all $x \in X$

$g(x) = h(x)$.

Hence, $g = h$.

$\boxed{}$

this little square indicates end of a proof.

2. <u>SURJECTIVE</u>

A map $f: X \longrightarrow Y$ is called surjective (or "onto") if for every $y \in Y$ there exists $x \in X$ s.t.

$f(x) = y$.

If f is both <u>injective</u> and <u>surjective</u> then f is called <u>bijective</u>.

3. Identity map

For any set X we will denote by $Id_X : X \longrightarrow X$ the map that takes each element of X to itself. In other ~~ese~~ words,

$$Id_X(x) = x \quad \text{for all } x \in X.$$

4. Bijective maps have inverses

If $f : X \longrightarrow Y$ is bijective then there exists a (unique) map $g : Y \rightarrow X$ s.t. $f \circ g = Id_Y$, $g \circ f = Id_X$.

Proof: (We define g first).

~~Furthermore~~ Let $y \in Y$. Since f is surjective there exists $x \in X$ s.t. $f(x) = y$. Moreover, since f is injective there ~~ex~~ is exactly one such x. Let $g(y) = x$.

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This defines the map $g : Y \to X$.

Now we check the two properties.

$f \circ g = Id_Y$:

Let $y \in Y$. Then $f \circ g(y) = f(g(y))$.

By definition $g(y)$ is the element of X s.t. $f(g(y)) = y$. This shows that

$$f \circ g = Id_Y.$$

$g \circ f = Id_X$.

Let $x \in X$. Then $g \circ f(x) = g(f(x))$

Let $y = f(x)$. Then $g(y)$ is by definition equal to the unique $x \in X$ s.t. $y = f(x)$.

Hence $g(y) = g(f(x)) = x$.

This shows that $g \circ f = Id_X$. ☐

<u>Notation</u>: We call the function g in the above discussion the "inverse" of f and denote $g = f^{-1}$.

If $X = Y$ then notice that

$$f^{-1} \circ f = f \circ f^{-1} = Id_X .$$

Thus f^{-1} is simultaneously a "left" as well as a "right" inverse of f.

This will be important when we will talk about "groups".

6. Binary relations and equivalence relations

A binary relation R on a set X is just a subset of $X \times X$. If $(x, x') \in R$ we will say $x R x'$ or "x is related by R to x'".

The ~~equivalence~~ relations that will be important for us are of a very special kind.

Equivalence relations

A binary relation $R \subset X \times X$ is said to be an equivalence relation if it satisfies

(1) $(x,x) \in R$ for every $x \in X$. (Reflexivity)

(2) $(x,y) \in R$ if and only if $(y,x) \in R$ for every $x,y \in X$. (Symmetry).

(3) $(x,y) \in R$ and $(y,z) \in R$ implies $(x,z) \in R$ for every $x,y,z \in X$. (Transitivity)

Question: If the empty relation $\phi \subset X \times X$ an equivalence relation?

What about the "full" relation $R = X \times X$?