453 Lecture 8

Groups and first examples

Def. A *group* G is a set with a binary operation $\cdot : G \times G \to G$ and an element $e \in G$ satisfying the following axioms.

(i) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in G$ (in other words $\cdot$ is associative),

(ii) For every element $a \in G$
$$a \cdot e = e \cdot a = a.$$

(iii) For every element $a \in G$, there exists $b \in G$ (denoted a^{-1}) such that
$$a \cdot b = b \cdot a = e.$$

(If moreover the operation $\cdot$ satisfies

(iv) $a \cdot b = b \cdot a$ for all $a, b \in G$

then we say that G is an abelian (or commutative) group.)

2. First examples.

$(\mathbb{Z}, +, 0)$ $(\mathbb{Q}, +, 0)$, $(\mathbb{R}, +, 0)$

$(\mathbb{Q}^* = \mathbb{Q} \setminus \{0\}, \cdot, 1)$, $(\mathbb{R}^* = \mathbb{R} \setminus \{0\}, \cdot, 1)$

are infinite commutative gps.

$GL(n, \mathbb{R})$ = group of $n \times n$ invertible real matrices under multiplication.

$SL(n, \mathbb{R})$ = group of $n \times n$ real matrices having determinant 1.

$\to$ these are infinite non-abelian groups for $n \geq 2$.

For any set X, the set S_X of bijections $X \to X$ is a group with the group operation being "composition".

S_X is abelian if $|X| \leq 2$ but non-abelian if $|X| > 2$.

If $X = \{1, \dots, n\}$ we denote $S_X = S_n$ and call it the "symmetric group" on n letters.