A2 $G = S_3 \quad H = \{\varepsilon, \alpha\}$

let $\alpha = (12)$, $\beta = (13)$, $\delta = (23)$, $\gamma = (123)$, $\kappa = (132)$, $\varepsilon = \binom{123}{123}$

$H\alpha = \{\alpha, \alpha^2\} = \{\alpha, (12)(12)\} = \{\alpha, \varepsilon\}$

$H\beta = \{\beta, \alpha\beta\} = \{\beta, (12)(13)\} = \{\beta, (132)\} = \{\beta, \kappa\}$

$H\delta = \{\delta, \alpha\delta\} = \{\delta, (12)(23)\} = \{\delta, (123)\} = \{\delta, \gamma\}$

$H\gamma = \{\gamma, \alpha\gamma\} = \{\gamma, (12)(123)\} = \{\gamma, (2,3)\} = \{\gamma, \delta\}$

$H\kappa = \{\kappa, \alpha\kappa\} = \{\kappa, (12)(132)\} = \{\kappa, (13)\} = \{\kappa, \beta\}$

$H\varepsilon = \{\varepsilon, \alpha\}$

$H\alpha = H\varepsilon$, $H\beta = H\kappa$, $H\delta = H\gamma$

$\{\alpha, \varepsilon\}$ $\{\beta, \kappa\}$ $\{\delta, \gamma\}$ there are 3 cosets

B5 $H = \{(x,y) : x = y\}$ of $\mathbb{R} \times \mathbb{R}$

for any point $a \in \mathbb{R} \times \mathbb{R}$, Ha is all points on ~~the~~ straight line passing through origin and this point a.

if $a = e$, then He is the origin.

C1. If G has order n, then $x^n = e$ for every x in G

according to theorem 5, let $\mathrm{ord}(x) = m$

then m divides n

so let $n = km$ for some integer k where $k > 0$

then $x^m = e$

so $x^n = x^{km} = (x^m)^k = e^k = e$

C3. if G is cyclic with a generator a, then for $a^m, a^n \in G$

$a^m a^n = a^{m+n} = a^{n+m} = a^n a^m$

if ~~$a = a^{-1}$~~ any element $b = b^{-1} \in G$

4/4

then for 2 arbitrary elements $x, y \in G$

$xy = x^{-1}y^{-1}$ and since $xy \in G$, then $xy = (xy)^{-1} = y^{-1}x^{-1} = x^{-1}y^{-1}$

$\therefore$ every group of order 4 is abelian

D3. $ord(H) = m$ $ord(K) = n$ $\gcd(m, n) = 1$

$H \cap K = \{e\}$ prove

let $a \in H \cap K$, then according to Theorem J

$ord(a)$ divides m and n, since $a \in H$ and $a \in K$

since m and n are relative prime,

their greatest common divisor is 1

then $ord(a) = 1$ $\therefore a = e$

$\therefore H \cap K = \{e\}$ 3/3

E1.

$Ha = Hb$ iff $ab^{-1} \in H$

- if $Ha = Hb$

then there is $h_1 \in H$ such that $h_1 a \in Ha$ and $h_1 a \in Hb$

there is $h_2 \in H$ such that $h_2 b \in H_2 b$ and $h_2 b = h_1 a$

~~$h_2 b$~~ $h_1 a = h_2 b$

$h_1 a b^{-1} = h_2 b b^{-1}$

$h_1 a b^{-1} = h_2 e$

$h_1^{-1} h_1 a b^{-1} = h_1^{-1} h_2$

$ab^{-1} = h_1^{-1} h_2 \in H$

- if $ab^{-1} \in H$

there exists some $h_3 \in H$ such that $h_3 = ab^{-1} \Rightarrow a = h^3 b \Rightarrow a \in Hb$

so let $x \in G$ and $x \in Ha$, then for some $h_4, h_5 \in H$

$x = h_4 a$, $a = h_5 b \Rightarrow$

$x = h_4 h_5 b = (h_4 h_5) b$ ✓ 3/3

since $h_4 h_5 \in H$, $x = Hb$

$\therefore Ha = Hb$

E5 $(ab)H = (ac)H \rightarrow bH = cH$ prove

$(ab)H = (ac)H \Rightarrow h_1, h_2 \in H$ such that

$\cancel{abH_1}$ $abh_1 = ach_2$

$a^{-1}abh_1 = a^{-1}ach_2$

$bh_1 = ch_2$

$bh_1h_1^{-1} = ch_2h_1^{-1}$

$b = c(h_2h_1^{-1})$

since $h_2h_1^{-1} \in H$, $b \in cH$

$\therefore bH = cH$ 3/3

E6

$a \in G$, $H \leq G$, $h \in H$

$(ah)^{-1} = h^{-1}a^{-1} \in Ha^{-1}$ establish a function $Ha \mapsto a^{-1}H$

well defined:

$Ha = Hb \longleftrightarrow ab^{-1} \in H \longleftrightarrow b^{-1} \in a^{-1}H \longleftrightarrow b^{-1}H = a^{-1}H$

injective:

$a^{-1}H = b^{-1}H \longleftrightarrow a^{-1} \in b^{-1}H \longleftrightarrow ba^{-1} \in H \longleftrightarrow b \in Ha \longleftrightarrow Ha = Hb$

surjective: ✓ 4/4

$\forall\, bH$, $Hb^{-1} \mapsto bH$

$\therefore$ number of left coset and right cosets are the same

CH 14

A3 $f(x) = x \bmod 5$ $\ker(f) = \{0, 5, 10\}$

~~K = {0}~~

A4. $f: D_4 \to S_2 = \mathbb{Z}_2$

let a be (changes sides) reflection

b be (keeps sides) rotation.

$f: D_4 \to S_2 = \mathbb{Z}_2$

$\hookrightarrow \langle a, b \mid a^2 = e, b^4 = e, ba = ab^3 \rangle = \{e, a, b, b^2, b^3, ab, ab^2, ab^3\}$

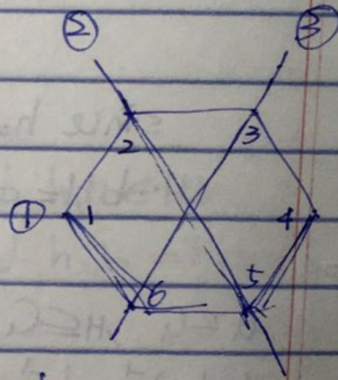
$e \mapsto 0$	$b^3 \to 0$
$a \to 1$	$ab \to 1$
$b \to 0$	$ab^2 \to 1$
$b^2 \to 0$	$ab^3 \to 1$

$Ker(f) = \{e, b, b^2, b^3\}$

A5. let a be some rotation clockwise $60°$

let b be some reflection

then $D_6 = \langle a, b \mid a^6 = e, b^2 = e, ab = ba^5 \rangle$



every symmetry of the hexagon either

- ε: keep all diagnals where they originally at
- α: keep diagonal ① where it is, interchange ② and ③
- β: keep digonal ② where it is, interchange ① and ③
- δ: diagonal ① moves to ~~becomes~~ diagonal ②, ② → ③, and ③ → ①
- γ: keep diagonal ③ where it is, interchans ① and ②
- κ: diagonal ① moves to ~~becomes~~ diagonal ③, ② → ①, and ③ → ②

$\therefore f: D_6 \to S_3$ is a hormormorphism

$f\begin{pmatrix}1&2&3&4&5&6\\1&2&3&4&5&6\end{pmatrix} = \varepsilon$ $\quad f\begin{pmatrix}1&2&3&4&5&6\\5&6&1&2&3&4\end{pmatrix} = \delta$ $\quad f\begin{pmatrix}1&2&3&4&5&6\\3&2&1&6&5&4\end{pmatrix} = \beta$

$f\begin{pmatrix}1&2&3&4&5&6\\2&3&4&5&6&1\end{pmatrix} = \delta$ $\quad f\begin{pmatrix}1&2&3&4&5&6\\6&1&2&3&4&5\end{pmatrix} = \kappa$ $\quad f\begin{pmatrix}1&2&3&4&5&6\\4&3&2&1&6&5\end{pmatrix} = \alpha$

$f\begin{pmatrix}1&2&3&4&5&6\\3&4&5&6&1&2\end{pmatrix} = \kappa$ $\quad f\begin{pmatrix}1&2&3&4&5&6\\1&6&5&4&3&2\end{pmatrix} = \alpha$ $\quad f\begin{pmatrix}1&2&3&4&5&6\\5&4&3&2&1&6\end{pmatrix} = \gamma$

$f\begin{pmatrix}1&2&3&4&5&6\\4&5&6&1&2&3\end{pmatrix} = \varepsilon$ $\quad f\begin{pmatrix}1&2&3&4&5&6\\2&1&6&5&4&3\end{pmatrix} = \gamma$ $\quad f\begin{pmatrix}1&2&3&4&5&6\\6&5&4&3&2&1\end{pmatrix} = \beta$

B5 let $a+bi$, $c+di \in \mathbb{C}^*$

$f((a+bi)(c+di)) = f(ac + (ad)i + (bc)i - bd)$

$= f((ac-bd) + (ad+bc)i)$

$= \sqrt{(ac-bd)^2 + (ad+bc)^2} = \sqrt{(a^2c^2 - 2abcd + b^2d^2) + (a^2d^2 + 2abcd + b^2c^2)}$

$= \sqrt{a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2}$

$f(a+bi)f(c+di) = (\sqrt{a^2+b^2})(\sqrt{c^2+d^2})$

$= \sqrt{(a^2+b^2)(c^2+d^2)}$

$= \sqrt{a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2}$

$= \sqrt{a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2} = f((a+bi)(c+di))$

$\ker(f) = \{a+bi \in \mathbb{C}^* : \sqrt{a^2+b^2} = 1\}$ ✓ 3/3