Problem 8.F.2

Let $\alpha = (a_1 a_2 \dots a_s)$ be a cycle of length s

Then $\alpha^2 = \begin{pmatrix} a_1 & a_2 & \cdots & a_s \\ a_2 & a_3 & \cdots & a_1 \end{pmatrix}\begin{pmatrix} a_1 & a_2 & \cdots & a_s \\ a_2 & a_3 & \cdots & a_1 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & \cdots & a_s \\ a_3 & a_4 & \cdots & a_2 \end{pmatrix}$

$\alpha^3 = \begin{pmatrix} a_1 & a_2 & \cdots & a_s \\ a_2 & a_3 & \cdots & a_1 \end{pmatrix}\begin{pmatrix} a_1 & a_2 & \cdots & a_s \\ a_2 & a_3 & \cdots & a_1 \end{pmatrix}\begin{pmatrix} a_1 & a_2 & \cdots & a_s \\ a_2 & a_3 & \cdots & a_1 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & \cdots & a_s \\ a_4 & a_5 & \cdots & a_3 \end{pmatrix}$

$\Rightarrow \alpha$ takes a_1 to a_2, α^2 takes a_1 to a_3, α^3 takes a_1 to a_4

…… …

α^{s-1} takes a_1 to a_s, α^s takes a_1 to a_1

$\Rightarrow s$ is the least positive integer such that $\alpha^s = \varepsilon$

Hence, the order of $\alpha = s$

Problem 8.F.3

(a) (12)(345): disjoint and of lengths 2 and 3

$\Rightarrow$ the order of permutation is $lcm(2,3) = \mathbf{6}$

(b) (12)(3456): disjoint and of lengths 2 and 4

$\Rightarrow$ the order of permutation is $lcm(2,4) = \mathbf{4}$

(c) (1234)(56789): disjoint and of lengths 4 and 5

$\Rightarrow$ the order of permutation is $lcm(4,5) = \mathbf{20}$

Problem 8.G.4

Let A_n be the set of even permutations in S_n.

(1) Let $a, b \in A_n$ where $a = a_1 a_2 \dots a_{2k}$ and $b = b_1 b_2 \dots b_{2k}$

$\Rightarrow a^{-1} = a_{2k} a_{2k-1} \dots a_2 a_1$ and $b^{-1} = b_{2k} b_{2k-1} \dots b_2 b_1$

$\Rightarrow (ab)^{-1} = b^{-1} a^{-1} = b_{2k} b_{2k-1} \dots b_2 b_1 a_{2k} a_{2k-1} \dots a_2 a_1$

$\Rightarrow$ the inverse of a product of even permutation is the product of the same transpositions in reverse order.

$\Rightarrow$ $(ab)^{-1} \in S_n$ since $a, b \in S_n$ where S_n is closed under products and inverse.

$\Rightarrow$ $(ab)^{-1} \in A_n$

(2) Take $a = b, (ab)^{-1} = (aa)^{-1} = a^{-1}a^{-1} = aa$

$\Rightarrow e \in A_n$

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(3) The product of two even permutations will have an even number of factors

$\Rightarrow$ A_n is closed

Therefore, we conclude that A_n is a subgroup of S_n.

Problem 9.C.2

Table for G				
	I	V	H	D
I	I	V	H	D
V	V	I	D	H
H	H	D	I	V
D	D	H	V	I

Table for $\mathbb{Z}_4 = \{0,1,2,3\}$				
	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

G and H are not isomorphic because in G every element is its own inverse ($VV = I$, $HH = I$, and $DD = I$), whereas in H there are elements not equal to their inverse. For example, $(1)(1) = 2 \neq 1$. Thus, $G \not\cong H$.

Problem 9.C.3

Table for $P_2 = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}$				
	$\emptyset$	$\{a\}$	$\{b\}$	$\{a,b\}$
$\emptyset$	$\emptyset$	$\{a\}$	$\{b\}$	$\{a,b\}$
$\{a\}$	$\{a\}$	$\emptyset$	$\{a,b\}$	$\{b\}$
$\{b\}$	$\{b\}$	$\{a,b\}$	$\emptyset$	$\{a\}$
$\{a,b\}$	$\{a,b\}$	$\{b\}$	$\{a\}$	$\emptyset$

Table for H				
	1	*-1*	*i*	*-i*
1	*1*	*-1*	*i*	*-i*
-1	*-1*	*1*	*-i*	*i*
i	*i*	*-i*	*-1*	*1*
-i	*-i*	*i*	*1*	*-1*

G and H are not isomorphic because in G every element is its own inverse ($aa = \emptyset$, $bb = \emptyset$, and $(ab)(ab) = \emptyset$), whereas in H there are elements not equal to their inverse. For example, $(1)(1) = 2 \neq 1$. Thus, $G \not\cong H$. 4/4

Problem 9.C.6

$e =$ Identity transformation

$a =$ Reflection by the midpoint of the long side

$b =$ Reflection by the midpoint of the short side

$c =$ Rotation through 180°

Table for rectangle				
	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Table for H				
	1	*-1*	*i*	*-i*
1	*1*	*-1*	*i*	*-i*
-1	*-1*	*1*	*-i*	*i*
i	*i*	*-i*	*-1*	*1*
-i	*-i*	*i*	*1*	*-1*

G and H are not isomorphic because in G every element is its own inverse ($aa = e$, $bb = e$, and $cc = e$), whereas in H there are elements not equal to their inverse. For example, $(i)(i) = -1 \neq 1$. Thus, $G \not\cong H$.

$G = \{e, a, b, c\}$, a = reflect y-axis, b = reflect x-axis, c = rotate $180°$ clockwise $\qquad H = \{1, -1, i, -i\}$

*	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

*	1	-1	i	-i
1	1	-1	i	-i
-1	-1	1	-i	i
i	i	-i	1	-1
-i	-i	i	-1	1

$$f = \begin{pmatrix} e & a & b & c \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & -1 & i & -i \end{pmatrix} \qquad f: G \to H$$

f is a bijection s.t. $f: G \to H$. The operation tables are similar.

Therefore, $G \cong H$

D 2. S_3, $\mathbb{Z}_6$, $\mathbb{Z}_3 \times \mathbb{Z}_2$, $\mathbb{Z}_7 \qquad \mathbb{Z}_7 = \{1, 2, 3, 4, 5, 6\}$

$S_3 = (1\ 2\ 3)$ degree $= 3$ order $= n! = 6$

S_3 *	()	(1 2)	(1 3)	(1 3 2)	(2 3)	(1 2 3)
()	()	(1 2)	(1 3)	(1 3 2)	(2 3)	(1 2 3)
(1 2)	(1 2)	()	(1 3 2)	(1 3)	(1 2 3)	(2 3)
(1 3)	(1 3)	(1 2 3)	()	(2 3)	(1 3 2)	(1 2)
(1 3 2)	(1 3 2)	(2 3)	(1 2)	(1 2 3)	(1 3)	()
(2 3)	(2 3)	(1 3 2)	(1 2 3)	(1 2)	()	(1 3)
(1 2 3)	(1 2 3)	(1 3)	(2 3)	()	(1 2)	(1 3 2)

$\mathbb{Z}_6$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$
$\bar{0}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$
$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{0}$
$\bar{2}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{0}$	$\bar{1}$
$\bar{3}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{0}$	$\bar{1}$	$\bar{2}$
$\bar{4}$	$\bar{4}$	$\bar{5}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
$\bar{5}$	$\bar{5}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$

$\mathbb{Z}_7^*$ *	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{6}$
$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{6}$
$\bar{2}$	$\bar{2}$	$\bar{4}$	$\bar{6}$	$\bar{1}$	$\bar{3}$	$\bar{5}$
$\bar{3}$	$\bar{3}$	$\bar{6}$	$\bar{2}$	$\bar{5}$	$\bar{1}$	$\bar{4}$
$\bar{4}$	$\bar{4}$	$\bar{1}$	$\bar{5}$	$\bar{2}$	$\bar{6}$	$\bar{3}$
$\bar{5}$	$\bar{5}$	$\bar{3}$	$\bar{1}$	$\bar{6}$	$\bar{4}$	$\bar{2}$
$\bar{6}$	$\bar{6}$	$\bar{5}$	$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{1}$

$$f = \begin{pmatrix} \bar{0} & \bar{1} & \bar{2} & \bar{3} & \bar{4} & \bar{5} \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 3 & 2 & 6 & 4 & 5 \end{pmatrix}$$

$f: \mathbb{Z}_6 \to \mathbb{Z}_7^*$

↑ I know the numbers shouldn't have a bar, but I don't want to rewrite the table. Sorry!

$\mathbb{Z}_3 \times \mathbb{Z}_2$	$(\bar{0},\bar{0})$	$(\bar{0},\bar{1})$	$(\bar{1},\bar{0})$	$(\bar{1},\bar{1})$	$(\bar{2},\bar{0})$	$(\bar{2},\bar{1})$
$(\bar{0},\bar{0})$	$(\bar{0},\bar{0})$	$(\bar{0},\bar{1})$	$(\bar{1},\bar{0})$	$(\bar{1},\bar{1})$	$(\bar{2},\bar{0})$	$(\bar{2},\bar{1})$
$(\bar{0},\bar{1})$	$(\bar{0},\bar{1})$	$(\bar{0},\bar{0})$	$(\bar{1},\bar{1})$	$(\bar{1},\bar{1})$	$(\bar{2},\bar{1})$	$(\bar{2},\bar{0})$
$(\bar{1},\bar{0})$	$(\bar{1},\bar{0})$	$(\bar{1},\bar{1})$	$(\bar{2},\bar{0})$	$(\bar{2},\bar{1})$	$(\bar{0},\bar{0})$	$(\bar{0},\bar{1})$
$(\bar{1},\bar{1})$	$(\bar{1},\bar{1})$	$(\bar{1},\bar{0})$	$(\bar{2},\bar{1})$	$(\bar{2},\bar{0})$	$(\bar{0},\bar{1})$	$(\bar{0},\bar{0})$
$(\bar{2},\bar{0})$	$(\bar{2},\bar{0})$	$(\bar{2},\bar{1})$	$(\bar{0},\bar{0})$	$(\bar{0},\bar{1})$	$(\bar{1},\bar{0})$	$(\bar{1},\bar{1})$
$(\bar{2},\bar{1})$	$(\bar{2},\bar{1})$	$(\bar{2},\bar{0})$	$(\bar{0},\bar{1})$	$(\bar{0},\bar{0})$	$(\bar{1},\bar{1})$	$(\bar{1},\bar{0})$

$$g = \begin{pmatrix} \bar{0} & \bar{1} & \bar{2} & \bar{3} & \bar{4} & \bar{5} \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ (\bar{0},\bar{0}) & (\bar{1},\bar{1}) & (\bar{1},\bar{0}) & (\bar{0},\bar{1}) & (\bar{2},\bar{0}) & (\bar{2},\bar{1}) \end{pmatrix}$$

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$g = \mathbb{Z}_6 \to \mathbb{Z}_3 \times \mathbb{Z}_2$

if $\mathbb{Z}_6 \cong \mathbb{Z}_7^*$ and $\mathbb{Z}_6 \cong \mathbb{Z}_3 \times \mathbb{Z}_2$, then $\mathbb{Z}_7^* \cong \mathbb{Z}_3 \times \mathbb{Z}_2$

Then, $\mathbb{Z}_6 \cong \mathbb{Z}_7^* \cong \mathbb{Z}_3 \times \mathbb{Z}_2$, $S_3 \ncong$ any of the groups

E 2. $G = \{10^n \mid n \in \mathbb{Z}\}$ Prove that $G \cong \mathbb{Z}$

Need (i) f is a bijection s.t. $f: G \to \mathbb{Z}$ and (ii) Show that $f(a*b) = f(a) * f(b)$, $f: G \to \mathbb{Z}$

Groups are $(G, \times)$ and $(\mathbb{Z}, +)$, $f(ab) = f(a) + f(b)$

$f(a*b) = f(10^a 10^b) = f(10^a) + f(10^b)$

(i) $f(x) = \log(x)$ $\quad \log_{10}(x)$

(ii) $f(a*b) = f(10^a 10^b) = f(10^{a+b}) = \log(10^{a+b}) = (a+b)\log 10$

$= (a+b)(1) = a+b$

$f(a) + f(b) = \log(10^a) + \log(10^b) = a\log 10 + b\log 10 = a(1) + b(1)$

$= a+b$

Therefore, $G = \{10^n \mid n \in \mathbb{Z}\} \cong \mathbb{Z}$

E 4. Prove that $(\mathbb{R}, +) \ncong (\mathbb{R}^*, \times)$

Torsion element $Tor(G) = \{g_1, \dots g_n\}$, $g \in G$ s.t. $g^n = e$

For $(\mathbb{R}, +)$, there are no torsion elements

For $(\mathbb{R}^*, \times)$, the torsion element is $\langle -1 \rangle$

This means that $(\mathbb{R}, +)$ is infinite while $(\mathbb{R}^*, \times)$ is finite.

Problem 9.E.2

Suppose $f: G \to \mathbb{Z}, f(10^n) = n$

Let $x, y \in G$ such that $f(x) = f(y)$ and let $x = 10^n$ and $y = 10^m$ for some $m, n \in \mathbb{Z}$

$f(x) = f(y) \Rightarrow f(10^n) = f(10^m) \Rightarrow n = m \Rightarrow$ **f is injective**

Let $x \in G, a \in \mathbb{Z}$, thus $10^a \in G$.

Now let $x = 10^a$, then $f(x) = f(10^a) = a \Rightarrow$ **f is surjective**

Let $x, y \in G$ such that $x = 10^n$ and $y = 10^m$ for some $m, n \in \mathbb{Z}$

Thus $f(xy) = f(10^n 10^m) = f(10^{n+m}) = n + m = f(10^n) + f(10^m) \Rightarrow$ **f is homomorphism**

Since f is injective, surjective and homomorphism, we conclude that $G \cong \mathbb{Z}$.

Problem 9.E.4

Let $\mathbb{R}^{pos} \cong \mathbb{R}^* \Rightarrow \mathbb{R}^* \cong \mathbb{R}^{pos} \Rightarrow f: \mathbb{R}^{pos}$ is bijective and homomorphim $\Rightarrow f(1) = 1$

Let $f(-1) = x \Rightarrow f(1) = f((-1)(-1)) = f(-1)f(-1) = x^2 = 1 \Rightarrow x = \pm 1$

Since $-1 \notin \mathbb{R}^{pos}, x = 1 \Rightarrow f(-1) = 1$

$\Rightarrow \begin{cases} f(1) = 1 \\ f(-1) = 1 \end{cases}$ which contradict to the assumption that f is injective.

Therefore, $\mathbb{R}^{pos} \ncong \mathbb{R}^*$ 2/2

Problem 9.E.6

Let $\mathbb{Q}^{pos} \cong \mathbb{Q} \Rightarrow f: \mathbb{Q} \to \mathbb{Q}^{pos}$ is bijective and homomorphim

For example , $2 \in \mathbb{Q}^{pos} \Rightarrow \exists\, x \in \mathbb{Q}$ such that $f(x) = 2$

Since $x \in \mathbb{Q}$, then $\frac{x}{2} \in \mathbb{Q}$

Let $f\left(\frac{x}{2}\right) = a$, then $f(x) = f\left(\frac{x}{2} + \frac{x}{2}\right) = f\left(\frac{x}{2}\right) f\left(\frac{x}{2}\right) = a^2 = 2 \Rightarrow a \notin \mathbb{Q}$

Since there is no $a \in \mathbb{Q}$ sucha that $a^2 = 2$ which is contradict to the assumption

Therefore, $\mathbb{Q}^{pos} \ncong \mathbb{Q}$ 2/2

Problem 9.G.3

Let $x, y, z \in G, \begin{cases} x * (y * z) = x * \left(\frac{yz}{2}\right) = \frac{xyz}{4} \\ (x * y) * z = \left(\frac{xy}{2}\right) * z = \frac{xyz}{4} \end{cases} \Rightarrow x * (y * z) = (x * y) * z \Rightarrow$ Associative

Let $e = 2, \begin{cases} x * 2 = \frac{x(2)}{2} = x \\ 2 * x = \frac{2x}{2} = x \end{cases} \Rightarrow x * e = e * x = x \Rightarrow$ Identity $= 2$

Let $x * x' = 2 \Rightarrow \frac{xx'}{2} = 2 \Rightarrow x' = \frac{4}{x} \Rightarrow \begin{cases} x * \frac{4}{x} = \frac{x\left(\frac{4}{x}\right)}{2} = 2 \\ \frac{4}{x} * x = \frac{\left(\frac{4}{x}\right)x}{2} = 2 \end{cases} \Rightarrow$ Inverse $= \frac{4}{x}$

Therefore, G is a group.

Let $f: \mathbb{R}^* \longrightarrow \mathbf{G}, \mathbf{G} = \mathbb{R} - \{\mathbf{0}\}, \mathbb{R}^* = \mathbb{R} - \{\mathbf{0}\}, f(x) = \frac{2}{x}$

Now let $x, y \in \mathbb{R}^* \Rightarrow f(x) = f(y) \Rightarrow \frac{2}{x} = \frac{2}{y} \Rightarrow x = y \Rightarrow f$ is injective

Let $\frac{2}{x} \in \mathbf{G} \Rightarrow f\left(\frac{2}{x}\right) = \frac{2}{\frac{2}{x}} = x \Rightarrow f$ is surjective

Hence $f: \mathbb{R}^* \longrightarrow \mathbf{G}$ is isomorphism.

Problem 9.H.3

Let $x, y \in G$, then $f(x) = f(y) \Rightarrow x^{-1} = y^{-1} \Rightarrow \frac{1}{x} = \frac{1}{y} \Rightarrow x = y \Rightarrow f$ is injective

Let $x \in G$, then $f(x^{-1}) = x \Rightarrow f$ is surjective

Hence, f is bijective.

($\Longrightarrow$) Suppose that G is abelian.

Let $x, y \in G$, then $f(xy) = (xy)^{-1} = y^{-1}x^{-1} = x^{-1}y^{-1} = f(x)f(y)$

$\Rightarrow f$ is homomorphism and bijective, thus isomorphism from G to G.

($\Longleftarrow$) Suppose that $f(x) = x^{-1}$ is isomorphism from G to G $\Rightarrow f$ is homomorphism

Let $x, y \in G$, then $\begin{cases} f(x^{-1}y^{-1}) = f(x^{-1})f(y^{-1}) = (x^{-1})^{-1}(y^{-1})^{-1} = xy \\ f(x^{-1}y^{-1}) = (x^{-1}y^{-1})^{-1} = (y^{-1})^{-1}(x^{-1})^{-1} = yx \end{cases} \Rightarrow$ G is abelian

3/3