Chap 6

Prove that if $g \circ f$ is injective, then f is injective.

Given $g \circ f$ injective, to prove $f: A \to B$ injective

suppose $f(x) = f(y)$ for some $x, y \in A$.

then $g(f(x)) = g(f(y)) \Leftrightarrow g \circ f(x) = g \circ f(y)$

$\because$ $g \circ f$ is injective, then $x = y$.

$\therefore$ f is injective.

If $g \circ f$ is surjective, the g is surjective. $f: A \to B$, $g: B \to C$

Given $g \circ f$ is surjective, suppose $\exists\ y \in C$,

$\because$ $g \circ f$ surjective, $\exists\ a \in A$, st $g \circ f(a) = y$.

$g(f(a)) = y$.

and suppose $b = f(a) \in B$

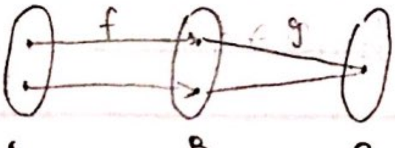
then $g(b) = g(f(a)) = y$.

$\therefore$ g is surjective.

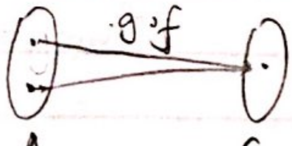
If f is injective and g surjective, $g \circ f$ bijective ?

Give $f: A \to B$ injective. $g: B \to C$ surjective

that is $A \xrightarrow{f} B \xrightarrow{g} C$



A B C



A C

3/3

$\downarrow$

$g \circ f$ is not 1-to 1 correspondence

f: $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

g: $\exists x \in A$ s.t $f(x) = y$, $\forall y \in B$

$\therefore$ $g \circ f$ is not bijective

Chap 7.

A3 $g \circ h^{-1}$

$$g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 6 & 5 & 4 \end{pmatrix}$$

$$h = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 6 & 4 & 5 & 2 \end{pmatrix}$$

$$g \circ h^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 2 & 6 & 5 & 1 \end{pmatrix}$$ 3/3

B1 $\varepsilon = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$ $f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$

$g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$ $h = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$

$\circ$	ε	f	g	h
ε	ε	f	g	h
f	f	ε	h	g
g	g	h	ε	f
h	h	g	f	ε

3/3

C3 $G = \{\varepsilon, f, g, h, j, k\}$ $f(x) = 1-x$, $A \subseteq \mathbb{R}$ $x \neq 0, 1$.

$g(x) = 1/x$

$h(x) = 1/(x-1)$

$j(x) = (x-1)/x$

$k(x) = x/(x-1)$

$\circ$	ε	f	g	h	j	k
ε	ε	f	g	h	j	k
f	f	ε	j	k	g	h
g	g	h	ε	f	k	j
h	h	g	k	j	ε	f
j	j	k	f	ε	h	g
k	k	j	k	g	f	ε

$\because$ $\varepsilon, f, g, h, j, k$ are permutations of A and $G \subseteq S_A$.

(1) associativity satisfy with composition of function.

(2) $\exists$ the identity element in the operatio which is ε

(3) there is a unique inverse element

C.3)

$\circ$	ε	f	g	h	j	k
ε	ε	f	g	h	j	k
f	f	ε	j	k	g	h
g	g	h	ε	f	k	j
h	h	g	k	j	ε	f
j	j	k	f	ε	h	g
k	k	j	h	g	f	ε

$f = 1-x \quad g = 1/x \quad h = 1/1-x \quad j = x-1/x \quad k = x/x-1$

$f\circ f = 1-(1-x) = 1-1+x = x \quad g\circ f = \frac{1}{1-x} \quad h\circ f = \frac{1}{1-(1-x)} = \frac{1}{x}$

$j\circ f = 1-x-1/1-x = \frac{-x}{1-x} = \frac{x}{x-1} \quad k\circ f = 1-x/1-x-1 = \frac{1-x}{-x} = \frac{x-1}{x}$

$f\circ g = 1-\frac{1}{x} = \frac{x-1}{x} \quad g\circ g = \frac{1}{\frac{1}{x}} = x \quad h\circ g = \frac{1}{1-\frac{1}{x}} = \frac{1}{\frac{x-1}{x}} = \frac{x}{x-1} \quad j\circ g = \frac{\frac{1}{x}-1}{1/x} = 1-x$

$k\circ g = \frac{\frac{1}{x}}{\frac{1}{x}-1} = \frac{\frac{1}{x}}{\frac{1-x}{x}} = \frac{1}{1-x} \quad f\circ h = 1-\frac{1}{1-x} = \frac{1-x-1}{1-x} = \frac{-x}{1-x} = \frac{x}{x-1} \quad g\circ h = \frac{1}{\frac{1}{1-x}} = 1-x$

$h\circ h = \frac{1}{1-\frac{1}{1-x}} = \frac{1}{\frac{1-x-1}{1-x}} = \frac{1-x}{-x} = \frac{x-1}{x} \quad j\circ h = \frac{\frac{1}{1-x}-1}{\frac{1}{1-x}} = \frac{1-1+x}{1-x}\cdot(1-x) = x$

$k\circ h = \frac{\frac{1}{1-x}}{\frac{1}{1-x}-1} = \frac{1}{1-x}\cdot\left(\frac{1-x}{1-1+x}\right) = \frac{1}{x} \quad f\circ j = 1-\frac{x-1}{x} = \frac{x-x+1}{x} = \frac{1}{x}$

$g\circ j = \frac{1}{\frac{x-1}{x}} = \frac{x}{x-1} \quad h\circ j = \frac{1}{1-\frac{x-1}{x}} = \frac{1}{\frac{x-x+1}{x}} = x \quad j\circ j = \frac{\frac{x-1}{x}-1}{\frac{x-1}{x}} = \frac{x-1-x}{x}\cdot\frac{x}{x-1} = \frac{-1}{x-1} = \frac{1}{1-x}$

$k\circ j = \frac{\frac{x-1}{x}}{\frac{x-1}{x}-1} = \frac{x-1}{x}\cdot\frac{x}{x-1-x} = \frac{x-1}{-1} = 1-x \quad f\circ k = 1-\frac{x}{x-1} = \frac{x-1-x}{x-1} = \frac{-1}{x-1} = \frac{1}{1-x}$

$g\circ k = \frac{1}{\frac{x}{x-1}} = \frac{x-1}{x} \quad h\circ k = \frac{1}{1-\frac{x}{x-1}} = \frac{1}{\frac{x-1-x}{x-1}} = \frac{x-1}{-1} = 1-x \quad j\circ k = \frac{\frac{x}{x-1}-1}{\frac{x}{x-1}} = \frac{x-x+1}{x-1}\cdot\frac{x-1}{x} = \frac{1}{x}$

$k\circ k = \frac{\frac{x}{x-1}}{\frac{x}{x-1}-1} = \frac{x}{x-1}\cdot\frac{x-1}{x-x+1} = \frac{x}{1} = x$

E1. $f_{a,b}(x) = ax+b$

$\because f_{a,b}(x) = ax+b$ for $\forall\ x \in \mathbb{R}$

$\Rightarrow f_{a,b}(x_1) = f_{a,b}(x_2)$

$ax_1 + b = ax_2 + b$

$ax_1 = ax_2$

$x_1 = x_2$

$\therefore f_{a,b}$ is a one-to-one function

Let $y \in \mathbb{R}$, then $(y-b)/a \in \mathbb{R}$

Let $x = (y-b)/a$

then $f_{a,b}(x) = a \cdot \frac{(y-b)}{a} + b = y - b + b = y.$

$\therefore f_{a,b}$ is also onto.

$\therefore f_{a,b}$ is a bijective function on $\mathbb{R}$

therefore $f_{a,b}$ is a permutation.

E2. $f_{a,b} \circ f_{c,d} = f_{a,b}(f_{c,d}(x))$

$= f_{a,b}(cx+d)$

$= a(cx+d) + b.$ 3|3

$= acx + ad + b.$

$f_{ac,\, ad+b}(x) = acx + ad+b$

$\therefore f_{a,b} \circ f_{c,d} = f_{ac,\, ad+b}.$

Suppose $f_{a,b}^{-1} = f_{\frac{1}{a}, \frac{-b}{a}}$

it must satisfy the inverse property

$f_{a,b} \circ f_{\frac{1}{a}, \frac{-b}{a}} = \varepsilon$ and $f_{\frac{1}{a}, \frac{-b}{a}} \circ f_{a,b} = \varepsilon$

$f_{a,b} \circ f_{\frac{1}{a}, \frac{-b}{a}} = f_{\frac{1}{a} \cdot a,\ a \cdot \frac{-b}{a} + b} = f_{1,0}$

$f_{\frac{1}{a}, \frac{-b}{a}} \circ f_{a,b} : f_{a \cdot \frac{1}{a},\ \frac{1}{a} \cdot b + (-\frac{b}{a})} = f_{1,0}$

and $f_{1,0} \circ f_{a,b} = f_{1 \cdot a,\ 1 \cdot b + 0} = f_{a,b}$

$f_{a,b} \circ f_{1,0} = f_{a \cdot 1,\ a \cdot 0 + b} = f_{a,b}.$

therefore $f_{1,0}$ is the identify element

$f_{a,b}^{-1} = f_{\frac{1}{a}, \frac{-b}{a}} = f_{1,0}$

$f_{a,b} \circ f_{\frac{1}{a}, \frac{-b}{a}}(x) = f_{a,b}(\frac{1}{a}x - \frac{b}{a})$

$= a \cdot (\frac{1}{a}x - \frac{b}{a})$

$= a \cdot \frac{1}{a}x - a$

$= x - b + b$

$= x$

3|3

H.3 Let $f, g \in G$. Since $f(x), g(x) \in B$ $\forall x \in B$, $f(g(x)) \in B$ $\forall x \in B$ as well. So G is closed under composition.

We also know that f^{-1} exists and is in G since the function that maps $f(x)$ to x is also a permutation, and both x and $f(x)$ are elements of B.

So f^{-1} is also a permutation of A s.t. $f^{-1}(x) \in B$ $\forall x \in B$. Hence G is closed under inverses and G is a subgroup of S_A. SKS

H.4 Let $A = \mathbb{R}^+$, $B = \mathbb{Z}^+$, and $f(x) = x^2$

$f(x) \in B$ $\forall x \in B$, and every $k \in \mathbb{Z}^+$ has a $\sqrt{k} \in \mathbb{R}^+$, so G is still the subset of S_A consisting of all the permutations f of A s.t. $f(x) \in B$ $\forall x \in B$.

However $f^{-1} = \sqrt{x}$ is not always a non-negative integer, so G is not closed under inverses and is not a subgroup of S_A.

① $\forall p \in G$, $\forall x \in B$, $p(x) \in B$ $\therefore \{p(x) : x \in B\} \subseteq B = \{x : x \in B\}$

$\because$ B is finite, p is bijective, $\therefore |\{p(x) : x \in B\}| = |\{x : x \in B\}|$

thus $\{p(x) : x \in B\} = \{x : x \in B\}$

$p^{-1}(p(x)) = x$ $\because \{p(x) : x \in B\} = B$

thus $\forall p(x) \in B$, $p^{-1}(p(x)) = B$

thus $\forall y \in B$, $p^{-1}(y) = B$ 5/5

thus $p^{-1} \in G$

② $\forall p, a_e \in G$, $\forall x \in B$, $p(x) \in B$, thus $a_e(p(x)) \in B$.

thus $a_e \circ p \in G$

Chapter 7, Problem H.4

Let $A = \mathbb{R}$, B be the open interval $(-1, 1)$, and G be the same as in H.3. E.1 tells us that $f(x) = \frac{1}{2}x$ is a permutation on $\mathbb{R}$. Since $f(B) = \{(-\frac{1}{2}, \frac{1}{2})\}$, this shows us that f is in G. However, consider $f^{-1}(x) = 2x$. We see that $f^{-1}(B) = \{(-2, 2)\}$, which tells us there is some $x \in B$ such that $f^{-1}(x) \notin B$. Therefore, if A is an infinite set, then G is NOT closed with respect to inverses, and G is NOT necessarily a subgroup of S_A.