ASSIGNMENT 8. DUE IN CLASS OCT 27, 2017.

1. Let G be a group. For $a, b \in G$, we denote by $[a, b]$ the element $aba^{-1}b^{-1}$ (called the *commutator* of a and b) of G. Let $[G, G]$ denote the set of elements of G which are each a product of a finite number of commutators. Thus, every element of $[G, G]$ is of the form $[a_1, b_1] \cdots [a_m, b_m]$ for some $m \geq 0$, and $a_1, b_1, \ldots, a_m, b_m \in G$.
   (a) Prove that the inverse of a commutator is again a commutator.
   (b) Prove that $[G, G]$ is a normal subgroup of G (this subgroup is called the *commutator* subgroup of G, or sometimes the *first derived group, $D^0(G)$, of* G).
   (c) What is the subgroup $[G, G]$ in the case G is abelian ?
   (d) Prove that the quotient group $G/[G, G]$ is always abelian. (The group $G/[G, G]$ is often called the *abelianization* of G. The next two exercises show that $[G, G]$ is the *smallest* normal subgroup of G such that quotienting by it gives an abelian group.)
   (e) Prove that if $\phi : G \to A$ is a group homomorphism of G to an *abelian* group A, then $[G, G] \subset \ker(\phi)$.
   (f) Suppose that N is a normal subgroup of G such that G/N is abelian. Prove that $[G, G] \subset N$.
   (g) Let G be the dihedral group, D_8, of order 8. Compute $[G, G]$ and $G/[G, G]$.
2. Let U be the subset of $\mathrm{GL}(3, \mathbb{R})$ consisting of all elements which are upper triangular and with 1's on the diagonal. Thus,

$$U = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\},$$

   and let

$$V = \left\{ \begin{bmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mid b \in \mathbb{R} \right\}.$$

   (a) Prove that U is a subgroup of $\mathrm{GL}(3, \mathbb{R})$.
   (b) Prove that V is a normal subgroup of U, and the quotient group U/V is isomorphic to the additive group $\mathbb{R}^2$. (Hint. Use the first isomorphism theorem).
3. Consider the action by conjugation of the group D_8 on itself. Thus, using the notation used in class, $G = D_8$, $X = D_8$, and the action is defined by $g \cdot x = gxg^{-1}$ for all $g \in G, x \in x$. List all the orbits of this action.