MATH 453 LECTURE 3 AUG 25, 2017

1. EQUIVALENCE CLASSES

Let X be a set, $\sim$ an equivalence relation and $x \in X$. The equivalence class of x, denoted $[x]$ is defined as

$$[x] = \{y \in X \mid x \sim y\} = \{y \in X \mid y \sim x\}$$

equal because of symmetry of $\sim$.

2. Properties of equivalence classes

(1) $x \in [x]$.

(2) If $y \in [x]$ then $[x] = [y]$

Proof of (1): By reflexivity $x \sim x$. Hence, from the definition of the equivalence class of x, $x \in [x]$. ☒

Proof of (2):

(We need to show under some hypothesis that two sets [x] and [y] are equal. This involves proving two inclusions separately. $[x] \subset [y]$ and $[y] \subset [x]$).

We first show that $[x] \subset [y]$:

Let $z \in [x]$. From the definition of equivalence class this implies that $x \sim z$.

But by hypothesis $y \in [x]$. Hence, $y \sim x$.

Thus, $y \sim x$ and $x \sim z$. By transitivity of $\sim$, $y \sim z$. Hence, $z \in [y]$. We have proved that $[x] \subset [y]$.

We next prove that $[y] \subset [x]$:

Let $z \in [y]$. Then $z \sim y$. But again from hypothesis $y \in [x]$. Hence $y \sim x$.

$z \sim y$ and $y \sim x$ implies by transitivity of $\sim$ that $z \sim x$. Hence, $z \in [x]$ and we have shown that $[y] \subset [x]$. Hence, $[x] = [y]$. ∎

(3). If $y \notin [x]$, then $[y] \cap [x] = \phi$.

Proof. We prove this using <u>contradiction</u>.

Suppose that $[y] \cap [x] \neq \phi$.

Then there exists an element $z \in X$ such that

$z \in [y] \cap [x]$.

"the contradiction".

In other words,

$z \in [y]$ and $z \in [x]$.

But using (2) (proved in the last page)

$[z] = [y]$ and $[z] = [x]$.

Hence, $[x] = [y]$.

But since by (1), $y \in [y] = [x]$ this implies ~~that~~ $y \in [x]$, CONTRADICTING THE ASSUMPTION THAT $y \notin [x]$.

Thus, the assumption $[y] \cap [x] \neq \phi$ leads to a contradiction. Hence, $[y] \cap [x] = \phi$.

$\blacksquare$

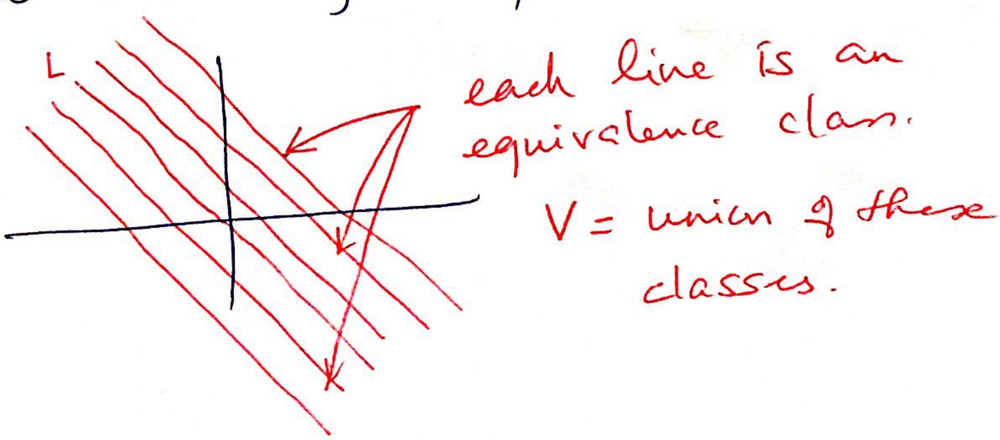
Properties (1), (2) and (3) imply together that the different equivalence classes of $\sim$ give a partition of X into disjoint subsets.

Key examples to remember:

(1) Let $V = \mathbb{R}^2$ and L a fixed linear subspace of dimension 1 (i.e. a line through the origin). Define an equivalence relation $\sim$ on V by

$u \sim v$ if and only if $u - v \in L$.

The equivalence class of a point $u \in V$ is the line containing u parallel to L.



Example 2

$X = \mathbb{Z}$ and define equivalence relation $\sim$ by $m \sim n$ if and only if $m-n$ is divisible by 4.

What are the equivalence classes of $\sim$?

How many equivalence classes are there?