Chapter 8. Permutations of a finite set.

S_n = group of all the permutations of $\{1, 2, \cdots, n\}$

Every permutation can be decomposed into simple parts called "cycles"

Ex:

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 6 & 8 & 2 & 3 & 1 & 5 & 9 & 7 \end{pmatrix} \iff \begin{matrix} 1 \to 4 \to 2 \to 6 \\ 3 \to 8 \to 9 \to 7 \to 5 \end{matrix}$$

cycle $(a_1\, a_2 \cdots a_s) \iff a_1 \to a_2 \to a_3 \to \cdots \to a_{s-1} \to a_s$ s: length of the cycle.

Ex: in S_6, $(1\,3\,2\,5) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 2 & 4 & 1 & 6 \end{pmatrix}$

$$(3\,4\,6) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 4 & 6 & 5 & 3 \end{pmatrix}$$

composition:

$$(1\,3\,2\,5)(3\,4\,6) = 1 \to 3 \to 4 \to 6 \to 2 \to 5 = (1\,3\,4\,6\,2\,5)$$

$$(3\,4\,6)(1\,3\,2\,5) = 1 \to 4 \to 6 \to 3 \to 2 \to 5 = (1\,4\,6\,3\,2\,5)$$

$$(2\,3\,4)(1\,3\,2\,5) = (1 \to 4 \to 2 \to 5)\ (3 \to 3)\ (6 \to 6) = (1\,4\,2\,5)$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 4 & 2 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 2 & 4 & 1 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 3 & 2 & 1 & 6 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 3 & 2 & 1 & 6 \end{pmatrix}$$

- Disjoint cycles commute.

<u>Thm</u>: Every permutation is either the identity, a single cycle, or a product of disjoint cycles.

A cycle of length 2 is called a transposition. Every cycle can be expressed as a product of one or more transpositions:

$$(a_1\, a_2 \cdots a_r) = (a_1\, a_r)(a_1\, a_{r-1}) \cdots (a_1\, a_3)(a_1\, a_2)$$

$$= (a_r\, a_{r-1})(a_r\, a_{r-2}) \cdots (a_r\, a_2)(a_r\, a_1)$$

$$= \cdots$$

in many other ways.

Every permutation $\longrightarrow$ decompose into cycles $\longrightarrow$ decomposition into transpositions.

decomposition is not unique, but the parity of the number of transpositions is unique.

even permutation   odd permutation.

Thm 2: No matter how ε is written as a product of transpositions, the number of transpositions is even.

Thm 3: If $\pi \in S_n$, then π cannot be both an odd permutation and an even permutation.

<u>Exer</u>: A. 1.(e) $(147)(1678)(74132) = (132846)(5)(7) = (132846)$

2.(a): $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 9 & 2 & 5 & 1 & 7 & 6 & 8 & 3 \end{pmatrix} = (145)(293)(67)(8)$

3.(b) $(416)(8235) = (46)(41)(53)(52)(58)$ odd permutation

6.(b) $(12345) = \alpha$. $\alpha^5 = \epsilon \Rightarrow \alpha = \alpha^{-4} = (\alpha^{-2})^2$

$\alpha^{-2} = (\alpha^{+2})^{-1} = (13524)^{-1} = (14253)$

check: $(14253)^2 = (12345) = \alpha$.

4.(f) $\gamma^3 \alpha^{-1} = (24135)^3 \cdot (3714)^{-1} = (12345)(1734) = (174235)$

B.4 $\alpha = (\alpha_1 \alpha_2 \cdots \alpha_s)$. $s = \text{even} \Rightarrow \alpha^2 = (\alpha_1 \alpha_3 \cdots \alpha_{s-1}) \cdot (\alpha_2 \alpha_4 \cdots \alpha_s)$

$s = \text{odd} \Rightarrow \alpha^2 = (\alpha_1 \alpha_3 \cdots \alpha_{s-2} \alpha_s \alpha_2 \alpha_4 \cdots \alpha_{s-3} \alpha_{s-1})$

F. 3. (a) $\alpha=(12)(345) \Rightarrow \alpha^6=\epsilon$ order of α is $6=\text{lcm}(2,3)$

H. 2. $T_1=\{(12),(13),\cdots,(1n)\}$ generates S_n:

$\begin{matrix} a\neq 1 \\ b\neq 1 \end{matrix} \Rightarrow (ab)=(1a)(1b)(1a) \quad (*)$

every permutation is generated by cycles. each cycle is generated by transpositions. each transposition is generated by elements of T_1, So T_1 generates S_n.

C.1.(d) $\underset{\text{odd}}{(1276)}\underset{\text{odd}}{(3241)}\underset{\text{odd}}{(7812)}$ is odd.

C.1.(a) $\pi=\begin{pmatrix} 1&2&3&4&5&6&7&8 \\ 7&4&1&5&6&2&3&8 \end{pmatrix} = \underset{\text{even}}{(173)}\underset{\text{odd}}{(2456)}\underset{\text{even}}{(8)}$ is odd