!!! WRITE YOUR NAME, STUDENT ID. BELOW !!!

NAME : PURDUE ID :

1(30pts) Is each of following groups a cyclic group or not? Explain the reason. If it is a cyclic group, then write down an element that generates the whole group.

(1) $\mathbb{Z}_2 \times \mathbb{Z}_2$;
(2) $\mathbb{Z}_2 \times \mathbb{Z}_3$;
(3) $\mathbb{Z}_2 \times \mathbb{Z}$.

(1). Not a cyclic group: The group has order 4 but each element has order at most 2. So no element can generate the whole group. 10

(2) It is cyclic group. $\underset{\overset{\shortparallel}{\mathrm{ord}((\bar{1},\bar{2}))}}{\mathrm{ord}((\bar{1},\bar{1}))}=6$ and

$$\underset{\overset{\shortparallel}{\langle(\bar{1},\bar{2})\rangle}}{\langle(\bar{1},\bar{1})\rangle} = \{(\bar{0},\bar{0}), (\bar{1},\bar{1}), \underset{(\bar{0},\bar{2})}{(\bar{2},\bar{2})}, \underset{(\bar{1},\bar{0})}{(\bar{3},\bar{3})}, \underset{(\bar{0},\bar{1})}{(\bar{4},\bar{4})}, \underset{(\bar{1},\bar{2})}{(\bar{5},\bar{5})}\}, \underset{(0,0)}{(\bar{6},\bar{6})}$$

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(3) Not a cyclic group: $\forall n \in \mathbb{Z}$ $(\bar{0}, n)$ can not generate the group

$$\overset{\shortparallel}{\{(\bar{0}, k\cdot n);\ k\in\mathbb{Z}\}}.$$

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$\langle(\bar{1},\bar{n})\rangle = \{(\bar{1},(2k+1)n), (\bar{0}, 2kn)\} \neq \mathbb{Z}_2\times\mathbb{Z}$ since

$(\bar{1}, 2kn) \notin \langle(\bar{1},\bar{n})\rangle$. $(\bar{0},(2k+1)n) \notin \langle(\bar{1},\bar{n})\rangle$

So no element can generate the whole group. and $\mathbb{Z}_2\times\mathbb{Z}$ is not a cyclic group.

2(30pts) Let $G = \langle a, b \mid a^2 = b^4 = e, ba = ab^3 \rangle$ be the group of symmetries of a square. Write down a table for G. Is G isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_4$?

	e	a	b	b^2	b^3	ab	ab^2	ab^3
e	e	a	b	b^2	b^3	ab	ab^2	ab^3
a	a	e	ab	ab^2	ab^3	b	b^2	b^3
b	b	ab^3	b^2	b^3	e	a	ab	ab^2
b^2	b^2	ab^2	b^3	e	b	ab^3	a	ab
b^3	b^3	ab	e	b	b^2	ab^2	ab^3	a
ab	ab	b^3	ab^2	ab^3	a	e	b	b^2
ab^2	ab^2	b^2	ab^3	a	ab	b^3	e	b
ab^3	ab^3	b	a	ab	ab^2	b^2	b^3	e

$b^2a = ab^6 = ab^2$

$b^3a = bab^2 = ab^5$
$= ab$

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G is not abelian
$\mathbb{Z}_2 \times \mathbb{Z}_4$ is abelian $\Rightarrow G$ is not isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_4$.

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3(20pts) G is the set of positive real numbers with the operation $x * y = 2xy$. Find an isomorphism of $(\mathbb{R}_{>0}, \times)$ to G.

w.r.t. $*$

$$\left(x * y = e \Leftrightarrow 2xy = \tfrac{1}{2} \Rightarrow y = x^{-1} = \tfrac{1}{4x}\right)$$

First find the identity of G: $x * e = x \quad \forall x$

$$\Updownarrow$$

$$2xe = x \quad \forall x \Rightarrow e = \frac{1}{2}.$$

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isomorphism maps identity to identity. natural to try:

$$f: (\mathbb{R}_{>0}, \times) \to G = (\mathbb{R}_{>0}, *).$$

$$x \mapsto \frac{1}{2}x.$$

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- Then f is clearly bijective: $f^{-1}(y) = 2y$.

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- f is a homomorphism:

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$$f(x \cdot y) = \frac{1}{2}x \cdot y = 2 \cdot \left(\frac{1}{2}x\right) \cdot \left(\frac{1}{2}y\right) = 2 f(x) \cdot f(y) = f(x) * f(y).$$

so f is an isomorphism.

4(20pts) Calculate the product $(123)(234)(123)^{-1}$ in S_4. Is the resulting permutation an odd or even permutation?

$$(123)^{-1} = \begin{pmatrix} 1 \rightarrow 2 \\ \nwarrow \;\; \swarrow \\ 3 \end{pmatrix}^{-1} = \begin{pmatrix} 1 \leftarrow 2 \\ \searrow \;\; \nearrow \\ 3 \end{pmatrix} = (1\,3\,2).$$ 3

$$(123)(234)(123)^{-1} = (123)(234)(132) = (143)(2) = (143).$$ 9

$(143) = (13)(14)$ is an even permutation. 8

$$(123)^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix} = (132)(4) = (132).$$