Kyle Manion CH14 C2,D3,5 CH15 A1,3,4,5 B3 C3,7 CH16 A3,5

C2 Assume f is injective, then if $x \in K$, then
$f(x)=e_H \Rightarrow f(xe)=f(x)f(e)$ since f is homomorphic
$\Rightarrow f(x)=e_H f(e) \Rightarrow f(x)=f(e)$ thus $x=e$
Hence $K=\{e\}$
Now assume $K=\{e\}$, then if $x \in K$, $x=e$
$f(x)=e_H \Rightarrow f(e)=e_H$

D3 Given a group G, H is a subgroup and the center of G if $H=\{h \in G, \forall g \in G, hg=gh\}$
assuming $ghgg^{-1}=gh \in H$, thus H is normal

D5 Assume H is not normal, then for $a \in G$ assume by contradiction
$aH \neq Ha$, thus $aHa^{-1} \neq Haa^{-1}$
$\Rightarrow aHa^{-1} \neq H$ However H is normal a contradiction
Now assume $aH=Ha \Rightarrow aHa^{-1}=Haa^{-1} \Rightarrow aHa^{-1} \in H$
Thus H is normal

3/3

CH15

A1 ~~$\mathbb{Z}_5$ and $\mathbb{Z}_{10}/\langle 5\rangle$~~
~~$f=\left(0\ 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ 13\ 14\ 15\ 16\ 17\ 18\ 19\right)$~~

A1 $G=\mathbb{Z}_{10}$ $H=\{0,5\}$ $G/H \cong \mathbb{Z}_5$
$G/H = \{0,5\}\ \{1,6\}\ \{2,7\}\ \{3,8\}\ \{4,9\}$
H $H+1$ $H+2$ $H+3$ $H+4$

+	H	H+1	H+2	H+3	H+4
H	H	H+1	H+2	H+3	H+4
H+1	H+1	H+2	H+3	H+4	H
H+2	H+2	H+3	H+4	H	H+1
H+3	H+3	H+4	H	H+1	H+2
H+4	H+4	H	H+1	H+2	H+3

4/4

B3 C3,7 $G/H \cong \mathbb{Z}_5$ since we can define a function that is bijective and homomorphic so long as

$$f = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ H & H+1 & H+2 & H+3 & H+4 \end{pmatrix}$$

A3 $G = D_4$ $H = \{R_0, R_2\}$ $R_0 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$ $R_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$

$D_4 = \{e, b, b^2, b^3, a, ab, ab^2, ab^3\} = \{R_0, R_1, R_2, R_3, R_4, R_5, R_6, R_7\}$

$G/H = \{R_0, R_2\}$ $\{R_1, R_3\}$ $\{R_4\, R_5\}$ $\{R_6\, R_7\}$

$\circ$	$\{R_0 R_2\}$	$\{R_1 R_3\}$	$\{R_4 R_5\}$	$\{R_6 R_7\}$
$\{R_0 R_2\}$	$(R_0 R_2)$	$\{R_1 R_3)$	$(R_4 R_5)$	$(R_6 R_7)$ =?
$\{R_1 R_3\}$	$(R_1 R_3)$	$(R_0 R_2)$	$(R_6 R_7)$	$(R_4 R_5)$ =?
$\{R_4 R_5\}$	$(R_4 R_5)$	$(R_6 R_7)$	$(R_0 R_2)$	$(R_1 R_3)$
$\{R_6 R_7\}$	$(R_6 R_7)$	$(R_4 R_5)$	$(R_1 R_3)$	$(R_0 R_2)$

need more details?

3/4

need to show the rotations in a table

A4 $G = D_4$ $H = \{R_0\, R_2\, R_4\, R_5\}$ NTS HR_1 HR_4 HR_6 compositions.

$G/H = \{R_0\, R_2\, R_4\, R_5\}$ $\{R_1\, R_3\, R_6\, R_7\} = H, \alpha$

G/H	H	α
H	H	α
α	α	H

(0,0) (0,1) (1,0) (1,1) (2,0) (2,1) (3,0) (3,1)

A5 $G = \mathbb{Z}_4 \times \mathbb{Z}_2$ $H = \langle(0,1)\rangle = \{(0,0)\ (0,1)\}$

$G/H = \{(0,0)\ (0,1)\}$ $\{(1,0)\ (1,1)\}$ $\{(2,0)\ (2,1)\}$ $\{(3,0)\ (3,1)\}$

G/H	H	a	b	c
H	H	a	b	c
a	a	b	c	H
b	b	c	H	a
c	c	H	a	b

33 a) $H = \{(x,y) : y = 2x\}$ say (a,b) $(c,d) \in H$

then $(a,b)+(c,d) = (a+c, b+d) \in H$

furthermore $(-x,-2x) \in H$, thus H is a subgroup and is normal since it is a subgroup of an abelian group

b) G/H

c)

13 for every $x \in G$, $\exists$ an integer n s.t. $x^n \in H$; then every element of G/H has finite order. Conversely if every element G/H has finite order, then for every $x \in G$ there is an integer n s.t $x^n \in H$

If $x \in G$, $(xH)^n = x^n H = H$, thus we can find some integer k s.t. $(x^n)^k = e$, thus $x^{nk} H = H$. Hence G/H is at most nk or finite

Say , let $xH \in G/H$ of finite order. Thus for some integer n $x^n H = (xH)^n = H$, thus $x^n \in H$.

Hence x^n is finite 8/13

C7 a) Suppose G/H is abelian. Let $a,b \in G$, and $x = aH$ $y = bH$
$xy = yx$, so $abH = baH$ and $ab = bah$ for some $h \in H$.
then $h = a^{-1}b^{-1}ab \in H$

b) $H \trianglelefteq K$, $K \trianglelefteq G \Rightarrow H \trianglelefteq G$ $G/H \leq H \leq K \leq G$, thus
G is abelian since abelian groups yeild abelian subgroups
$\underbrace{G/K \leq K \text{ and } K/H \leq H}$, thus G/K, K/H are abelian

need to show it! 2/3

CH16 A3 $\mathbb{Z}_2$ and $S_3/\{\varepsilon, \beta, \delta\}$ $f = \left\{ \begin{matrix} \varepsilon & \alpha & \beta & \gamma & \delta & \kappa \\ 0 & 0 & 0 & 1 & 1 & 1 \end{matrix} \right\}$
$\ker(f) = \{\varepsilon, \alpha, \beta\}$ = any rotation
by FHT $\mathbb{Z}_2 \cong S_3/\{\varepsilon, \beta, \delta\}$

A5 $\mathbb{Z}_3$ and $\mathbb{Z}_3 \times \mathbb{Z}_3/K$ $K = \{(0,0)\ (1,1)\ (2,2)\}$ $\mathbb{Z}_3 \times \mathbb{Z}_3 = \begin{matrix} (00) & (01) & (02) \\ (10) & (11) & (12) \\ (20) & (21) & (22) \end{matrix}$
$f(a,b) = a - b$
$$f(a,b) = \begin{pmatrix} (00) & (01) & (02) & (10) & (11) & (12) & (20) & (21) & (22) \\ 0 & 1 & 2 & 1 & 2 & 0 & 2 & 0 & 1 \end{pmatrix}$$
$\ker f = \{(0,0)(1,2)(2,1)\}$
f generats $\mathbb{Z}_3$, thus $\mathbb{Z}_3 \times \mathbb{Z}_3/K \cong \mathbb{Z}_3$ 3/3

Question C2

($\rightarrow$) Clearly $e \in K$ (theorem 1). However if $\exists a \in K$ where $a \neq e$, $f(a) = e$, f is not injective. Thus $K = \{e\}$.

($\leftarrow$) Let $a, b \in K$ and $f(a) = f(b)$. Thus $f(a) \cdot [f(b)]^{-1} = e$. By theorem 1, $[f(b)]^{-1} = f(b^{-1})$. Then $f(a) \cdot f(b^{-1}) = e$. Since f is homomorphic, $f(ab^{-1}) = e$, $ab^{-1} \in K$. Since $K = \{e\}$. Thus $ab^{-1} = e$. $a = b$

Question D3

Denote the center of the group C.

Lemma C forms a subgroup.

(1) $e \in C$

(2) If $a, b \in C$, then $\forall x \in G$ $ax = xa$, $bx = xb$. $abx = axb = xab$. Thus $ab \in C$. C is closed under multiplication.

(3) If $a \in C$ then $\forall x \in G$ $ax = xa$, $x = a^{-1}xa$, $xa^{-1} = a^{-1}x$. Thus $a^{-1} \in C$. C is closed under inverses. ■

$\forall a \in C$, $\forall g \in G$ we have $gag^{-1} = agg^{-1} = a \in C$. C is a normal subgroup.

Question D5

($\rightarrow$) H is normal. $a \in G$ $h \in H$ then $aha^{-1} \in H$ namely $h' = aha^{-1}$ then $h'a = ah$. Thus $Ha = aH$.

3/3

($\leftarrow$) $aH = Ha$. Then $aHa^{-1} = H$. Hence $\forall h \in H$, $aha^{-1} \in H$. H is normal.

Question A1

$G/H = \{\{0,5\},\{1,6\},\{2,7\},\{3,8\},\{4,9\}\}$. $G/H \cong \mathbb{Z}_5$ because we can find a bijection $f : G/H \to \mathbb{Z}_5$ such that $f(\{a,b\}) = a$ need to demonstrate in a table.

2/4

Question A3

$G/H = \{\{R_0, R_2\}, \{R_1, R_3\}, \{R_4, R_5\}, \{R_6, R_7\}\}$ Same here 2/4

Question A4

$G/H = \{\{R_0, R_2, R_4, R_5\}, \{R_1, R_3, R_7, R_6\}\}$

Question A5

$G/H = \{\{(0,0),(0,1)\},\{(1,0),(1,1)\},\{(2,0),(2,1)\},\{(3,0),(3,1)\}\}$

Question B3

(a) Since $\mathbb{R} \times \mathbb{R}$ is abelian, H is normal.

(b) G/H contains all lines on the plane parallel to $y = 2x$

(c) Shift a line which is parallel to $y = 2x$ up and down and preserve the slope.

Question C3

$\forall x \in G$ we have $xH \in G/H$. $x^n \in H$ thus $x^n H = H$ by theorem 5. Also $x^n H = (xH)^n$ by definition. Thus $xH \in G/H$ has finite orders.

Conversely, if $(xH)^n = H$, we know that $x^n H = H$. Hence $x^n \in H$ by theorem 5 (ii). 3/3

Question C7

(a) Let $a, b \in G$ $abH = baH$. Since $e \in H$, $\exists n \in H$ such that $abe = ban$, $n = a^{-1}b^{-1}ab$, $n \in H$

(b) Let $a, b \in K$, K is a subgroup of G so $a, b \in G$. G/H is abelian then, $abH = baH$. Therefore K/H is abelian.

why? need more details to prove it.

what about G/K 1.5/3