1(30pts) Is each of following groups a cyclic group or not? Explain the reason. If it is a cyclic group, then write down an element that generates the whole group.

(1) $\mathbb{Z}_2 \times \mathbb{Z}_2$; no
(2) $\mathbb{Z}_2 \times \mathbb{Z}_3$; yes
(3) $\mathbb{Z}_2 \times \mathbb{Z}$. no

(1) no

$(00)^2 = (00)$
$(01)^2 = (00)$
$(10)^2 = (00)$
$(11)^2 = (00)$

thus no element could generate a group of size 4.

(2) yes

$\langle(1,1)\rangle$ is $\mathbb{Z}_2 \times \mathbb{Z}_3$.

(3) no.

suppose $e = \langle a, b\rangle$ generates $\mathbb{Z}_2 \times \mathbb{Z}$
then b must generate $\mathbb{Z}$.
there are 2 elements generate $\mathbb{Z}$, -1 and 1
thus e can only be $(0,-1)(0,1)(1,-1)(1,1)$
try them one by one, none of them generates $\mathbb{Z}_2 \times \mathbb{Z}$
thus $\mathbb{Z}_2 \times \mathbb{Z}$ is not cyclic.

2(30pts) Let $G = \langle a, b \mid a^2 = b^4 = e, ba = ab^3 \rangle$ be the group of symmetries of a square. Write down a table for G. Is G isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_4$?

	ε	b	b^2	b^3	a	ab	ab^2	ab^3
ε	ε	b	b^2	b^3	a	ab	ab^2	ab^3
b	b	b^2	b^3	ε	ab	ab^2	ab^3	a
b^2	b^2	b^3	ε	b	ab^2	ab^3	a	ab
b^3	b^3	ε	b	b^2	ab^3	a	ab	ab^2
a	a	ab^3	ab^2	ab	ε	b^3	b^2	b
ab	ab	a	ab^3	ab^2	b	ε	b^3	b^2
ab^2	ab^2	ab	a	ab^3	b^2	b	ε	b^3
ab^3	ab^3	ab^2	ab	a	b^3	b^2	b	ε

not isomorphic

$\mathbb{Z}_2 \times \mathbb{Z}_4$ is communcative, G is not

3(20pts) G is the set of positive real numbers with the operation $x * y = 2xy$. Find an isomorphism of $(\mathbb{R}_{>0}, \times)$ to G.

let $f(x)$ be $G \to (\mathbb{R}_{>0}, \times)$

$f(x * x) = f(x) f(x)$

$f(x) = \sqrt{f(2x^2)}$

guess: $f(x) = ax^b$.

$f(1) = \sqrt{f(2)}$ $\quad a \cdot 1^b = \sqrt{a \cdot 2^b}$ $\quad a^2 = a \cdot 2^b$ $\quad a = 2^b$

try $b = 1$ $\quad f(x) = 2^1 \cdot x^1 = 2x$, clearly it is bijective.

$f(x * y) = f(2xy) = 4xy = 2x * 2y = f(x) f(y)$, it works.

thus $f(x) = 2x$ is an isomorphism.

actually any $f(x) = (2x)^b$ should work.

4(20pts) Calculate the product $(123)(234)\boxed{(123)^{-1}}$ in S_4. Is the resulting permutation an odd or even permutation?

(321)

1 2 3 4
3 1 2 4
4 1 3 2
4 2 1 3

it is $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}$, which is $(1\ 4\ 3)$.

since $(1\ 4\ 3) = (1\ 3)(1\ 4)$, it is even.