453 **Lecture 11**

1. Cosets and Lagrange's theorem

Let G be a group and H a subgroup of G.

Consider the relation on G defined by

$$a \sim b \iff ab^{-1} \in H.$$

This is an equivalence relation.

$a \sim a$ because $aa^{-1} = e \in H$

$a \sim b \Rightarrow ab^{-1} \in H \Rightarrow (ab^{-1})^{-1} \in H$

because H is closed under inverses

$\Rightarrow ba^{-1} \in H \Rightarrow b \sim a$.

$a \sim b$ & $b \sim c \Rightarrow ab^{-1} \in H$, $bc^{-1} \in H$

$\Rightarrow (ab^{-1})(bc^{-1}) \in H$

$\Rightarrow ac^{-1} \in H \Rightarrow a \sim c$.

because H is closed under taking products

The equivalence class

$$[b] = \{ a \in G \mid a \sim b \}$$

$$= \{ a \in G \mid ab^{-1} \in H \}$$

$$= Hb \longleftarrow \text{right coset}$$

It follows from the properties of equivalence relation that the distinct right cosets give a partition of G into disjoint subsets.

Def. The number of right cosets of H is called the index of H in G (denoted $[G:H]$).

Thm Suppose G is a finite group and H a subgroup of G. Then:

(1) $|Hb| = |H|$ for all $b \in G$

(2) $[G:H] \times |H| = |G|$

In particular, $|H| \mid |G|$.

Pf: We prove (1). (2) and (3) are then simple consequences of (1) and the previous observations.

We prove (1) by giving a bijection between H and Hb.

Let $f : H \longrightarrow Hb$ defined by

$$f(h) = hb \quad \text{for all } h \in H.$$

We prove that f is bijective by proving that f is surjective and injective.

(1) f is surjective. Given $hb \in Hb$, clearly $f(h) = hb$.

(2) f is injective:

Suppose $f(h) = f(h')$ for some $h, h' \in H$.

Then $hb = h'b$. Multiplying on the right by b^{-1} we obtain $h = h'$. This shows that f is injective. ☒