Chapter 16

Question B

1. $\alpha(f) \cdot \alpha(g) = f(1) \cdot g(1) = f \cdot g(1) = \alpha(f \cdot g)$

$\beta(f) \cdot \beta(g) = f(2) \cdot g(2) = f \cdot g(2) = \beta(f \cdot g)$

2. J is the kernel of α. α is onto because $\forall x \in R$ we can define a function $f : R \to R$, $f(1) = x$. By FHT, $\mathbb{R} \cong \mathscr{F}(\mathbb{R})/J$.

K is the kernel of β. β is onto because $\forall x \in R$ we can define a function $f : R \to R$, $f(2) = x$. By FHT, $\mathbb{R} \cong \mathscr{F}(\mathbb{R})/K$. 2/2

3. $\mathbb{R} \cong \mathscr{F}(\mathbb{R})/J$ and $\mathbb{R} \cong \mathscr{F}(\mathbb{R})/K$. Thus $\mathscr{F}(\mathbb{R})/J \cong \mathscr{F}(\mathbb{R})/K$

Question E

1. $f(x_1, y_1) \cdot f(x_2, y_2) = (Jx_1, Ky_1) \cdot (Jx_2, Ky_2) = (Jx_1x_2, Ky_1y_2) = f(x_1x_2, y_1y_2)$ 1.5/2

2. $\ker(f) = J \times K$. why? need some details. 1.5/2

3. By FHT, $(G \times H)/(J \times K) = (G/J) \times (H/K)$

2/2

Question H

1. $cis(x + y) = \cos(x + y) + i \sin(x + y)$

$(cis(x))(cis(y)) = (\cos(x) + i\sin(x))(\cos(y) + i\sin(y)) = \cos(x)\cos(y) + i\cos(x)\sin(y) + i\cos(y)\sin(x) - \sin(x)\sin(y)$

$\cos(x)\cos(y) = \frac{1}{2}[\cos(x+y)+\cos(x-y)]$, $\cos(x)\sin(y) = \frac{1}{2}[\sin(x+y)-\sin(x-y)]$

$\cos(y)\sin(x) = \frac{1}{2}[\sin(x+y)+\sin(x-y)]$, $\sin(x)\sin(y) = -\frac{1}{2}[\cos(x+y)-\cos(x-y)]$

By expanding the formulas, we get $cis(x)cis(y) = \cos(x + y) + i\sin(x + y)$

Thus $cis(x)cis(y) = cis(x + y)$

2. $(cis(x)cis(y))cis(z) = cis(x+y)cis(z) = cis(x+y+z)$

$cis(x)(cis(y)cis(z)) = cis(x)cis(y+z) = cis(x+y+z)$

$(T,*)$ is associative.

$e = cis(0)$, $(cis(x))^{-1} = cis(-x)$ because $cis(x)cis(-x) = cis(x-x) = cis(0)$

$(T,*)$ has identity $cis(0)$, inverse of $cis(x)$ is $cix(-x)$ 2|2

3. $f(x)f(y) = cis(x)cis(y) = cis(x+y) = f(x+y)$

4. Since we know the property of trig functions: $\sin(x+2\pi) = \sin(x)$ and $\cos(x+2\pi) = \cos(x)$. Thus $cis(2n\pi) = \cos(2n\pi) + i\sin(2n\pi) = 0$ where $n \in \mathbb{Z}$. Hence, $\ker f = \langle 2\pi \rangle$ 2|2

5. Since f is a homomorphism from $\mathbb{R}$ onto T. $\langle 2\pi \rangle$ is the kernel of f. By FHT, $T \cong \mathbb{R}/\langle 2\pi \rangle$

6. $g(x)g(y) = (cis2\pi x)(cis2\pi y) = cis(2\pi(x+y)) = g(x+y)$. From the image we know that $cis(2\pi x) = cis(0)$ iff $x \in \mathbb{Z}$. Hence the kernel of g is $\mathbb{Z}$. 3|3

7. Since g is a homomorphism from $\mathbb{R}$ onto T with kernel $\mathbb{Z}$. By FHT, $T \cong \mathbb{R}/\mathbb{Z}$

Chapter 17

Question A

2. First, we need to prove that $(A, \oplus)$ is an abelian group.

$a \oplus b = a+b+1 = b+a+1 = b \oplus a$ Commutative.

$(a \oplus b) \oplus c = (a+b+1)+c+1 = a+(b+c+1)+1 = a \oplus (b \oplus c)$. Associative.

$a \oplus e = a$, $a+e+1 = a$, $e = -1$. Identity is -1.

$a \oplus a^{-1} = e$. $a + a^{-1} + 1 = -1$. $a^{-1} = -2-a$. Inverse of a is $-2-a$.

Then we need to show $\odot$ is associative.

$(a \odot b) \odot c = (ab+a+b) \odot c = (ab+a+b)c+ab+a+b+c = abc+ac+bc+ab+a+b+c$

$a \odot (b \odot c) = a \odot (bc+b+c) = abc+ab+ac+a+bc+b+c = abc+ac+bc+ab+a+b+c$

Thus $\odot$ is associative.

Then we need to show $a \odot (b \oplus c) = a \odot b + a \odot c$

$a \odot (b \oplus c) = a(b+c+1) + a + (b+c+1) = ab+ac+2a+b+c+1$

$a \odot b \oplus a \odot c = ab+a+b+ac+a+c+1$

Hence $a \odot (b \oplus c) = a \odot b \oplus a \odot c$

$\odot$ is commutative, $a \odot b = ab + a + b = ba + b + a = b \odot a$

$(b \oplus c) \odot a = b \odot a \oplus b \odot c$ automatically holds.

$a \odot 0 = a$. $(A, \oplus, \odot)$ has a unity 0

$\therefore (A, \oplus, \odot)$ is a commutative ring with unity 0. The *zero* element is -1.

$\ominus a = -2 - a$

6. zero is -1. Let $a \odot a^{-1} = 0$, $aa^{-1} + a + a^{-1} = 0$, $a^{-1}(a+1) + a = 0$. Hence $a^{-1} = -\dfrac{a}{a+1}$. There exists an inverse for all nonzero elements.

Question C

1. Addition is commutative.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} r & s \\ t & u \end{pmatrix} = \begin{pmatrix} a+r & b+s \\ c+t & d+u \end{pmatrix} = \begin{pmatrix} r+a & s+b \\ t+c & u+d \end{pmatrix} = \begin{pmatrix} r & s \\ t & u \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

$M_2(\mathbb{R}) \cong \mathbb{R}^4$. R^4 is a group. Thus $M_2(\mathbb{R})$ is an abelian group.

$$\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\right) \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \left(\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix}\right).$$

It's easy to show but too difficult for me to calculate.

why need ~~us~~ to show them

It's also easy to show that

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \left(\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} + \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix}\right) =$$

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} + \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix}$$

need to write some details.

1.5/3

and

$$\left(\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} + \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix}\right) \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} =$$

$$\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$$

2. $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

However, $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$

Ch. 16 B.1-3, E.1-3, H.1-7 MA 45300

Ch. 17 A.2, A.6, C.1-3

B.1 $\alpha : \mathcal{F}(\mathbb{R}) \longrightarrow \mathbb{R}$

$\alpha(fg) = (fg)(1) = f(1)g(1) = \alpha(f)\alpha(g)$

$\forall c \in \mathbb{R}$, let $f(x) = cx$, so $f(1) = c$. Then every $x \in \mathbb{R}$ is an image of some $f(1)$, $f \in \mathcal{F}(\mathbb{R})$.

Similarly, $\beta : \mathcal{F}(\mathbb{R}) \to \mathbb{R}$, $\beta(fg) = (fg)(2) = f(2)g(2) = \beta(f)\beta(g)$

$\forall c \in \mathbb{R}$, let $f(x) = \frac{1}{2}cx$, so $f(2) = c$. Then every $x \in \mathbb{R}$ is an image $f(2)$, some $f \in \mathcal{F}(\mathbb{R})$

Hence α and β are homomorphisms from $\mathcal{F}(\mathbb{R})$ onto $\mathbb{R}$.

B.2 J is the kernel of α, and $K = \ker(\beta)$.

Then $\alpha : \mathcal{F}(\mathbb{R}) \underset{J}{\twoheadrightarrow} \mathbb{R} \Rightarrow \mathbb{R} \cong \mathcal{F}(\mathbb{R})/J$ 2|2

And $\beta : \mathcal{F}(\mathbb{R}) \underset{K}{\twoheadrightarrow} \mathbb{R} \Rightarrow \mathbb{R} \cong \mathcal{F}(\mathbb{R})/K$

B.3 $\mathbb{R} \cong \mathcal{F}(\mathbb{R})/J$

$\mathbb{R} \cong \mathcal{F}(\mathbb{R})/K$

$\Rightarrow \mathcal{F}(\mathbb{R})/J \cong \mathcal{F}(\mathbb{R})/K$.

E.1 $f : G \times H \longrightarrow G/J \times H/K$.

$f((x,y) * (t,u)) = f(x*t, y*u) = (J(x*t), K(y*u))$

$= (J(xt), K(yu)) = ((Jx)*(Jt), (Ky)*(Ku)) = f(x,y) * f(t,u)$

$\cancel{1}$ .5|2 is it onto?

E.2 $\mathrm{Ker}(f) = \{(x,y);\ x \in J, y \in K\}$. $x \in J, y \in K \Leftrightarrow Jx = J, Ky = K$. ($\ker = J \times K$)

2|2

E.3 $f : G \times H \underset{J \times K}{\twoheadrightarrow} G/J \times H/K$

$\Rightarrow G \times H / J \times K \cong G/J \times H/K$ by FHT

2|2

H.1 $cis(x+y) = \cos(x+y) + i\sin(x+y) = \cos x\cos y - \sin x \sin y + i \sin x \cos y + i \sin y \cos x$

$= \cos x \cos y + i \cos x \sin y + i \sin x \cos y + i^2 \sin x \sin y$

$= (\cos x + i \sin x)(\cos y + i \sin y)$.

H.2 $cis(x+(y+z)) = cis\, x\,(cis\, y\, cis\, z) = cis\, x\, cis\, y\, cis\, z = (cis\, x\, cis\, y)\, cis\, z = cis((x+y)+z)$

$e = 0$; $cis(x+0) = cis\, x\, cis\, 0 = cis\, x\,(1) = cis\, x$

$a^{-1} = -a$; $cis(x+(-x)) = cis\, 0 = 1$ ✓

$cis(x+y) = cis\, x\, cis\, y = cis\, y\, cis\, x = cis(y+x) \Rightarrow$ abelian

2/2

H.3 $f(a+b) = cis(a+b) = cis\, a\, cis\, b = f(a)f(b)$

$\sin \pi z = 0 \Rightarrow z \in \mathbb{Z}$ or $z = 2\pi n$

H.4 $Ker(f) = \{x;\ cis\, x = 1 \Leftrightarrow \cos x = 1\} \overset{\text{or}}{\Leftrightarrow} \{2\pi n;\ n \in \mathbb{Z}\} = \langle 2\pi \rangle$

need some details if $z = \pi n \to \cos z = -1$ 2/2

H.5 $f: \mathbb{R} \xrightarrow[\langle 2\pi \rangle]{} T$ to according.

$\Rightarrow T \cong \mathbb{R}/\langle 2\pi \rangle$.

H.6 $g(x+y) = cis(2\pi(x+y)) = cis(2\pi x + 2\pi y) = cis\, 2\pi x + cis\, 2\pi y = g(x)g(y)$

$Ker(g) = \{x;\ cis\, 2\pi x = 1 \Longleftrightarrow \cos 2\pi x = 1\} \Leftrightarrow x \in \mathbb{Z} \Leftrightarrow ker(g) = \mathbb{Z}$.

3/3

H.7 Then $g: \mathbb{R} \xrightarrow[\mathbb{Z}]{} T$, so $T \cong \mathbb{R}/\mathbb{Z}$.

A.2 $(\mathbb{Q}, \oplus)$: $(a \oplus b) \oplus c = a+b+c+2 = a \oplus (b \oplus c)$ ✓

$\underline{e = -1}$: $a \oplus -1 = a+1-1 = a$, $-1 \oplus a = -1+a+1 = a$ ✓

$\underline{a^{-1} = -(a+2)}$: $a \oplus a^{-1} = a+1-(a+2) = -1 = e$, $a^{-1} \oplus a = -a-2+a+1 = -1$ ✓

$a \oplus b = a+b+1 = b+a+1 = b \oplus a$ ✓

$(a \odot b) \odot c = (ab+a+b) \odot c = abc + ac + bc + ab + a + b + c$

$= a \odot (bc + b + c) = a \odot (b \odot c)$ ✓

$a \odot (b \oplus c) = a \odot (b+c+1) = ab + ac + a + a + b + c + 1 = (a \odot b) \oplus (a \odot c)$ ✓

$\underline{unity = 0}$: $a \odot 0 = 0 + a + 0 = a$

A.6 $\underline{a^{-1} = -\frac{a}{a+1}}$: $a^{-1} \odot a = -\frac{a^2}{a+1} + a - \frac{a}{a+1} = \frac{-a^2+a^2+a-a}{a+1} = 0$

$\Rightarrow$ and commutative

$\Rightarrow$ is field

C.1 $\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} r & s \\ t & u \end{pmatrix}\right) + \begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} a+r & b+s \\ c+t & d+u \end{pmatrix} + \begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} a+r+w & b+s+x \\ c+t+y & d+u+z \end{pmatrix}$

$= \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} r+w & s+x \\ t+y & u+z \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \left(\begin{pmatrix} r & s \\ t & u \end{pmatrix} + \begin{pmatrix} w & x \\ y & z \end{pmatrix}\right)$ ✓

$e = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$a^{-1} = -a : \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ 3|3

$a+b = b+a \implies \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} r & s \\ t & u \end{pmatrix} = \begin{pmatrix} r & s \\ t & u \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\left[\begin{pmatrix} r & s \\ t & u \end{pmatrix}\begin{pmatrix} w & x \\ y & z \end{pmatrix}\right] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} rw+sy & rx+sz \\ tw+uy & tx+uz \end{pmatrix} = \begin{pmatrix} arw+asy+btw+buy & arx+asz+btx+buz \\ crw+csy+dtw+duy & crx+csz+dtx+duz \end{pmatrix}$

$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} r & s \\ t & u \end{pmatrix}\right]\begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} ar+bt & as+bu \\ cr+dt & cs+du \end{pmatrix}\begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} arw+btw+asy+buy & arx+btx+asz+buz \\ crw+dtw+csy+duy & crx+dtx+csz+duz \end{pmatrix}$ ✓

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\left[\begin{pmatrix} r & s \\ t & u \end{pmatrix} + \begin{pmatrix} w & x \\ y & z \end{pmatrix}\right] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} r+w & s+x \\ t+y & u+z \end{pmatrix} = \begin{pmatrix} ar+aw+bt+by & as+ax+bu+bz \\ cr+cw+dt+dy & cs+cx+du+dz \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} r & s \\ t & u \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} w & x \\ y & z \end{pmatrix}$

C.2 $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} r & s \\ t & u \end{pmatrix} = \begin{pmatrix} ar+bt & as+bu \\ cr+dt & cs+du \end{pmatrix} \neq \begin{pmatrix} ra+sc & rb+sd \\ ta+uc & tb+ud \end{pmatrix} = \begin{pmatrix} r & s \\ t & u \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

unity $= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} : \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} ; \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

C.3 Not all matrices are invertible, so $M_2(\mathbb{R})$ cannot be a field.
Additionally, if a matrix does not have full rank, $a \cdot x = 0$ may have infinitely many solutions, so is not an integral domain. 2|2