Chap 7: groups of permutations.

- $f: A \to B$, $g: A \to B$ are equal iff $f(x) = g(x)$ $\forall x \in A$.

- $f: A \to A$, $g: A \to A$ $\quad (g \circ f)(x) = g(f(x))$. $(f \circ g)(x) = f(g(x))$.

  $g \circ f \neq f \circ g$ in general: $\quad f(x) = 2x$, $g(x) = x+1$ $\quad (g \circ f)(x) = 2x+1$, $(f \circ g)(x) = 2(x+1) = 2x+2$.

- composition of functions is associative:

  $f, g, h : A \to A \Rightarrow f \circ (g \circ h) = (f \circ g) \circ h$.

  $f \circ (g \circ h)(x) = f((g \circ h)(x)) = f(g(h(x))) = (f \circ g)(h(x)) = (f \circ g) \circ h\,(x)$.

Def: a permutation of a set A is a bijective function from A to A
i.e. a one-to-one correspondence between A and itself.

($\Leftrightarrow$ a permutation is a rearrangement of elements of a set.)

Composite of any two permutations of A is a permutation of A.

$\Longrightarrow$ the operation $\circ$ of composition is an operation on the set of all the permutations of A.

Thm: The set of all the permutations of A, with the operation $\circ$ of composition, is a group.

Pf: G1: $(f \circ g) \circ h = f \circ (g \circ h)$

G2: identity function on A: $\varepsilon: A \to A$, $\varepsilon(x) = x$ $\forall x \in A$ $\quad$: $\underline{f \circ \varepsilon = \varepsilon \circ f = f}$

G3: inverse of a permutation f is f^{-1}. $\forall x \in A$

$(f \circ f^{-1})(x) = f(f^{-1}(x)) = x = \varepsilon(x)$. $(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x = \varepsilon(x)$

$\Rightarrow \quad f \circ f^{-1} = f^{-1} \circ f = \varepsilon$.

Def: For any set A, the group of all the permutations of A is called the symmetric group on A, and ~~it~~ is represented by S_A.

The symmetric group on the set $\{1, 2, 3, \cdots, n\}$ is called the symmetric group on n elements, and is denoted by S_n.

Ex: S_3: $\varepsilon = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$, $\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$, $\beta = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$

$\gamma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$, $\delta = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, $\kappa = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$.

$[\beta \circ \gamma](1) = \beta(\gamma(1)) = \beta(2) = 1$

$[\beta \circ \gamma](2) = \beta(\gamma(2)) = \beta(1) = 3$ $\quad \Rightarrow \quad \beta \circ \gamma = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = \alpha$.

$[\beta \circ \gamma](3) = \beta(\gamma(3)) = \beta(3) = 2$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \gamma$$

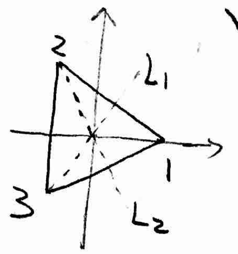
$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \beta$$

$\beta^2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \delta$

$\alpha\beta^2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \beta\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \kappa$

$\Rightarrow \boxed{S_3 = \langle \alpha, \beta \mid \alpha^2 = \varepsilon, \beta^3 = \varepsilon. \ \beta\alpha = \alpha\beta^2 \rangle}$

$\beta^2\alpha = \alpha\beta^4 = \alpha\beta$

Ex: 

$S_3 \cong$ group of symmetries of the equilateral triangle:

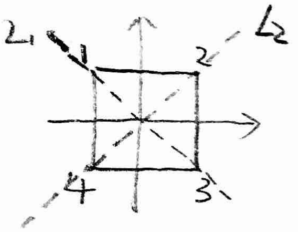
- reflection about the x-axis: $\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$
- rotation by $120°$ clockwise: $\beta = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$

$\gamma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$: reflection about L_1

$\delta = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$: rotation by $120°$ counter-clockwise

$\kappa = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$: reflection about L_2

Ex: group of symmetries of the square.

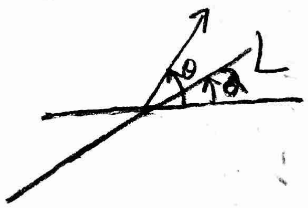


a: reflection about the x-axis $= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$

b: rotation of $90°$ counterclockwise $= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$

$b^2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$ rotation of $180°$. $b^3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$ rotation of $270°$

relation: $a^2 = e = b^4$ Another relation: $\boxed{a^{-1} \cdot b \cdot a = b^{-1} = b^3}$



$$\theta \xrightarrow[a]{\text{reflection about } L} 2\alpha - \theta \xrightarrow[b]{\text{rotation by } \beta} 2\alpha - \theta + \beta$$

$$\xrightarrow[a^{-1}]{\text{reflection about } L} 2\alpha - (2\alpha - \theta + \beta) = \theta - \beta$$

rotation by $-\beta$.

$\Rightarrow$ Other elements:

$$ab = \begin{matrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} \\ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \end{matrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix} \text{ reflection about } L_1 = \overline{13}.$$

$$ab^2 = \begin{pmatrix}1&2&3&4\\3&4&1&2\end{pmatrix}\begin{pmatrix}1&2&3&4\\4&3&2&1\end{pmatrix} = \begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix}$$ reflection about y-axis

$$ab^3 = \begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix}\begin{pmatrix}1&2&3&4\\4&3&2&1\end{pmatrix} = \begin{pmatrix}1&2&3&4\\3&2&1&4\end{pmatrix}$$ reflection about $L_2 = \overline{24}$

$$G = D_4 = \langle a, b \mid a^2 = e,\ b^4 = e,\ ba = ab^3 \rangle.$$

More generally $D_n = \langle a, b \mid a^2 = e,\ b^n = e,\ ba = ab^{n-1} \rangle$ is the group of symmetries of the regular polygon with n sides, and is called the n-th dihedral group. $|D_n| = 2n$

$$D_n = \{e, a, b, \cdots, b^{n-1}, ab, \cdots, ab^{n-1}\}$$

- every plane figure which exhibits regularities has a group of symmetries artificial as well as natural objects often have a surprising number of symmetries
- Modern-day crystallography and crystal physics rely heavily on group theory of symmetries of 3-dim. shapes.
- Groups of symmetries are widely employed in the theory of electron structure and of molecular vibrations. In elementary particle physics such groups have been used to predict the existence of certain elementary particles before they were found experimentally.
- Symmetries and their groups in nature: quantum physics, flower petals, cell division, the work habits of bees in the hive, snowflakes, music, Romanesque cathedrals.

Exer. A.1 $f=\begin{pmatrix}1&2&3&4&5&6\\6&1&3&5&4&2\end{pmatrix}$ $g=\begin{pmatrix}1&2&3&4&5&6\\2&3&1&6&5&4\end{pmatrix}$

$f^{-1}=\begin{pmatrix}1&2&3&4&5&6\\2&6&3&5&4&1\end{pmatrix}$. $f\circ g^{-1}=\begin{pmatrix}1&2&3&4&5&6\\3&6&1&2&4&5\end{pmatrix}$

C. 2: $A=\mathbb{R}^*$ = set of all nonzero real numbers.

$G=\{\varepsilon, f, g, h\}$. $f(x)=\frac{1}{x}$, $g(x)=-x$. $h(x)=-\frac{1}{x}$

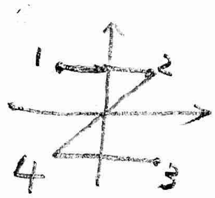
	ε	f	g	h
ε	ε	f	g	h
f	f	ε	h	g
g	g	h	ε	f
h	h	g	f	ε

$f\circ f=\frac{1}{\left(\frac{1}{x}\right)}=x$. $f\circ g=\frac{1}{-x}=-\frac{1}{x}$

$f\circ h=\frac{1}{-\frac{1}{x}}=-x=g$. $g\circ f=-\frac{1}{x}=h$

$g\circ g=-(-x)=x$ $g\circ h=-\left(-\frac{1}{x}\right)=\frac{1}{x}=f$

F. 3 symmetries of the letter Z



$G=\{e, a\}$ a: rotation by $180°$

G. symmetries of polynomials

2. $p=(x_1-x_2)(x_2-x_3)(x_1-x_3)$. gp. of symmetries $\subset S_3$:

$\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}$, $\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}$, $\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}$, $\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}$, $\begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}$

$\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix} \to (x_1-x_3)(x_3-x_2)(x_1-x_2) = -p$

$\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix} \to (x_2-x_1)(x_1-x_3)(x_2-x_3) = -p$

$\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix} \to (x_2-x_3)(x_3-x_1)(x_2-x_1) = (x_2-x_3)(x_1-x_3)(x_1-x_2) = p$

$\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix} \to p$

$\begin{pmatrix}1&2&3\\3&2&1\end{pmatrix} \to -p$

$\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix} \to p$

$G=\left\{\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}, \begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}, \begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}\right\} \cong A_3$

even permutations