Chapter 4

Question G.1

(G1) $(x_1, y_1)[(x_2, y_2)(x_3, y_3)] = (x_1, y_1)(x_2x_3, y_2y_3) = (x_1x_2x_3, y_1y_2y_3)$

$[(x_1, y_1)(x_2, y_2)](x_3, y_3) = (x_1x_2, y_1y_2)(x_3, y_3) = (x_1x_2x_3, y_1y_2y_3)$

$(x_1, y_1)[(x_2, y_2)(x_3, y_3)] = [(x_1, y_1)(x_2, y_2)](x_3, y_3)$

(G2) (e_G, e_H) is the identity element.

$(e_G, e_H)(x, y) = (x, y)$, $(x, y)(e_G, e_H) = (x, y)$

(G3) Inverse of (a, b) is (a^{-1}, b^{-1})

So $G \times H$ is a group
write details 4/4

Question G.2

$+$	$(0,0)$	$(0,1)$	$(0,2)$	$(1,0)$	$(1,1)$	$(1,2)$
$(0,0)$	$(0,0)$	$(0,1)$	$(0,2)$	$(1,0)$	$(1,1)$	$(1,2)$
$(0,1)$	$(0,1)$	$(0,2)$	$(0,0)$	$(1,1)$	$(1,2)$	$(1,0)$
$(0,2)$	$(0,2)$	$(0,0)$	$(0,1)$	$(1,2)$	$(1,0)$	$(1,1)$
$(1,0)$	$(1,0)$	$(1,1)$	$(1,2)$	$(0,0)$	$(0,1)$	$(0,2)$
$(1,1)$	$(1,1)$	$(1,2)$	$(1,0)$	$(0,1)$	$(0,2)$	$(0,0)$
$(1,2)$	$(1,2)$	$(1,0)$	$(1,1)$	$(0,2)$	$(0,0)$	$(0,1)$

Question G.3

$(x, y) \in G \times H$, $(x', y') \in G \times H$

$(x, y)(x', y') = (xx', yy')$. Since $x, x' \in G$ and $y, y' \in H$,

$(xx', yy') = (x'x, y'y) = (x', y')(x, y)$ ■

Chapter 5

Question D.3

Let $a, b \in C$, $ax = xa$ and $bx = xb$. Then $axbx = xaxb$, then $xabx = xxab$. Multiply both sides by x^{-1}, we get $abx = xab$, also $(ab)x = x(ab)$. Thus $ab \in C$. C is closed under multiplication.

Let $a \in C$, $ax = xa$. Multiply both sides a^{-1} on the left, $a^{-1}ax = a^{-1}xa$, then $x = a^{-1}xa$. Multiply both sides a^{-1} on the right, $xa^{-1} = a^{-1}x$. a^{-1} commutes with x. $a^{-1} \in C$. Thus C is closed under inverses.

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H is a subgroup of G. ■

Question E.5

$7 + 7 = 14$ then $14 \in Z$, then $-14 \in Z$, $-14 + 5 + 5 + 5 = 1$ then $1 \in Z$, also $-1 \in Z$. Then if $a \in Z$, $a - 1 \in Z$, $a + 1 \in Z$ Thus generates the group. ■

Question E.6

Start with $(1,1)$,

$(1,1)+(1,1)=(0,2)$

$(1,1)+(1,1)+(1,1)=(1,0)$

$(1,1)+(1,1)+(1,1)+(1,1)=(0,1)$

$(1,1)+(1,1)+(1,1)+(1,1)+(1,1)=(1,2)$

$(1,1)+(1,1)+(1,1)+(1,1)+(1,1)+(1,1)=(0,0)$ Is it cyclic??.

Start with $(1,1)$,

$(1,1)+(1,1)=(2,2)$

$(1,1)+(1,1)+(1,1)=(0,3)$

$(1,1)+(1,1)+(1,1)+(1,1)=(1,0)$

$(1,1)+(1,1)+(1,1)+(1,1)+(1,1)=(2,1)$

$(1,1)+(1,1)+(1,1)+(1,1)+(1,1)+(1,1)=(0,2)$ $\frac{3.5}{4}$

$7(1,1)=(1,3)$

$8(1,1)=(2,0)$

$9(1,1)=(0,1)$

$10(1,1)=(1,2)$ is it cyclic??.

$11(1,1)=(2,3)$

$12(1,1)=(0,0)$

6) $Z_2 \times Z_3$

Show one element generated all element

Consider $(1,1)$

$(1,1) = (1,1)$

$2(1,1) = (1,1) + (1,1) = (0,2)$

$3(1,1) = (1,1) + \overset{3 \text{ times}}{\cdots} (1,1) = (1,0)$

$4(1,1) = (1,1) + \cdots (1,1) = (0,1)$

$5(1,1) = (1,1) + \cdots (1,1) = (1,2)$

$6(1,1) = (1,1) + \underset{\text{times}}{6} (1,1) = (0,0)$

$\therefore$ All 6 elements are generated by $(1,1)$

$\therefore$ Cyclic

$Z_3 \times Z_4$

Consider $(1,1)$

$(1,1) = (1,1)$

$2(1,1) = (1,1) + (1,1) = (2,2)$

$3(1,1) = (1,1) + \cdots (1,1) = (0,3)$

$4(1,1) = (1,1) + \cdots (1,1) = (1,0)$

$5(1,1) =$ 5 times $= (2,1)$

$6(1,1) = (0,2)$

$7(1,1) = (1,3)$

$8(1,1) = (2,0)$

$9(1,1) = (0,1)$

$10(1,1) = (1,2)$

$11(1,1) = (2,3)$

$12(1,1) =$ 12 times $= (0,0)$

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$\therefore$ All 12 elements are generated by $(1,1)$

$\therefore$ Cyclic

F2) $G = \{e, a, b, b^2, b^3, ab, ab^2, ab^3)$

$a^2 = e, \; b^4 = e, \; ba = ab^3$

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	e	a	b	b^2	b^3	ab	ab^2	ab^3
e	e	a	b	b^2	b^3	ab	ab^2	ab^3
a	a	$a^2(e)$	ab	ab^2	ab^3	b	b^2	b^3
b	b	ab^3	b^2	b^3	e	a	ab	ab^2
b^2	b^2	ab^2	b^3	e	b	ab^3	a	ab
b^3	b^3	ab	e	b	b^2	ab^2	ab^3	a
ab	ab	b^3	ab^2	ab^3	a	e	b	b^2
ab^2	ab^2	b^2	ab^3	a	ab	b^3	e	b
ab^3	ab^3	b	a	ab	ab^2	b^2	b^3	e

Question F.2

$*$	e	a	b	b^2	b^3	ab	ab^2	ab^3
e	e	a	b	b^2	b^3	ab	ab^2	ab^3
a	a	e	ab	ab^2	ab^3	b	b^2	b^3
b	b	ab^3	b^2	b^3	e	a	ab	ab^2
b^2	b^2	ab^2	b^3	e	b	ab^3	a	ab
b^3	b^3	ab	e	b	b^2	ab^2	ab^3	a
ab	ab	b^3	ab^2	ab^3	a	e	b	b^2
ab^2	ab^2	b^2	ab^3	a	ab	b^3	e	b
ab^3	ab^3	b	a	ab	ab^2	b^2	b^3	e

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Question F.2

$*$	e	a	b	c	ab	bc	ac	abc
e	e	a	b	c	ab	bc	ac	abc
a	a	e	ab	ac	b	abc	c	bc
b	b	ab	e	bc	a	c	abc	ac
c	c	ac	bc	e	abc	b	a	ab
ab	ab	b	a	abc	e	ac	bc	c
bc	bc	abc	c	b	ac	e	ab	a
ac	ac	c	abc	a	bc	ab	e	b
abc	abc	bc	ac	ab	c	a	b	e

Chapter 6

Question B.5

$f : \mathbb{Z} \to E$ defined by $f(x) = 2 \cdot x$

Question C.5

$f(x) = x^{-1}$

f is injective,

Proof: Suppose $f(a) = f(b)$, i.e. $a^{-1} = b^{-1}$, by uniqueness of inverses, $a = b$ ■

f is surjective,

$\frac{3.5}{4}$

Proof: Take any element $y \in G$, then $y = (y^{-1})^{-1} = f(y^{-1})$. Thus every y is equal to $f(x)$ for $x = y^{-1}$. ■

why?. Since we are in group and every element has inverse

Question D.5

$$f \circ g = \begin{pmatrix} a & b & c & d \\ c & a & c & a \end{pmatrix}, g \circ f = \begin{pmatrix} a & b & c & d \\ b & b & b & b \end{pmatrix}$$