ASSIGNMENT 10. DUE IN CLASS DEC 1, 2017.

The goal of this multi-part assignment is to learn about *semi-direct* products of groups.

1. Let N and H be groups. Write down the definition of the automorphism group, $\mathrm{Aut}(N)$, of N.
2. Let $\phi : H \to \mathrm{Aut}(N)$ be a group homomorphism. Let $\cdot_\phi$ denote the binary operation on the *set* of ordered pairs $N \times H$ defined by:
$$(n_1, h_1) \cdot_\phi (n_2, h_2) = (n_1\phi(h_1)(n_2), h_1h_2).$$
Prove that the binary operation $\cdot_\phi$ gives a structure of a group on the *set* of ordered pairs $N \times H$. (We denote this group by $N \rtimes_\phi H$ (semi-direct product of N and H). Notice that the group structure depends on the homomorphism ϕ, and different homomorphisms can produce non-isomorphic groups. Notice also the assymmetric nature of the symbol $\rtimes$ indicating the different roles of N and H (in contrast with the case of the ordinary direct product).)
3. Prove that the subset $\{(n, e) \in N \times H \mid n \in N\}$ is a *normal* subgroup of $N \rtimes_\phi H$.
4. Prove that if ϕ is the trivial homomorphism, then $N \rtimes_\phi H$ is isomorphic to the ordinary direct product $N \times H$ of the groups $N$and H that we have discussed in class.
5. Let $N = Z_4$ and $H = Z_2$.
   (a) Identify the group $\mathrm{Aut}(N)$.
   (b) How many distinct homomorphisms $\phi : H \to \mathrm{Aut}(N)$ are there in this case ?
   (c) Realize the two non-isomorphic groups, $Z_4 \times Z_2$ and D_8, as semi-direct products of Z_4 and Z_2 (by choosing different homomorphisms ϕ).