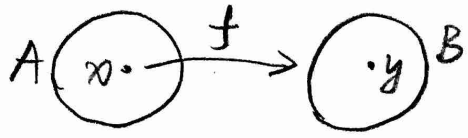
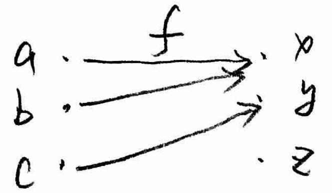
Chap 6: Functions



$f: A \to B$
↑ domain ↑ target

Range of $f = \{$images of elements of $A\}$
$= \{f(a);\ a \in A\}$.

<u>Ex</u>: $A = \{a, b, c\}$. $B = \{x, y, z\}$



$f = \begin{pmatrix} a & b & c \\ x & x & y \end{pmatrix}$. Range of $f = \{x, y\}$

<u>Def</u>: $f: A \to B$ is injective if different elements of A are mapped to different elements of B.

$\Leftrightarrow$ each element of B is the image of no more than one element of A

$\Leftrightarrow$ $f(x_1) = f(x_2)$ implies $x_1 = x_2$

<u>Def</u>: $f: A \to B$ is called surjective if each element of B is the image of at least one element of A

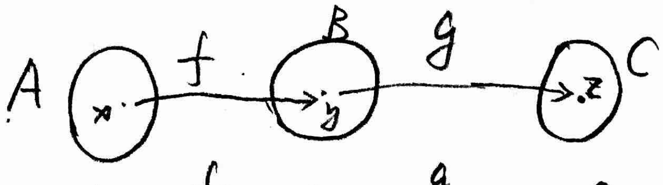
$\Leftrightarrow$ Range of f is equal to B. $\Leftrightarrow$ $\forall y \in B,\ \exists x \in A$, s.t. $f(x) = y$.

<u>Def</u>: $f: A \to B$ is called bijective if it is both injective and surjective

Def: composition of 2 functions: Given $f: A \to B$, $g: B \to C$ functions.

The composite function, denoted by $g \circ f$ is a function from A to C defined as:

$$[g \circ f](x) = g(f(x)) \quad \text{for every } x \in A$$



f: wife. $g =$ mother

Ex: $A \xrightarrow{f} B \xrightarrow{g} C$
A = married man, B = married woman, C = mothers

$g \circ f =$ mother-in-law.

Ex: $f(x) = x-1$, $g(y) = y^2+1 \Rightarrow (g \circ f)(x) = (x-1)^2+1$.

Prop:
1. f & g injective $\Rightarrow$ $g \circ f$ is injective
2. f & g surjective $\Rightarrow$ $g \circ f$ surjective
3. f & g bijective $\Rightarrow$ $g \circ f$ bijective

Def: Inverse of a function $f: A \to B$ is a function $f^{-1}: B \to A$ satisfying $x = f^{-1}(y)$ if and only if $y = f(x)$

$\Leftrightarrow f^{-1}(f(x)) = x$, $\forall x \in A$ and $f(f^{-1}(y)) = y$ $\forall y \in B$

Prop. f has an inverse if and only if f is bijective. In that case, $f^{-1}: B \to A$ is bijective and $(f^{-1})^{-1} = f$.

Exer. A.4 $f: \mathbb{R} \to \mathbb{R}$ $f(x) = x^3 - 3x$.

- $f(3) = f(0) = 0 \Rightarrow f$ is not injective.
- $\forall y \in \mathbb{R}$. $\underline{f(x) = y}$ has a solution $\Rightarrow f$ is surjective.

$\lim_{x \to -\infty} (f(x) - y) = -\infty$, $\lim_{x \to +\infty} (f(x) - y) = +\infty$

by intermediate value theorem, $\exists x_0 \in \mathbb{R}$, s.t. $f(x_0) - y = 0$.

C.4: G is a group. $f: G \to G$ $f(x) = ax$.

- $f(x_1) = f(x_2) \Leftrightarrow ax_1 = ax_2 \Leftrightarrow x_1 = x_2 \Rightarrow f$ is injective
- $\forall y \in G$, $y = a a^{-1} y = f(a^{-1}y) \Rightarrow f$ is surjective

so f is bijective and inverse of f is $f^{-1}: G \to G$. $f^{-1}(y) = a^{-1}y$.

E. 5. $A = \{a, b, c, d\}$. $B = \{1, 2, 3, 4\}$.

$$f = \begin{pmatrix} a & b & c & d \\ 3 & 1 & 2 & 4 \end{pmatrix} \Rightarrow f^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ b & c & a & d \end{pmatrix}$$

F. 5: If A has n elements, how many functions are there from A to A? n^n

How many bijective functions are there from A to A? $n!$

number of rearrangements

Prove:

G. 1: $g \circ f$ injective $\Longrightarrow$ f is injective.

Assume $f(x_1) = f(x_2)$, then $g \circ f(x_1) = g(f(x_1)) = g(f(x_2)) = g \circ f(x_2)$

because $g \circ f$ is injective, we know that $x_1 = x_2$. So we conclude that f is injective.