PRACTICE FINAL EXAM, 2017

Please write your name in the top left corner. Attempt all questions. Time 2 hrs

1. (40 pts)
   (i) How many conjugacy classes does the symmetric group S_5 have ?
   (ii) For each $n \geq 2$, is the alternating group A_n always a normal subgroup of S_n ?
   (iii) Is the element $(1,2,3)(4,5) \in S_n$ an even permutation ?
   (iv) How many elements are there in the conjugacy class of the element $(1,2,3)(4,5) \in S_5$?
   (v) How many elements are there in the conjugacy class of the element $(1,2,3)(4,5) \in S_6$?
   (vi) Is the center of a group always a normal subgroup ?
   (vii) Is the commutator subgroup of a group always a normal subgroup ?
   (viii) Is the commutator subgroup of a group necessarily abelian ?
   (ix) Let G be a group N a normal subgroup such that G/N is abelian. IS N necessarily contained in the commutator subgroup $[G,G]$?
   (x) Let p be a prime and G a p-group. Is it always true that G has a non-trivial center ?
   (xi) Is the group of S_3 a simple group ?
   (xii) Is $GL(n,\mathbb{R})$ a simple group for every n?
   (xiii) In the following questions, by a ring is always meant a commutative ring with 1. Let $R = \mathbb{R}[X]$. Is the subset $\mathbb{R}[X^2]$ an ideal of R ?
   (xiv) Let $R = \mathbb{R}[X]$. Is the subset $\mathbb{R}[X^2]$ an subring of R ?
   (xv) Let $R = \mathbb{R}[X]$, and I the ideal generated by the polynomial X^2. Is R/I an integral domain ?
   (xvi) Let $R = \mathbb{R}[X]$, and I the ideal generated by the polynomial X. Is R/I an integral domain ?
   (xvii) Let $R = \mathbb{R}[X]$, and I the ideal generated by the polynomial X. Is R/I a field?
   (xviii) Let $R = \mathbb{Z}[X]$, and I the ideal generated by the polynomial X. Is I a maximal ideal ?
   (xix) Let $R = \mathbb{Z}[X]$, and I the ideal generated by the polynomial X. Is I a prime ideal ?
   (xx) Let R be a ring which is also a field. Is the ring $R \times R$ necessarily a field ?
2. (10 pts) Let R be a commutative ring with 1, and M a maximal ideal of R. Prove that R/M is a field.
3. (10 pts) Describe the set of integers n satisfying the equations $n = 5 \mod 7, n = 7 \mod 5$.
4. (10 pts) Let R be a commutative ring with 1, and I, J two ideals of R which are co-prime. Prove that $IJ = J \cap J$.

5. (10 pts) Let p be a prime. Prove that every finite abelian group whose order is divisible by p must have a subgroup isomorphic to Z_p.
6. (10pts)
   (a) Define even and odd permutations in S_n.
   (b) Prove that the number of even permutations equal to the number of odd permutations in S_n.
7. (10pts) Let p be a prime. Prove that $\mathbb{Z}/(p)$ is a field.