Chap 19: Quotient Rings.

<u>Def</u>: Let A be a ring, and J an ideal of A. For any element $a \in A$, $J+a$ denotes the set of all sums $j+a$, as a remains fixed and j ranges over J.

$$J+a = \{j+a : j \in J\}$$ is called a coset of J in A.

<u>Thm</u>: Let J be an ideal of A. If $J+a=J+c$ and $J+b=J+d$, then

(i) $J+(a+b) = J+(c+d)$ (ii) $J+ab = J+cd$.

<u>Pf.</u> $J+a=J+c \Leftrightarrow a-c \in J$ $\Rightarrow$ $a+b-(c+d) = (a-c)+(b-d) \in J$
$J+b=J+d \Leftrightarrow b-d \in J$ $\Rightarrow J+(a+b) = J+(c+d)$ (i)

(ii) $ab-cd = ab-bc+bc-cd = b(a-c)+c(b-d) \in J$ because J is an ideal absorbing products.
$\Rightarrow J+ab = J+cd$. ■

$A/J = \{J+a, J+b, J+c, \cdots\}$.

<u>Thm</u>: A/J with the coset addition and multiplication is a ring.

<u>Pf</u>: · coset addition is an abelian group structure
· coset multiplication is associative.
· coset multiplication is distributive over the coset addition

$\Rightarrow$ A/J is a ring. (with zero element J, negative of $J+a$ is $J+(-a)$)

<u>Thm</u>. A/J is a homomorphic image of A

<u>Pf</u>: $f: A \to A/J$ is a homomorphism onto A/J
$a \mapsto J+a$

Ex: $\mathbb{Z}/(n) = \mathbb{Z}/n\mathbb{Z}$ is a quotient ring of $\mathbb{Z}$ by the principal ideal (n).

Thm: Let $f: A \to B$ a homomorphism from a ring A onto a ring B, and let K be the kernel of f. Then $B \cong A/K$.

Pf: Define a homomorphism: $f: A/K \longrightarrow B$

$$K+a \longmapsto f(a).$$

Show f is a $\underbrace{\text{bijective homomorphism}}_{\text{Similar to the case of groups}}$. So it's an isomorphism

The above theorem is called the fundamental homomorphism theorem for rings.

Prop: If an ideal J of A contains all the differences $ab-ba$ $\forall a,b \in A$ Then the quotient ring A/J is commutative.

Pf: $(J+a)\cdot(J+b) = J+ab \underset{ab-ba\in J}{=} J+ba = (J+b)\cdot(J+a)$.

Def: An ideal J of a commutative ring is said to be a prime ideal if the following property is satisfied:

If $ab \in J$, then $a \in J$ or $b \in J$.

Thm: If J is a prime ideal of a commutative ring with unity A, the quotient ring A/J is an integral domain.

Pf: A/J is commutative ring with unity $J+1$.

$$(J+a)\cdot(J+b) = J \Leftrightarrow ab \in J \overset{J \text{ prime}}{\Longrightarrow} a \in J \text{ or } b \in J \Longrightarrow J+a = J \text{ or } J+b = J$$

So A/J does not have divisors of zero, and hence is an integral domain.

Def: An ideal of a ring is called proper if it is not equal to the whole ring.

A proper ideal J of a ring A is called a maximal ideal if there exists no proper ideal K of A s.t. $J \subseteq K$ with $J \neq K$. In other words J is not contained in any strictly larger proper ideal.

Thm: If A is a commutative ring with unity, then J is a maximal ideal of A iff A/J is a field.

Pf: A field is a commutative ring with unity such that any nonzero element is invertible.

Let J be a maximal ideal. A/J is clearly a commutative ring with a unity. We show that if $J+a \neq J$ then $\exists\, J+b$ s.t. $(J+a)\cdot(J+b) = J+1$:

$J+a \neq J \Leftrightarrow a \notin J$. J is a maximal ideal $\Rightarrow$ $J+(a) = A$

$\Downarrow$

$(J+(a) = \{a\cdot x + j;\ j \in J,\ x \in A\}$ is an ideal.) $\exists\, j \in J$ and $x \in A$ s.t.

$J+a$ has the inverse $J+x$. $\Leftarrow$ $(J+a)(J+x) = J+ax$ $\Leftarrow$ $j + a\cdot x = 1$

$J+ax = J+1-j = J+1$

So A/J is indeed a field.

Conversely assume A/J is a field. Assume $J \subset J'$ an ideal and $J' \neq J$. Then $\exists\, a \in J' \setminus J \Rightarrow J+a \neq 0 \in A/J$. So there exists $J+b \in A/J$ s.t. $(J+a)\cdot(J+b) = J+ab = J+1 \Rightarrow ab-1 = j \in J \Rightarrow 1 = ab-j \in J'$ $\Rightarrow J' = R$. So J is a maximal ideal.

Exer A: $A = \mathbb{Z}_6$, $J = \{\bar{0}, \bar{3}\} = (\bar{3})$ $\quad A/J \cong \mathbb{Z}_3$

$$f: \mathbb{Z}_6 \twoheadrightarrow \mathbb{Z}_3 \qquad f = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 0 & 1 & 2 \end{pmatrix}$$

$$\bar{n}^6 \longmapsto \bar{n}^3 \qquad \ker(f) = (\bar{3}) \qquad \Rightarrow \mathbb{Z}_6/(\bar{3}) \cong \mathbb{Z}_3$$

Ex: $A = \mathbb{Z}_2 \times \mathbb{Z}_6$, $J = \{(0,0), (0,2), (0,4)\} = ((0,2))$. $\quad A/\mathbb{Z} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$

Exer D.4. $\forall a \in A$. let π_a be the function $\pi_a : A \to A$
$$x \mapsto a \cdot x.$$

Let $\bar{A} = \{\pi_a : a \in A\}$ define addition and multiplication on $\bar{A}$:

$$\pi_a + \pi_b = \pi_{a+b}, \quad \pi_a \cdot \pi_b = \pi_{ab}.$$

Then $\bar{A}$ is a ring. $f: A \to \bar{A}$, $a \mapsto \pi_a$ is a ring homomorphism:

$$f(a+b) = \pi_{a+b} = \pi_a + \pi_b = f(a) + f(b); \quad f(ab) = \pi_{ab} = \pi_a \cdot \pi_b = f(a) \cdot f(b).$$

$\ker(f) = \{a \in A : \pi_a(x) = a \cdot x = 0 \ \forall x \in A\} =$ annihilating ideal of A $= \operatorname{ann}(A)$

By FTH, we get $A/\operatorname{ann}(A) \cong \bar{A}$.