SOLUTION OF SYSTEM OF LINEAR EQUATIONS

Lecture 3: Crout's method or
LU decomposition method.

Crout's Method (LU Decomposition method)

It is a distinct method of solving a system of linear equations of the form $A\tilde{x} = b,$ where the matrix A is decomposed into a product of a lower triangular matrix L and an upper triangular matrix U, that is $A = LU$

Explicitly, we can write it as

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots\ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & a_{n3} & \ldots\ldots & a_{nn} \end{pmatrix} = \begin{pmatrix} l_{11} & 0 & 0 & \ldots\ldots & 0 \\ l_{21} & l_{22} & 0 & \ldots\ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ l_{n1} & l_{n2} & l_{n3} & \ldots\ldots & l_{nn} \end{pmatrix} \begin{pmatrix} 1 & u_{12} & u_{13} & \ldots\ldots & u_{1n} \\ 0 & 1 & u_{23} & \ldots\ldots & u_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots\ldots & 1 \end{pmatrix}$$

Therefore, by LU-decomposition, the system of linear equations $A\underset{\sim}{x} = \underset{\sim}{b}$ can be solved in three steps:

I. Construct the lower triangular matrix L and upper triangular matrix U.

II. Using forward substitution, solve $L\tilde{y} = \underset{\sim}{b}$

III. Solve $U\tilde{x} = \tilde{y}$, backward substitution.

We further elaborate the process by considering a 3×3 matrix A. We consider solving the system of equation of the form $A\underset{\sim}{x} = \underset{\sim}{b}$, where,

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad \tilde{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} and \quad b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

The matrix A is factorized as a product of two matrices L (lower triangular matrix) and U (upper triangular matrix) as follows:

$$\begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \begin{pmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12}+l_{22} & l_{21}u_{13}+l_{22}u_{23} \\ l_{31} & l_{31}u_{12}+l_{32} & l_{31}u_{13}+l_{32}u_{23}+l_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

This implies

$$l_{11} = a_{11},\ l_{21} = a_{21},\ l_{31} = a_{31};$$

$$l_{11}u_{12} = a_{12} \Rightarrow u_{12} = \frac{a_{12}}{l_{11}} = \frac{a_{12}}{a_{11}};$$

$$l_{11}u_{13} = a_{13} \Rightarrow u_{13} = \frac{a_{13}}{l_{11}} = \frac{a_{13}}{a_{11}};$$

$$l_{21}u_{12} + l_{22} = a_{22} \Rightarrow l_{22} = a_{22} - l_{21}u_{12};$$

$$l_{21}u_{13} + l_{22}u_{23} = a_{23} \Rightarrow u_{23} = \frac{1}{l_{22}}\left(a_{23} - l_{21}u_{13}\right);$$

$$l_{31}u_{12} + l_{32} = a_{32} \Rightarrow l_{32} = a_{32} - l_{31}u_{12};$$

$$l_{31}u_{13} + l_{32}u_{32} + l_{33} = a_{33} \Rightarrow l_{33} = a_{33} - l_{31}u_{13} - l_{32}u_{23}$$

Once all the value of l_{ij}'s and u_{ij}'s are obtained, we can write

$$A\tilde{x} = \underset{\sim}{b} \text{ as } LU\,\tilde{x} = \underset{\sim}{b}$$

$$Let\ U\,\tilde{x} = \underset{\sim}{y}, \text{ then } L\,\underset{\sim}{y} = \underset{\sim}{b}$$

$$\Rightarrow \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} l_{11}y_1 \\ l_{21}y_1 + l_{22}y_2 \\ l_{31}y_1 + l_{32}y_2 + l_{33}y_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\Rightarrow y_1 = \frac{b_1}{l_{11}}, y_2 = \frac{1}{l_{22}}\left(b_2 - l_{21}y_1\right) \text{ and } y_3 = \frac{1}{l_{33}}\left(b_3 - l_{31}y_1 - l_{32}y_2\right)$$

By forward substitution we obtain, $U\underset{\sim}{x}=\underset{\sim}{y}$

$$\Rightarrow\begin{pmatrix}1 & u_{12} & u_{13}\\ 0 & 1 & u_{23}\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=\begin{pmatrix}y_1\\ y_2\\ y_3\end{pmatrix}$$

By back substitution we get,

$$x_3 = y_3$$
$$x_2 + u_{23}x_3 = y_2 \Rightarrow x_2 = y_2 - u_{23}x_3$$
$$x_1 + u_{12}x_2 + u_{13}x_3 = y_1 \Rightarrow x_1 = y_1 - u_{12}x_2 - u_{13}x_3$$

Example 4. Solve the following system of linear equations, by Crout's method:

$$10x_1 + 3x_2 + 4x_3 = +15$$
$$2x_1 - 10x_2 + 3x_3 = 37$$
$$3x_1 + 2x_2 - 10x_3 = -10$$

Solution: In matrix form, the given system of equation can be written as

$$\begin{pmatrix}10 & 3 & 4\\ 2 & -10 & 3\\ 3 & 2 & -10\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=\begin{pmatrix}15\\ 37\\ -10\end{pmatrix}$$

which is of the form $A\underset{\sim}{x}=\underset{\sim}{b}$. Let $A=LU$, which implies

$$\begin{pmatrix}10 & 3 & 4\\ 2 & -10 & 3\\ 3 & 2 & -10\end{pmatrix}=\begin{pmatrix}l_{11} & 0 & 0\\ l_{21} & l_{22} & 0\\ l_{31} & l_{32} & l_{33}\end{pmatrix}\begin{pmatrix}1 & u_{12} & u_{13}\\ 0 & 1 & u_{23}\\ 0 & 0 & 1\end{pmatrix}$$

$$=\begin{pmatrix}l_{11} & l_{11}u_{12} & l_{11}u_{13}\\ l_{21} & l_{21}u_{12}+l_{22} & l_{21}u_{13}+l_{22}u_{23}\\ l_{31} & l_{31}u_{12}+l_{32} & l_{31}u_{13}+l_{32}u_{23}+l_{33}\end{pmatrix}$$

$$\Rightarrow l_{11}=10,\ l_{21}=2,\ l_{31}=3;\ u_{12}\frac{3}{10},\ u_{13}=\frac{4}{10};$$

$$l_{21}u_{12}+l_{22}=-10 \Rightarrow l_{22}=-10-2\times\frac{3}{10}=-\frac{106}{10};$$

$$l_{21}u_{13}+l_{22}u_{23}=3 \Rightarrow u_{23}=\frac{\left(3-2\times\frac{4}{10}\right)}{\left(-\frac{106}{10}\right)}=-\frac{11}{53};$$

$$l_{31}u_{12}+l_{32}=2 \Rightarrow l_{32}=2-l_{31}u_{12}=2-3\times\frac{3}{10}=\frac{11}{10};$$

$$l_{31}u_{13}+l_{32}u_{23}+l_{33}=-10 \Rightarrow l_{33}=-10-l_{31}u_{13}-l_{32}u_{23}$$
$$=-10-3\times\frac{4}{10}+\frac{11}{10}\times\frac{11}{53}=-\frac{1163}{106}$$

Therefore, we get,

$$L=\begin{pmatrix} 10 & 0 & 0 \\ 2 & \frac{-106}{10} & 0 \\ 3 & \frac{11}{10} & \frac{-1163}{106} \end{pmatrix} \text{ and } U=\begin{pmatrix} 1 & \frac{3}{10} & \frac{4}{10} \\ 0 & 1 & \frac{-11}{53} \\ 0 & 0 & 1 \end{pmatrix}$$

Now, let $U\underset{\sim}{x}=\underset{\sim}{y}$, then $L\underset{\sim}{y}=\underset{\sim}{b}$ implies

$$\begin{pmatrix} 10 & 0 & 0 \\ 2 & \frac{-106}{10} & 0 \\ 3 & \frac{11}{10} & \frac{-1163}{106} \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}=\begin{pmatrix} 15 \\ 37 \\ -10 \end{pmatrix}$$

This implies

$$10y_1 = 15 \Rightarrow y_1 = \frac{3}{2}$$

$$2y_1 - \frac{106}{10}y_2 = 37 \Rightarrow y_2 = \frac{-170}{53}$$

$$y_1 + \frac{11}{10}y_2 - \frac{1163}{106}y_3 = -10 \Rightarrow y_3 = 1$$

Thus, $y = \begin{pmatrix} 3/2 \\ \frac{-170}{53} \\ 1 \end{pmatrix}$ and $U\tilde{x} = \underset{\sim}{y}$ gives

$$\begin{pmatrix} 1 & \frac{3}{10} & \frac{4}{10} \\ 0 & 1 & \frac{-11}{53} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ \frac{-170}{53} \\ 1 \end{pmatrix}$$

, which implies

$$x_1 + \frac{3}{10}x_2 - \frac{4}{10}x_3 = \frac{3}{2}$$

$$x_2 - \frac{11}{53}x_3 = \frac{-170}{53}$$

$$x_3 = 1$$

By back substitution, we get,

$$x_3 = 1$$

$$x_2 = \frac{11 \times 1}{53} - \frac{170}{53} = -3$$

$$x_1 = \frac{3}{2} - \frac{3}{10}x_2 - \frac{4}{10}x_3 = \frac{3}{2} - \frac{3}{10} \times (-3) - \frac{4}{10} \times 1 = 2$$

Therefore, the required solution by Crout's method (LU decomposition method) is $x_1 = 2, \quad x_2 = -3, \quad x_3 = 1.$

Example 5. Solve the following system of linear equations by Crout's Method (LU factorization or decomposition method):

$$
\begin{aligned}
9x_1 + 3x_2 + 3x_3 + 3x_4 &= 24 \\
3x_1 + 10x_2 - 2x_3 - 2x_4 &= 17 \\
3x_1 - 2x_2 + 18x_3 + 10x_4 &= 45 \\
3x_1 - 2x_2 + 10x_3 + 10x_4 &= 29
\end{aligned}
$$

<u>**Solution:**</u> The given system of equation can be written in matrix form as

$$
\begin{pmatrix} 9 & 3 & 3 & 3 \\ 3 & 10 & -2 & -2 \\ 3 & -2 & 18 & 10 \\ 3 & -2 & 10 & 10 \end{pmatrix}
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}
=
\begin{pmatrix} 24 \\ 17 \\ 45 \\ 29 \end{pmatrix}
$$

Let
$$
\begin{pmatrix} 9 & 3 & 3 & 3 \\ 3 & 10 & -2 & -2 \\ 3 & -2 & 18 & 10 \\ 3 & -2 & 10 & 10 \end{pmatrix}
=
\begin{pmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ l_{31} & l_{32} & l_{33} & 0 \\ l_{41} & l_{42} & l_{43} & l_{44} \end{pmatrix}
\begin{pmatrix} 1 & u_{12} & u_{13} & u_{14} \\ 0 & 1 & u_{23} & u_{24} \\ 0 & 0 & 1 & u_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

Comparing, we get,

$$l_{11} = 9, \;\; l_{21} = 3, \;\; l_{31} = 3, \;\; l_{41} = 3;$$

$$l_{11}u_{12} = 3 \Rightarrow u_{12} = \frac{1}{3}. \text{ Similarly, } u_{13} = u_{14} = \frac{1}{3};$$

$$l_{21}u_{12} + l_{22} = 10 \Rightarrow l_{22} = 10 - l_{21}u_{12} = 10 - 3 \times \frac{1}{3} = 9$$

$$l_{21}u_{13} + l_{22}u_{23} = -2 \Rightarrow u_{23} = \frac{-2 - l_{21}u_{13}}{l_{22}} = -\frac{1}{3}$$

$$l_{21}u_{14} + l_{22}u_{24} = -2 \Rightarrow u_{24} = \frac{-2 - l_{21}u_{14}}{l_{22}} = -\frac{1}{3}$$

$$l_{31}u_{12} + l_{32} = -2 \Rightarrow l_{32} = -2 - l_{31}u_{12} = -3$$

$$l_{31}u_{13} + l_{32}u_{23} + l_{33} = 18 \Rightarrow l_{33} = 18 - l_{31}u_{13} - l_{32}u_{23} = 16$$

$$l_{31}u_{14} + l_{32}u_{24} + l_{33}u_{34} = 10 \Rightarrow u_{34} = \frac{10 - l_{31}u_{14} - l_{32}u_{24}}{l_{33}} = \frac{1}{2}$$

$$l_{41}u_{12} + l_{42} = -2 \Rightarrow l_{42} = -2 - l_{41}u_{12} = -3$$

$$l_{41}u_{13} + l_{42}u_{23} + l_{43} = 10 \Rightarrow l_{43} = 10 - l_{41}u_{13} - l_{42}u_{23} = 8$$

$$l_{41}u_{14} + l_{42}u_{24} + l_{43}u_{34} + l_{44} = 10$$

$$\Rightarrow l_{44} = 10 - l_{41}u_{14} - l_{42}u_{24} - l_{43}u_{34} = 4$$

Therefore, we get

$$L = \begin{pmatrix} 9 & 0 & 0 & 0 \\ 3 & 9 & 0 & 0 \\ 3 & -3 & 16 & 0 \\ 3 & -3 & 8 & 4 \end{pmatrix} \text{ and } U = \begin{pmatrix} 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & \frac{-1}{3} & \frac{-1}{3} \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Forward substitution gives

$$9y_1 = 24 \Rightarrow y_1 = \frac{8}{3}$$

$$3y_1 + 9y_2 = 17 \Rightarrow y_2 = 1$$

$$3y_1 - 3y_2 + 16y_3 = 45 \Rightarrow y_3 = \frac{5}{2}$$

$$3y_1 - 3y_2 + 8y_3 + 4y_4 = 29 \Rightarrow y_4 = 1$$

Thus, $y = \begin{pmatrix} 8/3 \\ 1 \\ 5/2 \\ 1 \end{pmatrix}$ and $U\,\tilde{x} = y$ gives

$$\begin{pmatrix} 1 & 1/3 & 1/3 & 1/3 \\ 0 & 1 & -1/3 & -1/3 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 8/3 \\ 1 \\ 5/2 \\ 1 \end{pmatrix}$$

By back substitution we get,

$$x_4 = 1$$

$$x_3 + \frac{1}{2}x_4 = \frac{5}{2} \Rightarrow x_3 = 2$$

$$x_2 - \frac{1}{3}x_3 - \frac{1}{3}x_4 = 1 \Rightarrow x_2 = 2$$

$$x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_3 + \frac{1}{3}x_4 = \frac{8}{3} \Rightarrow x_1 = 1$$

Therefore, the required solution by Crout's method is

$$x_1 = 1,\ x_2 = 2,\ x_3 = 2,\ x_4 = 1$$