Test Monte Carlo su approcci Monte Carlo: Least Median of Squares come caso di studio

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22 Luglio 2024

Least Median of Squares Algoritmo di regressione *robusta* resistente agli *outliers*, basato su campionamento Monte Carlo

$$\hat{\beta} = \arg\min_{\beta} \text{median}_n \left(y_n - (\beta_0 + \beta_1 x_{1n} + \cdots + \beta_q x_{qn}) \right)^2 \quad n = 1 \ldots N$$

Outliers Un *outlier* è un'osservazione che appare deviare marcatamente dagli altri membri del campione in cui si verifica.

Robustezza Uno stimatore o un procedimento statistico è *robusto* se la sua capacità di stimare correttamente i parametri rimane valida nonostante alcune delle assunzioni che giustificano il metodo impiegato vengano meno

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Regressione semplice ai minimi quadrati

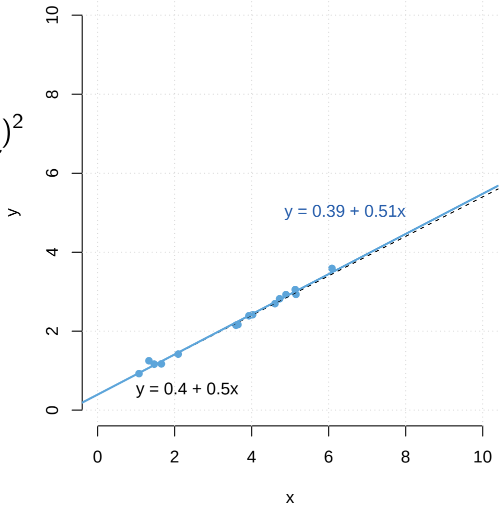
$y = \beta_0 + \beta_1 x$

N coppie di punti (x_n, y_n), $n = 1 \dots N$ con
$y_n = \beta_0 + \beta_1 x_n + \varepsilon_n, \;\; \varepsilon \sim \mathrm{N}(0, \sigma_e^2)$

$$\hat{\beta} = [\hat{\beta}_0 \;\; \hat{\beta}_1]^\top = \arg\min_{\beta} \sum_{n=1}^{N} (\underbrace{y_n - (\beta_0 + \beta_1 x_n)}_{r_n})^2$$

$$\hat{\beta} = \arg\min_{\beta} \frac{\sum_n r_n^2}{N} = \arg\min_{\beta} \mathrm{mean}_n\{r_n^2\}$$

```r
set.seed(111); b0 = 0.4; b1 = 0.5
N = 15; x = sort(runif(N, 1, 8))
sigma_e = 0.1; e = rnorm(N,0,sigma_e)
y = b0 + b1*x + e
ls = lm(y ~ x)
```



Test Monte Carlo su LMedS

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La mappa d'errore
Intermezzo matematico

$$y = \beta_0 + \beta_1 x \iff \beta_1 x - y + \beta_0 = 0$$

$$\mathbf{l} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \propto \begin{bmatrix} \beta_1 \\ -1 \\ \beta_0 \end{bmatrix} \Rightarrow [x\ y\ 1]^\top \mathbf{l} = 0$$

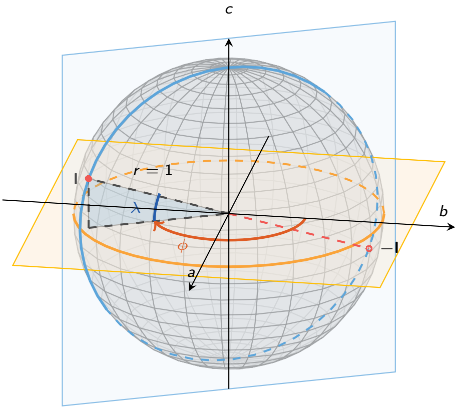
senza perdere di generalità, $\|\mathbf{l}\| = 1$

Per $r = 1$,

$$a = r \cos \lambda \sin \phi$$

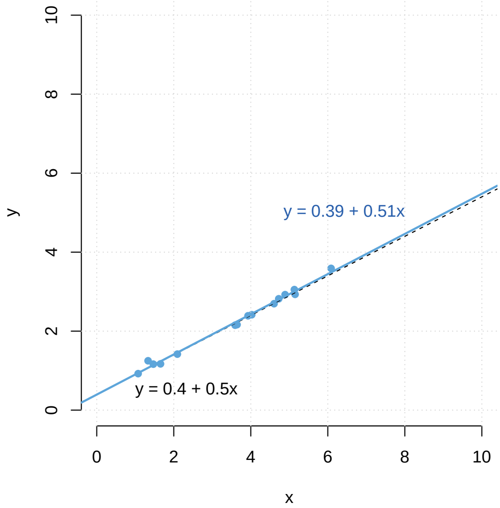
$$b = r \cos \lambda \cos \phi$$

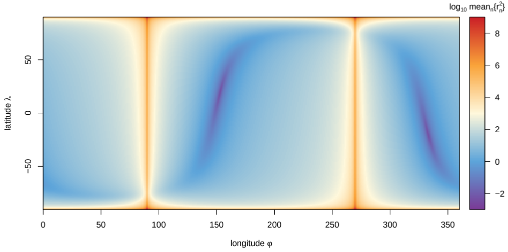
$$c = r \sin \lambda$$



La mappa d'errore

Least Mean of Squares





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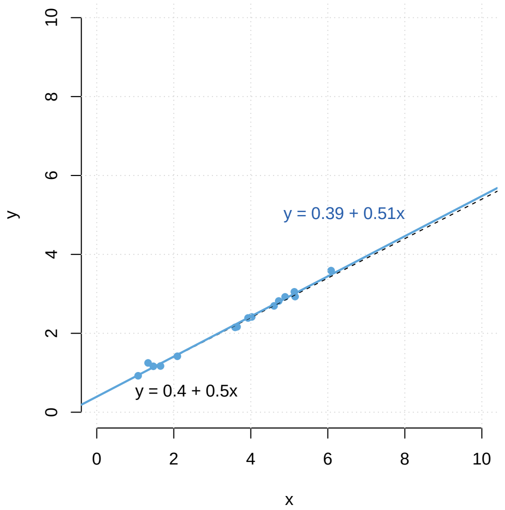

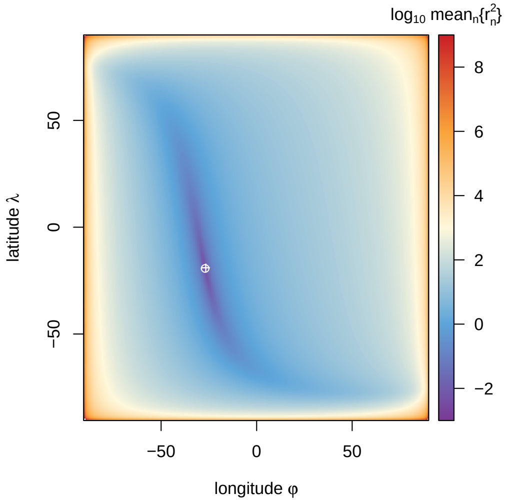

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Introduzione

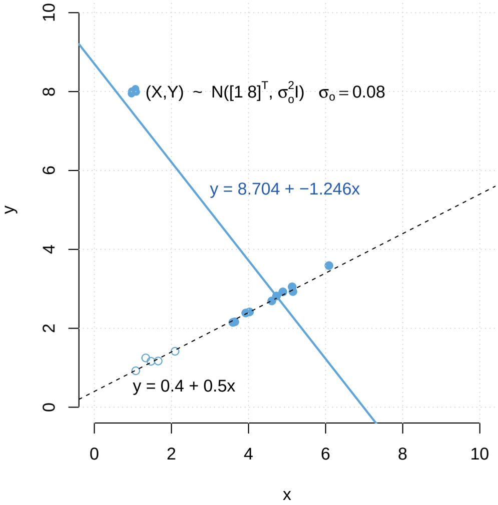
Least Median of Squares + RLS

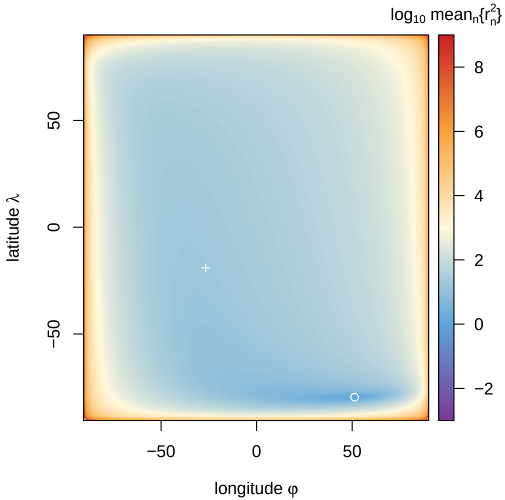
Monte Carlo

Chiusura

La mappa d'errore

Least Mean of Squares





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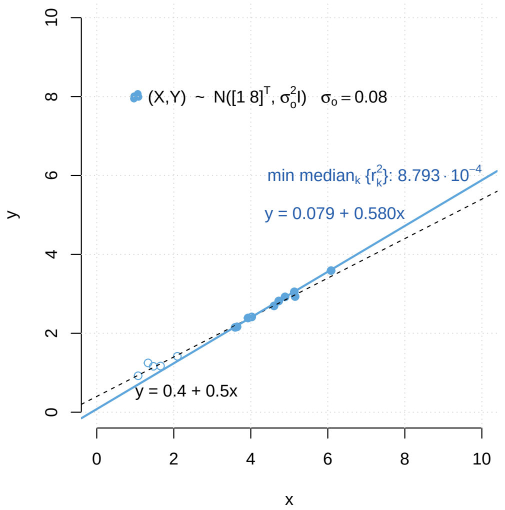

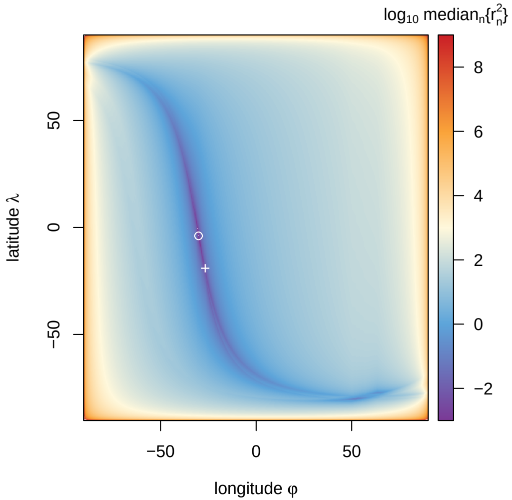

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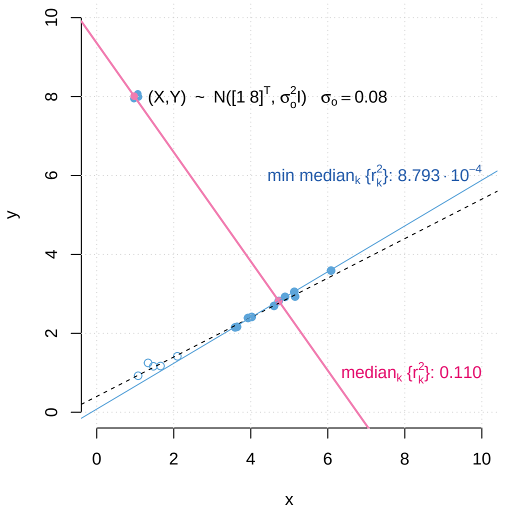
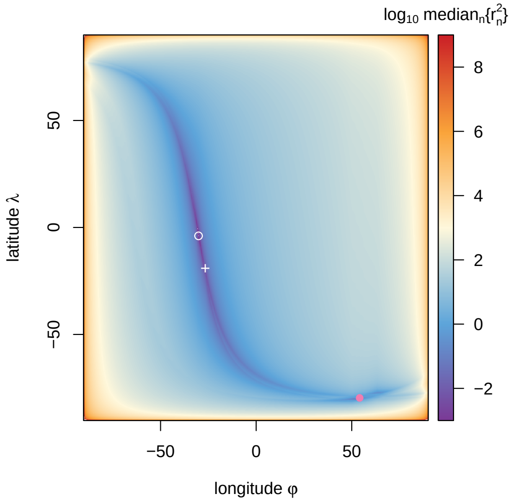
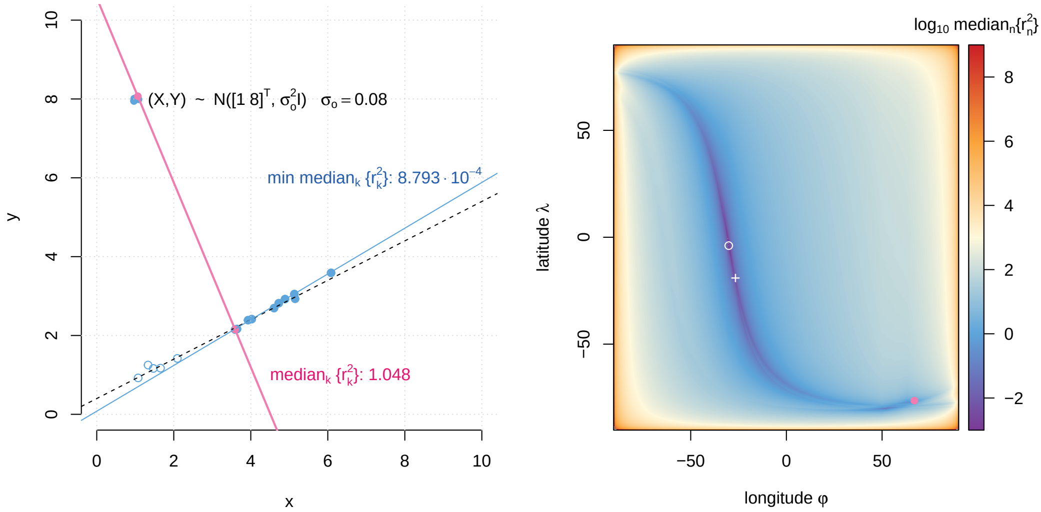


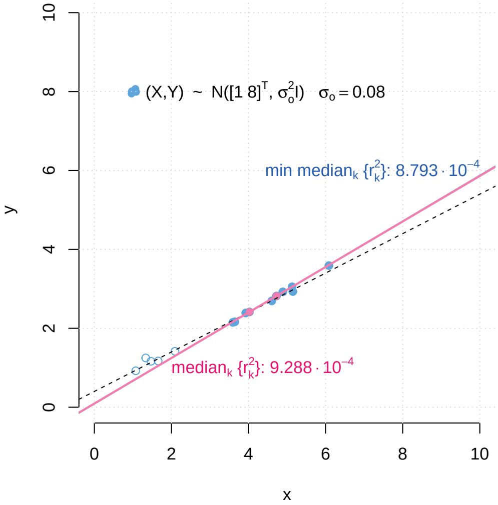
(X,Y) ~ N([1 8]^T, σ_o^2 I)   σ_o = 0.08
min median_k {r_k^2}: 8.793 · 10^−4
median_k {r_k^2}: 0.110
y
x
0
2
4
6
8
10

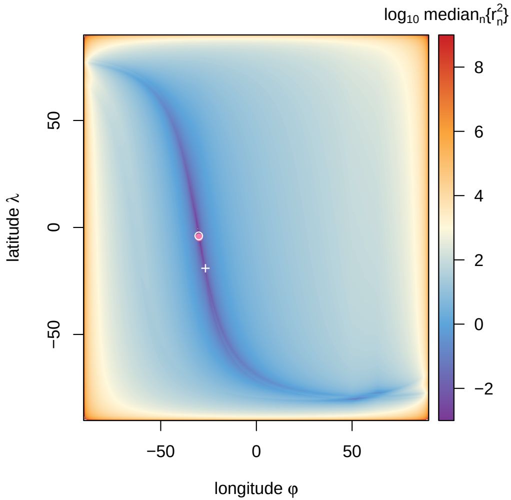


log10 median_n{r_n^2}
latitude λ
longitude φ
50
0
−50
8
6
4
2
0
−2



(X,Y) ~ N([1 8]^T, σ_o^2 I)   σ_o = 0.08
min median_k {r_k^2}: 8.793 · 10^−4
median_k {r_k^2}: 1.048
x
y
log10 median_n{r_n^2}
latitude λ
longitude φ


(X,Y) ~ N([1 8]^T, σ_o^2 I)   σ_o = 0.08
min median_k {r_k^2}: 8.793 · 10^−4
median_k {r_k^2}: 9.288 · 10^−4
y
x
0
2
4
6
8
10


log10 median_n{r_n^2}
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Algoritmo LMedS

Input: tentativi T; $\mathcal{D} = \{(y_n, \mathbf{x}_n)\}$, $y_n \in \mathbb{R}$, $\mathbf{x}_n \in \mathbb{R}^q$

1. Poni $t := 1$; med $:= +\infty$; $\hat{\beta} = [\,]$
2. Seleziona casualmente un sottoinsieme minimale $\mathcal{D}_q^{(t)} = \{(y_i, \mathbf{x}_i)\} \subset \mathcal{D}$ di $p = q+1$ dati per cui $\beta^{(t)}$ è t.c. $y_i - (\beta_0^{(t)} + \beta_1^{(t)} x_{1i} + \ldots + \beta_q^{(t)} x_{qi}) = 0$
3. Calcola $r_n := y_n - ([1\ \mathbf{x}_k^\top]\beta^{(t)})\ \forall (y_n, \mathbf{x}_n) \in \mathcal{D}$
4. Se med $> \text{med}_n r_n^2$
   $\hat{\beta} := \beta^{(t)}$; med $:= \text{med}_n r_n^2$
5. $t := t + 1$
6. Se $t < T$, vai a 2
7. Restituisci $\hat{\beta}$ e med

		T	
p	τ_N	$N \le \tau_N$	$N > \tau_N$
2	50	$c_{N,p}$	1000
3	22	$c_{N,p}$	1500
4	17	$c_{N,p}$	2000
5	15	$c_{N,p}$	2500
6	14	$c_{N,p}$	3000
≥ 7		3000	3000

$$N = \|\mathcal{D}\|$$

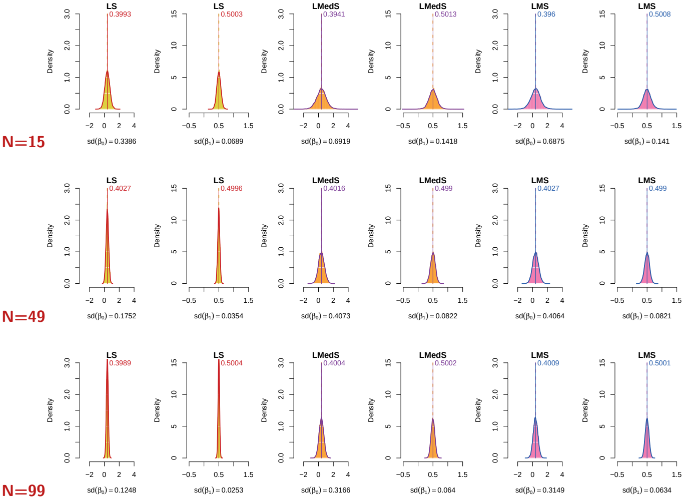
$$c_{N,p} = \binom{N}{p} = \frac{N!}{p!(N-p)!}$$

Least Median of Squares vs. Least Squares

E in assenza di outliers?

```
nsim=10000
for N %in% c(15,49,99){
# DGP:
b0 = 0.4; b1 = 0.5
x = runif(N, 1, 8)
sigma_e = 0.5
e = rnorm(N,0,sigma_e)
y = b0 + b1*x + e
...}
```

- LS: Least Squares
- LMedS: Least Median of Squares
- LMS: lmsreg(...)



Reweighted Least Squares (RLS)

Incremento dell'efficienza

$$\widehat{|r|} = \sqrt{\mathrm{med}_n r_n^2}$$

$$s^{(0)} = 1.4826\left(1+\frac{5}{N-p}\right)\widehat{|r|}$$

$$w_n^{(0)} = \mathbb{1}(|r_n/s_0| \leq 2.5)$$

$$\hat{\sigma} = \sqrt{\frac{\sum_n w_n^{(0)} r_n^2}{\sum_n w_n^{(0)} - p}}$$

$$w_n = \mathbb{1}(|r_n/\hat{\sigma}| \leq 2.5)$$

$$\hat{\beta} = \arg\min_{\beta} \sum_n w_n r_n^2$$

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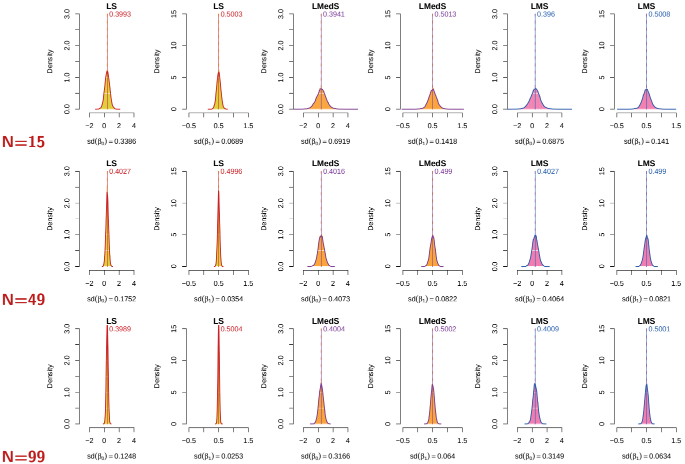

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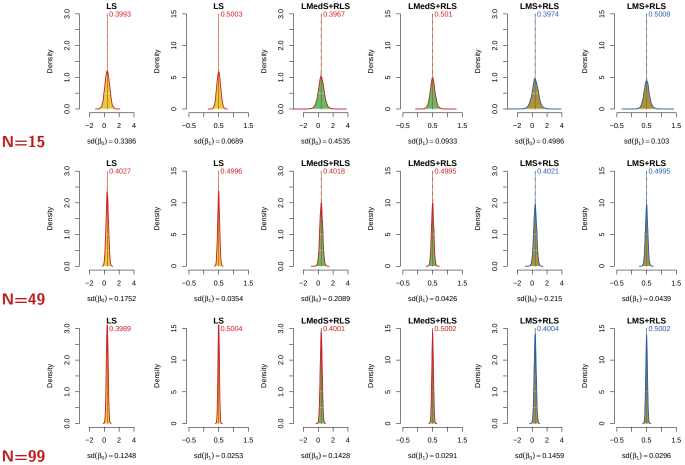

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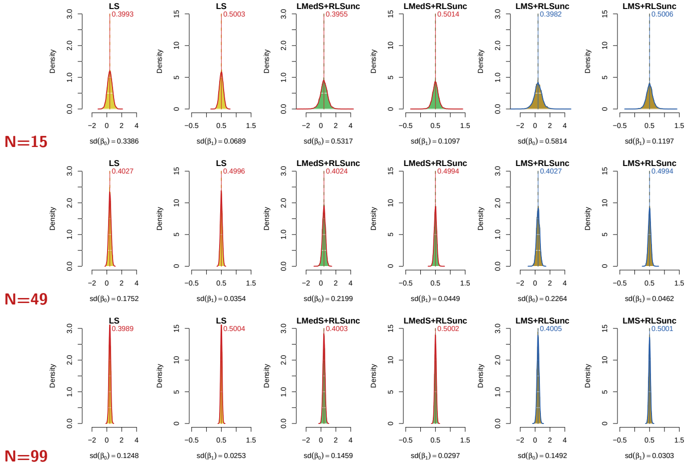
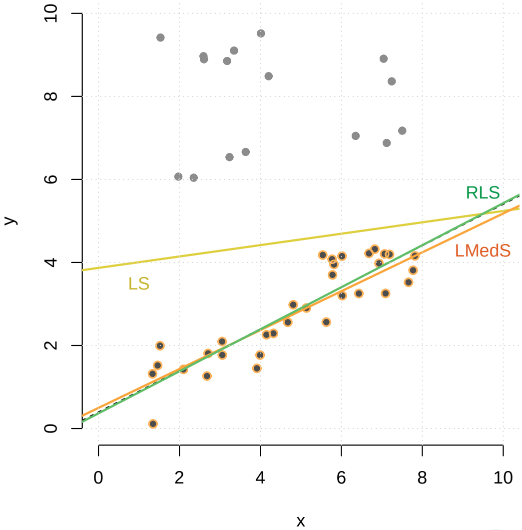


LMedS+RLS vs LS

Con outliers, scenario #1

$x \sim \mathrm{U}(a=1, b=8)$
$\varepsilon \sim \mathrm{N}(\mu=0, \sigma=0.5)$
$b_0 = 0.4, \quad b_1 = 0.5$
$y_n = b_0 + b_1 x_n + \varepsilon_n \quad (n = 1 \ldots N)$
$N \in \{15, 49, 99\}$

$n_{\text{sims}} = 10\,000$
$p_o = 0.333$
$N_o = \mathrm{round}(p_o N)$
$n_o = \mathrm{sample}(1:N, N_o, \mathrm{repl.} = F)$
$y_{n_o} \sim \mathrm{U}(a=6, b=10)$



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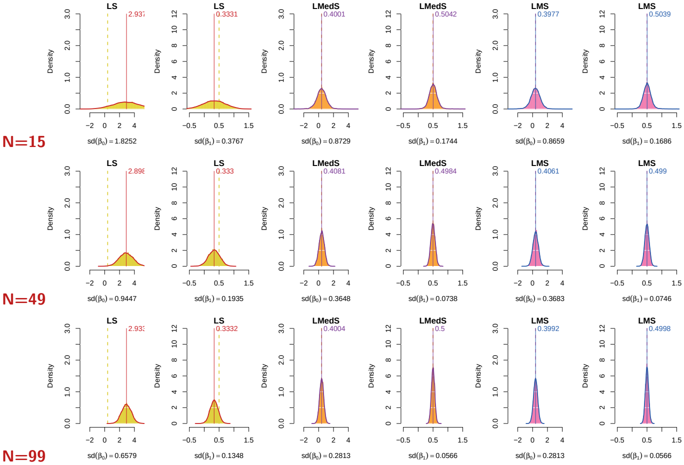

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N=15

N=49

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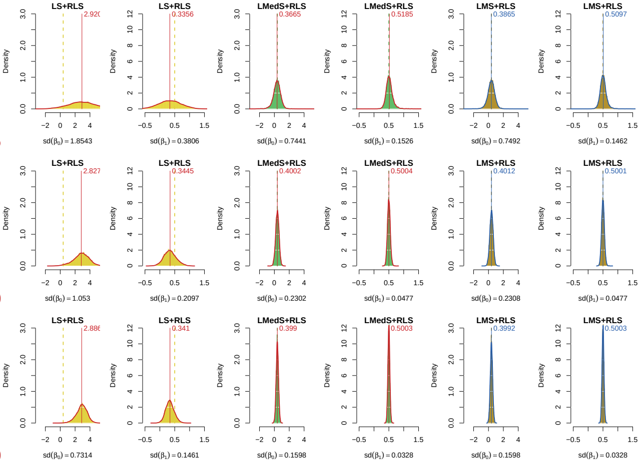

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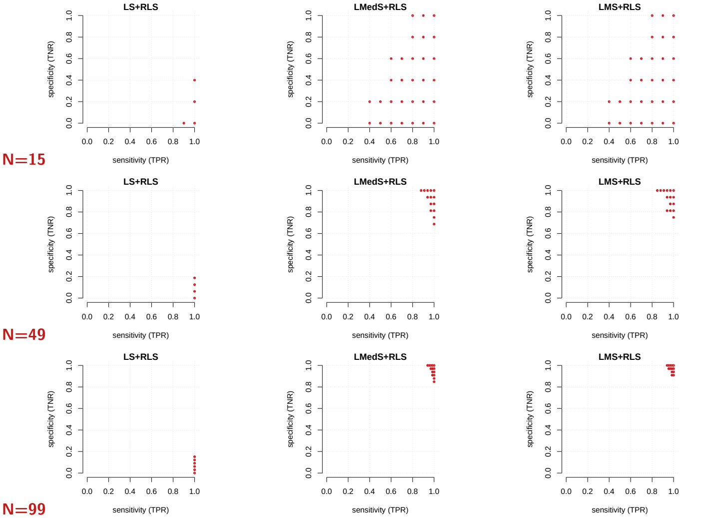

LMedS+RLS vs LS

Con outliers, scenario #2

$x \sim \mathrm{U}(a=1, b=8)$
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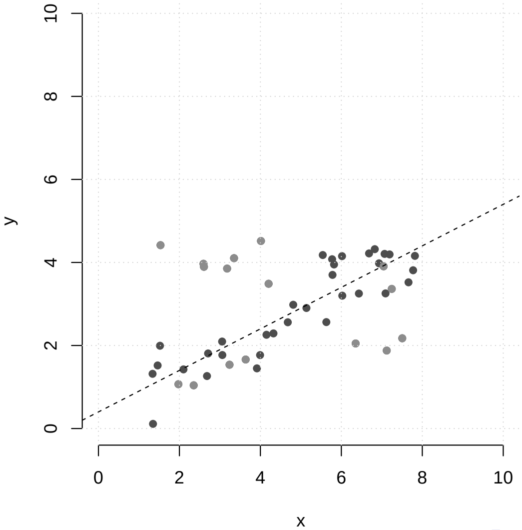

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N=49

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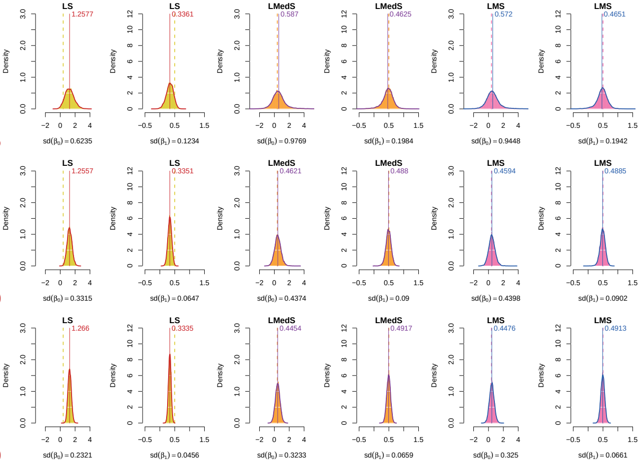

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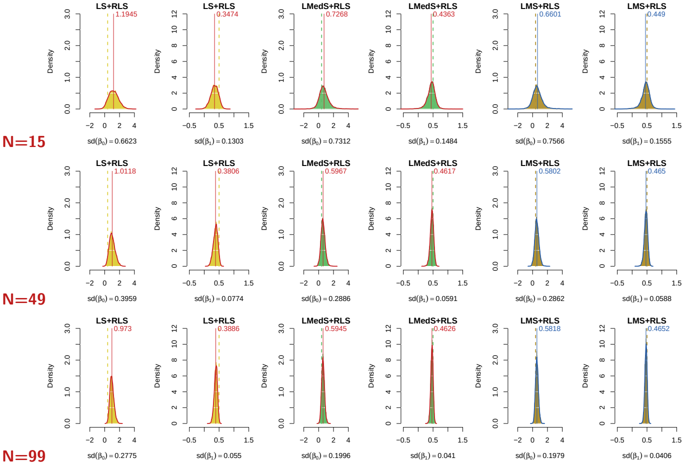

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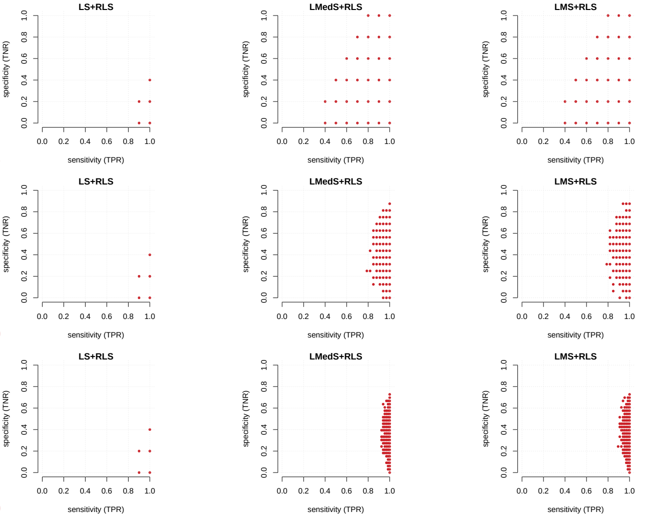

LMedS+RLS vs LS

Con outliers, scenario #3

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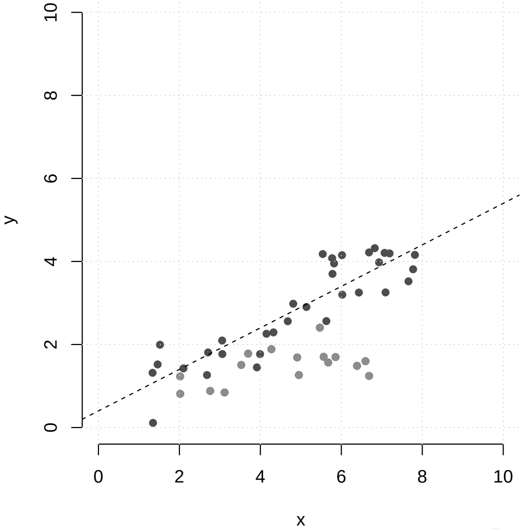
$p_o = 0.333$
$N_o = \text{round}(p_o N)$
$n_o = \text{sample}(1 : N, N_o, \text{repl.} = F)$
$y_{n_o} \sim \mathrm{N}(\mu = 1.5, \sigma = 0.4)$



LMedS+RLS vs LS

Con outliers, scenario #3

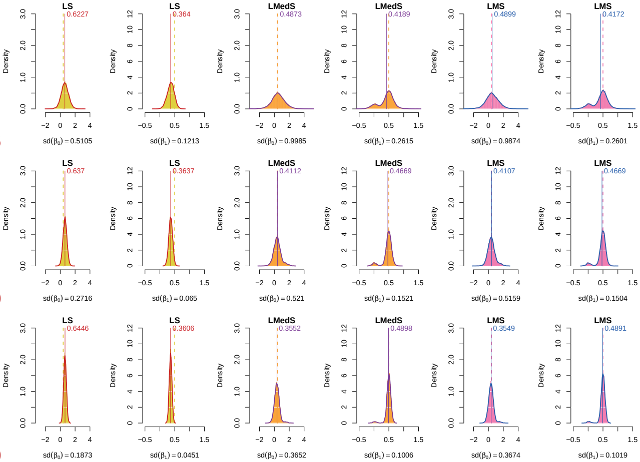
$x \sim \mathrm{U}(a=1, b=8)$
$\varepsilon \sim \mathrm{N}(\mu=0, \sigma=0.5)$
$b_0=0.4, \quad b_1=0.5$
$y_n=b_0+b_1x_n+\varepsilon_n \quad (n=1\ldots N)$
$N \in \{15, 49, 99\}$

$n_{\text{sims}} = 10\,000$
$p_o=0.333$
$N_o=\mathrm{round}(p_o N)$
$n_o=\mathrm{sample}(1:N, N_o, \mathrm{repl.}=F)$
$y_{n_o} \sim \mathrm{N}(\mu=1.5, \sigma=0.4)$

N=15

N=49

N=99



LMedS+RLS vs LS

Con outliers, scenario #3

$x \sim \mathrm{U}(a=1, b=8)$

$\varepsilon \sim \mathrm{N}(\mu=0, \sigma=0.5)$

$b_0 = 0.4, \quad b_1 = 0.5$

$y_n = b_0 + b_1 x_n + \varepsilon_n \quad (n = 1 \ldots N)$

$N \in \{15, 49, 99\}$

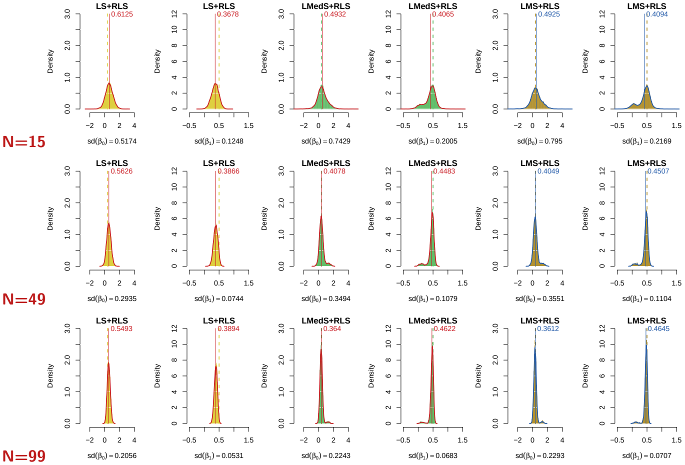
$n_{\text{sims}} = 10\,000$

$p_o = 0.333$

$N_o = \text{round}(p_o N)$

$n_o = \text{sample}(1:N, N_o, \text{repl.} = F)$

$y_{n_o} \sim \mathrm{N}(\mu=1.5, \sigma=0.4)$

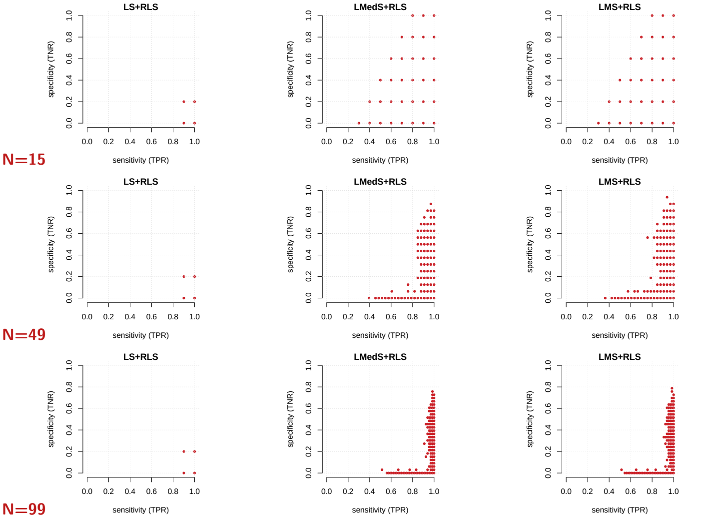


LMedS+RLS vs LS

Con outliers, scenario #3

$x \sim \mathrm{U}(a=1, b=8)$
$\varepsilon \sim \mathrm{N}(\mu=0, \sigma=0.5)$
$b_0=0.4, \quad b_1=0.5$
$y_n=b_0+b_1 x_n+\varepsilon_n \quad (n=1\ldots N)$
$N \in \{15, 49, 99\}$

$n_{\text{sims}} = 10\,000$
$p_o=0.333$
$N_o=\text{round}(p_o N)$
$n_o=\text{sample}(1:N, N_o, \text{repl.}=F)$
$y_{n_o} \sim \mathrm{N}(\mu=1.5, \sigma=0.4)$

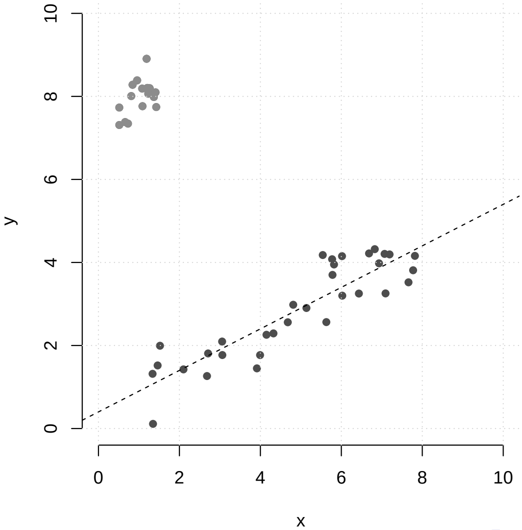


LMedS+RLS vs LS

Con outliers, scenario #4

$x \sim \mathrm{U}(a=1, b=8)$
$\varepsilon \sim \mathrm{N}(\mu=0, \sigma=0.5)$
$b_0=0.4, \quad b_1=0.5$
$y_n=b_0+b_1 x_n+\varepsilon_n \quad (n=1\ldots N)$
$N \in \{15, 49, 99\}$

$n_{\text{sims}} = 10\,000$
$p_o=0.333$
$N_o=\text{round}(p_o N)$
$n_o=\text{sample}(1:N, N_o, \text{repl.}=F)$
$x_{n_o} \sim \mathrm{N}(\mu=1.5, \sigma=0.35)$
$y_{n_o} \sim \mathrm{N}(\mu=8, \sigma=0.4)$

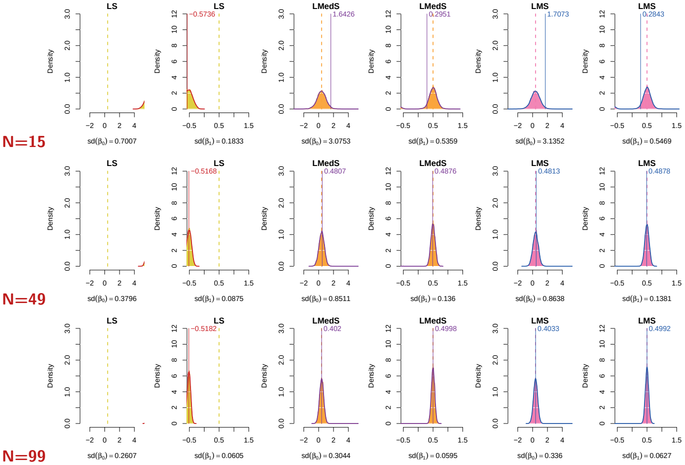


LMedS+RLS vs LS

Con outliers, scenario #4

$x \sim \mathrm{U}(a=1, b=8)$
$\varepsilon \sim \mathrm{N}(\mu=0, \sigma=0.5)$
$b_0 = 0.4, \quad b_1 = 0.5$
$y_n = b_0 + b_1 x_n + \varepsilon_n \quad (n = 1 \ldots N)$
$N \in \{15, 49, 99\}$

$n_{\text{sims}} = 10\,000$
$p_o = 0.333$
$N_o = \text{round}(p_o N)$
$n_o = \text{sample}(1 : N, N_o, \text{repl.} = F)$
$x_{n_o} \sim \mathrm{N}(\mu = 1.5, \sigma = 0.35)$
$y_{n_o} \sim \mathrm{N}(\mu = 8, \sigma = 0.4)$

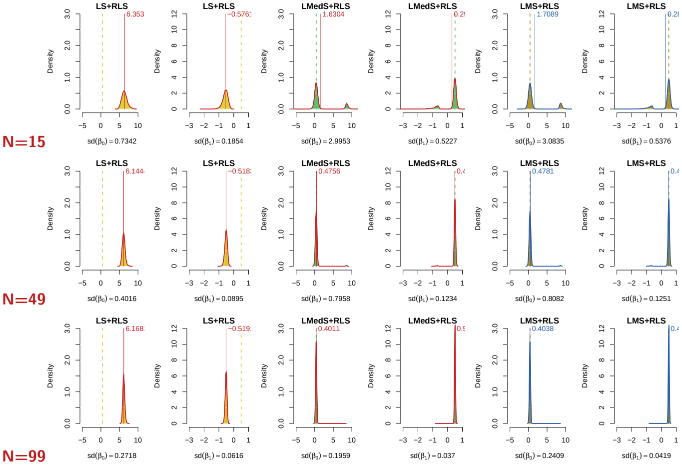


LMedS+RLS vs LS

Con outliers, scenario #4

$$x \sim \mathrm{U}(a=1, b=8)$$
$$\varepsilon \sim \mathrm{N}(\mu=0, \sigma=0.5)$$
$$b_0=0.4, \quad b_1=0.5$$
$$y_n=b_0+b_1 x_n+\varepsilon_n \quad (n=1\ldots N)$$
$$N \in \{15, 49, 99\}$$

$$n_{\text{sims}} = 10\,000$$
$$p_o=0.333$$
$$N_o=\text{round}(p_o N)$$
$$n_o=\text{sample}(1:N, N_o, \text{repl.}=F)$$
$$x_{n_o} \sim \mathrm{N}(\mu=1.5, \sigma=0.35)$$
$$y_{n_o} \sim \mathrm{N}(\mu=8, \sigma=0.4)$$



LMedS+RLS vs LS

Con outliers, scenario #4

$x \sim \mathrm{U}(a=1, b=8)$
$\varepsilon \sim \mathrm{N}(\mu=0, \sigma=0.5)$
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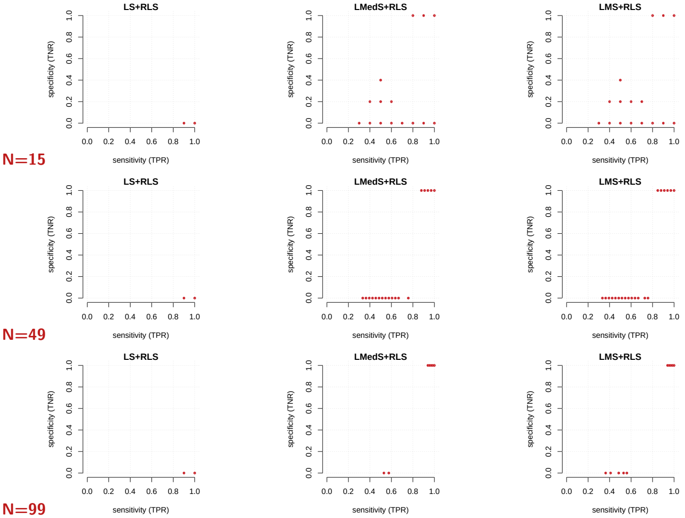
$x_{n_o} \sim \mathrm{N}(\mu=1.5, \sigma=0.35)$
$y_{n_o} \sim \mathrm{N}(\mu=8, \sigma=0.4)$



Conclusioni su Least Median of Squares

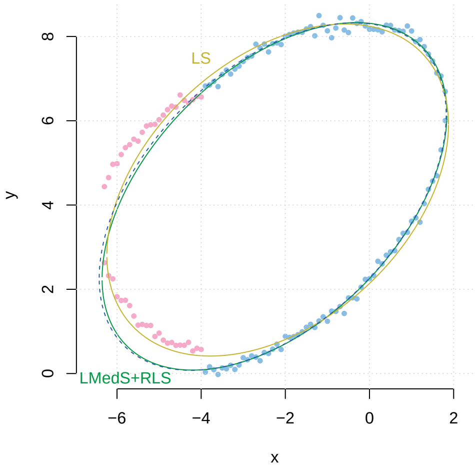
Take away

- Least Median of Squares (LMedS) è un algoritmo robusto ideato per resistere agli outliers
- Least Median of Squares (LMedS) è scarsamente efficiente
- Utile per stima iniziale da raffinare, ad esempio con Reweighted Least Squares (RLS) o M-estimators
- Test Monte Carlo applicati sul 33% di outliers hanno mostrato che la robustezza di LMedS+RLS dipende da come gli outliers sono distribuiti e dal numero di dati a disposizione
- La robustezza aumenta all'aumentare del numero di dati
- L'approccio di LMedS è generalizzabile anche al di fuori della regressione

Oltre la regressione

Esempio: stima di coniche

- $ax^2 + bxy + cy^2 + dx + ey + f = 0$
- $p = 5$ punti bastano per definire una conica
- residui: distanza geometrica punti-conica o Sampson Error



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