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EXAM ROLL - CSE224002
REGISTRATION NO. - 153160

Graph Theory & Combinatorics Exam

SECTION - A

(1) Let G be a connected graph and v_1 and v_2 be the 2 vertices of G with odd degree.
We know that no. of odd degree vertices is always even
So there is a ~~pa~~ path connecting v_1 and v_2 since G is connected.

Let G be disconnected and let us consider that there is no $v_1 - v_2$ path

By definition G is disconnected graph and v_1 and v_2 must lie in separate connected components of G. Thus vertex v_1 lies in H, a connected subgraph of G and all other vertices in H have even ~~odd~~ degree. So sum of degrees of vertices in H is an odd no.
But by handshaking lemma, the sum of degrees of the ~~vertex~~ vertices in H must be an even no. So we have a contradiction and there must be a path from v_1 to v_2.

(2) Let G be a maximal planar graph of order n, size m and faces f. G must be connected. So by Euler's formula $n - m + f = 2$. Since G is a maximal planar graph, each face is of size 3 in ~~any~~ ~~any~~ any planar embedding of G. So the no. of edges of G is 3f. But each edge is shared by exactly 2 faces, we have $2m = 3f$, so $m = n + f - 2 = n + \frac{2}{3}m - 2$, so $m = 3n - 6$

3. Let the graph has n vertices. The possibilities for the degree are 0, 1, ..., n−1. But a graph cannot have one ~~degree of~~ vertex of degree 0 and one vertex of degree n−1, because these 2 vertices would need to be adjacent to satisfy degree n−1.

By using Pigeonhole Principle we must pick the n degrees (one for each vertex), from a set of n−1 options (since we cannot have both 0 and n−1). Hence 2 vertices are of same degree.

4. Order of G $= n = 13$

$\therefore |E| = n-1 = 12$ [$\because$ G is a ~~graph~~ tree]

Let G has 'a' degree 1 vertex, b degree 5 and 3 degree 2 vertices

$a \times 1 + 5 \times b + 2 \times 3 = 2 \times 12$ [handshake lemma]

$\Rightarrow a + 5b = 18$

also, $a + b + 3 = n = 13 \Rightarrow a + b = 10$

so $\boxed{b = 2,\ a = 8}$

so no. of terminals $= 8$

5. Let G be a connected graph, a vertex $V \in G$ is called a cut vertex of G, if $G - V$ results in a disconnected graph, Removing a cut vertex from a graph breaks it into 2 or more graphs.

Cut Set — Let G be a connected graph, a subset E' of E is called a cut set of G if deletion of all edges of E' from G makes G disconnected.

They help to find connectedness of graph, articulation points and bridges.

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(6) $x+y+z=17$.

$a = x-1$
$b = y-1$
$c = z-1$

$a+b+c$
$= 17-3$
$= 14$

Positive integers solutions $= {}^{n+r-1}C_{r-1}$

$n+r-1$
$= 14+3-1$
$= 16$.

$\boxed{r=3}$

$= {}^{17-1}C_{3-1}$

$= {}^{16}C_2 = 120$

(7) We can arrange 8 trophies out of 30 by 1st choosing 8 trophies in ${}^{30}C_8$ ways and arrange in $8!$ ways
so total no. of ways $= 8! \times {}^{30}C_8 = {}^{30}P_8$ ways.

(8) We can choose Ram in 3 ways (can be any one of 1st, 2nd, 3rd) so now we have 29 ~~positions~~ ~~and~~ runners and 7 trophies left which can be given in ${}^{29}P_7$ ways
so total no. of ways $= 3 \times {}^{29}P_7$ ways.

(9) We have k different boxes and r different objects
Now to keep no box empty we choose k objects from r, which can be done in ${}^{r}C_K$ ways
these k objects can be arranged in k boxes in $k!$ ways.
Now, $r-k$ objects can be placed in any of the k boxes, so they can be arranged in $K^{(r-K)}$ ways
so total no. of ways $= {}^{r}C_K \times K! \times K^{r-K}$
$= r!$

(9) By inclusion-exclusion principle,

$$\text{no. of ways} = k^r - {}^kc_1(k-1)^r + {}^kc_2(k-2)^r - {}^kc_3(k-3)^r + \dots + (-1)^{k-1}\cdot {}^kc_{k-1}(k-(k-1))^r$$

$$= k^r - {}^kc_1(k-1)^r + {}^kc_2(k-2)^r - {}^kc_3(k-3)^r + \dots + (-1)^{k-1}\, {}^kc_{k-1}(1)^r$$

(10) We want no 3 consequtive people with increasing heights.

Let the heights be 1, 2, 3, 4, 5

i) case 1   All 5 people in increasing order

no. of ways = 1 (1, 2, 3, 4, 5)

ii) case 2   4 consequtive in one order

no. of ways = ${}^5c_4 \cdot 2!$ [4 selected people in ascending, 1 remaining]

$= 5\times2 = 10$

no. of ways $= 10 - 2 \times$ case-1

$= 10 - 2\times1 = 8$

[if we select (1, 2, 3, 4) as 4 people and 5 as remainder or (2, 3, 4, 5) and 1 as remainder]

iii) case-3   3 consequtive in line

No. of ways $= {}^5c_3 \cdot 3! - 2\times$case-2 $- 3\times$case-1

$= 41$

Total ways $= 120\,(5!) - (41+8+1) = 70$

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SECTION - B

(1) $a_n = -4a_{n-1} - 4a_{n-2} \quad n \geq 2 \quad a_0 = 4, a_1 = 8$

We get,

$r^n = -4r^{n-1} - 4r^{n-2}$

$\Rightarrow r^2 = -4r - 4$

$\Rightarrow r^2 + 4r + 4 = 0$

$\Rightarrow r = -2, -2$

let $a_n = A(-2)^n + Bn(-2)^n$

$a_0 = A + 0 = 4 = A = 4$

$a_1 = A(-2) + B(-2) = -2(A+B) = 8$

$\Rightarrow A + B = -4$

$\therefore B = -8$

so $a_n = 4(-2)^n - 8n(-2)^n$.

(2) We have ~~to~~ to take n-bit strings that do not contain 2 consequtive 1's.

For $n=1$, 1 bit string, no. of strings $= \{0, 1\} = 2$

For $n=2$, 2 bit string, no. of strings $= \{00, 01, 10\} = 3$

For $n=3$, 3 bit string, no. of strings $= \{000, 001, 010, 100, 101\} = 5$

For $n=4$, no. of strings $= \{0000, 0001, 0010, 0100, 1000, 0101, 1010, 1001\} = 8$

These values are forming Fibonacci series.

So Recurrence relation is $T(n) = T(n-1) + T(n-2)$

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so, no. of 6 bit strings $= T(6) = T(5) + T(4)$

$= T(4) + T(3) + T(4)$

$= T(3) + 2T(4)$

$= 5 + 2 \times 8$

$= 5 + 16 =$

$= \underline{21}$

Ans

(3) For each size we must select one ball, so we have to start from power of 1

so $g(x) = (x^1 + x^2 + \cdots + x^{k_1})(x^1 + x^2 + \cdots + x^{k_2}) \cdots (x^1 + x^2 + \cdots + x^{k_n})$

~~where $k_1 + k_2 + \cdots + k_n = k$~~ $= (x + x^2 + \cdots + x^k)^n$

This is the generating ~~equation~~ function.

so $g(x) = \left(\frac{x(x^k - 1)}{x - 1}\right)^n = \cancel{x^n (1-x^k)^n (1 + k c_1 x + k c_2 x^2}$

$= x^n (1 - x^k)^n (1 + {}^nc_1 x + {}^{(n+1)}c_2 x^2 + \cdots)$

$\boxed{\text{No. of ways} = {}^{k-1}c_{k-n} = {}^{k-1}c_{n-1}}$ Ans.

(4) We have 4 children who can get at least 2 mables and not more than 3 marbles so our generating function is

$g(x) = (x^2 + x^3)^4$

Now, we have to distribute 10 identical marbles so we have to find coefficient of x^{10} in $g(x)$

so $[x^{10}]$ in $x^8(1+x)^4$

$[x^{10}]$ in $x^8(1+4c_1x+4c_2x^2+4c_3x^3+4c_4x^4)$

$\Rightarrow [x^2]$ in $(1+4x+4c_2x^2+4c_3x^3+x^4)$

so coefficient of x^2 is $4c_2 = 6$ ways.

(5) $a+b+c=7$ $\quad a,b,c \geq 0$

$a \in \{0,1,3\}$

$1 \leq b \leq 3$

$c \in \{4,6\}$

so, generating function

$$g(x) = \underbrace{(x^0+x^1+x^3)}_{a} \cdot \underbrace{(x^1+x^2+x^3)}_{b} \cdot \underbrace{(x^4+x^6)}_{c}$$

$$= (1+x+x^3)\cdot(x+x^2+x^3)(x^4+x^6)$$

$$= (1+x+x^3)\cdot x(1+x+x^2)\cdot x^4(1+x^2)$$

$$= x^5(1+x+x^3)(1+x+x^2)(1+x^2)$$

Now, $[x^7]$ in $g(x)$

$\Rightarrow [x^2]$ in $(1+x+x^3)(1+x+x^2)(1+x^2)$

$=(1+x+x^3)(1+2x^2+x+x^3+x^4)$

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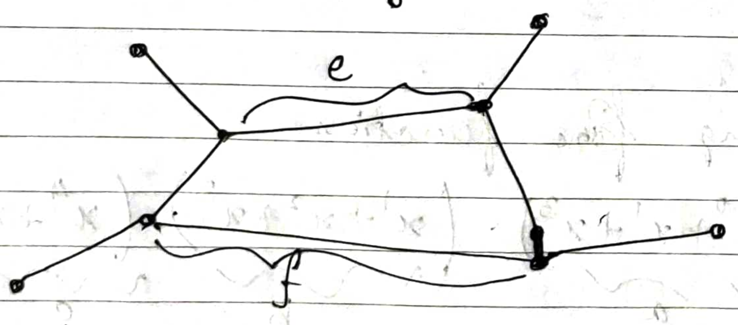
so, $[x^2]$ in $(1+x+x^3)(1+2x^2+x+x^3+x^4)$

so no. of ways = $1+2 = 3$ (Ans)

SECTION - C

(1) Kruskal's algorithm - We sort the edges according to their path lengths. Now, we take from the shortest length such that there is no previous connection between the nodes, this algorithm is greedy and traverses on grouping the nodes according to the shortest edge.

If we take ~~other~~ other edges that are not of the shortest lengths then, the graph could still be connected but because simply the nodes are connected.



suppose e has a very high value (edge weight is high) but f is smaller. we can see that f connects the 2 portions of the graph and so we can always reach everywhere, on this connected part, but f has a shorter length so idea of Kruskal holds.

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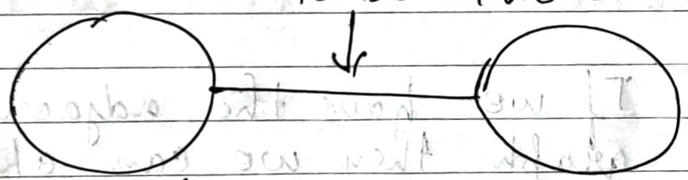
(2) Take an arbitrary node and connect n other nodes so that there can be n edges. There will be (n+1) nodes in ~~this~~ these groups. Now, out of the other (n−1) nodes, when we take a node from these (n−1) nodes. We have to assign atleast n edges ~~but~~ it is clear to see that we cannot keep the edges in this group as clearly $n-1 < n$

So we have to take atleast 1 node from the 1st group. so these points become connected one by one.

This edge/connection has to be there



A group with n+1 nodes

B group with (n−1) nodes

so the graph has to be connected, because this is the worst case scenario, Now if there are ~~are~~ less nodes or more connections then it is easily possible.

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(3) As G(V, E) is a connected graph where number of edges are less than the number of vertices, it is quite deducible that this is tree. We can simply see that, if we keep adding edges to the graph then we have to add them in the tree like order. Like take 1 node then there will be no edge, now add another node then add it to the graph with an edge.

So as this is a tree, this has to have leaves (nodes with no children), now the leaves have a degree of 1.

(4) If we have the adjacency matrix of the graph then we can apply Floyd Warshall's algorithm on the graph to find out the distance between every 2 points then we can say that the highest distance is actually the diameter of the graph.

Floyd Warshall algorithm is a ~~Dynammic~~ Dynammic Programming algorithm.

We take every 2 pairs of points and then try to reduce the dist between them using all the other nodes.

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```
for( k : 1 -> n)
    for( i : 1 -> n)
        for( j : i+1 -> n)
            dist[i][j] = min(dist[i][j],
                             dist[i][k] +
                             dist[k][j])
```

diameter of the graph = max(dist). [Ans]

(6) Let D be a digraph and V = vertex set of the digraph D and A = Arc family.

Let $f: A \to V$, $f(a) = v$, if a is directed towards v.

Let $A = \bigcup_{i=1}^{n} F_i$, $F_i = \{a \mid f(a) = v_i, v_i \in V\}$

clearly $F_i \cap F_j = \phi \ \forall \ 1 \le i < j \le n$ and

$|F_i| = \text{indegree}(v_i)$ [by definition of F]

$\therefore |A| = \left| \bigcup_{i=2}^{n} F_i \right| = \sum_{i=1}^{n} |F_i|$ [$\because F_i \cap F_j = \phi$]

$\Rightarrow |A| = \sum_{i=1}^{n} \text{indegree}(v_i)$

similarly $|A| = \sum_{i=1}^{n} \text{outdegree}(v_i)$

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Section - D

1) n objects are divided into classes $c_1, c_2, \ldots, c_K$ with $c_1 \leq c_2 \leq c_3 \leq c_4 \cdots \leq c_K$

i) no. of distinct arrangements of r objects chosen from these n, when $r \leq c_1$

Since $r \leq c_1$, there is no restriction on no. of objects from each class

$\therefore$ Number of choices for the ith object

$= k$ (since, there are k classes ~~$k_1, k_2, \ldots k$~~ $c_1, c_2, \cdots c_K$)

So total number of distinct arrangement

$= k^r$.

ii) No. of distinct arrangements of r objects when $r \leq n$.

As $r \leq n$, then there are ~~chanches~~ chances that we take more than possible no. of elements from each class.

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$\therefore$ Total number of distinct arrangements

$$= \sum_{\substack{0 \le a_i \le c_i}}^{a_1+a_2+\cdots+a_k=r} \frac{r!}{a_1!\, a_2! \cdots a_k!}$$