MATH 347 The Poisson model (one-parameter)

(Based on Hoff Section 3.2)

1. Poisson distribution

$$Pr(Y = y|\theta) = \text{dpois}(y, \theta) = \theta^y \exp(-\theta)/y!, \quad \text{for } y \in \{0, 1, 2, \cdots\}$$

   - $E(Y|\theta) = \theta$, $Var(Y|\theta) = \theta$

2. Gamma distribution

$$p(\theta) = \text{dgamma}(\theta, a, b) = \frac{b^a}{\Gamma(a)}\theta^{a-1} \exp(-b\theta), \quad \text{for } \theta, a, b > 0$$

   - $E(\theta) = a/b$, $Var(\theta) = a/b^2$
   - $\text{mode}(\theta) = (a - b)/b$, if $a > 1$; 0, if $a <= 1$

3. Now if we have a Poisson data model and a Gamma prior
   - Prior Gamma(a, b): complete the prior density
     $\pi(\theta) =$
   - Likelihood: complete the joint pdf
     $Pr(Y_1 = y_1, \cdots, Y_n = y_n|\theta) = \prod_{i=1}^n p(y_i|\theta) =$
   - Posterior Gamma$(a + \sum_{i=1}^n y_i, b + n)$: derive and recognize the hyper-parameters
     $\pi(\theta|y_1, \cdots, y_n) =$

4. Results: the gamma distribution is conjugate prior for Poisson sampling model (i.e. data model)

$$\left.\begin{aligned} \theta &\sim \text{Gamma}(a, b) \\ Y_1, \cdots, Y_n|\theta &\sim \text{Poisson}(\theta) \end{aligned}\right\} \rightarrow \{\theta|Y_1, \cdots, Y_n\} \sim \text{Gamma}(a + \sum_{i=1}^n y_i, b + n)$$

   - $E(\theta|y_1, \cdots, y_n) = \frac{a+\sum_{i=1}^n y_i}{b+n} = \frac{b}{b+n}\frac{a}{b} + \frac{n}{b+n}\frac{\sum y_i}{n}$,
     weighted average of the prior mean and sample mean

5. Prediction (Hoff page 47)