MATH 347 Gibbs sampler for Gamma-Gamma-Poisson (Solutions)

In class, we have demonstrated Gibbs sampling for a two-parameter Normal model, where both μ and σ are unknown. In fact, the Gibbs sampling algorithm works for any two-parameter model, or multi-parameter model when the number of parameters is more than 2.

Recall the Gamma-Poisson conjugate model we have discussed before, where the sampling model is a $Y_1, \cdots, Y_n \overset{i.i.d.}{\sim} \text{Poisson}(\lambda)$, and the conjugate prior for proportion λ is a Gamma, where $\lambda \sim \text{Gamma}(a, b)$. We know that the posterior distribution of λ is also a Gamma, where $\lambda \mid Y_1 = y_1, \cdots, Y_n = y_n \sim \text{Gamma}(a + \sum_{i=1}^{n} y_i, b + n)$.

Note that the Poisson sampling model has the following density function:

$$P(Y = y \mid \lambda) = \lambda^y \exp(-\lambda)/y!, \text{ for } y \in \{0, 1, 2, \cdots\}. \tag{1}$$

And the Gamma prior distribution has the following density function:

$$\pi(\lambda) = \frac{b^a}{\Gamma(a)} \lambda^{a-1} \exp(-b\lambda), \text{ for } \lambda, a, b > 0. \tag{2}$$

Now consider that $a = 1$ and b is unknown, and we give a Gamma prior for b, as in $b \sim \text{Gamma}(1, 1)$.

We come to the following model:

$$\begin{aligned}
Y_1, \cdots, Y_n \mid \lambda, b &\sim \text{Poisson}(\lambda), && (3)\\
\lambda \mid b &\sim \text{Gamma}(1, b), && (4)\\
b &\sim \text{Gamma}(1, 1). && (5)
\end{aligned}$$

How to sample the joint posterior distribution of (λ, b)? Gibbs sampler!

- Step 1: Write out the likelihood function $L(\lambda, b)$.

$$\begin{aligned}
L(\lambda, b) &= \prod_{i=1}^{n} \frac{\lambda^{y_i} \exp(-\lambda)}{y_i!} \\
&\propto \lambda^{\sum_{i=1}^{n} y_i} \exp(-n\lambda).
\end{aligned}$$

- Step 2: Write out the joint prior distribution $\pi(\lambda, b)$.

$$\begin{aligned}
\pi(\lambda, b) = \pi(\lambda \mid b)\pi(b) &= \frac{b^1}{\Gamma(1)} \lambda^{1-1} \exp(-b\lambda) \frac{1^1}{\Gamma(1)} b^{1-1} \exp(-b) \\
&\propto b \exp(-(\lambda b + b)).
\end{aligned}$$

- Step 3: Write out the joint posterior distribution $\pi(\lambda, b \mid y_1, \cdots, y_n)$.

$$\begin{aligned}
\pi(\lambda, b \mid y_1, \cdots, y_n) &\propto L(\lambda, b)\pi(\lambda, b) \\
&\propto \lambda^{\sum_{i=1}^{n} y_i + 1 - 1} b \exp(-(n\lambda + \lambda b + b)).
\end{aligned}$$

- Step 4: Derive the full conditional posterior distribution for each parameter: (λ, b)
  - Full conditional posterior distribution for λ: $\pi(\lambda \mid y_1, \cdots, y_n, b)$

    $$\pi(\lambda \mid y_1, \cdots, y_n, b) \propto \lambda^{\sum_{i=1}^n y_i + 1 - 1} \exp(-(n+b)\lambda), \tag{6}$$

    which means $\lambda \mid y_1, \cdots, y_n, b$ follows

    $$\lambda \mid y_1, \cdots, y_n, b \sim \text{Gamma}(\sum_{i=1}^n y_i + 1, n + b).$$

  - Full conditional posterior distribution for b: $\pi(b \mid y_1, \cdots, y_n, \lambda)$

    $$\pi(b \mid y_1, \cdots, y_n, \lambda) \propto b \exp(-(\lambda + 1)b), \tag{7}$$

    which means $b \mid y_1, \cdots, y_n, \lambda$ follows

    $$b \mid y_1, \cdots, y_n, \lambda \sim \text{Gamma}(2, \lambda + 1).$$

- Step 5: Code your Gibbs sampler.