Unit II

Non-Uniform flows-

Gradually Varied Flow (GVF)

Governing equation

Types of Flow Profiles

Computation of Gradually varied Flow profiles

by

a)Direct Integration

b)Graphical methods

and

c)Numerical methods.

4.1 INTRODUCTION

A steady non-uniform flow in a prismatic channel with gradual changes in its water surface elevation is termed as *gradually varied flow* (GVF). The backwater produced by a dam or weir across a river and the drawdown produced at a sudden drop in a channel are few typical examples of GVF. In a GVF, the velocity varies along the channel and consequently the bed slope, water surface slope, and energy slope will all differ from each other. Regions of high curvature are excluded in the analysis of this flow.

The two basic assumptions involved in the analysis of GVF are the following:

1. The pressure distribution at any section is assumed to be hydrostatic. This follows from the definition of the flow to have a gradually-varied water surface. As gradual changes in the surface curvature give rise to negligible normal accelerations, the departure from the hydrostatic pressure distribution is negligible. The exclusion of the region of high curvature from the analysis of GVF, as indicated earlier, is only to meet this requirement.
2. The resistance to flow at any depth is assumed to be given by the corresponding uniform flow equation, such as the Manning's formula, with the condition that the slope term to be used in the equation is the energy slope and not the bed slope. Thus, if in a GVF the depth of flow at any section is y, the energy slope S_f is given by

$$S_f = \frac{n^2 V^2}{R^{4/3}} \tag{4.1}$$

where R = hydraulic radius of the section at depth y.

4.2 DIFFERENTIAL EQUATION OF GVF

Consider the total energy H of a gradually varied flow in a channel of small slope and $\alpha = 1.0$ as

$$H = Z + E = Z + y + \frac{V^2}{2g} \tag{4.2}$$

where E = specific energy.

A schematic sketch of a gradually varied flow is shown in Fig. 4.1. Since the water surface, in general, varies in the longitudinal (x) direction, the depth of flow and total energy are functions of x. Differentiating Eq. 4.2 with respect to x

$$\frac{dH}{dx} = \frac{dZ}{dx} + \frac{dE}{dx} \tag{4.3}$$

i.e.

$$\frac{dH}{dx} = \frac{dZ}{dx} + \frac{dy}{dx} + \frac{d}{dx}\left(\frac{V^2}{2g}\right) \tag{4.4}$$

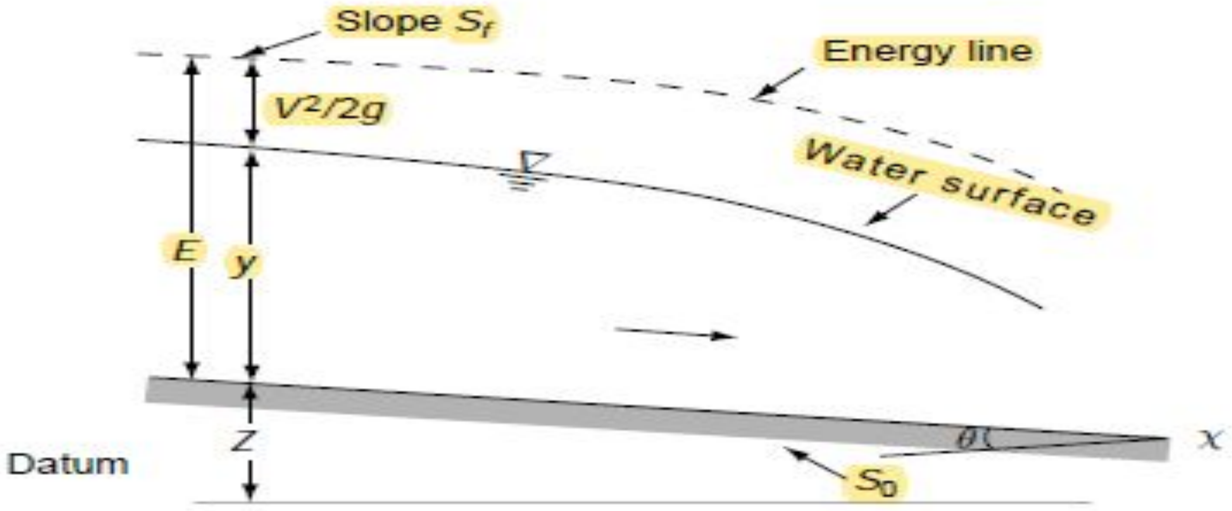

Fig. 4.1 *Schematic sketch of GVF*

In equation 4.4, the meaning of each term is as follows:

1. $\dfrac{dH}{dx}$ represents the energy slope. Since the total energy of the flow always decreases in the direction of motion, it is common to consider the slope of the decreasing energy line as positive. Denoting it by S_f, we have

$$\frac{dH}{dx} = -S_f \tag{4.5}$$

2. $\dfrac{dZ}{dx}$ denotes the bottom slope. It is common to consider the channel slope with bed elevations decreasing in the downstream direction as positive. Denoting it as S_0, we have

$$\frac{dZ}{dx} = -S_0 \tag{4.6}$$

3. $\dfrac{dy}{dx}$ represents the water surface slope relative to the bottom of the channel.

Other Forms of Eq. 4.8 (a) If K = conveyance at any depth y and K_0 = conveyance corresponding to the normal depth y_0, then

$$K = Q/\sqrt{S_f} \text{ (By assumption 2 of GVF)} \tag{4.9}$$

and $$K_0 = Q/\sqrt{S_0} \text{ (Uniform flow)}$$

$$S_f/S_0 = K_0^2/K^2 \tag{4.10}$$

Similarly, if Z = section factor at depth y and Z_c = section factor at the critical depth y_c,

$$Z^2 = A^3/T$$

and $$Z_c^2 = \frac{A_c^3}{T_c} = \frac{Q^2}{g}$$

4. $$\frac{d}{dx}\left(\frac{V^2}{2g}\right)=\frac{d}{dy}\left(\frac{Q^2}{2gA^2}\right)\frac{dy}{dx}$$

$$=-\frac{Q^2}{gA^3}\frac{dA}{dy}\frac{dy}{dx}$$

Since $dA/dy = T$,

$$\frac{d}{dx}\left(\frac{V^2}{2g}\right)=-\frac{Q^2 T}{gA^3}\frac{dy}{dx} \tag{4.7}$$

Equation 4.4 can now be rewritten as

$$-S_f=-S_0+\frac{dy}{dx}-\left(\frac{Q^2 T}{gA^3}\right)\frac{dy}{dx}$$

Re-arranging

$$\frac{dy}{dx}=\frac{S_0-S_f}{1-\dfrac{Q^2 T}{gA^3}} \tag{4.8}$$

This forms the basic differential equation of GVF and is also known as the *dynamic equation* of GVF. If a value of the kinetic-energy correction factor α greater than unity is to be used, Eq. 4.8 would then read as

$$\frac{dy}{dx}=\frac{S_0-S_f}{1-\dfrac{\alpha Q^2 T}{gA^3}} \tag{4.8a}$$

Hence,
$$\frac{Q^2T}{gA^3} = \frac{Z_c^2}{Z^2} \tag{4.11}$$

Using Eqs 4.10 and 4.11, Eq. 4.8 can now be written as

$$\frac{dy}{dx} = S_0 \frac{1 - \dfrac{S_f}{S_0}}{1 - \dfrac{Q^2T}{gA^3}}$$

$$= S_0 \frac{1 - \left(\dfrac{K_0}{K}\right)^2}{1 - \left(\dfrac{Z_c}{Z}\right)^2} \tag{4.12}$$

This equation is useful in developing direct integration techniques.

(b) If Q_n represents the normal discharge at a depth y and Q_c denotes the critical discharge at the same depth y,

$$Q_n = K\sqrt{S_0} \tag{4.13}$$

and
$$Q_c = Z\sqrt{g} \tag{4.14}$$

Using these definitions, Eq. 4.8 can be written as

$$\frac{dy}{dx} = S_0 \frac{1 - (Q/Q_n)^2}{1 - (Q/Q_c)^2} \tag{4.15}$$

(c) Another form of Eq. 4.8 is Eq. 4.3 and can be written as

$$\frac{dE}{dx} = S_0 - S_f \tag{4.16}$$

This equation is called the differential-energy equation of GVF to distinguish it from the GVF differential equations (Eqs (4.8), (4.12) and (4.15)). This energy

4.3 CLASSIFICATION OF FLOW PROFILES

In a given channel, y_0 and y_c are two fixed depths if Q, n and S_0 are fixed. Also, there are three possible relations between y_0 and y_c as (i) $y_0 > y_c$, (ii) $y_0 < y_c$ and (iii) $y_0 = y_c$. Further, there are two cases where y_0 does not exist, i.e. when (a) the channel bed is horizontal, ($S_0 = 0$), (b) when the channel has an adverse slope,

(S_0 is –ve). Based on the above, the channels are classified into five categories as indicated in Table 4.1.

For each of the five categories of channels, lines representing the critical depth and normal depth (if it exists) can be drawn in the longitudinal section. These would divide the whole flow space into three regions as:

Region 1: Space above the top most line

Region 2: Space between top line and the next lower line

Region 3: Space between the second line and the bed

Figure 4.2 shows these regions in the various categories of channels.

Table 4.1 *Classification of Channels*

Sl. No	*Channel category*	*Symbol*	*Characteristic condition*	*Remark*
1	*Mild slope*	*M*	$y_0 > y_c$	*Subcritical flow at normal depth*
2	*Steep slope*	*S*	$y_c > y_0$	*Supercritical flow at normal depth*
3	*Critical slope*	*C*	$y_c = y_0$	*Critical flow at normal depth*
4	*Horizontal bed*	*H*	$S_0 = 0$	*Cannot sustain uniform flow*
5	*Adverse slope*	*A*	$S_0 < 0$	*Cannot sustain uniform flow*

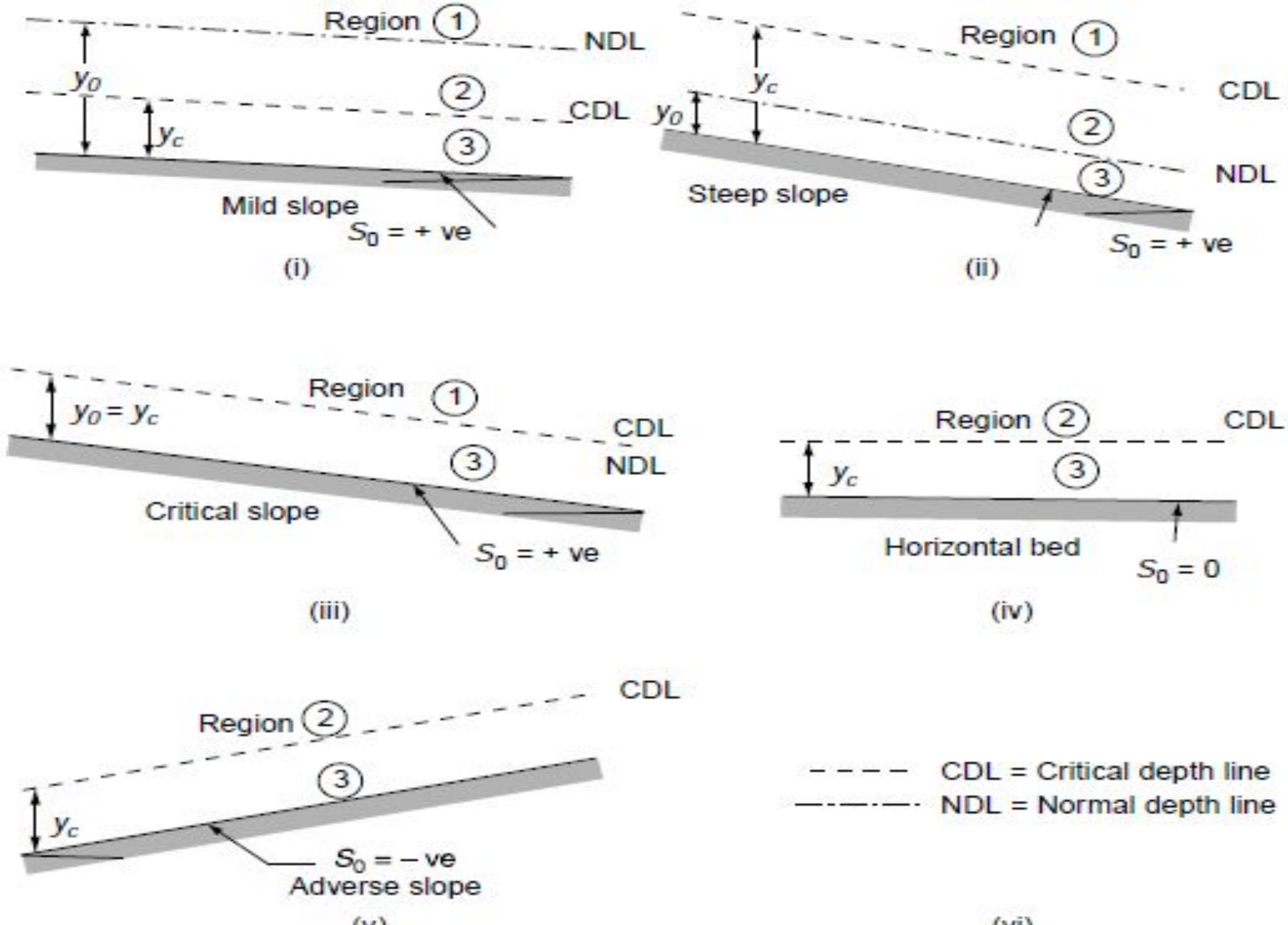

Fig. 4.2 *Regions of flow profiles*

Depending upon the channel category and region of flow, the water surface profiles will have characteristic shapes. Whether a given GVF profile will have an increasing or decreasing water depth in the direction of flow will depend upon the term dy/dx in Eq. 4.8 being positive or negative.

It can be seen from Eq. 4.12 that $\frac{dy}{dx}$ is positive

(i) if the numerator > 0 and the denominator > 0

or (ii) if the numerator < 0 and the denominator < 0.

i.e. $\frac{dy}{dx}$ is positive if (i) $K > K_0$ and $Z > Z_c$ or

(ii) $K < K_0$ and $Z > Z_c$

For channels of the first kind, K is a single-valued function of y, and hence

$\frac{dy}{dx} > 0$ if (i) $y > y_0$ and $y > y_c$ or

(ii) $y < y_0$ and $y < y_c$

Similarly, $\frac{dy}{dx} < 0$ if (i) $y_c > y > y_0$ or

(ii) $y_0 > y > y_c$

Further, to assist in the determination of flow profiles in various regions, the behaviour of dy/dx at certain key depths is noted by studying Eq. 4.8 as follows:

1. As $y \to y_0, \dfrac{dy}{dx} \to 0,$ i.e. the water surface approaches the normal depth line asymptotically.
2. As $y \to y_c, \dfrac{dy}{dx} \to \infty,$ i.e. the water surface meets the critical depth line vertically. This information is useful only as indicative of the trend of the profile. In reality, high curvatures at critical depth zones violate the assumption of gradually-varied nature of the flow and as such the GVF computations have to end or commence a short distance away from the critical-depth location.
3. $y \to \infty, \dfrac{dy}{dx} \to S_0$, i.e. the water surface meets a very large depth as a horizontal asymptote.

Based on this information, the various possible gradually varied flow profiles are grouped into twelve types (Table 4.2). The characteristic shapes and end conditions of all these profiles are indicated in Fig. 4.3.

In Fig. 4.3, an exaggerated vertical scale is adopted to depict the nature of curvature. In reality the GVF profiles, especially M_1, M_2 and H_2 profiles, are very flat. The longitudinal distances are one to two orders of magnitude larger than the depths. It is evident from Fig. 4.3 that all the curves in region 1 have positive slopes; these

are commonly known as backwater curves. Similarly, all the curves in region 2 have negative slopes and are referred to as drawdown curves. At critical depth the curves are indicated by dashed lines to remind that the GVF equation is strictly not applicable in that neighbourhood.

Table 4.2 *Types of GVF Profiles*

Channel	Region	Condition	Type
Mild slope	1	$y > y_0 > y_c$	M_1
	2	$y_0 > y > y_c$	M_2
	3	$y_0 > y_c > y$	M_3
Steep slope	1	$y > y_c > y_0$	S_1
	2	$y_c > y > y_0$	S_2
	3	$y_c > y_0 > y$	S_3
Critical slope	1	$y > y_0 = y_c$	C_1
	3	$y < y_0 = y_c$	C_3
Horizontal bed	2	$y > y_c$	H_2
	3	$y < y_c$	H_3
Adverse slope	2	$y > y_c$	A_2
	3	$y < y_c$	A_3

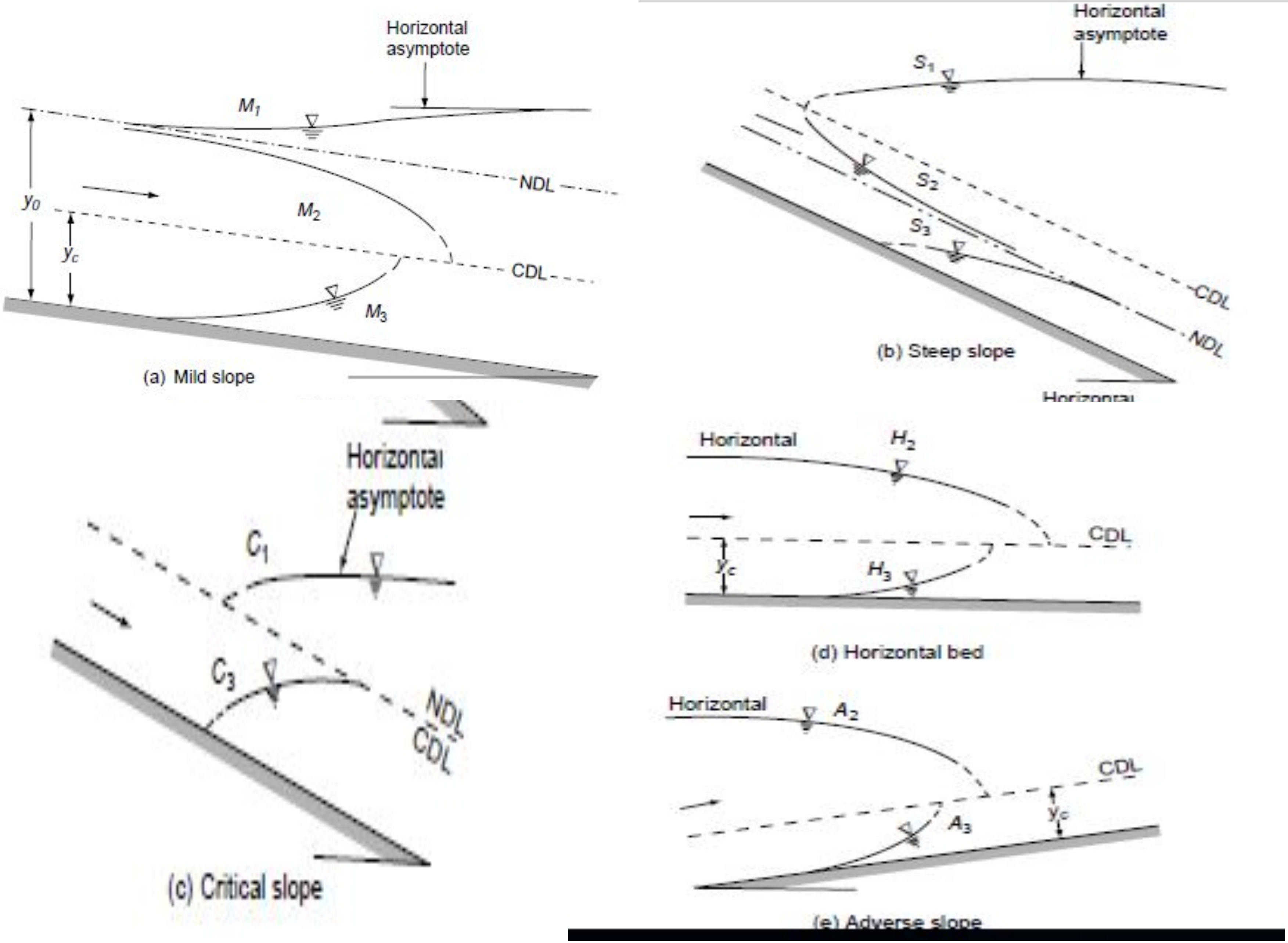

Fig. 4.3 *Various GVF Profiles (C*

4.4 SOME FEATURES OF FLOW PROFILES

(a) Type-M Profiles The most common of all GVF profiles is the M_1 type, which is a subcritical flow condition. Obstructions to flow, such as weirs, dams, control structures and natural features, such as bends, produce M_1 backwater curves Fig. 4.4 (a). These extend to several kilometres upstream before merging with the normal depth.

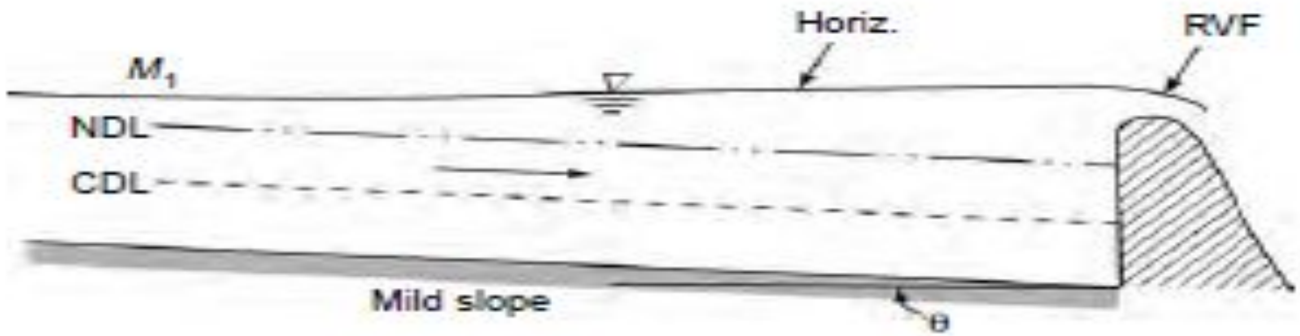

Fig. 4.4 (a) *M_1 profile*

The M_2 profiles occur at a sudden drop in the bed of the channel, at constriction type of transitions and at the canal outlet into pools Fig. 4.4 (b).

Fig. 4.4(b) *M_2 profile*

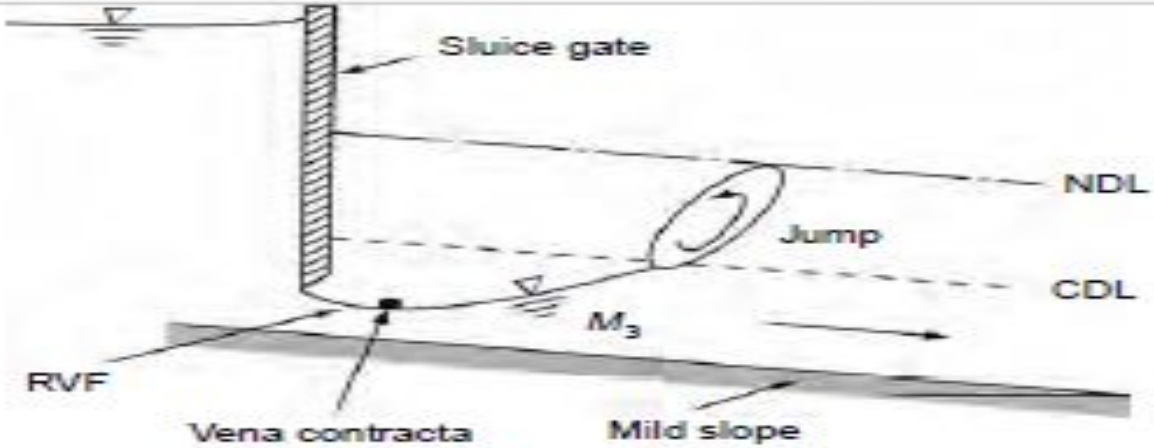

Fig. 4.4 (c) *M_3 profile*

Where a supercritical stream enters a mild-slope channel, the M_3 type of profile occurs. The flow leading from a spillway or a sluice gate to a mild slope forms a typical example (Fig. 4.4(c)). The beginning of the M_3 curve is usually followed by a small stretch of rapidly-varied flow and the down stream is generally terminated by a hydraulic jump. Compared to M_1 and M_2 profiles, M_3 curves are of relatively short length.

(b) Type-S Profiles The S_1 profile is produced when the flow from a steep channel is terminated by a deep pool created by an obstruction, such as a weir or dam (Fig. 4.4 (d)). At the beginning of the curve, the flow changes from the normal depth (supercritical flow) to subcritical flow through a hydraulic jump. The profiles extend downstream with a positive water surface slope to reach a horizontal asymptote at the pool elevation.

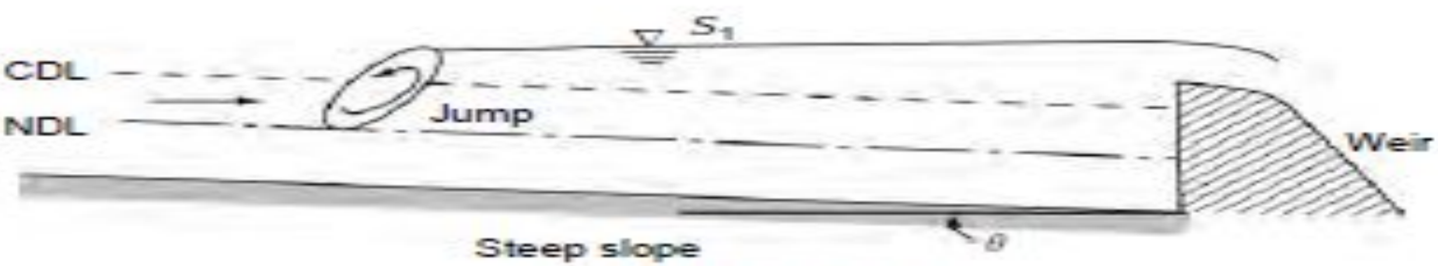

Fig. 4.4(d) *S_1 profile*

Profiles of the S_2 type occur at the entrance region of a steep channel leading from a reservoir and at a break of grade from mild slopes to steep slope (Fig. 4.4(e)). Generally S_2 profiles are of short length.

Free flow from a sluice gate with a steep slope on its downstream is of the S_3 type (Fig. 4.4(f)). The S_3 curve also results when a flow exists from a steeper slope to a less steep slope (Fig. 4.4.(g)).

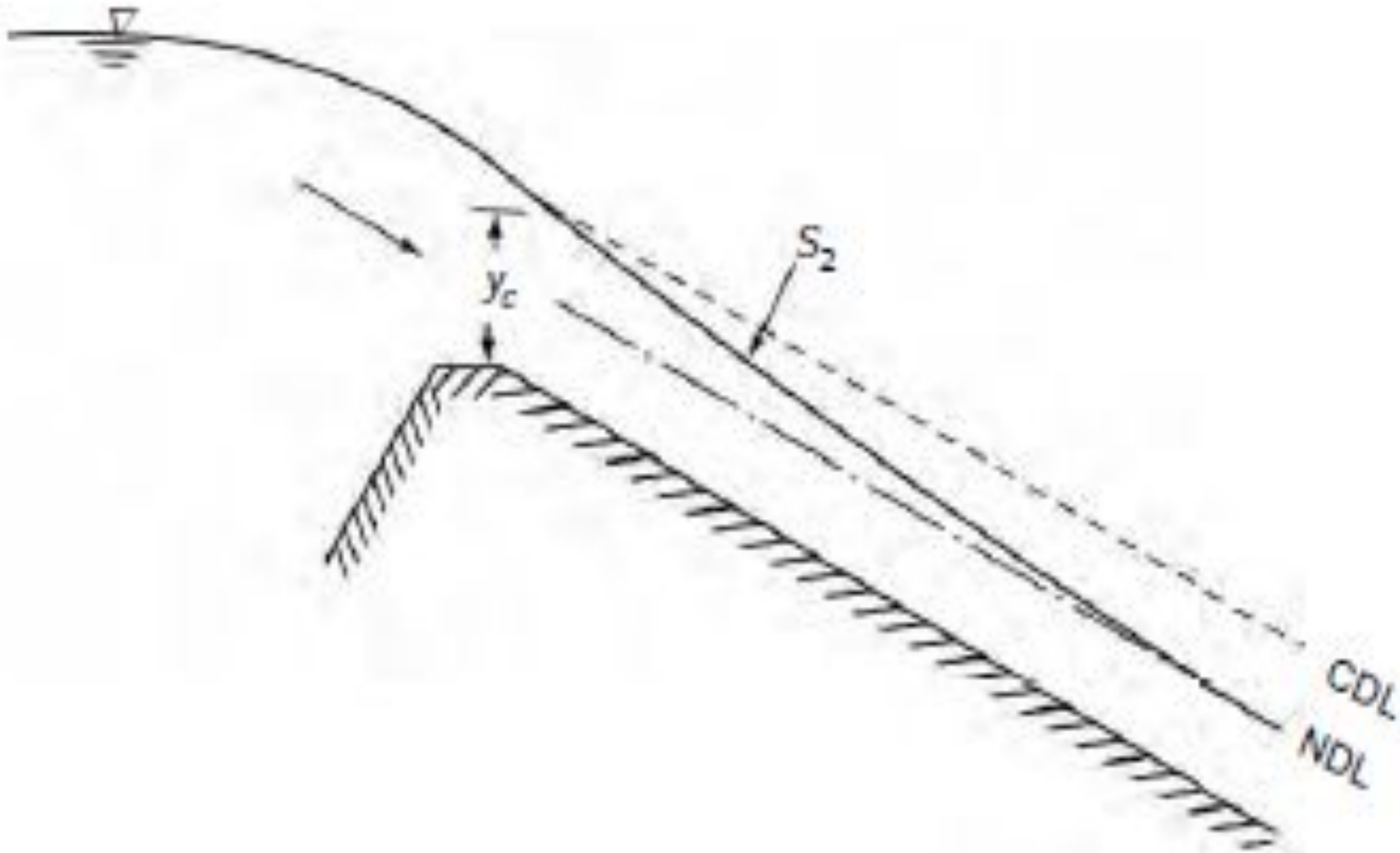

Fig. 4.4(e) S_2 *profile*

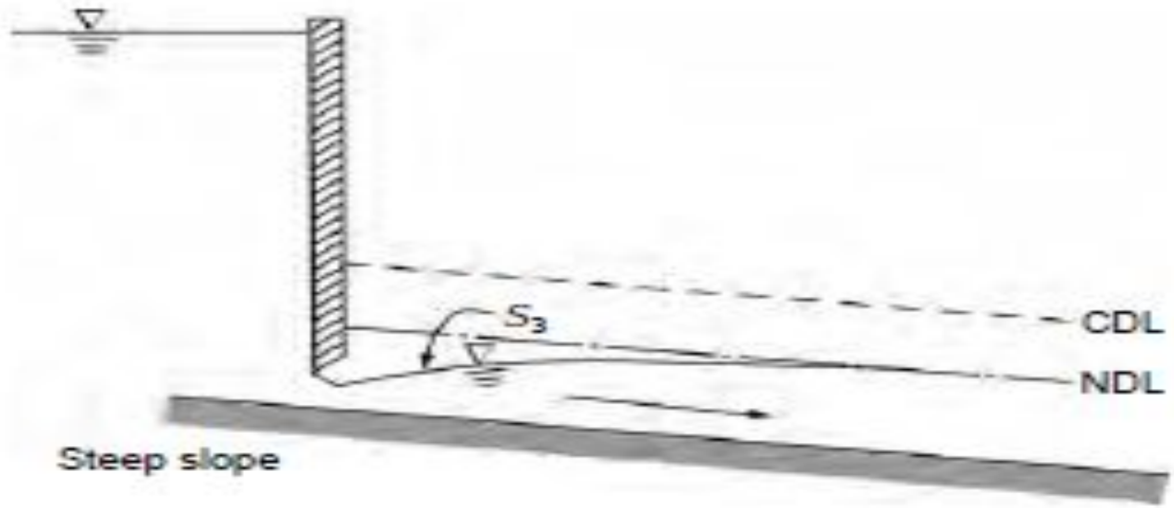

Fig. 4.4(f) *S_3 profile*

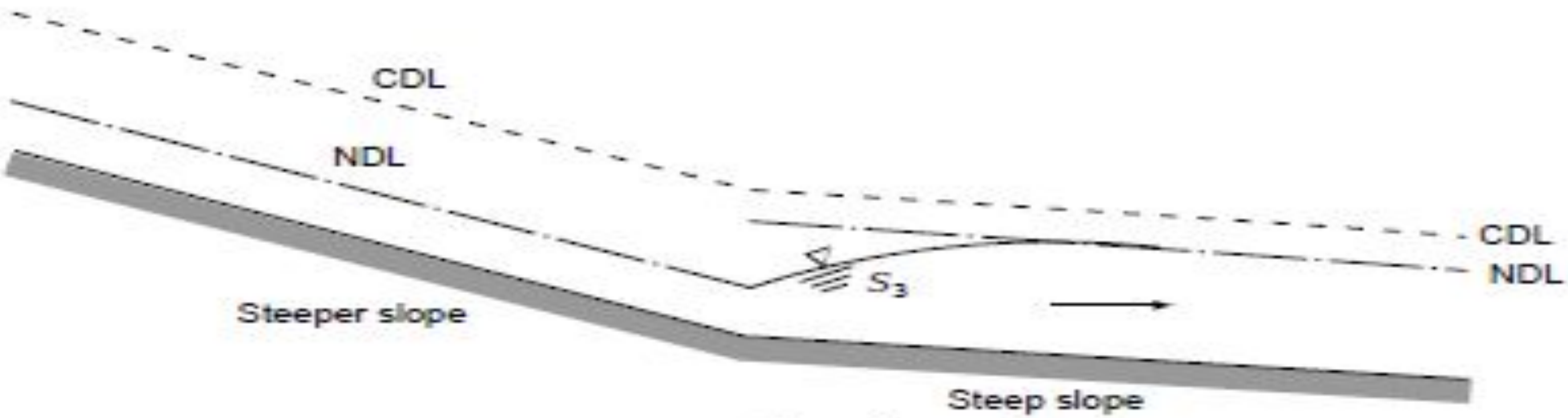

Fig. 4.4(g) *S_4 profile*

(c) Type C Profiles C_1 and C_3 profiles are very rare and are highly unstable.

(d) Type H Profiles A horizontal channel can be considered as the lower limit reached by a mild slope as its bed slope becomes flatter. It is obvious that there is no region 1 for a horizontal channel as $y_0 = \infty$. The H_2 and H_3 profiles are similar to M_2 and M_3 profiles respectively [Fig. 4.4(h)]. However, the H_2 curve has a horizontal asymptote.

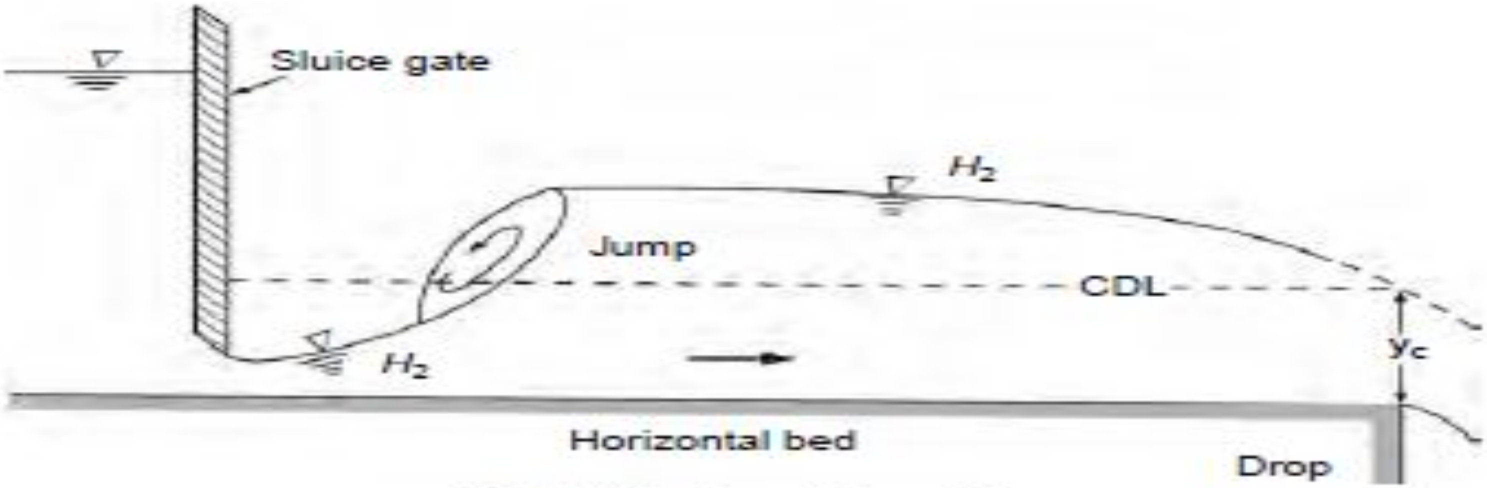

Fig. 4.4(h) *H_2 and H_3 profiles*

(e) Type A Profiles Adverse slopes are rather rare and A_2 and A_3 curves are similar to H_2 and H_3 curves respectively (Fig. 4.4 (i)). These profiles are of very short length.

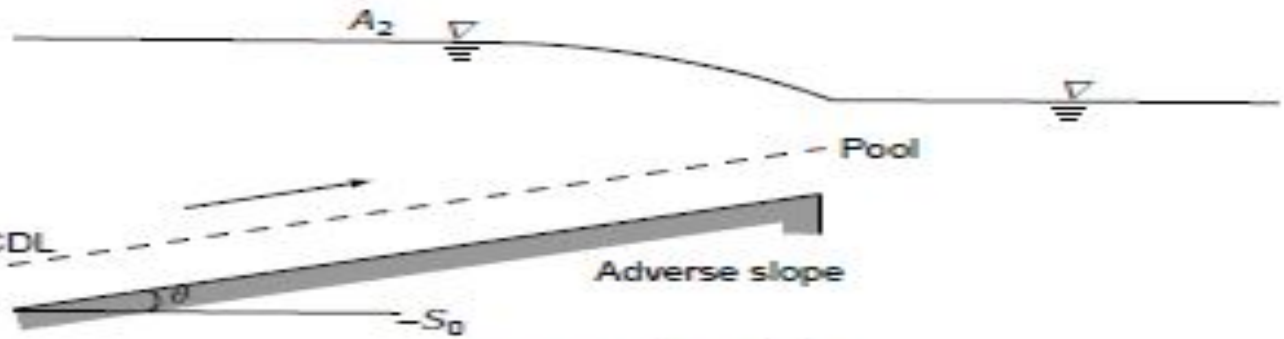

Fig. 4.4(i) *A_2 profile*

The various available procedures for computing GVF profiles can be classified as:

1. Graphical method
2. Direct integration methods
3. Numerical methods

Out of these the graphical method is practically obsolete and is seldom used. The direct integration technique is essentially of academic interest.

With the development of direct integration, to meet the practical needs, various solution procedures involving graphical and numerical methods were evolved for use by professional engineers.

The advent of high-speed computers has given rise to general programmes utilizing sophisticated numerical techniques for solving GVF in natural channels . Further, the numerical method is the most extensively used technique. In the form of a host of available comprehensive soft wares, it is the only method available to solve practical problems in natural channels.

The computation of gradually-varied-flow profiles involves basically the solution of the dynamic equation of gradually varied flow. The main objective of the computation is to determine the shape of the flow profile. Broadly classified, there are three methods of computation; namely, the graphical-integration method, the direct-integration method, and the step method. The development and procedure of several typical methods will be described in this chapter.

10-1. The Graphical-integration Method. This method is to integrate the dynamic equation of gradually varied flow by a graphical procedure. Consider two channel sections (Fig. 10-1*a*) at distances x_1 and x_2, respectively, from a chosen origin and with corresponding depths of flow y_1 and y_2. The distance along the channel floor is

$$x = x_2 - x_1 = \int_{x_1}^{x_2} dx = \int_{y_1}^{y_2} \frac{dx}{dy}\,dy \qquad (10\text{-}1)$$

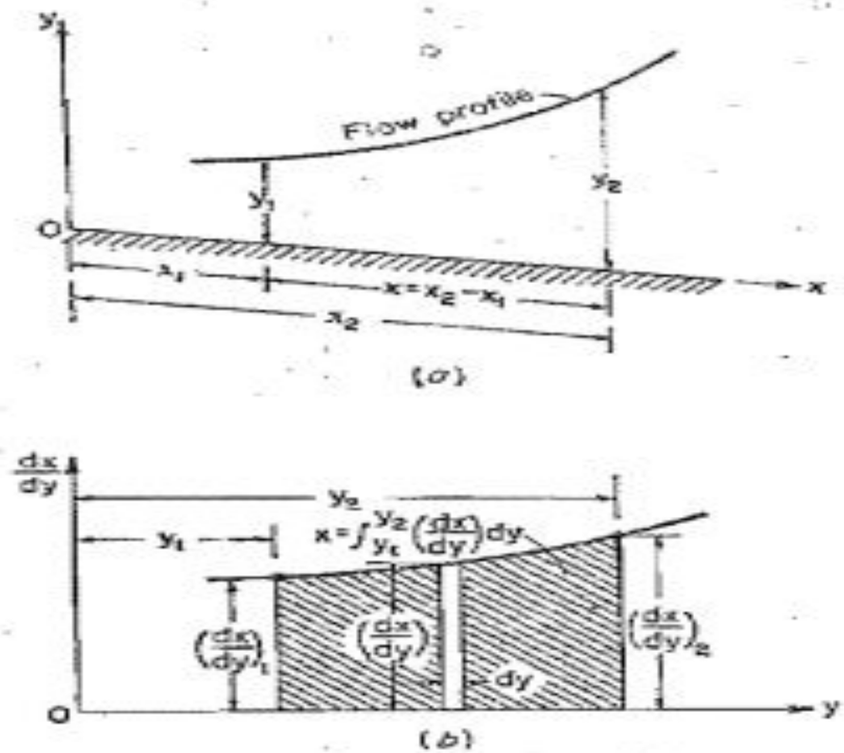

Fig. 10-1. Principle of the graphical-integration method.

Assume several values of y, and compute the corresponding values of dx/dy, which is the reciprocal of the right-side member of a gradually-varied-flow equation, say Eq. (9-13). A curve of y against dx/dy is then constructed (Fig. 10-1*b*). According to Eq. (10-1), it is apparent that the value of x is equal to the shaded area formed by the curve, the y axis, and the ordinates of dx/dy corresponding to y_1 and y_2. This area can be measured and the value of x determined.

This method has broad application. It applies to flow in prismatic as .well as non prismatic channels of any Shape and slope. The procedure is straightforward and easy to follow. It may, *however, become very* laborious when applied to actual problems.

5.2 DIRECT INTEGRATION OF GVF DIFFERENTIAL EQUATION

The differential equation of GVF for the prismoidal channel, from Eq. 4.12, given by

$$\frac{dy}{dx} = S_0 \frac{1-\left(K_0^2 / K^2\right)}{1-\left(Z_c^2 / Z^2\right)} = F(y)$$

is a non-linear, first order, ordinary differential equation. This can be integrated by analytical methods to get closed from solutions only under certain very restricted conditions. A methods due to Chow[2], which is based on certain assumptions but applicable with a fair degree of accuracy to a wide range of field conditions, is presented here.

Let it be required to find $y = f(x)$ in the depth range y_1 to y_2. The following two assumptions are made:

1. The conveyance at any depth y is given by

$$K^2 = C_2 y^N \tag{5.1}$$

and at the depth y_0 by

$$K_0^2 = C_2 y_0^N \tag{5.2}$$

This implies that in the depth range which includes y_1, y_2 and y_0, the coefficient C_2 and the second hydraulic exponent N are constants.

This implies that in the depth range which includes y_1, y_2 and y_0, the coefficient C_2 and the second hydraulic exponent N are constants.

2. The section factor Z at any depth y is given by

$$Z_2 = C_1 y^M \tag{5.3}$$

and at the critical depth y_0 by

$$Z_c^2 = C_1 y_c^M \tag{5.4}$$

implying that in the depth range which includes y_1, y_2 and y_c, the coefficient C_1 and the first hydraulic exponent M are constants.

Substituting the relationships given by Eqs 5.1 through 5.4 in Eq. 4.12,

$$\frac{dy}{dx} = S_0 \frac{1-(y_0 / y)^N}{1-(y_c / y)^M} \tag{5.5}$$

Putting $u = y/y_0$, $dy = y_0\ du$ and Eq. 5.5 simplifies to

$$\frac{du}{dx} = \frac{S_0}{y_0}\left[\frac{1-1/u^N}{1-\left(y_c^M / y_0^M\right)\frac{1}{u^M}}\right]$$

i.e. $$dx = \frac{y_0}{S_0}\left[1 - \frac{1}{1-u^N} + \left(\frac{y_c}{y_0}\right)^M \frac{u^{N-M}}{1-u^N}\right] du$$

Integrating

$$x = \frac{y_0}{S_0}\left[u - \int_0^u \frac{du}{1-u^N} + \left(\frac{y_c}{y_0}\right)^M \int_0^u \frac{u^{N-M}}{1-u^N}\, du\right] + \text{ Const.} \tag{5.6}$$

Calling $$\int_0^u \frac{du}{1-u^N} = F(u,N)$$

the second integral can be simplified as follows:

Put $$v = u^{N/J} \text{ where } J = \frac{N}{(N-M+1)}$$

to get $$dv = \frac{N}{J} u^{\frac{N}{J}-1}\, du$$

$$= (N - M + 1)\; u^{N-M}\, du$$

$\therefore$ $$\int_0^u \frac{u^{N-M}}{1-u^N}\, du = \frac{1}{(N-M+1)} \int_0^v \frac{dv}{1-v^J}$$

$$= \frac{J}{N} F(v,J) \tag{5.8}$$

It may be noted that $F(v, J)$ is the same function as $F(u, N)$ with u and N replaced by v and J respectively.
Eq. 5.6 can now be written as

$$x = \frac{y_0}{S_0}\left[u - F(u,N) + \left(\frac{y_c}{y_0}\right)^M \frac{J}{N} F(v,J)\right] \tag{5.9}$$

Using Eq. 5.9 between two sections (x_1, y_1) and (x_2, y_2) yields

Using Eq. 5.9 between two sections (x_1, y_1) and (x_2, y_2) yields

$$(x_2 - x_1) = \frac{y_0}{S_0}\Big[(u_2 - u_1) - \{F(u_2, N) - F(u_1, N)\}$$

$$+ \left(\frac{y_c}{y_0}\right)^M \frac{J}{M}\{F(v_2, J) - F(v_1, J)\}\Big] \tag{5.10}$$

The function $F(u, N)$ is known as the *varied-flow function*. Extensive tables of $F(u, N)$ are readily available[1,2,3] and a table showing $F(u, N)$ for a few values of N is presented in Table 5A.1 in Appendix 5A at the end of this chapter.

A method of obtaining the exact analytical solutions of $\int_0^u \frac{du}{1-u^N}$ for integral and non-integral values of N is given by Gill[4]. Numberical integration of $\int_0^u \frac{du}{1-u^N}$ can be performed easily on a computer to obtain tables of varied-flow functions. Bakhmeteff gives a procedure for this in the appendix of his treatise[1].

In practical applications, since the exponents N and M are likely to depend on the depth of flow, though to a smaller extent, average values of the exponents applicable to the ranges of values of depths involved must be selected. Thus the appropriate range of depths for N includes y_1, y_2, and y_0; and for M it includes y_1, y_2, and y_c. In computing water-surface profiles that approach their limits asymptotically (e.g. $y \to y_0$), the computations are usually terminated at y values which are within 1 per cent of their limit values.

5.3 BRESSE'S SOLUTION

For a wide rectangular channel, if the Chezy formula with C = constant is used the hydraulic exponents take the value $M = 3.0$ and $N = 3.0$. By putting these values of $M = 3.0$ and $N = 3.0$ in Eq. (5.9) the GVF profile would be

$$x = \frac{y_0}{S_0}\left[u - \left(1 - \left(\frac{y_c}{y_0}\right)^3\right) F(u,3)\right] + \text{ a constant} \tag{5.11}$$

And from Eq. (5.7)

$$F(u,3) = \int_0^u \frac{du}{1-u^3}$$

The function $F(u, 3)$ was first evaluated by Bresse in 1860 in a closed form as

$$F(u,3) = \frac{1}{6}\ln\left[\frac{u^2+u+1}{(u-1)^2}\right] - \frac{1}{\sqrt{3}}\arctan\left[\frac{\sqrt{3}}{2u+1}\right] + \text{ a constant} \tag{5.12}$$

$F\,(u,\,3)$ is known as Bresse's function. Apart from historical interest, values of Bresse's function being based on an exact solution are useful in comparing the relative accuracies of various numerical schemes of computation.

[*Note:* Table 5A–1 has a constant of value 0.6042 added to all values of $F(u, N)$.

Bresse's solution is useful in estimating approximately the length of GVF profiles between two known depths. The length of M_1 profile from 150% of normal depth downstream to 101% of normal depth upstream can be shown to be given by

$$\frac{LS_0}{y_0} = 1.654 - 1.164\,F^2 \tag{5.13}$$

where F is the Froude number of the normal flow in the channel.

In general the length of the GVF profile between two feasible depths are given by

$$\frac{LS_0}{y_0} = A + BF^2 \tag{5.14}$$

where values of A and B for some ranges are as given below:

Value of A	*Value of B*	*Range of percentage of y/y_0 values*	*Typical case*
0.599	*– 0.869*	*97% to 70%*	*M_2 curve*
0.074	*– 0.474*	*70% to 30%*	*M_2 curve*
1.654	*– 1.164*	*101% to 150%*	*M_1 curve*
1.173	*– 0.173*	*150% to 250%*	*M_1 curve*
–1.654	*1.164*	*150% to 101%*	*S_2 curve*

5.5 DIRECT INTEGRATION FOR CIRCULAR CHANNELS

5.5.1 Keifer and Chu's method

The direct integration of the differential equation of GVF by Chow's method is very inconvenient to use in the computation of GVF profiles in circular channels.

A different approach of integration of the differential equation of GVF for circular channels, developed by Keifer and Chu[5], simplifies the calculation procedure considerably.

Let Q be the actual discharge in a circular channel of diameter D and bed slope S_0. Then

$$Q = K\sqrt{S_f} \tag{5.21}$$

and

$$Q = K_0\sqrt{S_0} \tag{5.22}$$

where K and K_0 are the conveyance at depths y and y_0 respectively, y_0 = normal depth, S_f = energy slope at depth y. Let Q_D = a hypothetical discharge corresponding to uniform flow with the channel flowing full.

Then

$$Q_D = K_D\sqrt{S_0} \tag{5.23}$$

where K_D = conveyance at depth D.

i.e.

$$K_D = \frac{1}{n}\left(\pi D^2/4\right)\left(D/4\right)^{2/3}$$

$$\left(\frac{K_0}{K}\right)^2 = \left(\frac{K_0}{K_D}\right)^2\left(\frac{K_D}{K}\right)^2$$

But

$$\left(\frac{K_D}{K}\right)^2 = \left[\frac{(\pi D^2/4)(D/4)^{2/3}}{AR^{2/3}}\right] = f_1\,(y/D)$$

and $$\left(\frac{K_0}{K_D}\right)^2 = \left(\frac{Q}{Q_D}\right)^2 = (Q_r)^2 \tag{5.24}$$

$\therefore$ $$\left(\frac{K_0}{K}\right)^2 = Q^2{}_r\, f_1(y/D) \tag{5.25}$$

The differential equation of GVF, Eq. 4.12, becomes

$$\frac{dy}{dx} = S_0 \frac{1 - Q_r^2\, f_1(y/D)}{1 - Q^2\, T/gA^3}$$

Noting that $$\frac{Q^2 T}{gA^3} = \frac{Q^2}{g}\,\frac{1}{D^5}\,\frac{T/D}{(A^3/D^6)} = \frac{Q^2}{gD^5}\, f_2(y/D) \tag{5.26}$$

and putting $$y/D = \eta$$

$$\frac{d\eta}{dx} = \frac{S_0}{D}\left[\frac{1 - Q_r^2\, f_1(\eta)}{1 - \dfrac{Q^2}{gD^5}\, f_2(\eta)}\right]$$

$$dx = \frac{D}{S_0}\left[\frac{d\eta}{1 - Q_r^2\, f_1(\eta)} - \frac{Q^2}{gD^5}\,\frac{f_2(\eta)\, d\eta}{1 - Q_r^2\, f_1(\eta)}\right]$$

Integrating,

$$x = \frac{D}{S_0}\left[\int_0^{\eta}\frac{d\eta}{1 - Q_r^2\, f_1(\eta)} - \frac{Q^2}{gD^5}\int_0^{\eta}\frac{f_2(\eta)\, d\eta}{1 - Q_r^2\, f_1(\eta)}\right] + \text{Const.} \tag{5.27}$$

$\therefore$ $$x = \frac{D}{S_0}\left[I_1 - \frac{Q^2}{gD^5}\, I_2\right] + \text{Const.} \tag{5.28}$$

where $$I_1 = \int_0^{\eta}\frac{d\eta}{1 - Q_r^2\, f_1(\eta)} = I_1(Q_r, \eta)$$

and $$I_2 = \int_0^{\eta}\frac{f_2(\eta)\, d\eta}{1 - Q_r^2\, f_1(\eta)} = I_2(Q_r, \eta)$$

Functions I_1 and I_2 are known as *Keifer and Chu functions* and are available in

Functions I_1 and I_2 are known as *Keifer and Chu functions* and are available in slightly different forms in References 3, 6 and 7. The computation of GVF profiles in circular channels is considerably simplified by the use of these functions. Since $y/D = \eta = 1/2(1 - \cos\theta) = f(\theta)$ where $2\theta =$ angle subtended by the water surface at the centre of the section the functions I_1 and I_2 can also be represented as $I_1(Q_r, \theta)$, and $I_2(Q_r, \theta)$. Tables 5A.2(a) and 5A.2(b) in Appendix 5A show the functions I_1 and I_2 respectively, expressed as functions of Q_r and θ/π. Reference 7 gives details of evaluating the integrals to get I_1 and I_2. Example 5.6 illustrates the use of Keifer and Chu method [Eq. 5.28] for circular channels.

It may be noted that the functions I_1 and I_2 are applicable to circular channels only. However, a similar procedure of non-dimensionalising can be adopted to any other channel geometry, e.g. oval and elliptic shapes, and functions similar to I_1 and I_2 can be developed. Applications of the above procedure for use in rectangular channels is available in literature[8].

5.6 SIMPLE NUMERICAL SOLUTIONS OF GVF PROBLEMS

The numerical solution procedures to solve GVF problems can be broadly classified into two categories as:

(a) Simple Numerical Methods These were developed primarily for hand computation. They usually attempt to solve the energy equation either in the form of the differential energy equation of GVF or in the form of the Bernoulli equation.

(b) Advanced Numerical Methods These are normally suitable for use in digital computers as they involve a large number of repeated calculations. They attempt to solve the differential equation of GVF [Eq. (4.8)].

The above classification is a broad one as the general availability of personal computers (PCs) have made many methods under category (b) available for desk-top calculations. Two commonly used simple numerical methods to solve GVF problems, viz. (i) *Direct-step method* and (ii) *Standard-step method* are described in this section.

5.6.1 Direct-Step Method

This method is possibly the simplest and is suitable for use in prismatic channels. Consider the differential-energy equation of GVF [Eq. (4.16)].

$$\frac{dE}{dx} = S_0 - S_f$$

Writing this in the finite-difference form

$$\frac{\Delta E}{\Delta x} = S_0 - \overline{S}_f \tag{5.29}$$

Where $\overline{S}_f$ = average-friction slope in the reach Δx

$$\therefore \qquad \Delta x = \frac{\Delta E}{S_0 - \overline{S}_f} \tag{5.30}$$

and between two Sections 1 and 2

$$(x_2 - x_1) = \Delta x = \frac{(E_2 - E_1)}{S_0 - \frac{1}{2}(S_{f1} + S_{f2})} \tag{5.31}$$

Equation (5.30) is used as indicated below to calculate the GVF profile.

Procedure Referring to Fig. 5.3, let it be required to find the water-surface profile between two Sections 1 and $(N + 1)$ where the depths are y_1 and y_{N+1} respectively. The channel reach is now divided into N parts of known depths, i.e., values of y_i $i = 1, N$ are known. It is required to find the distance Δx_i between y_i and y_{i+1}. Now, between the two Sections i and $i + 1$,

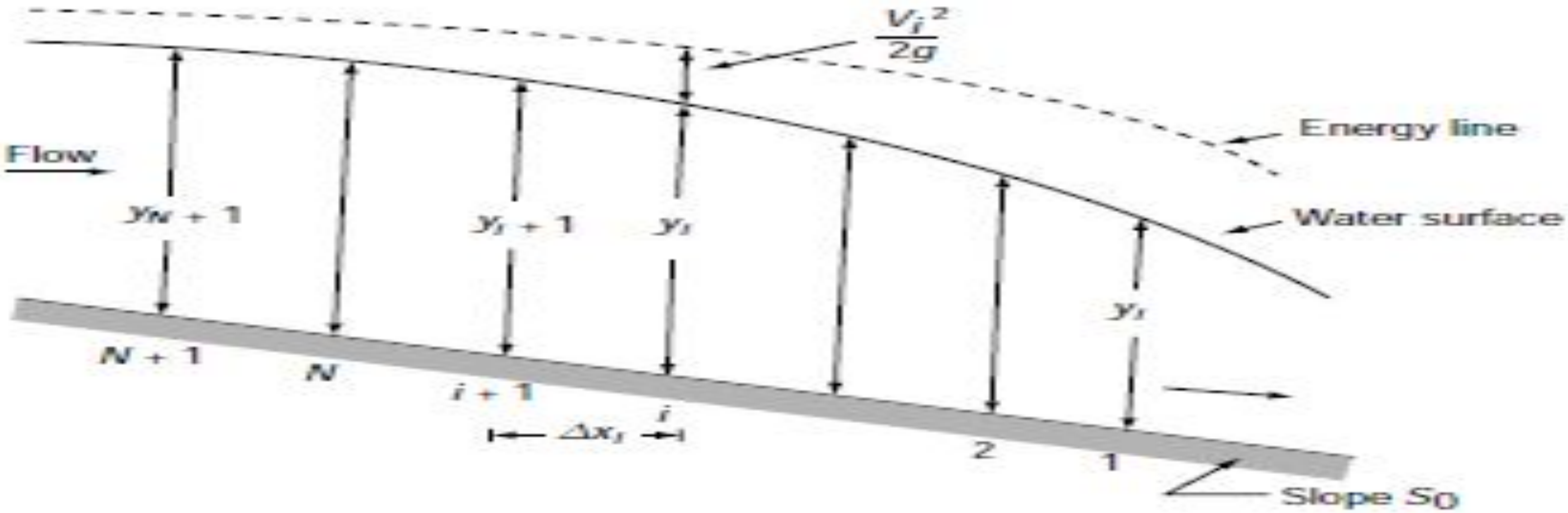

Fig. 5.3 *Direct step method*

$$\Delta E = \Delta\left(y + \frac{V^2}{2g}\right) = \Delta\left(y + \frac{Q^2}{2gA^2}\right)$$

$$\Delta E = E_{i+1} - E_i = \left[y_{i+1} + \frac{Q^2}{2gA_{i+1}^2}\right] - \left[y_i + \frac{Q^2}{2gA_i^2}\right] \quad (5.32)$$

and
$$\overline{S}_f = \frac{1}{2}\left(\overline{S}_{fi+1} + S_{fi}\right) = \frac{n^2 Q^2}{2}\left[\frac{1}{A^2_{i+1} R_{i+1}^{4/3}} + \frac{1}{A_i^2 R_i^{4/3}}\right] \quad (5.33)$$

From Eq. (5.30), $\Delta x_i = \frac{E_{i+1} - E_i}{S_0 - \overline{S}_f}$ .Using Eqs (5.32) and (5.33), Δx_i can be evaluated in the above expression. The sequential evaluation of Δx_i starting from $i = 1$ to N, will give the distances between the N sections and thus the GVF profile. The process is explicit and is best done in a tabular manner if hand computations are used. Use of spread sheet such as MS Excel is extremely convenient.

5.6.2 Standard-step Method

While the direct-step method is suitable for use in prismatic channels, and hence applicable to artificial channels, there are some basic difficulties in applying it to natural channels. As already indicated, in natural channels the cross-sectional shapes are likely to vary from section to section and also the cross-section information is known only at a few locations along the channel. Thus, the problem of computation of the GVF profile for a natural channel can be stated as: Given the cross-sectional information at two adjacent sections and the discharge and stage at one section, it is required to determine the stage at the other section. The sequential determination of the stage as a solution of the above problem will lead to the GVF profile.

The solution of the above problem is obtained by a trial-and-error solution of the basic-energy equation. Consider Fig. 5.4 which shows two Sections 1 and 2 in a natural channel. Section 1 is downstream of Section 2 at a distance Δx. Calculation are assumed to proceed upstream. Equating the total energies at Sections 1 and 2,

$$Z_2 + y_2 + \alpha_2 \frac{V_2^2}{2g} = Z_1 + y_1 + \alpha_1 \frac{V_1^2}{2g} + h_f + h_e \tag{5.34}$$

where h_f = friction loss and h_e = eddy loss. The frictional loss h_f can be estimated as

$$h_f = \overline{S}_f \Delta x = \frac{1}{2}(S_{f1} + S_{f2})$$

where $$S_f = \frac{n^2 V^2}{R^{4/3}} = \frac{n^2 Q^2}{A^2 R^{4/3}} \tag{5.35}$$

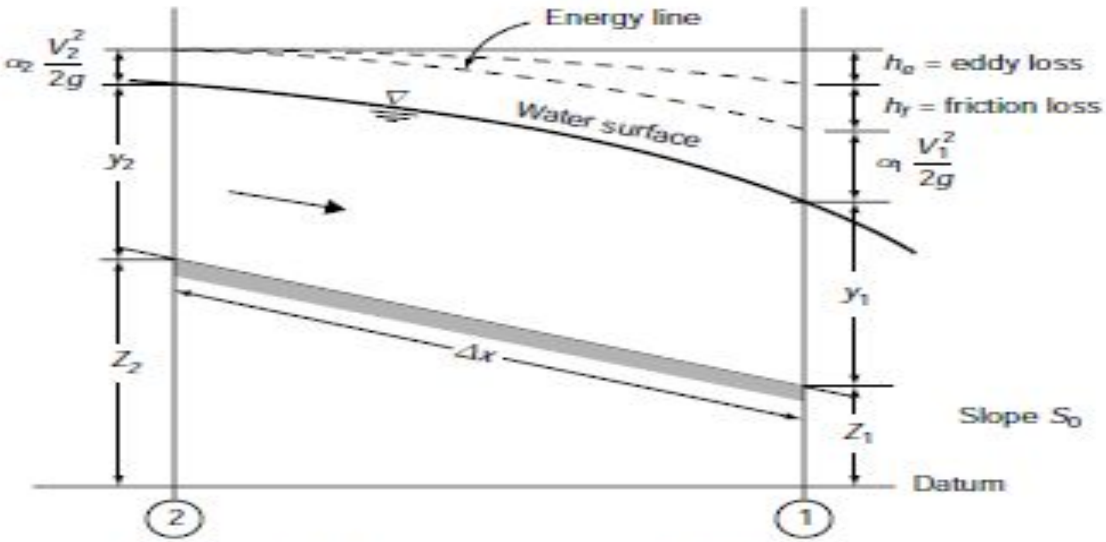

Fig. 5.4 *Definition sketch for the Standard-Step method*

There is no rational method for estimating the eddy loss but it is usually expressed as,

$$h_e = C_e \left| \frac{\alpha_1 V_1^2 - \alpha_2 V_2^2}{2g} \right| \tag{5.36}$$

where C_e is a coefficient having the values as below[9].

Nature of Transition	*Value of Coefficient C*	
	Expansion	*Contraction*
1. No transition (Prismatic channel)	*0.0*	*0.0*
2. Gradual transition	*0.3*	*0.1*
3. Abrupt transition	*0.8*	*0.6*

An alternative practice of accounting for eddy losses is to increase the Manning's n by a suitable small amount. This procedure simplifies calculations in some cases.

Denoting the stage $= Z + y = h$ and the total energy by H, and using the suffixes 1 and 2 to refer the parameters to appropriate sections,

$$H = h + \alpha \frac{V^2}{2g} \qquad \text{and Eq. (5.34) becomes}$$

$$H_2 = H_1 + h_f + h_e \tag{5.37}$$

The problem can now be stated as: Knowing H_1 and the geometry of the channel at Sections 1 and 2 it is required to find h_2. This is achieved in the standard-step method by the trial-and-error procedure outlined below.

Procedure Select a trial value of h_2 and calculated H_2, h_f and h_e and check whether Eq. (5.37) is satisfied. If there is a difference, improve the assumed value of h_2 and repeat calculations till the two sides of Eq. (5.37) match to an acceptable degree of tolerance.

On the basis of the i th trial, the $(i + 1)$ th trial value of h_2 can be found by the following procedure suggested by Henderson[11]. Het H_E be the difference between the left-hand side and right-hand side of Eq. (5.37) in the i th trial, i.e.

$$H_E = \left[H_2 - \left(H_1 + h_f + h_e\right)\right] \text{ in the } i \text{ th trial.}$$

The object is to make H_E vanish by changing the depth y_2.

Hence
$$\frac{dH_E}{dy_2} = \frac{d}{dy_2}\left[y_2 + Z_2 + \alpha_2 \frac{V_2^2}{2g} - Z_1 - y_1 - \alpha_1 \frac{V_1^2}{2g}\right.$$

$$\left. - \frac{1}{2}\Delta x (S_{f1} + S_{f2}) - C_e \left\| \frac{\alpha_1 V_1^2}{2g} - \frac{\alpha_2 V_2^2}{2g} \right\| \right]$$

Since y_1, Z_1, Z_2 and V_1 are constants,

$$\frac{dH_E}{dy_2} = \frac{d}{dy_2}\left[y_2 + (1 + C_e)\frac{\alpha_2 V_2^2}{2g} - \frac{1}{2}\Delta x\, S_{f2}\right]$$

$$= 1 - (1 + C_e)F_2^2 - \frac{1}{2}\Delta x . \frac{dS_{f2}}{dy_2} \tag{5.38}$$

Where $$F_2^2 = \frac{\alpha_2 Q^2 T_2}{g A_2^3}$$

For a wide rectangular channel,

$$\frac{dS_f}{dy} = \frac{d}{dy}\left(\frac{n^2 q^2}{y^{10/3}}\right) = 3.33\ S_f / y$$

Hence $$\frac{dS_{f2}}{dy_2} \simeq -\frac{3.33\ S_{f2}}{y_2} = -\frac{3.33\ S_{f2}}{R_2},$$ leading to

$$\frac{dH_E}{dy_2} = \left[1 - (1 - C_e)F_2^2 + \frac{1.67 S_{f2} \Delta x}{R_2}\right]$$

If $\frac{dH_E}{dy_2} = \frac{\Delta H_E}{dy_2}$ and Δy_2 is chosen such that $\Delta H_E = H_E$.

$$\Delta y_2 = -H_E \Big/ \left[1 - (1 + C_e)F_2^2 + \frac{1.67 S_{f2} \Delta x}{R_2}\right] \tag{5.39}$$

The negative sign denotes that Δy_2 is of opposite sign to that of H_E. It may be noted that if the calculations are performed in the downward direction, as in supercritical flow, the third term in the denominator will be negative. The procedure is illustrated in the following example. Spread sheets, such as MS Excel, are extremely convenient to calculate GVF profile through the use of the standard step method.

5.7 ADVANCED NUMERICAL METHODS

The basic differential equation of GVF [Eq. (4.8)] can be expressed as

$$\frac{dy}{dx} = F(y) \tag{5.56}$$

in which $F(y) = \dfrac{S_0 - S_f}{1 - \left(Q^2 T / gA^3\right)}$ and is a function of y only for a given S_0, n, Q and channel geometry. Equation 5.56 is non-linear and a class of methods which is particularly suitable for numerical solution of the above equation is the *Runge–Kutta* method. There are different types of Runge–Kutta methods and all of them evaluate y at $(x + \Delta x)$ given y at x. Using the notation $y_i = y(x_i)$ and $x_i + \Delta x = x_{i+1}$ and hence $y_{i+1} = y(x_{i+1})$, the various numerical methods for the solution of Eq. 5.56 are as follows:

(a) Standard Fourth Order Runge–Kutta Methods (SRK)

$$y_{i+1} = y_i + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) \tag{5.57}$$

where
$$K_1 = \Delta x\, F(y_i)$$

$$K_2 = \Delta x\, F\left(y_i + \frac{K_1}{2}\right)$$

$$K_3 = \Delta x\, F\left(y_i + \frac{K_2}{2}\right)$$

$$K_4 = \Delta x\, F(y_i + K_3)$$

(b) Kutta–Merson Method (KM)[15]

$$y_{i+1} = y_i + \frac{1}{2}(K_1 + 4K_4 + K_5) \tag{5.58}$$

where
$$K_1 = \frac{1}{3}\Delta x\, F(y_i)$$

$$K_2 = \frac{1}{3}\Delta x\, F(y_i + K_1)$$

$$K_3 = \frac{1}{3}\Delta x\, F\left(y_i + \frac{K_1}{2} + \frac{K_2}{2}\right)$$

$$K_4 = \frac{1}{3}\Delta x F\left(y_i + \frac{3}{8}K_1 + \frac{9}{8}K_3\right)$$

$$K_5 = \frac{1}{3}\Delta x F\left(y_i + \frac{3}{2}K_1 - \frac{9}{2}K_3 + 6K_4\right)$$

An estimate of the truncation error in Eq. 5.58 is given by

$$\varepsilon = 0.2\,K_1 - 0.9\,K_3 + 0.8\,K_4 - 0.1\,K_5 \tag{5.59}$$

In using the above methods, the channel is divided into N parts of known length interval Δx. Starting from the known depth, the depths at other sections are systematically evaluated. For a known y_i and Δx, the coefficients K_1, K_2, ..., etc. are determined by repeated calculations and then by substitution in the appropriate main equation [Eq. 5.57 or Eq. 5.58], the value of y_{i+1} is found. The SRK method involves the determination of $F(y)$ four times while the KM method involves $F(y)$ to be evaluated five times for each depth determination. These two methods are direct methods and no iteration is involved. The KM method possesses an important advantage in the direct estimate of its truncation error, which can be used to provide automatic interval and accuracy control in the computations[17].

(c) Trapezoidal Method (TRAP) This is an iteration procedure with

$$y_{i+1} = y_i + \frac{1}{2}\,\Delta x\{F(y_i) + F(y_{i+1})\} \tag{5.60}$$

The calculation starts with the assumption of $F(y_{i+1}) = F(y_i)$ in the right hand side of Eq. 5.60. The value of y_{i+1} is evaluated from Eq. 5.60 and substituted in Eq. 5.56 to get $F(y_{i+1})$. This revised $F(y_{i+1})$ is then substituted in Eq. 5.60. The process is repeated. Thus the rth iteration will have

$$y_{i+1}^{(r)} = y_i + \frac{1}{2}\,\Delta x\left\{F(y_i) + F(y_{i+1})^{(r-1)}\right\} \tag{5.61}$$

The iteration proceeds till two successive values of $F(y_{i+1})$ or y_{i+1} agree to a desirable tolerance.

Comparison of Various Methods Studies have been reported by Apelt[15] and Humpridge and Moss[16] on the SRK method; by Apelt[17] on the KM and TRAP methods and by Prasad[17] on the TRAP method. It has been found that all these three methods are capable of direct determination of the GVF profile in both upstream and downstream directions irrespective of the nature of flow, i.e., whether the flow is subcritical or supercritical. Apelt[17] in his comparative study of the three methods has observed that the SRK and KM methods possess better stability characteristics and require less computational effort than the TRAP method. Also, while the SRK method is slightly more efficient than the KM method, the possibility of providing automatic control of the step size and accuracy in the KM process makes it a strong contender for any choice.

All the three methods are well-suited for computer applications and can easily be adopted to GVF calculations in natural channels. In these three methods when the calculations involve critical depth, care should be taken to avoid $dy/dx = \infty$ at $y = y_c$ by terminating the calculations at a depth slightly different from y_c.