GROUND WATER ENGINEERING

Unit 2

Lectures 7 to 9

Prof. M.L. Narasimham
Retd Professor of Civil Engg.,
Andhra University

UNIT 2

- ***Governing Equations for Groundwater Flow:***
  - ✓ ***Dupuit- Forchheimer assumptions***
  - ✓ ***General differential equations governing groundwater flows,***
  - ✓ ***Analytical solutions.***

ELEMENTARY GROUNDWATER FLOW

- *For some of the 2D flow problems, one component of the flow can be neglected with respect to the other.*
- *In particular, in some unconfined flows with a free surface, the vertical component of the flow can be neglected.*
- *This approximation pioneered by Dupuit (1863) and utilized later by Forchheimer (1930) is known as the Dupuit–Forchheimer assumption.*
- *It gives reasonable results when the depth of the unconfined flow is shallow and the slope of the free surface is small.*
- *These assumptions are summarized as follows:*

ELEMENTARY GROUNDWATER FLOW

1) *The flow is horizontal at any vertical cross-section.*

2) *The velocity is constant over the depth.*

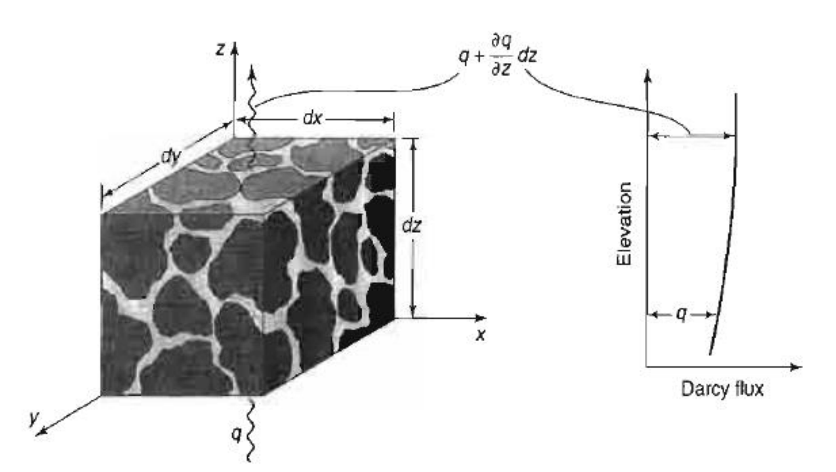
3) *The velocity is calculated using the slope of the free surface as the hydraulic gradient.*

4) *The slope of the water table is relatively small.*

GENERAL GROUNDWATER FLOW EQUATIONS

- *Consider the control volume (CV) for a saturated flow as shown in the fig.*
- *The sides, which define the control surface (CS), have lengths dx, dy and dz in the coordinate directions.*
- *The total volume of the CV is dxdydz and the volume of water flowing into or out of the CV is θ dxdydz, where 'θ' is the moisture content.*
- *Rate of outflow of GW thro' the CS is = Rate of change of GW stored in the CV. V = Velocity Vector and the volume of flow rate past a given area 'dA' is V.dA, where 'A' is the area vector.*

Control Volume for development of the continuity equation in porous medium.



2003

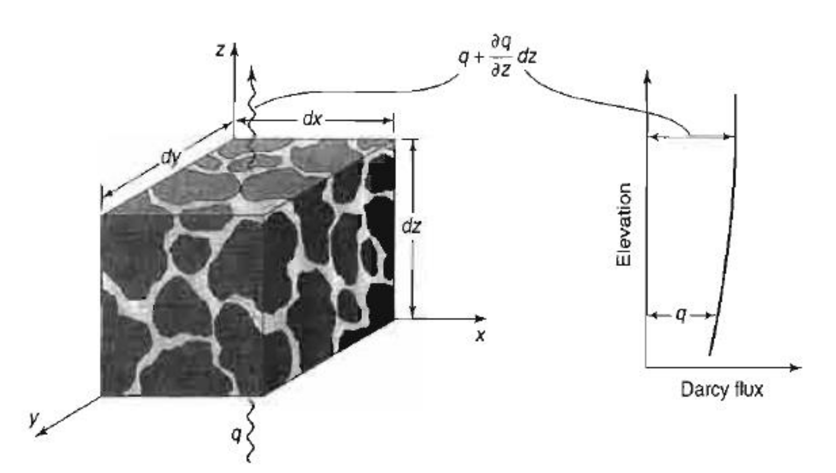
GENERAL GROUNDWATER FLOW EQUATIONS

- ***The general control volume equation for continuity is applicable and is as Follows:***

$$0 = \frac{d}{dt}\int_{CV} \rho . d\forall + \int_{CS} \rho\vec{V} . dA \text{ - - (1) or}$$

$$\int_{CS} \rho V . dA = -\frac{d}{dt}\int_{CV} \rho . d\forall$$

where d∀ isthe elemental volume given by dxdydz.



- ***The time rate of change of mass stored in the CV is defined as the rate of change of fluid mass storage. Expressed as,***

$$\frac{d}{dt}\int_{CV} \rho . d\forall = \rho S_S \frac{\partial h}{\partial t}(dxdydz) + \rho W(dxdydz)$$

where,SS is the specific storage and W is the flow into or out of the CV. W is given by: $W = Q/(dxdydz)$

S. Hughes, 2003

GENERAL GROUNDWATER FLOW EQUATIONS

$\int_{CS} \rho V . dA = -\frac{d}{dt}\int_{CV} \rho . d\forall$ RHS - Time rate of change of mass stored in CV:

$\frac{d}{dt}\int_{CV} \rho . d\forall = \rho S_S \frac{\partial h}{\partial t}(dxdydz) + \rho W(dxdydz)$ - (2) $W = Q/(dxdydz)$

- The term $\rho S_S \frac{\partial h}{\partial t}(dxdydz)$ indicates the mass rate of water produced by:
  1) An expansion of the water under change in density and
  2) The compaction of the porous medium due to change in the porosity.
- LHS: The inflow of water through the CS at the bottom of CV = qdxdy and the outflow at the top of CV "[q+(∂q/ ∂z)dz]dxdy". So the net outflow in the vertical direction is "$\rho dxdydz \frac{\partial q}{\partial z}$" - - (3a).

"$\frac{\partial q}{\partial z}$" is denoted as "$q_z$"

- Since the inflow and out flow will occur in all three directions the term is sum of the net flows in all the 3 directions , x, y and z.

GENERAL GROUNDWATER FLOW EQUATIONS

- Net outflow in the vertical (z) direction is "$\rho dxdydz\frac{\partial q}{\partial z}$" - - (3a)

$$\int_{CS}\rho V.dA = \rho dxdydz\frac{\partial q}{\partial x} + \rho dxdydz\frac{\partial q}{\partial y} + \rho dxdydz\frac{\partial q}{\partial z} \text{ - - (3)}$$

- Substituting (2) & (3) in eq(1):

$$0=\rho S_S\frac{\partial h}{\partial t}(dxdydz) + \rho W(dxdydz) + \rho dxdydz\frac{\partial q}{\partial x} + \rho dxdydz\frac{\partial q}{\partial y} + \rho dxdydz\frac{\partial q}{\partial z} \text{ - - (4a)}$$

- Dividing by $\rho(dxdydz)$ (4a) becomes:

$$S_S\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} + \frac{\partial q}{\partial y} + \frac{\partial q}{\partial z} + W = 0 \text{ OR}$$

$$S_S\frac{\partial h}{\partial t} + q_x + q_y + q_z + W = 0 \text{ - - (4)}$$

GENERAL GROUNDWATER FLOW EQUATIONS

- *Darcy' Law:* $q_s = -K_s \frac{\partial h}{\partial s}$ *where q_s is the flux in s-direction and K_s is the coefficient of permeability in s-direction.*
- *Using Darcy's law eqn (4) can be written as:*

$$\frac{\partial}{\partial x}\left(K_x \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y \frac{\partial h}{\partial y}\right) + \frac{\partial}{\partial z}\left(K_z \frac{\partial h}{\partial z}\right) = S_s \frac{\partial h}{\partial t} + W \text{- - - (5)}$$

- *Eq(5) is the equation for three-dimensional transient flow through a saturated anisotropic porous medium.*
- *For a homogeneous, isotropic medium $K_x = K_y = K_z = K$*

$$S_S \frac{\partial h}{\partial t} - K\left[\frac{\partial}{\partial x}\left(\frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial h}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial h}{\partial z}\right)\right] + W = 0 \text{ - - (5a)}$$

$$\left[\frac{\partial}{\partial x}\left(\frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial h}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial h}{\partial z}\right)\right] = \frac{S_S}{K}\frac{\partial h}{\partial t} + \frac{W}{K} \text{ - - (5b)}$$

S. Hughes, 2003

GENERAL GROUNDWATER FLOW EQUATIONS

- For steady flow $\frac{\partial h}{\partial t} = 0$. Hence $\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} = \frac{W}{K}$ - - (5c)
- For a horizontal confined aquifer of thickness 'b', $S = S_S b$ and the transmissivity T = Kb. With W = 0, the equation for the unsteady flow through an isotropic homogeneous 2-dimensional flow through a confined aquifer will be:

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = \frac{S}{T}\frac{\partial h}{\partial t} \quad \text{- - (6)}$$

- Eqn (6) for radial flow will be: (using the relation $r = \sqrt{x^2 + y^2}$)

$\frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\frac{\partial h}{\partial r} = \frac{S}{T}\frac{\partial h}{\partial t}$ - - (7) where r is the radial distance from pumped well and t is the time since the beginning of pumping.

- Eqn (7) is known as "Diffusion Equation"

ANALYTICAL SOLUTIONS

- *Again for steady flow $\frac{\partial h}{\partial t} = 0$ and eqn (7) becomes:*

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial h}{\partial r}\right) = 0 \text{ - - } (8)$$

- *Eqn (8) can be adopted for different aquifers with proper boundary conditions.*
- *For solution of any problem, idealization of of aquifer and the boundary conditions of the flow system is necessary.*
- *Results may only approximate the field conditions.*
- *Known deviations from assumptions allow analytical solutions to be modified to obtain an answer that otherwise would not have been possible.*
- *A common assumption regarding the aquifer is that it is homogeneous and isotropic.*

S. Hughes, 2003

ANALYTICAL SOLUTIONS

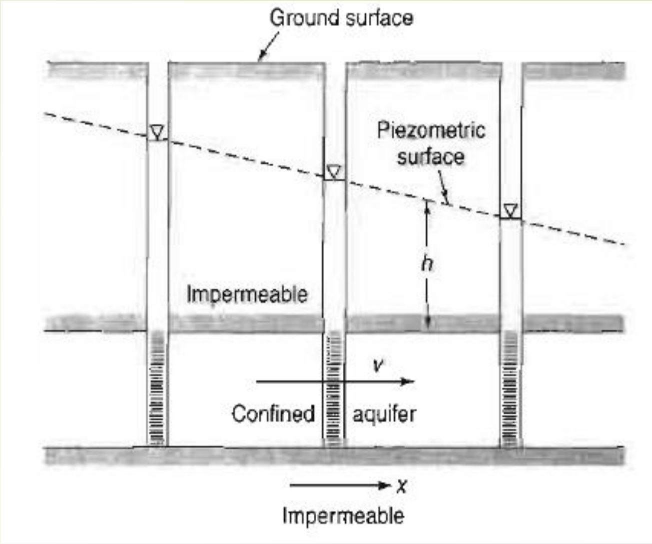
- *Often aquifers can be assumed to be infinite in areal extent; if not boundaries are assumed to be*
  1) *Impermeable, such as underlying or overlying rock or clay layers, dikes, faults, or valley walls; or*
  2) *Permeable, including surface water bodies in contact with the aquifer, ground surfaces where water emerges from underground, and wells.*
- *As a first attempt, we will consider Steady, Unidirectional flow in confined and unconfined aquifers.*
- *Flow conditions differ for confined and unconfined aquifers and hence they need to be considered separately with flow in one direction.*

ANALYTICAL SOLUTIONS

- *Often aquifers can be assumed to be infinite in areal extent; if not boundaries are assumed to be*
  1) *Impermeable, such as underlying or overlying rock or clay layers, dikes, faults, or valley walls; or*
  2) *Permeable, including surface water bodies in contact with the aquifer, ground surfaces where water emerges from underground, and wells.*
- *As a first attempt, we will consider Steady, Unidirectional flow in confined and unconfined aquifers.*
- *Flow conditions differ for confined and unconfined aquifers and hence they need to be considered separately with flow in one direction.*

CONFINED AQUIFER

$$\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}+\frac{\partial^2 h}{\partial z^2}=\frac{W}{K}$$



- ***The general equation for steady groundwater flow in isotropic homogeneous porous medium is:***

$$\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}+\frac{\partial^2 h}{\partial z^2}=\frac{W}{K} \text{ - - (5c)}$$

- ***Now the equation for steady groundwater flow with a velocity 'v' in the x-direction in the confined aquifer (with W = 0) will be:***

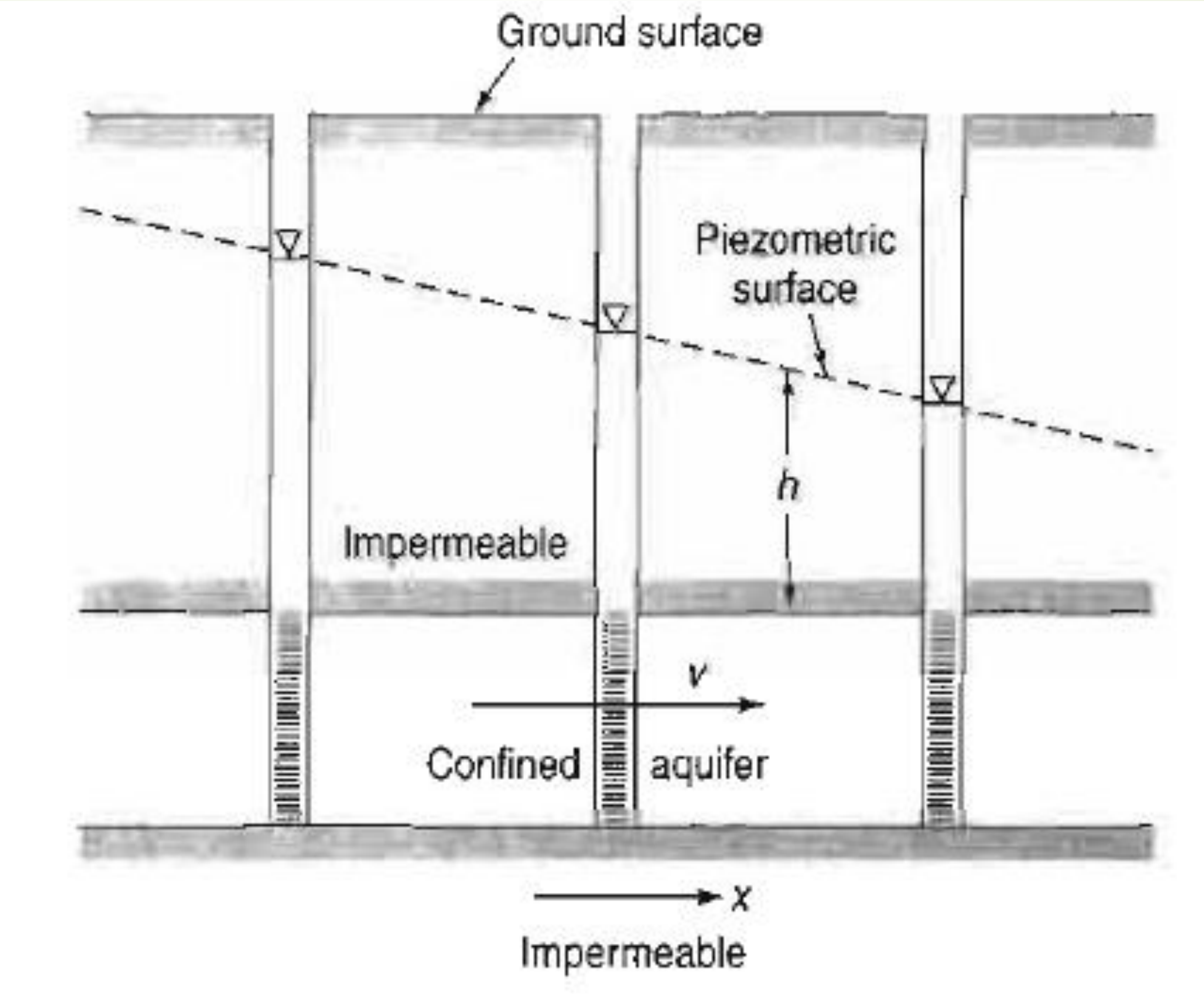
$\frac{\partial^2 h}{\partial x^2}=0$ - - (1), the solution of this equation is $h=C_1x+C_2$ - - (1a) where h = the head above a given datum and C_1 & C_2 are constants of integration.

- ***Assuming h = 0 when x = 0 we get C_2 = 0 and $C_1=\frac{\partial h}{\partial x}$.***
- ***Further, from Darcy' law we have $\frac{\partial h}{\partial x}=-(v/K)$ and thus $C_1=\frac{\partial h}{\partial x}$.***
- ***Substituting the value of the constants in (1a) we obtain $h=-\frac{vx}{K}$ - - (2)***

CONFINED AQUIFER

$$h = -\frac{vx}{K} \text{ - - (2)}$$

- *This states that the head decreases linearly with the flow in the x-direction.*



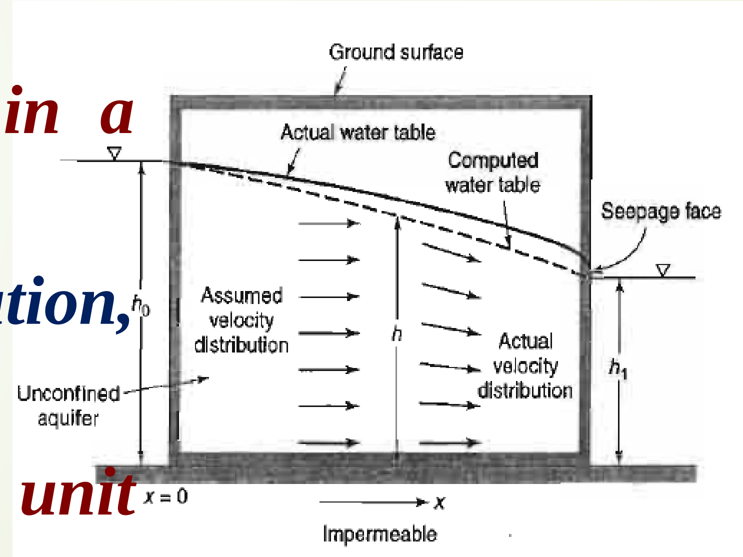
UNCONFINED AQUIFER

- Solution of Laplace eqn for this case is not possible since the water table which is a flow line is also an unknown bobdary.

- To obtain a solution, Dupuit has made the following assumptions:

i. The velocity of flow is proportional to the tangent of the hydraulicgradient instead of the sine as defned in the eqn : $v = -K\frac{\partial h}{\partial s}$

ii. The flow to be horizontal and uniform everywhere in a vertical section.



- These assumptions, although permit obtaining a solution, limit the applicability of the results.

- For a unidirectional flow as shown, the discharge per unit width q at any vertical section is given by: $v = -Kh\frac{dh}{dx}$

where K = hydraulic conductivity, h = ht. of the water table and x is the direction of flow.

UNCONFINED AQUIFER

$$v = -Kh\frac{dh}{dx}$$

- Integrating, we get $qx = -\frac{K}{2}h^2 + C$ and, if $h = h_0$ where $x = 0$, then the Dupuit eqn for unconfined aquifer will be:

$$qx = \frac{K}{2x}\left(h_0^2 - h^2\right) \quad \text{- - (3)}$$

- For flow between two fixed boundaries of water of constant heads h0 and h1 as shown, the water table slope at u/s boundary of the aquifer, (neglecting capillary zone) $\frac{dh}{dx} = -\frac{q}{Kh_0}$