Data ^{X}

About Me:

Underfitting / Overfitting, Polynomial Regression & Regularization

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Underfitting

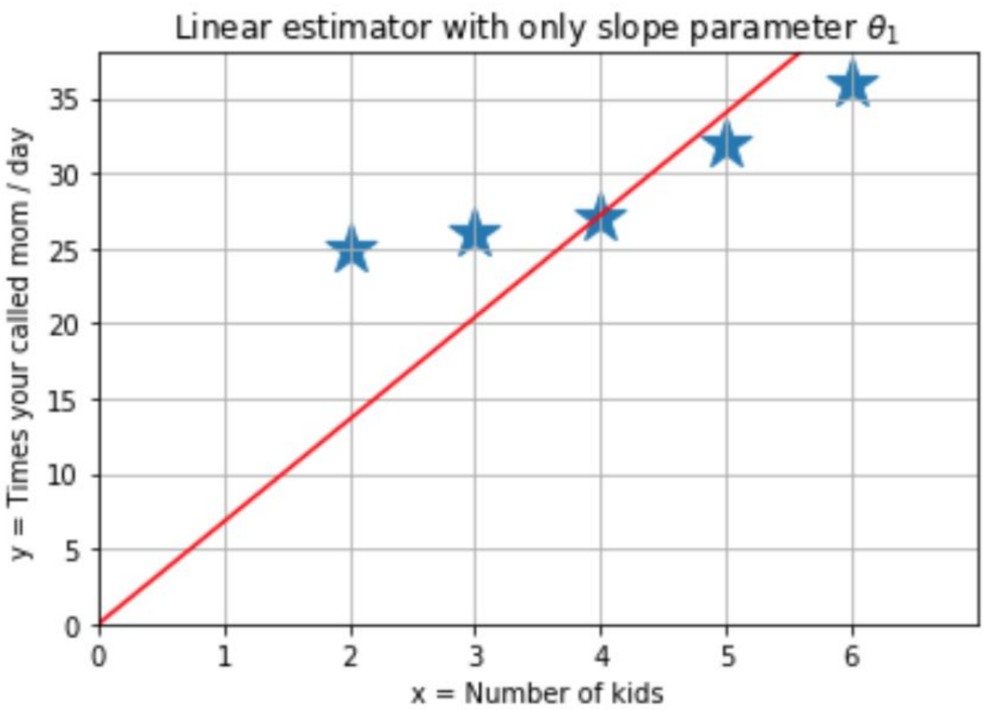
Underfitting / High bias

Characteristics of underfitting:

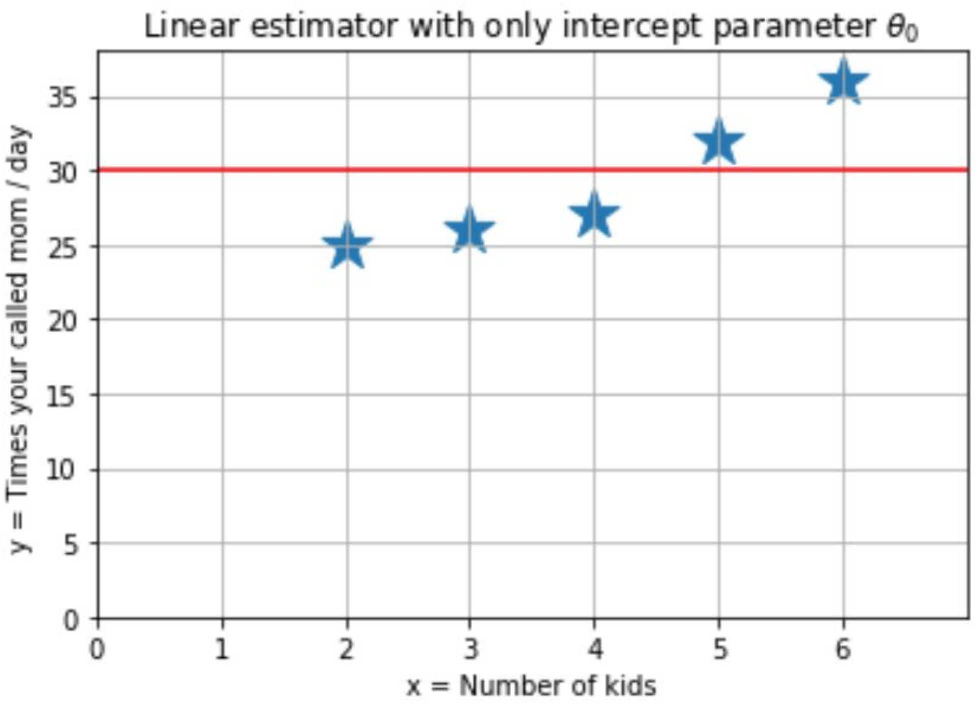
- Our model is too simple (few degrees of freedoom)
- Low variance, but high bias
- Strong preconception about model parameters

Underfitting / High bias

Examples of underfitting: One degree of freedom is not enough



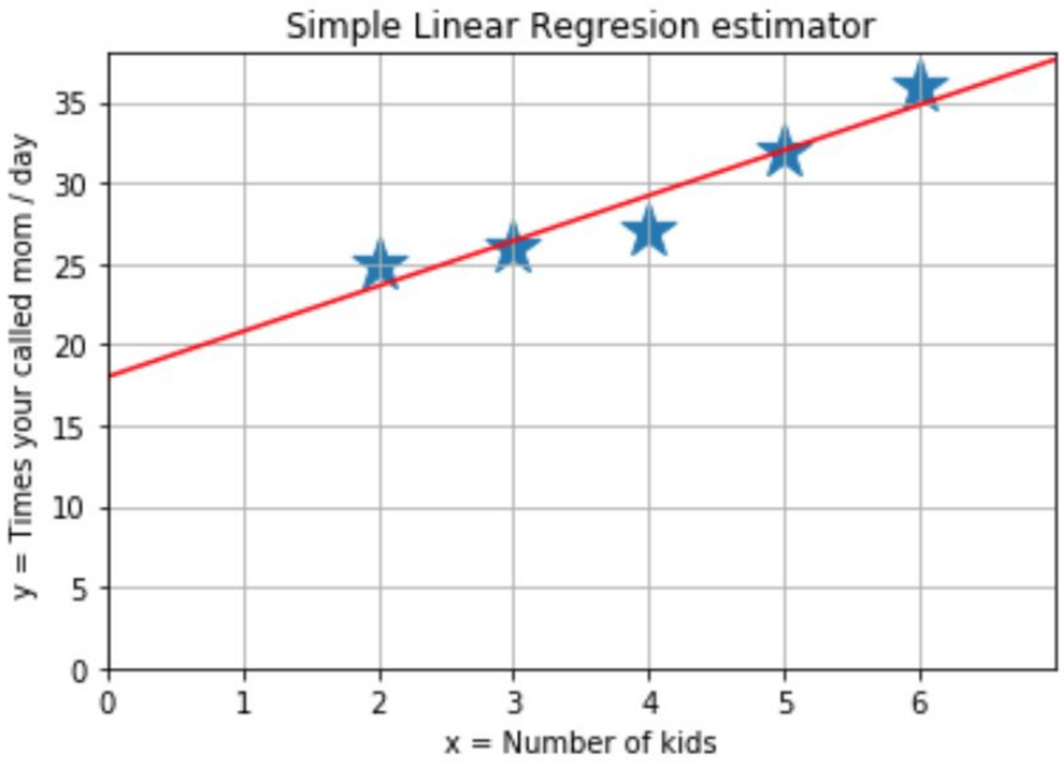
$$\hat{y} = h_\theta(x) = \theta_1 x$$



$$\hat{y} = h_\theta(x) = \theta_0$$

Data X

Good model approximation for our data



$$\hat{y} = h_\theta(x) = \theta_0 + \theta_1 x$$

Underfitting / High bias

How to prevent underfitting:

- Add more features to the model
- Transform features to increase complexity of the model
- Construct features from existing ones
- Fit a more complex model / algorithm

Polynomial Regression

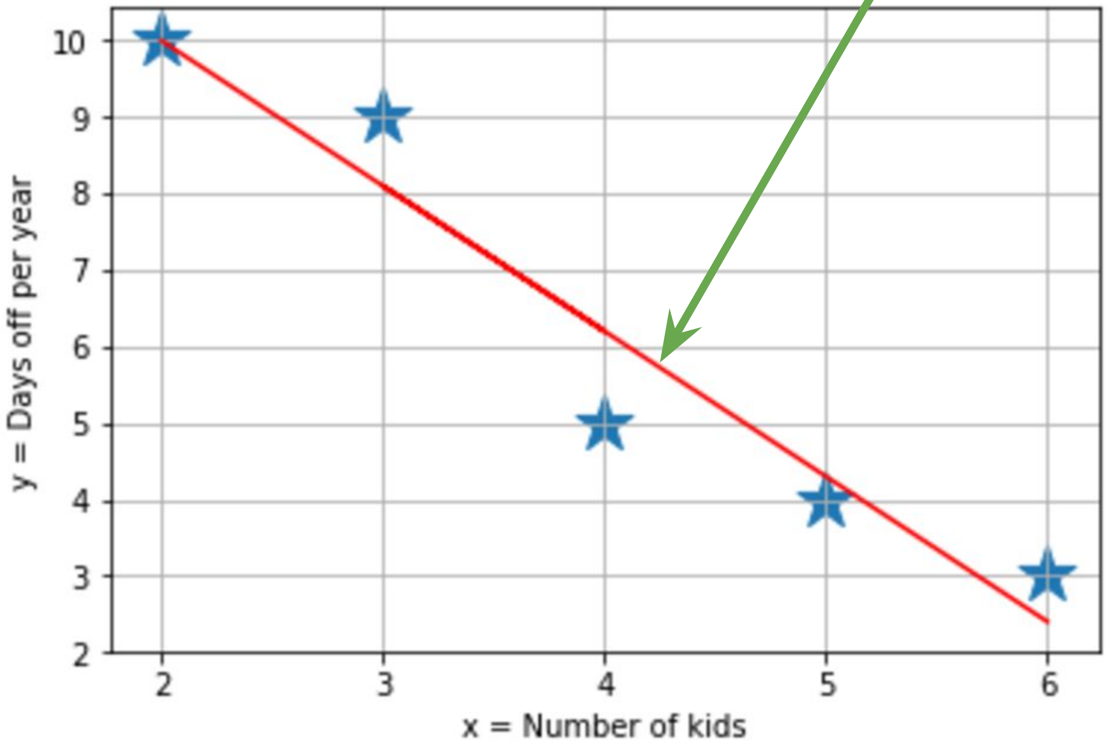
(Simple) Linear Regression

$h_\theta(x)$

Works when have a linear relation between dependent and independent variables

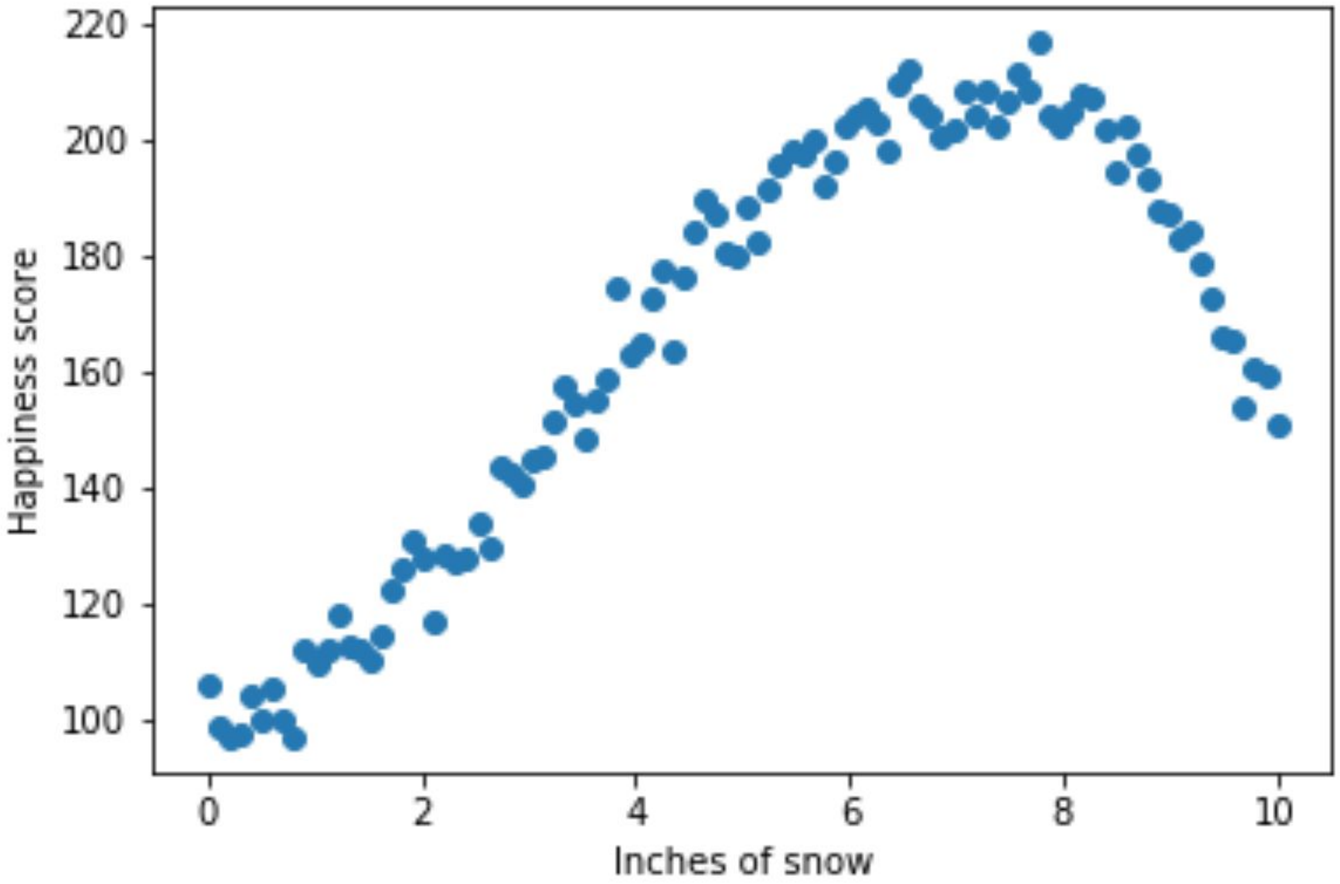
$$\hat{y} = f(x, \theta) = h_\theta(x) = \theta^T x = \theta_0 + \theta_1 x_1$$

$h_\theta(x)$



Modeling Non-linear relationships

What if we want to model this relation?



Modeling Non-linear relationships

The best Linear Estimator

Obtained by solving the Normal Equation

$$\theta = (X^T X)^{-1} X^T y$$

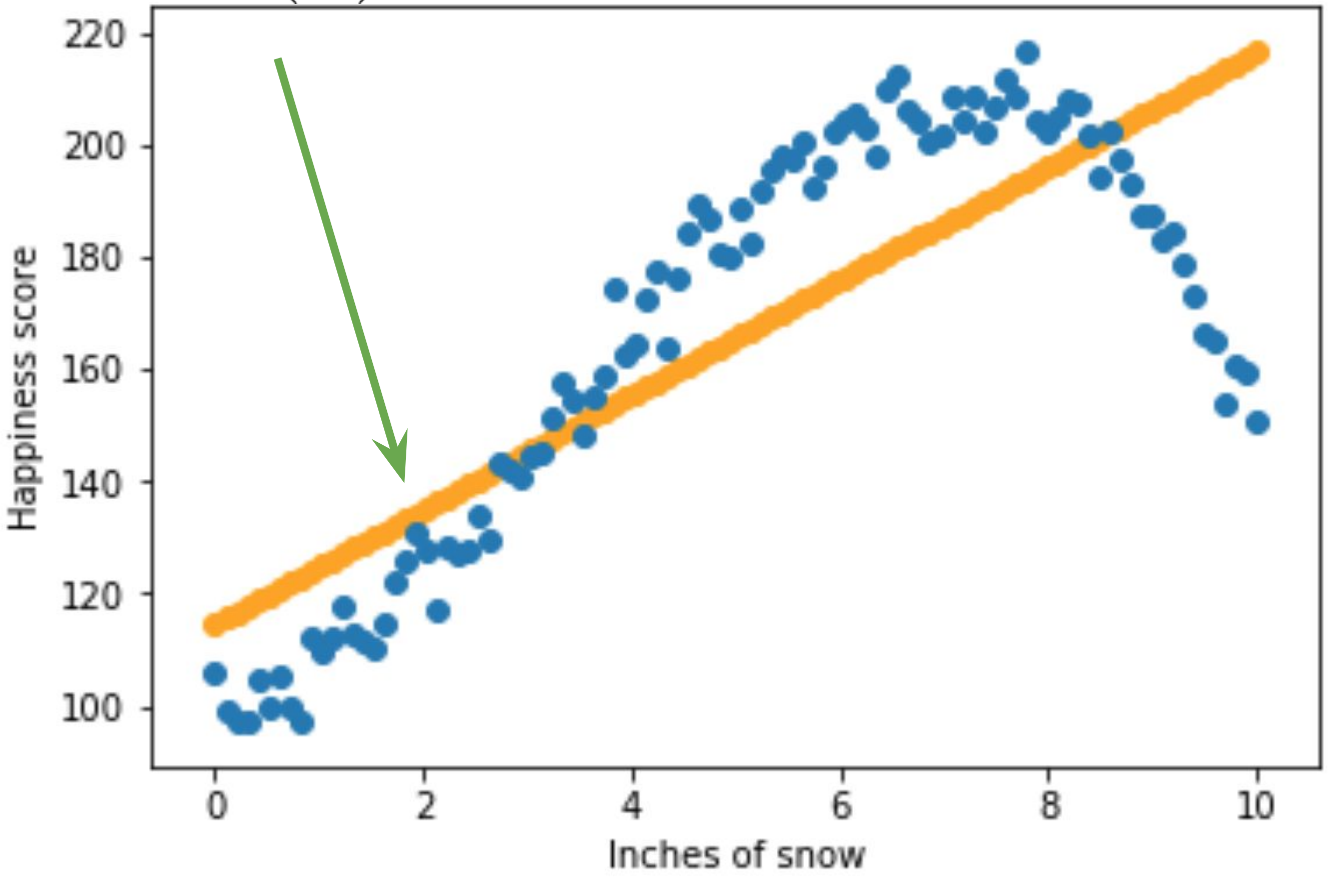
gives us:

$$\hat{y} = h_\theta(x) = \theta_0 + \theta_1 x_1$$
$$\approx 117 + 10x_1$$

We are clearly underfitting!

(Our model has high bias)

$h_\theta(x)$



Introducing: Polynomial Regression

(Simple) Polynomial Regression:

$$\hat{y} = h_\theta(x) = \theta_0 + \theta_1 x + \theta_2 x^2 + \ldots + \theta_n x^n$$

It is still a Linear Regression model, we have just transformed some of the predictors.

Rewrite the predictors, as:

$$\begin{aligned} x &\rightarrow x_1 \\ x^2 &\rightarrow x_2 \\ &\vdots \\ x^n &\rightarrow x_n \end{aligned}$$

to see that it's still a Linear function for the parameters.

Exponential / Logarithmic / Square root ... Regression

$$\hat{y} = h_\theta(x) = \theta_0 + \theta_1 \sqrt{x} + \theta_2 \log\left(x\right) + \theta_3 e^x$$

You can also transform the logarithmic, exponential, the square root of predictors etc.

$$\sqrt{x} \to x_1$$
$$\log(x) \to x_2$$
$$e^x \to x_3$$
$$\vdots$$

Since it still can be cast as a multiple linear regression problem we can find the optimal parameters by using the Normal Equations or Gradient Descent (as shown earlier)!

Polynomial Regression

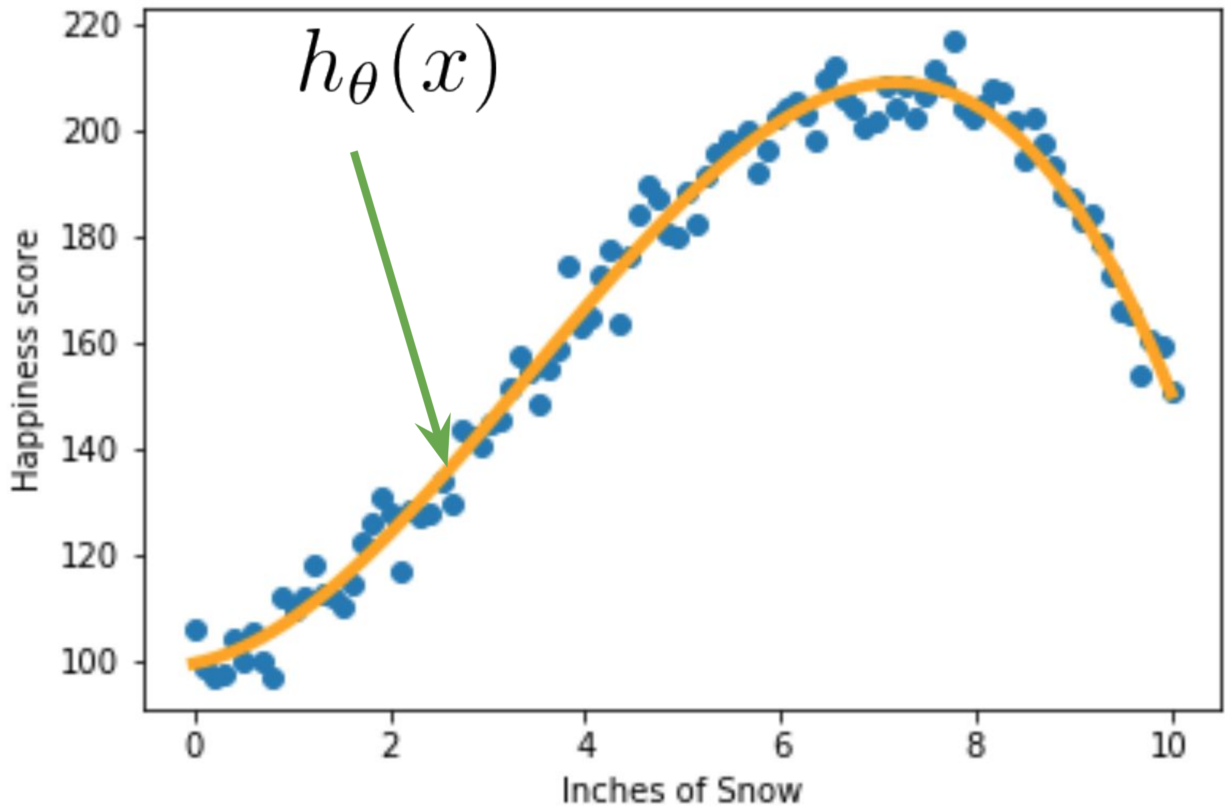
The best polynomial function for prediction (of degree 3)

Obtained by solving the Normal Equation

$$\theta = (X^T X)^{-1} X^T y$$

is given by:

$$\hat{y} = h_\theta(x) = \theta_0 + \theta_1 x + \theta_2 x^2 + \theta_3 x^3$$
$$\approx 98 + 7x + 4.6x^2 - 0.5x^3$$



which is a much better fit to our training data!

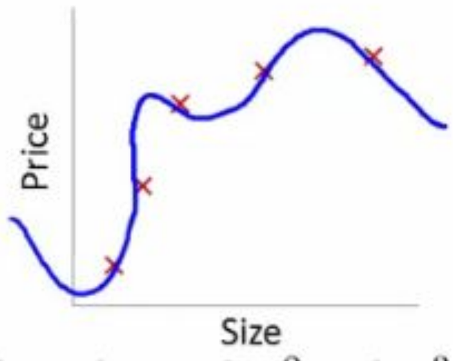
Overfitting

Overfitting

Why don't we fit polynomial functions of very high degrees that always fit our data perfectly so that we get an error that approaches zero?

$$\hat{y} = h_\theta(x) = \theta_0 + \theta_1 x + \theta_2 x^2 + \ldots + \theta_\infty x^\infty \qquad J(\theta) = \frac{1}{2m}\sum_{i=1}^{m}\left(h_\theta(x^{(i)}) - y^{(i)}\right)^2 \rightarrow \mathbf{0}$$

It leads to **overfitting** (we won't predict well on new data that our model hasn't seen)



$$\theta_0 + \theta_1 x + \theta_2 x^2 + \theta_3 x^3 + \theta_4 x^4$$

High variance
(overfit)

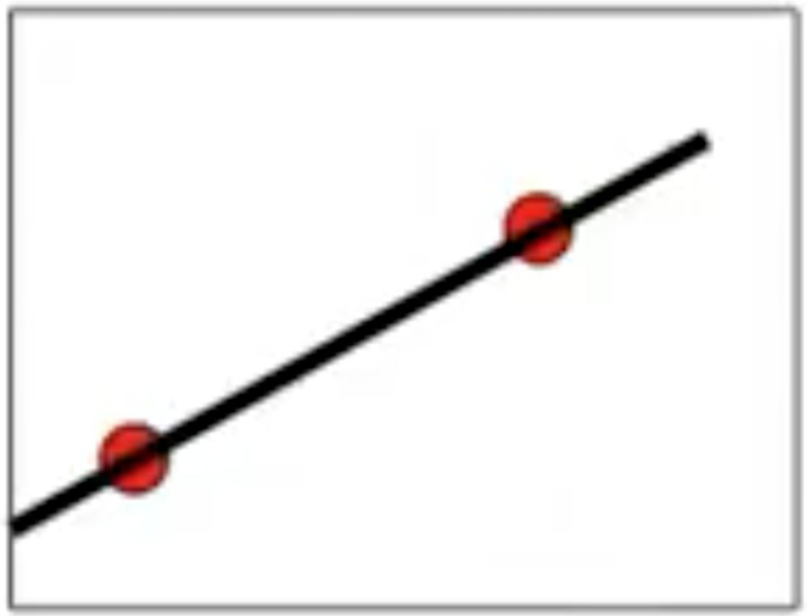
Overfitting / High variance

Characteristics of overfitting:

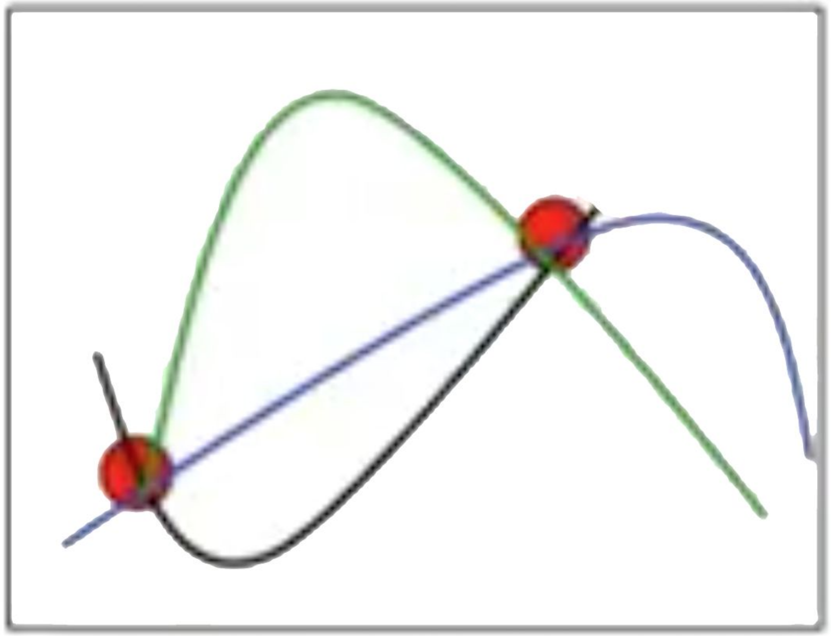
- Our model is too complex (picks up pattern in noise / sees patterns not in data)
- High variance, but low bias
- Use too many features, our model parameters are too big
- We are able to perfectly predict training data, but not test data
- Small changes in the test data, leads to big change in model

Overfitting / High variance

What model to choose? The simplest, Occam's razor theorem



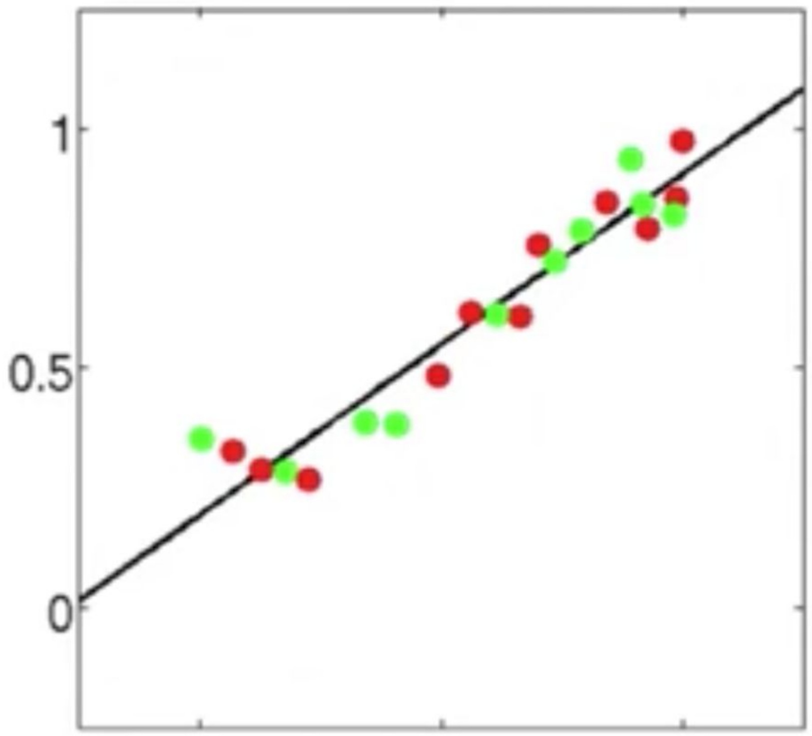
Best (Simple) Model!



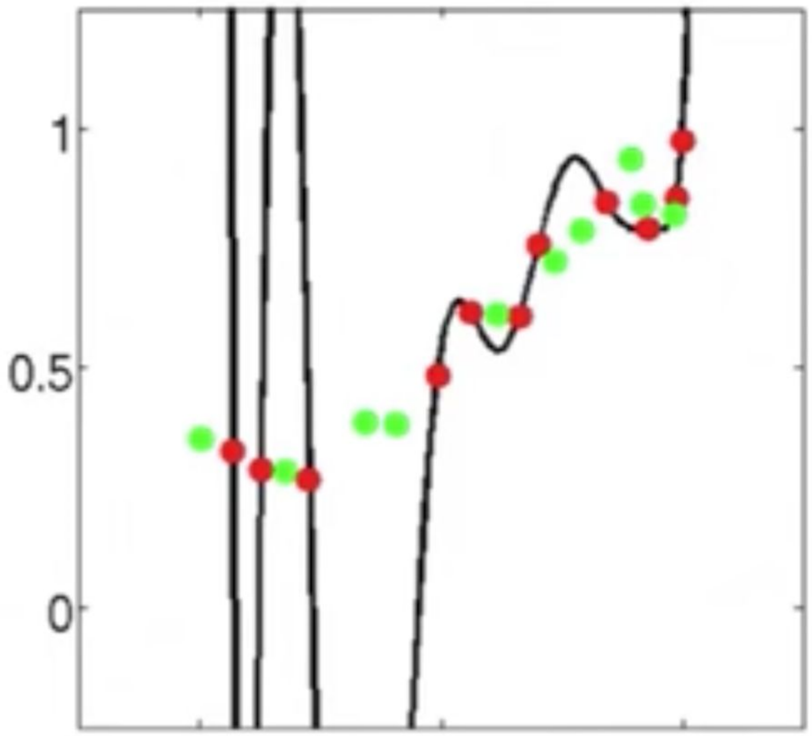
Worse (too complex) models!

Overfitting / High variance

Examples of overfitting: Red = Tranining data   Test data = green



Best (Simple) Model!



Extremely high variance, bad model!

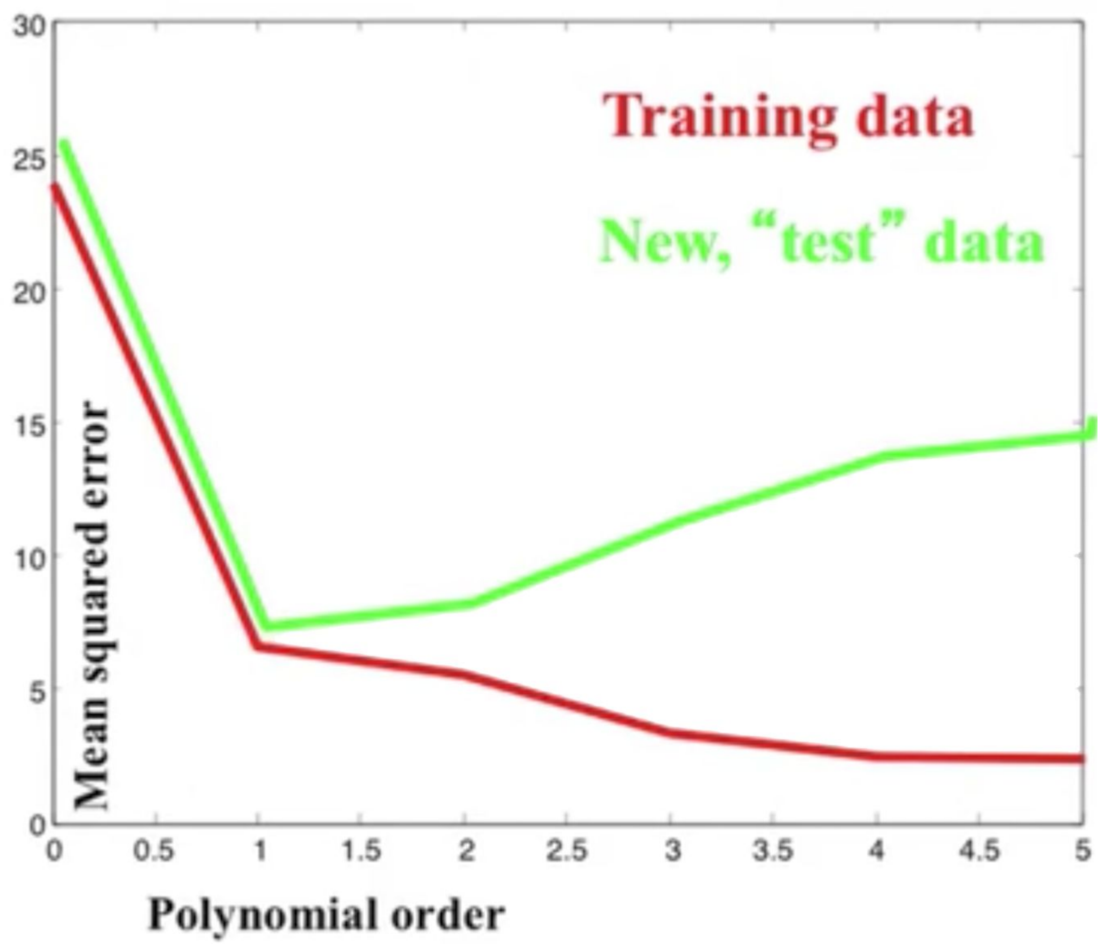
Check if you are overfitting

Plot MSE against model complexity

When MSE starts to increase for

the test data, that is the point where

we are starting to overfit.



Overfitting data / High variance

How to check if you're overfitting:

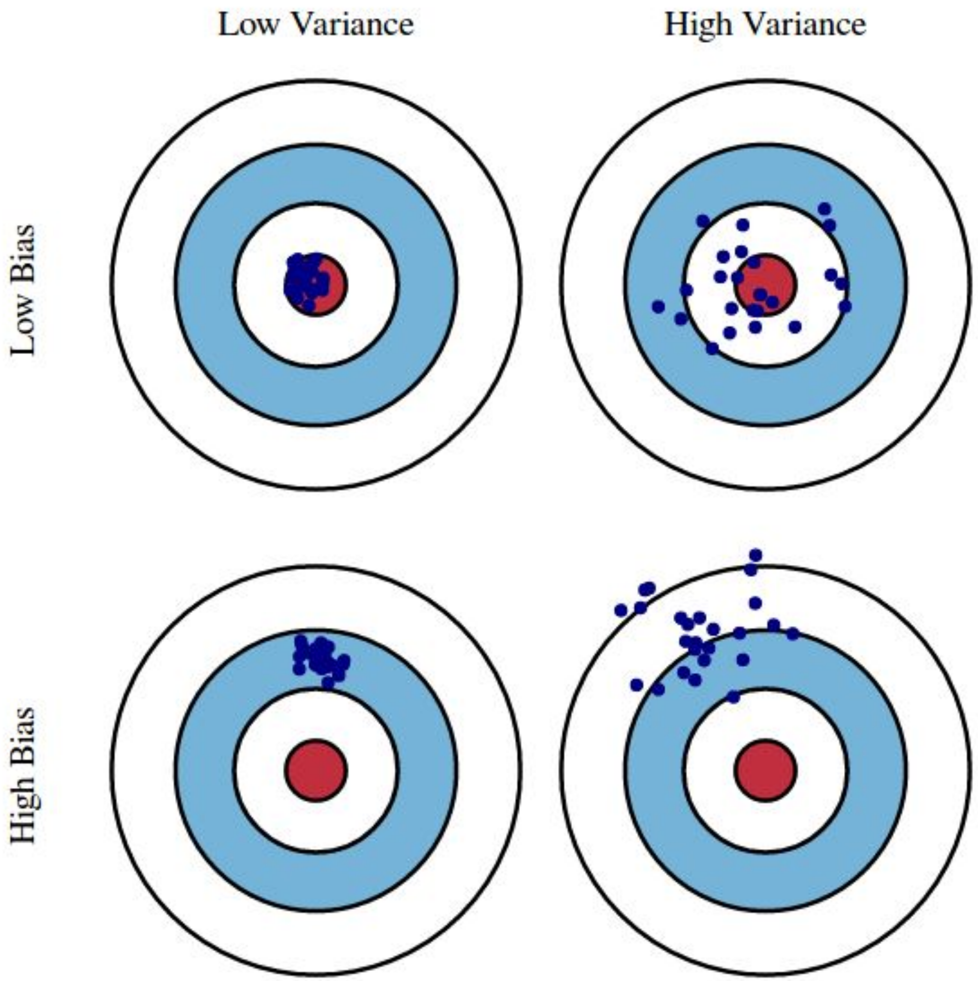
- Use cross-validation predict on test data
- Plot MSE against model complexity for training and test set, see how the error changes.

How to prevent overfitting:

- Exclude some features from the model (feature engineering)
- Introduce a stopping criteria in the optimization algorithm
- ***Regularization!***

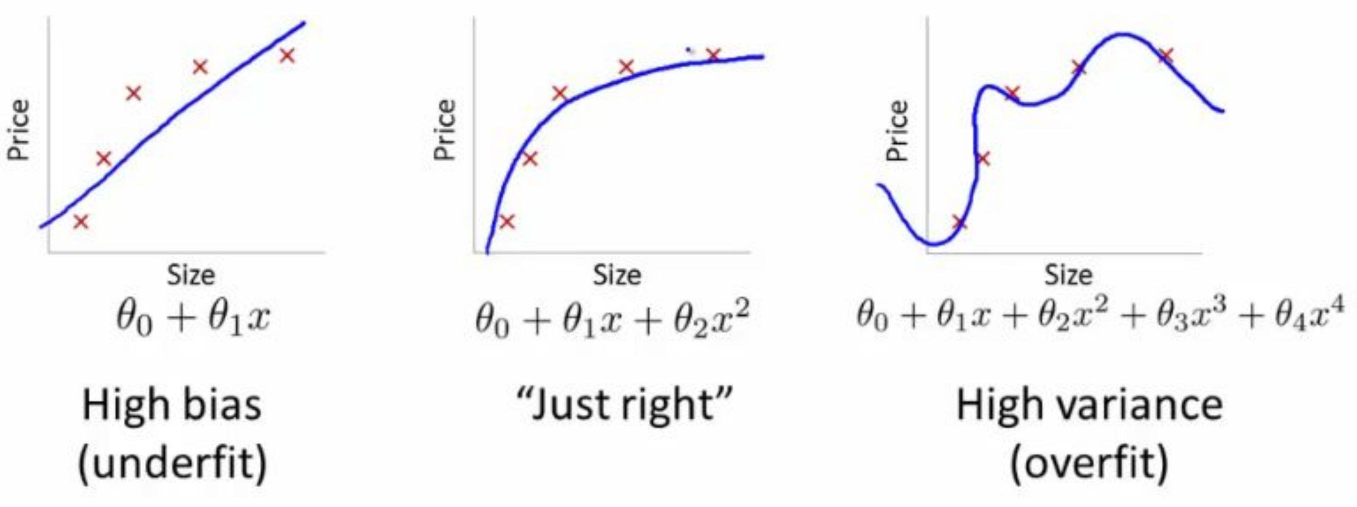
Bias (underfitting) - Variance (Overfitting) tradeoff

Bias-Variance Tradeoff:

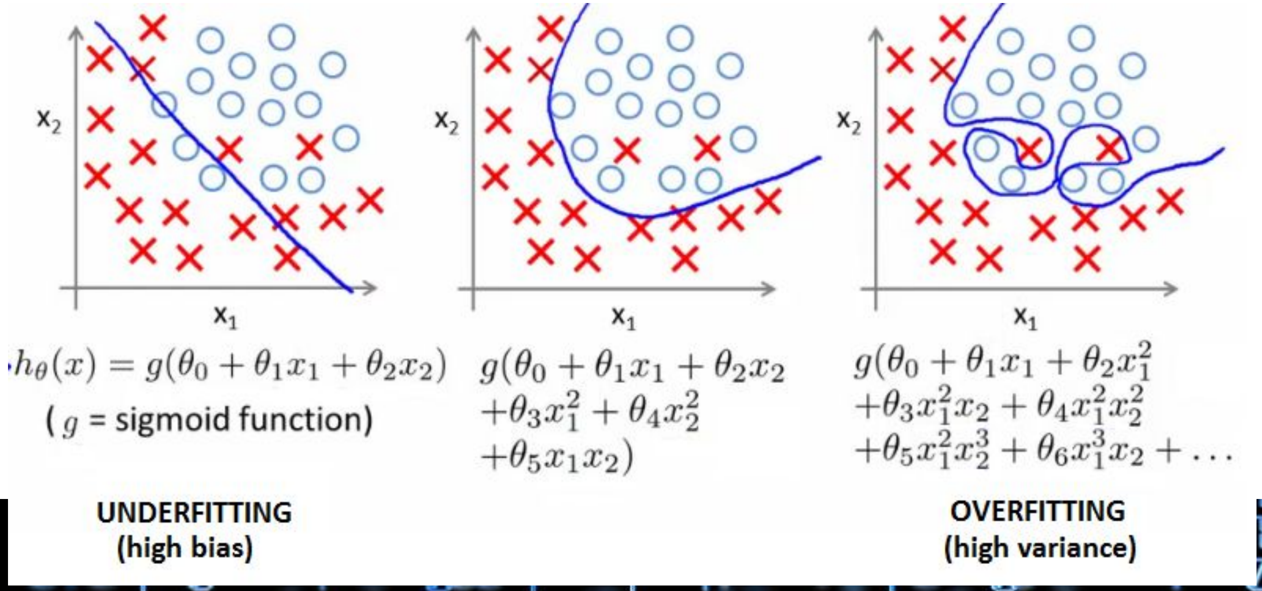


Bias-Variance Tradeoff:

REGRESSION CASE:



CLASSIFICATION CASE:



Mathematical Derivation

$$y = f(x) + \varepsilon$$

Our goal is to model:

$$\hat{f}(x) \approx f(x)$$

where $\epsilon \sim N(0, \sigma^2)$

- f(x) is the "true" function our data is generated from
- Ɛ is zero-mean Guassian noise that affects our samples

Bias-Variance Decomposition

$$\mathrm{E}\left[\left(y - \hat{f}(x)\right)^2\right] = \mathrm{Bias}\left[\hat{f}(x)\right]^2 + \mathrm{Var}\left[\hat{f}(x)\right] + \sigma^2$$

where

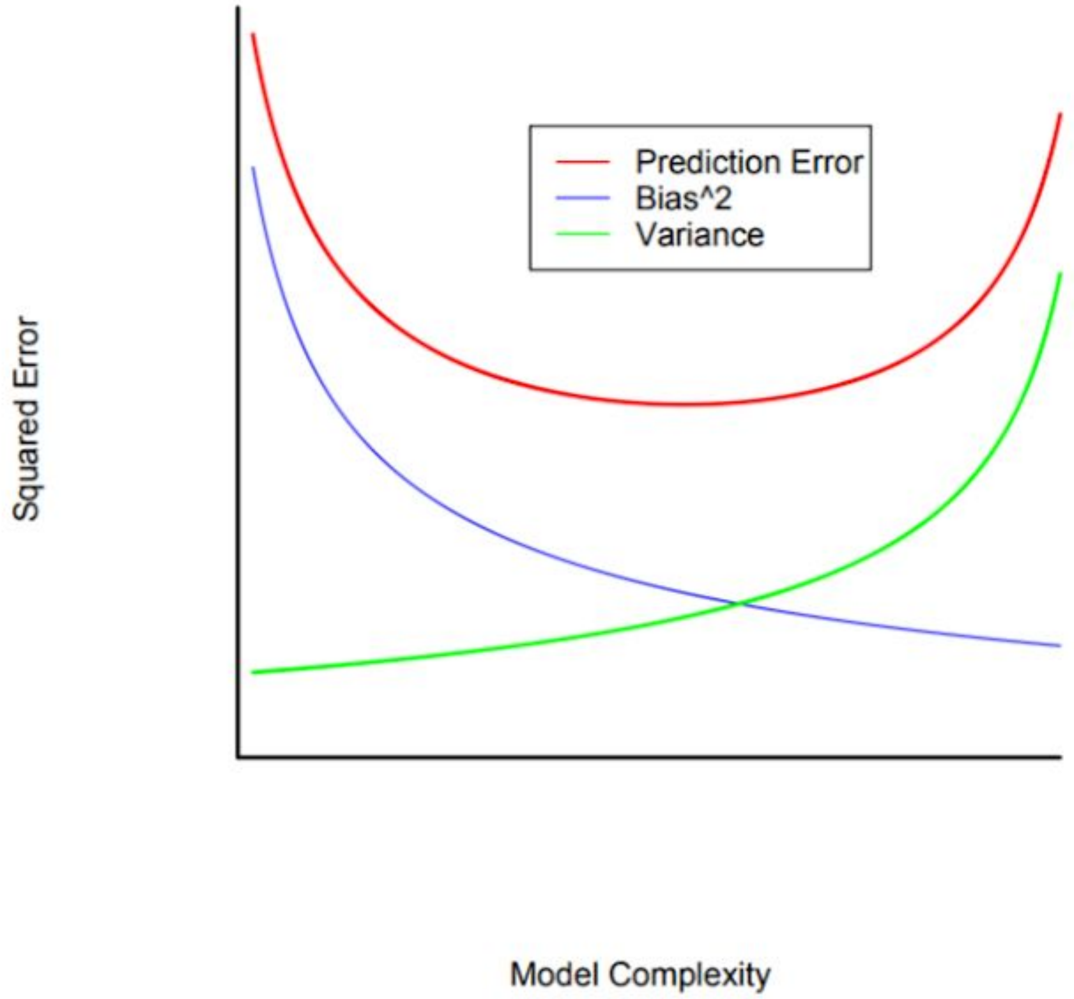
$$\mathrm{Var}\left[\hat{f}(x)\right] = \mathrm{E}[\hat{f}(x)^2] - \mathrm{E}[\hat{f}(x)]^2$$

$$\mathrm{Bias}\left[\hat{f}(x)\right] = \mathrm{E}\left[\hat{f}(x) - f(x)\right]$$

σ^2 Irreducible error (variance of the noise), lower bound for the model MSE

Blackboard proof of the Bias-Variance Decomposition

Bias-Variance plotted against model complexity



Data X

How to detect

Overfitting (high variance) / Underfitting (high bias)

Plot error

- Overfitting gives high test error, low training error
- Underfitting gives high train and test errors

Use k-fold Cross validation

- To get a good approximation of the error when there are few samples

Find the right tradeoff

- Combat underfitting with more complex model
- Combat overfitting by adding samples (error asymptotically -> 0)

or even better use **Regularization**

Regularization

Regularization

Why:

Avoid overfitting

(and LASSO - *Least Absolute Shrinkage and Selection Operator* - can perform auto feature selection)

How:

Increase bias by penalizing the model for many and large model parameters.

Add a multiple of an L1 (LASSO) or an L2 (Ridge) norm of the model parameters **θ** to the cost function

New cost function:

$$J(\theta) = \frac{1}{2m}\sum_{i=1}^{m}(h_\theta(x^{(i)}) - y^{(i)})^2 + \lambda \cdot ||\theta||_p^p$$

- *λ is the regularization parameter, basically a tuning parameter*
- $||\theta||_p^p$ *is the p:th matrix norm on the parameters*

Regularization (increase error if we have too many or too big parameters)

<u>Non-regularized COST FUNCTION:</u> $$J_{old}(\theta) = MSE(\theta) = \frac{1}{2m}\sum_{i=1}^{m}(h_\theta(x^{(i)}) - y^{(i)})^2$$

<u>RIDGE REGRESSION (L2 NORM):</u> $$J(\theta) = MSE(\theta) + \lambda\sum_{j=1}^{n}\theta_j^2$$

<u>LASSO (L1 NORM):</u> $$J(\theta) = MSE(\theta) + \lambda\sum_{j=1}^{n}|\theta_j|$$

Find optimal regularization term λ by tuning it and using <u>Cross-validation</u>:

- Divide your training data,
- Train your model for a fixed value of λ, test it on the remaining subset (unregularized cost function)
- Repeat this procedure while varying λ.
  Then choose the λ that performed best on the test sets.

Regularization (increase error if we have too many parameters)

The optimal estimates of the model parameters, β, could be denoted as shown below.

This shows us the difference between Ridge and Lasso Regression

$$\hat{\beta}^{\text{lasso}} = \text{argmin } \|y - X\beta|_2^2 + \lambda \, \|\beta\|_1$$

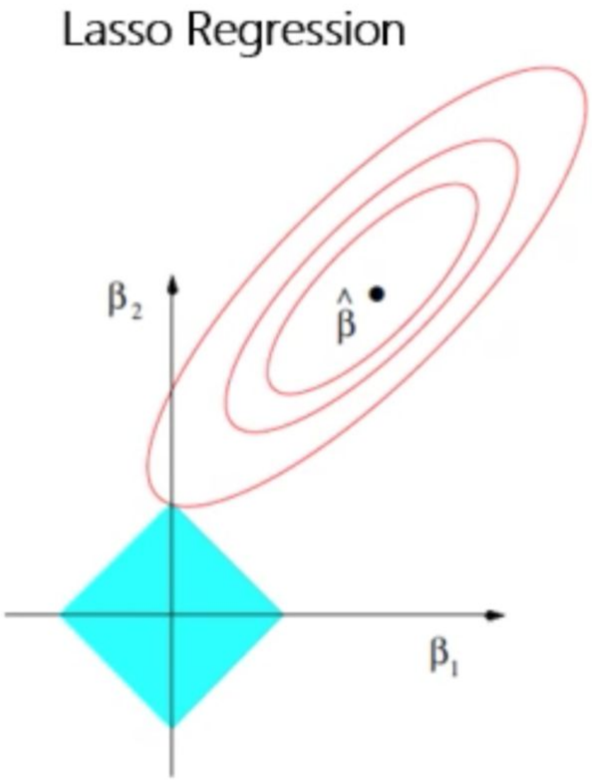
$$\hat{\beta}^{\text{ridge}} = \text{argmin } \|y - X\beta\|_2^2 + \lambda\|\beta\|_2^2$$

$$\hat{\beta}^{\text{lasso}} = \text{argmin } \|y - X\beta|_2^2 \quad \text{subject to } \|\beta\|_1 \leq t$$

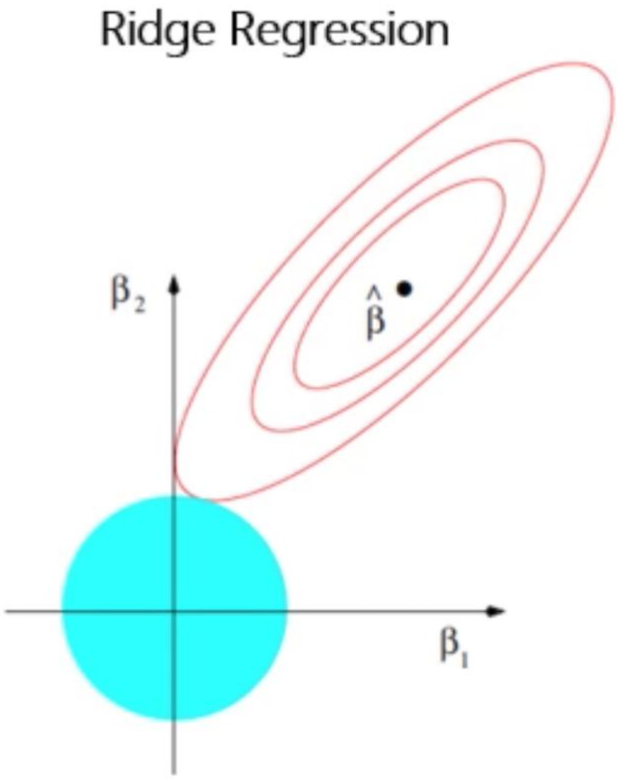
$$\hat{\beta}^{\text{ridge}} = \text{argmin } \|y - X\beta\|_2^2 \quad \text{subject to } \|\beta\|_2^2 \leq t$$

Regularization (increase error if we have too many parameters)

We can visualize the difference between Ridge and Lasso Regression for two parameters. Note, there is a trade-off between the Least Square error and the size of the parameters (which are constrained, to the blue areas).



$$\hat{\beta}^{\text{lasso}} = \text{argmin } \|y - X\beta\|_2^2 \quad \text{subject to } \|\beta\|_1 \leq t$$



$$\hat{\beta}^{\text{ridge}} = \text{argmin } \|y - X\beta\|_2^2 \quad \text{subject to } \|\beta\|_2^2 \leq t$$

$$t \propto \frac{1}{\lambda}$$

Example Code: Regularization

End

References

- The material presented in this lecture references lecture material draws on the materials the following courses:
- Derek Kane's Data Science Tutorials: https://www.youtube.com/channel/UC33qFpcu7eHFtpZ6dp3FFXw
- Stanford – CS229 (Machine Learning) & Andrew Ng's Machine Learning at Coursera: http://cs229.stanford.edu/ & https://www.coursera.org/learn/machine-learning
- Professor Alexander Ihler, UC Davis: youtube.com/watch?v=sO4ZirJh9ds