श्रीमद्भास्कराचार्यविरचिता लीलावती

व्याख्यानम् 14: प्रकीर्णकं (सप्तराशिकादिः भाण्डप्रतिभाण्डकञ्च) श्रेढीव्यवहारश्च

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प्रकीर्णकान्तर्गताः विषयाः — Topics under Miscellaneous operations

The following table presents the various topics discussed by Bhāskara to solve commonly encountered basic arithmetic problems, in the section titled *Prakīrṇaka*.

क्रमः	विषयः	अनुवादः
1	व्यस्तविधिः	Inverting mathematical processes
2	इष्टकर्म	Operations with assumed numbers
3	सङ्क्रमणं विषमकर्म च	Operations with sums and differences
4	वर्गकर्म	Operations with squares of numbers
5	गुणकर्म	Dealing with univariate quadratic equations
6	त्रैराशिकादिः	Rules of three etc.
7	भाण्डप्रतिभाण्डकम्	Barter of commodities

प्रमाणमिच्छा च समानजाती आद्यन्तयोस्स्तः फलमन्यजातिः ।
मध्ये तदिच्छाहतमाद्यहृत् स्यात् इच्छाफलं व्यस्तविधिर्विलोमे ॥७३॥ ।उपजातिः ।

Pramāṇa and *icchā* [which] are of the same kind (units), are [placed] in the first and last positions. The result (*pramāṇaphala*) [which is] of another kind is [placed] in the middle. That (*pramāṇaphala*), multiplied by the *icchā*, and divided by the *ādya* (*pramāṇa*) would be *icchāphala*. In [case of] inverse proportion, reverse process [is employed].

The relation given in the verse to determine the *icchāphala* in terms of *ādi* and *anta* can be interpreted as follows:

$$\textit{icchāphala} = \frac{\textit{pramāṇaphala} \times \textit{icchā}}{\textit{pramāṇa}}$$

व्यस्तत्रैराशिकम् – Inverse proportions

इच्छावृद्धौ फले ह्रासः ह्रासे वृद्धिः फलस्य तु ।
व्यस्तं त्रैराशिकं तत्र ज्ञेयं गणितकोविदैः ॥७७॥ ।अनुष्टुभ् ।

Wherein reduction happens in [*icchā*] *phala* when *icchā* increases and increment happens [in *phala*] when *icchā* decreases, there inverse proportion should be considered by experts of mathematics.

$$\textit{icchāphala} \propto \frac{1}{\textit{icchā}} \qquad \textit{pramāṇaphala} \propto \frac{1}{\textit{pramāṇa}}$$

And so,

$$\textit{icchāphala} \times \textit{icchā} = \textit{pramāṇaphala} \times \textit{pramāṇa}$$

$$\text{or,} \qquad \textit{icchāphala} = \frac{\textit{pramāṇaphala} \times \textit{pramāṇa}}{\textit{icchā}}$$

विस्तारे त्रिकराः कराष्टकमिता दैर्घ्ये विचित्राश्च चेत्
रूपैरुत्कटपट्टसूत्रपटिका अष्टौ लभन्ते शतम् । (पट्ट = कौशेय = कृमिकोषादिजातेवस्त्रे)
दैर्घ्ये सार्धकरत्रयाऽपरपटी हस्तार्धविस्तारिणी
तादृक् किं लभते द्रुतं वद वणिक् वाणिज्यकं वेत्सि चेत् ॥८५॥ ।शार्दूलविक्रीडितम् ।

If eight multi-coloured superior quality woven silk cloths which are of three *hastas* in breadth and eight *hastas* in length fetch a hundred [*niṣkas*], then O merchant! If you know trade, tell quickly how much [money] would another cloth which is of three and a half *karas* (*hastas*) in length and of half *hasta* breadth fetch.

Unit	*Pramāṇapakṣa*	*Icchāpakṣa*
Length	8	$\frac{7}{2}$
Breadth	3	$\frac{1}{2}$
Number of pieces	8	1
Price	100	

Illustrative example of सप्तराशिकम्

Here the *phala* is the price as it is directly proportional to the other units. Transposing the *phala* as well as the *chid*, we have:

Pramāṇapakṣa	*Icchāpakṣa*
8	7
2	1
3	1
2	100
8	
(*svalparāśi*)	(*bahurāśi*)

Then, we have

$$\text{Price} = \frac{7 \times 1 \times 1 \times 100}{8 \times 2 \times 3 \times 2 \times 8} = \frac{175}{192} \ \textit{niṣkas}.$$

The above quantity is equal to 14 *drammas*, 9 *paṇas*, 1 *kākiṇī*, and $6\frac{2}{3}$ *varāṭakas*.

Illustrative example of नवराशिकम्

पिण्डे येऽर्कमिताङ्गुलाः किल चतुर्वर्गाङ्गुला विस्तृतौ
पट्टा दीर्घतया चतुर्दशकराः त्रिंशल्लभन्ते शतम् ।
एता विस्तृतिपिण्डदैर्घ्यमितयो येषां चतुर्वर्जिताः
पट्टास्ते वद मे चतुर्दश सखे मौल्यं लभन्ते कियत् ॥८६॥ ।शार्दूलविक्रीडितम् ।

If thirty wooden planks which are twelve *aṅgulas* in thickness (height), sixteen *aṅgulas* in breadth, and fourteen *hastas* in length then fetch a hundred [*niṣkas*], O friend! Tell me what value would those fourteen planks fetch, of which these measurements of breadth, thickness and length have been reduced by four [respective units].

This problem is an example of *navarāśika.* The five units in this problem are (i) number of planks, (ii) width, (iii) breadth, (iv) length, and (v) price, with the price being directly proportional to the rest.

Unit	*Pramāṇapakṣa*	*Icchāpakṣa*
Width	12	8
Breadth	16	12
Length	14	10
Number of planks	30	14
Price	100	

Transposing the *phala*, and dividing the product of the quantities in the *bahurāśi* (*icchāpakṣa*) with the product of the quantities in the *svalparāśi* (*pramāṇapakṣa*), we determine the price of the fourteen planks. We have

$$\text{Price} = \frac{8 \times 12 \times 10 \times 14 \times 100}{12 \times 16 \times 14 \times 30} = \frac{50}{3} \textit{ niṣkas.}$$

The price for the fourteen planks ($= 16\frac{2}{3}$) in terms of lower currencies or 16 *niṣkas*, 10 *drammas*, 10 *paṇas*, 2 *kākiṇīs*, and $13\frac{1}{3}$ *varāṭakas.*

Illustrative example of एकादशराशिकम्

पट्टा ये प्रथमोदितप्रमितयो गव्यूतिमात्रे स्थिताः
तेषामानयनाय चेच्छकटिनां[a] द्रम्माष्टकं भाटकम् ।
अन्ये ये तदनन्तरं निगदिता मानैश्चतुर्वर्जिताः
तेषां का भवतीह भाटकमितिः गव्यूतिषट्के वद ॥८७॥

Unit	*Pramāṇapakṣa*	*Icchāpakṣa*
Width	12	8
Breadth	16	12
Length	14	10
Number of planks	30	14
Distance	1	6
Hire	8	

Solving as before, we obtain

$$\text{Price} = \frac{8 \times 12 \times 10 \times 14 \times 6 \times 8}{12 \times 16 \times 14 \times 30 \times 1} = 8 \textit{ drammas.}$$

[a] शक्नोति भारं वोढुमिति .शक् +“ शकादिभ्योऽटन् .” इतिअटन्

- The term employed here is derived from the root

  भण् शब्दे *(to sound)* भ्वादिः परस्मैपदी अकर्मकः । भण्यते भणति वेति ।

- Here the suffix 'ड' is used in its own sense (स्वार्थे).
- The vessel that is given (with goods) भाण्ड and the one returned is प्रतिभाण्ड।
- Thus गृहीतस्य भाण्डस्य कृते भाण्डान्तरस्य पुनर्दाने (=प्रतिसमर्पणे) 'भाण्डप्रतिभाण्डकम्' इति व्यावहारः।
- The barter of goods is usually a problem of inverse proportions. That is,
- Here the quantity of a certain kind of goods to be bartered $\propto \dfrac{1}{\text{unit price}}$.
- That is, as the unit price of an item increases, fewer quantities of that item will be exchanged in a barter, and vice versa.
- Keeping this in mind, this problem is reformulated as a modified *pañcarāśika*.
- Thus it is simply solved by adding an extra step to the procedure outlined previously.

How to solve problems involving barter of goods

Bhāskara outlines how to modify *pañcarāśika* in the following verse:

तथैव भाण्डप्रतिभाण्डकेऽपि
विधिर्विपर्यस्य हरांश्च मूल्ये ॥८८॥ ।उपेन्द्रवज्रा ।

Similarly, in barter also, having transposed the denominators (of fractional values) and the price, the method [has to be applied].

Given the market price and the quantity obtained for two goods, as well as the quantity of one of these goods which one seeks to barter for the other, the problem can be stated as follows:

Unit	***Pramāṇapakṣa***	***Icchāpakṣa***
Market Price	*pramāṇa*$_1$	*icchā*$_1$
Market Quantity	*pramāṇa*$_2$	*icchā*$_2$
Quantity bartered	*pramāṇaphala*	

To solve this problem, in addition to transposing the *phala* as before, the price of the goods is also transposed between the *pramāṇapakṣa* and *icchāpakṣa*:

Unit	***Pramāṇapakṣa***	***Icchāpakṣa***
Market Price	$icch\bar{a}_1$	$pram\bar{a}\d{n}a_1$
Market Quantity	$pram\bar{a}\d{n}a_2$	$icch\bar{a}_2$
Quantity bartered		*pramāṇaphala*
	svalparāśi	*bahurāśi*

Then the *bahurāśi* and *svalparāśi* are determined and rest of the procedure is carried out as before. Therefore, we have

$$icch\bar{a}ph\bar{a}la = \frac{pram\bar{a}\d{n}a_1 \times icch\bar{a}_2 \times pram\bar{a}\d{n}aphala}{icch\bar{a}_1 \times pram\bar{a}\d{n}a}.$$

द्रम्मेण लभ्यत इहाऽऽम्रशतत्रयं चेत्
त्रिंशत् पणेन विपणौ वरदाडिमानि ।
आम्रैर्वदाशु दशभिः कति दाडिमानि
लभ्यानि तद्विनिमयेन भवन्ति मित्र ॥८९॥ ।वसन्ततिलका ।

If three hundred mangoes are obtained by a *dramma* in the market here, and thirty good pomegranates by a *paṇa*, then O friend! Tell quickly how many pomegranates can be obtained by exchanging ten of those mangoes.

Unit	*Pramāṇapakṣa*	*Icchāpakṣa*
Price	16	1
Quantity of fruit	300	30
Number of fruit bartered	10	

Usual approach to solve the problem

- Given that 300 mangoes can be procured for 1 *dramma* (or 16 *paṇas*), and that 30 pomegranates can be purchased for 1 *paṇa*, we need to determine how many pomegranates can be obtained by exchanging 10 mangoes.
- The unit price of a mango is $\frac{16}{300}$ *paṇas*,
- The unit price of a pomegranate is $\frac{1}{30}$ *paṇas*.
- As the number of pomegranates which can be exchanged for mangoes varies inversely with their unit prices, the given problem is one of inverse proportions.
- Here, the *pramāṇa* and *pramāṇaphala* are $\frac{16}{300}$ (unit price of mango) and 10 (number of mangoes) respectively, while the *icchā* equals $\frac{1}{30}$ (unit price of pomegranate). Then applying the inverse proportion rule, we can obtain the *icchāphala*.

Example of भाण्डप्रतिभाण्डकम् – Textual method

Unit	*Pramāṇapakṣa*	*Icchāpakṣa*
Price	16	1
Quantity of fruit	300	30
Number of fruit bartered	10	

We transpose the price as follows:

Pramāṇapakṣa	*Icchāpakṣa*
1	16
300	30
	10
svalparāśi	*bahurāśi*

The *bahurāśi* are the quantities in the *icchāpakṣa*, and the *svalparāśi* are the quantities in the *pramāṇapakṣa*. Dividing their respective products as before, we have

$$\text{Number of pomegranates} = \frac{16 \times 30 \times 10}{1 \times 300} = 16.$$

मिश्रव्यवहारः

Dealing with mixed quantities

Verses 90 – 116

श्रेढीव्यवहारः

Dealing with series

Verses 117 – 134

सैकपदघ्नपदार्धमथैकाद्यङ्कयुतिः किल सङ्कलिताख्या ।
सा द्वियुतेन पदेन विनिघ्नी स्यात् त्रिहृता खलु सङ्कलितैक्यम् ॥११७॥ । दोधकवृत्तम् ।

Now, the sum of the numbers starting with one is called *saṅkalita*, which is indeed half the number of terms (*pada*) [in the series] multiplied by the *pada* added by one. That [sum] multiplied by the *pada* [which is] added by two, [and] divided by three would indeed be the sum of the *saṅkalitas*.

$$S_n = \sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \frac{n}{2} \times (n+1)$$

$$V_n = \sum_{i=1}^{n} S_i = S_1 + S_2 + \cdots + S_n = \frac{S_n \times (n+2)}{3}$$

Problem in *saṅkalita* and *saṅkalitaikya* or *vārasaṅkalita*

एकादीनां नवान्तानां पृथक् सङ्कलितानि मे ।
तेषां सङ्कलितैक्यानि चाचक्ष्व सकलं द्रुतम् ॥११८॥ । अनुष्टुभ् ।

O mathematician! Tell me quickly the sums of the numbers starting from one up to nine separately, and [also] the sum of those summations.

p	1	2	3	4	5	6	7	8	9
S_n	1	3	6	10	15	21	28	36	45
V_n	1	4	10	20	35	56	84	120	165

वर्गसङ्कलितं घनसङ्कलितं च – Summation of squares and cubes

द्विघ्नपदं कुयुतं त्रिविभक्तं सङ्कलितेन हतं कृतियोगः ।
सङ्कलितस्य कृतेः सममेकाद्यङ्कघनैक्यमुदाहृतमाद्यैः ॥११९॥ । दोधकवृत्तम् ।

Twice the number of terms (*pada*) added by one, divided by three [and] multiplied by *saṅkalita* is the sum of squares (*kṛti*) [of natural numbers]. The sum of cubes of the numbers starting from one has been stated to be equal to the square of *saṅkalita* by the ancestors.

$$S_{n^2} = \sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{2n+1}{3} \times S_n$$

$$S_{n^3} = \sum_{i=1}^{n} i^3 = 1^3 + 2^3 + \cdots + n^3 = S_n^2 = \left[\frac{n(n+1)}{2}\right]^2$$

तेषामेव च वर्गैक्यं घनैक्यं च वद द्रुतम् ।
इति कृतिसङ्कलनामार्गे कुशला यदि ते मतिः ॥१२०॥ । अनुष्टुभ् ।

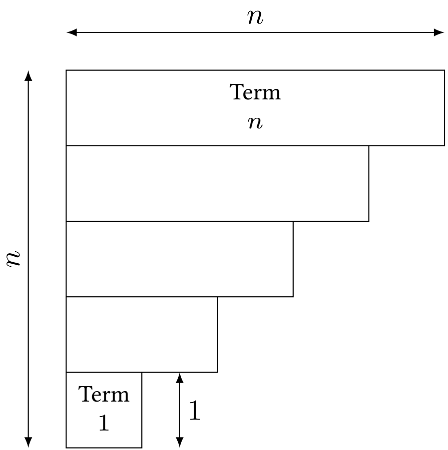
Tell quickly the sum of squares and the sum of cubes of those very numbers [in the previous example] if your intellect is sharp in the method of calculating the sum of the squares etc.

p	1	2	3	4	5	6	7	8	9
S_{n^2}	1	5	14	30	55	91	140	204	285
S_{n^3}	1	9	36	100	225	441	784	1296	2025

Proof for *saṅkalita* in Nīlakaṇṭha's *Āryabhaṭīya-bhāṣya*

Generally it is thought that Indian mathematics is bereft of proofs. This is not true!! We shall demonstrate this with some proofs!

- In *Āryabhaṭīya-bhāṣya*, Nīlakaṇṭha visualises the various terms of the sequence as rectangles of width equal to unity, and length equal to the value of the term.
- These rectangles are then stacked one upon another to form a *śreḍhīkṣetra* as depicted

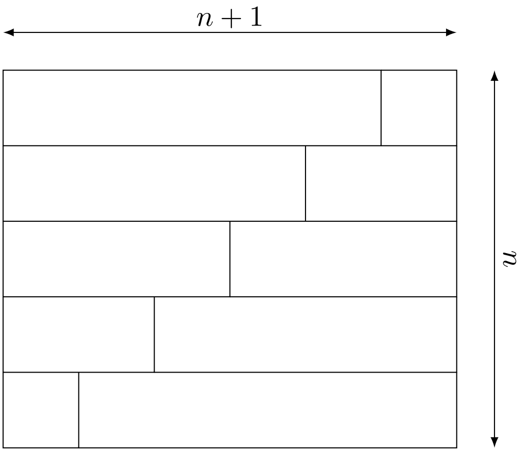


A single *śreḍhīkṣetra*

- This *śreḍhīkṣetra* is then joined with another similar but inverted *śreḍhīkṣetra* to form a rectangle.
- The area of this rectangle is then obviously equal to

$$n \times (n+1)$$

- Therefore, the area of one *śreḍhīkṣetra*, which represents the sum of the terms equals $\dfrac{n(n+1)}{2}$.



Two joined *śreḍhīkṣetras*

Proofs in Munīśvara’s

Nisṛṣṭārthadūtī

Munīśvara's proof for *Saṅkalita*

एकादिपदपर्यन्ताङ्कान् क्रमेण संस्थाप्य तन्मध्ये व्यस्तक्रमेण एकादयो युज्यन्ते । तदा प्रत्येकं सैकपदतुल्याङ्काः स्युः । तेषां योगे सैकपदेन गुणितं पदं स्यात् । अत्र सङ्कलितस्य द्विगुणतया पर्यवसानादेतदर्धं सैकपदतुल्याङ्कानां सङ्कलितमुपपन्नम् ।

Having placed the numbers beginning from one and ending with the last term (*pada*) sequentially, the numbers one etc. are added to them in reverse order. Then each one [i.e. sum of corresponding terms] would be equal to last term plus one (*saikapada*). When added, the sum would be the last term multiplied by last term plus one. Since the result happens to be twice the summation (*saṅkalita*), it is [indeed] proved that half of this is the [required] summation.

$$S_n = n + (n-1) + \cdots + 2 + 1$$

$$2S_n = (n+1) + (n+1) + \cdots + (n+1)$$

$$S_n = \frac{n(n+1)}{2}$$

- Munīśvara credits this proof to a certain Lakṣmīdāsa. As per the prescription given here, we first need to place the *pada* n in n places, and add them

$$n + n + \cdots + n = n^2$$

$$\begin{array}{cccccccccccccc}
\hline
 & n^2 & = & n & + & n & + & \ldots & + & n & + & n & + & n \\
- & S_{n-1} & = & n-1 & + & n-2 & + & \ldots & + & 2 & + & 1 & & \\
\hline
 & & = & 1 & + & 2 & + & \ldots & + & n-2 & + & n-1 & + & n \\
\hline
\end{array}$$

The quantities subtracted add up to S_{n-1}, whereas the resultant quantities add up to S_n.

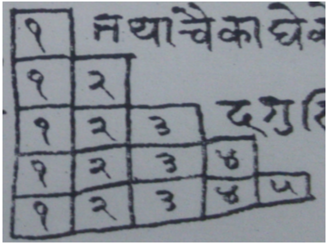
$$S_n + S_{n-1} = n^2$$

$$S_n + S_{n-1} + n = n^2 + n$$

As $S_{n-1} + n = S_n$, the above equation reduces to

$$2S_n = n^2 + n \qquad \text{or,} \qquad S_n = \frac{n(n+1)}{2}$$

Munīśvara's proof for *Saṅkalitaikya*



$$
\begin{array}{ccccccc}
\hline
1 & & & & & & \\
1 & 2 & & & & & \\
1 & 2 & 3 & & & & \\
1 & 2 & 3 & 4 & & & \\
1 & 2 & 3 & 4 & 5 & & \\
\vdots & & & & & & \\
1 & 2 & 3 & 4 & 5 & \dots & n \\
\hline
\end{array}
$$

$$
\begin{aligned}
V_n &= n \times 1 + (n-1) \times 2 + (n-2) \times 3 + \\
&\qquad \cdots + (n-(n-1)) \times n \\
&= n \times (1 + 2 + \cdots + n) - [1 \times 2 + \\
&\qquad 2 \times 3 + 3 \times 4 + \cdots + (n-1) \times n] \\
&= nS_n - 2 \times \left(1 + 3 + \cdots + \frac{(n-1)n}{2}\right) \\
&= nS_n - 2 \times (S_1 + S_2 + \cdots + S_{n-1}) \\
&= nS_n - 2 \times V_{n-1}.
\end{aligned}
$$

Since $V_{n-1} = V_n - S_n$, we get

$$V_n = (n+2)S_n - 2V_n.$$

$$V_n = \frac{S_n(n+2)}{3} = \frac{n(n+1)(n+2)}{1 \cdot 2 \cdot 3}$$

अत्रोपपत्तिस्तु लक्ष्मीदासोक्ता । सङ्कलितपदघाते पदपर्यन्तमेकादिजातवर्गाणामैक्यस्य रूपोनपदसङ्कलितैक्यस्य च योगोऽवश्यं भवति । कुत एवमिति चेच्छृणु । एकादिपदपर्यन्तानामङ्कानां योगः सङ्कलितम् । तस्मिन् पदगुणिते पदगुणितानामङ्कानां वा योगे समानत्वात् अन्तिमाङ्कः पदगुणितः पदवर्गः । ततः पदेन उपान्तिमाङ्को रूपोनपदमितो गुणितः । स रूपोनपदस्य वर्गो रूपोनपदेन एकगुणेन युतो भवति । पदस्य रूपतदूनपदात्मकखण्डद्वययोगात्मकत्वेन गुणकत्वाभ्युपगमात् । एवं व्युत्क्रमेण तृतीयादयोऽङ्काः पदगुणिताः सन्तो द्व्याद्यूनपदानां वर्गाद्या द्व्यूनपदैर्द्व्यादिगुणितैर्युक्ता भवन्ति, उक्तरीत्या द्व्याद्यूनपदद्व्यादिखण्डद्वययोगात्मकपदस्य गुणकत्वात् । एतेषां गुणितानां योगे एकाद्येकोत्तराङ्कानां वर्गयोगो, रूपोनपदपर्यन्तम् एकाद्येकोत्तराङ्कानां व्यस्तानां क्रमस्थैः एकाद्येकोत्तराङ्कैर्गुणितानां योगेन रूपोनपदस्य प्रागुक्तनिर्णीतसङ्कलितैक्यात्मकेन युतो भवति । तथा च सङ्कलितपदघाते रूपोनपदस्य सङ्कलितैक्योने कृते वर्गयोगः फलितः ।

Munīśvara's proof for *Varga-saṅkalita*

The procedure outlined above can be expressed in our mathematical notation as follows:

$$nS_n = n \times [1 + 2 + \cdots + (n-1) + n] = \sum_{i=1}^{n} n \cdot i.$$

Now, the last term of the above expansion is equal to n^2, while the penultimate term is $n(n-1)$. By rewriting the multiplier (*guṇaka*) n as

$$n = (n-1) + 1$$

the penultimate term reduces to

$$\begin{aligned} n \cdot (n-1) &= [(n-1)+1] \cdot (n-1) \\ &= (n-1)^2 + 1 \cdot (n-1) \end{aligned}$$

Similarly, rewriting n as $n = (n-2) + 2$, the third-last term becomes

$$\begin{aligned} n \cdot (n-2) &= [(n-2)+2] \cdot (n-2) \\ &= (n-2)^2 + 2 \cdot (n-2) \end{aligned}$$

Munīśvara's proof for *Varga-saṅkalita*

Therefore, in general, rewriting $n = (n-i) + i$, we have

$$nS_n = \sum_{i=1}^{n} [(n-i)+i] \cdot i \qquad = \sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i \cdot (n-i)$$

It is easily seen that the first term in the RHS of the above equation is sum of squares of natural numbers, and the second term is sum of sums of $n-1$ terms. So, the above equation reduces to:

$$nS_n = S_{n^2} + V_{n-1}.$$

It is obvious that

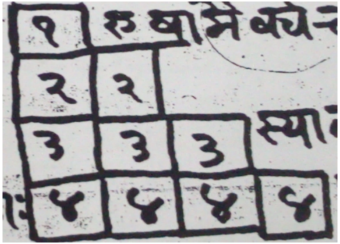
$$V_{n-1} = \frac{(n-1)\, n\, (n+1)}{1 \cdot 2 \cdot 3}.$$

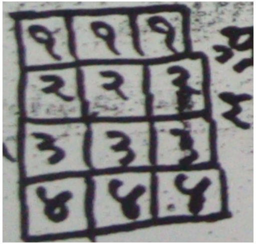
We know $S_n = \frac{n(n+1)}{2}$. Using this we get,

$$S_{n^2} = n \times \frac{n(n+1)}{2} - \frac{(n-1)\, n\, (n+1)}{1 \cdot 2 \cdot 3} \qquad = \frac{n(n+1)}{2} \times \frac{2n+1}{3}$$

Another proof for *Varga-saṅkalita*

Munīśvara also gives another visually-oriented proof, crediting Rāmacandra for the same.





1					
2	2				
3	3	3			
$\vdots$	$\vdots$	$\vdots$			
$n-1$	$n-1$	$n-1$	$\ldots$	$n-1$	
n	n	n	$\ldots$	n	n

1	1	$\ldots$	1	1
2	2	$\ldots$	2	2
3	3	$\ldots$	3	3
$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$
$n-1$	$n-1$	$\ldots$	$n-1$	$n-1$
n	n	$\ldots$	n	n

1					
2	2				
3	3	3			
$\vdots$	$\vdots$	$\vdots$			
$n-1$	$n-1$	$n-1$	$\dots$	$n-1$	
n	n	n	$\dots$	n	n

1	1	$\dots$	1	1
2	2	$\dots$	2	2
3	3	$\dots$	3	3
$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$
$n-1$	$n-1$	$\dots$	$n-1$	$n-1$
n	n	$\dots$	n	n

It can be easily seen that the sum of all numbers in the coloured region (originally empty) turns out to be

$$S_1 + S_2 + \cdots + S_{n-1} = V_{n-1}$$

As the sum of each column of numbers considering both the coloured and uncoloured cells together is equal to S_n, the sum of the all the uncoloured elements in the grid may be expressed as

$$(n-1)S_n - V_{n-1}$$

Another proof for *Varga-saṅkalita*

$$\begin{array}{cccccc}
\hline
1 & & & & & \\
2 & 2 & & & & \\
3 & 3 & 3 & & & \\
\vdots & \vdots & \vdots & & & \\
n-1 & n-1 & n-1 & \dots & n-1 & \\
n & n & n & \dots & n & n \\
\hline
\end{array}$$

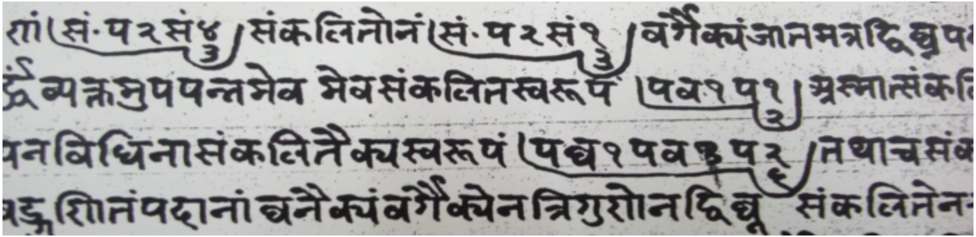
$$\begin{array}{ccccc}
\hline
1 & 1 & \dots & 1 & 1 \\
2 & 2 & \dots & 2 & 2 \\
3 & 3 & \dots & 3 & 3 \\
\vdots & \vdots & \dots & \vdots & \vdots \\
n-1 & n-1 & \dots & n-1 & n-1 \\
n & n & \dots & n & n \\
\hline
\end{array}$$

$$S_{n^2} = S_n + (n-1)S_n - V_{n-1}$$

We know

$$V_{n-1} = \frac{(n-1)\,n\,(n+1)}{1\cdot 2\cdot 3} = \frac{(n-1)S_n}{3}.$$

$$S_{n^2} = S_n + (n-1)S_n - \frac{(n-1)S_n}{3} = \frac{2n+1}{3} \times S_n.$$


गां (सं·प२सं४/३) संकलितोनं (सं·प२सं१/३) वर्गैक्यं जातमत्र द्विघ्न
व्यक्तमुपपन्नमेव मेवसंकलितस्वरूपं (पव१प१/२) अस्मात्संक
विधिना संकलितैक्यस्वरूपं (पघ१पव३प२/६) तथाच सं
गुणितं पदानां घनैक्यं वर्गैक्येन त्रिगुणेन द्विघ्न संकलितेन

- Here, the notations प, सं, पव, and पघ stand for *pada*, *saṅkalita*, *pada-varga*, and *pada-ghana*, which represent n, S_n, n^2, and n^3 in our notation system.
- Accordingly, the underlined terms in the second and third rows in the figure are equivalent to

$$S_n = \frac{n^2 + n}{2} \qquad \text{and} \qquad V_n = \frac{n^3 + 3n^2 + 2n}{6}.$$

- It may be noted that the number appearing below the last term is the denominator for the entire expression.

My Dear Sir,
I have great pleasure in communiating the Series, to which I alluded ...

$$C = 4D\left(1-\frac{1}{3}+\frac{1}{5}-\ldots\right), \tag{1}$$

$$C = \sqrt{12D^2}-\frac{\sqrt{12D^2}}{3.3}+\frac{\sqrt{12D^2}}{3^2.5}-\frac{\sqrt{12D^2}}{3^3.7}+\cdots, \tag{2}$$

$$C = 2D+\frac{4D}{(2^2-1)}-\frac{4D}{(4^2-1)}+\frac{4D}{(6^2-1)}-\cdots \tag{3}$$

$$C = 8D\left[\frac{1}{(2^2-1)}+\frac{1}{(6^2-1)}+\frac{1}{(10^2-1)}+\cdots\right]. \tag{4}$$

$$C = 8D\left[\frac{1}{2}-\frac{1}{(4^2-1)}-\frac{1}{(8^2-1)}-\frac{1}{(12^2-1)}-\cdots\right]. \tag{5}$$

$$C = 3D+\frac{4D}{(3^3-3)}-\frac{4D}{(5^3-5)}+\frac{4D}{(7^3-7)}-\cdots \tag{6}$$

$$C = 16D\left(\frac{1}{1^5+4.1}-\frac{1}{3^5+4.3}+\frac{1}{5^5+4.5}-\cdots\right) \tag{7}$$

I am, my dear Sir, most sincerely, your's,

Madras, 17th August 1825.

G. HYNE.

The dilemma of John Warren

In his *Kālasaṇkalita* John Warren observes (pp. 92-93):

> *Of their manner of resolving geometrically the ratio of the diameter to the circumference of a circle, I never saw any Indian demonstration: the common opinion, however is, that they approximate it in the manner of the ancients, by exhaustion; that is, be means of inscribed and circumscribed Polygons.*[1] *However, a Native Astronomer who was a perfect stranger to European Geometry, gave me the well known series* $1 - \frac{1}{3} + \frac{1}{5} \quad \ldots$
>
> *This proves at least, that the Hindus are not ignorant of the doctrine of series; but I could not understand whether he pretended to make out ...*

I join in substance Mr. Hyne's opinion, but do not admit that the circumstance that none of the Sastras mentioned by Mr. Whish, who used the series could demonstrate them, would alone be conclusive.

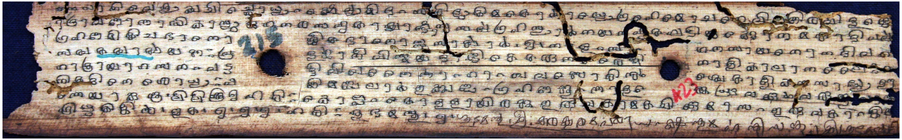
George Hyne's note to John Warren

I owe the following Note to Mr. Hyne's favour.

> *The Hindus never invented the series; it was communicated with many others, by Europeans, to some learned Natives in modern times. Mr. Whish sent a list of the various methods of demonstrating the ratio of the diameter and circumference of a Circle employed by the Hindus to the literary society, being impressed with the notion that they were the inventors. I requested him to make further inquiries, and his reply was, that he had reasons to believe them entirely modern and and derived from Europeans, observing that not one of those used the Rules could demonstrate them. Indeed the pretensions of the Hindus to such a knowledge of geometry, is too ridiculous to deserve refutation.*

Concluding Remarks

History vs. Myth-making



Finally, again, I would like to conclude with the words of Claude Alvares (*The Indian Science and Technology in the 18th Century, Other India Press, Goa.*) –

- All History is elaborate efforts in myth-making. ...
- If we must continue to live with myths, however, it is far better we choose to live with those of our own making rather than by those invented by others for their own purposes.
- That much at least we owe as an independent Society and Nation !

Thanks!

धन्यवादाः!