श्रीमद्भास्कराचार्यविरचिता लीलावती

व्याख्यानम् 11: प्रकीर्णकम् (वर्गकर्म गुणकर्म च

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प्रकीर्णकान्तर्गताः विषयाः — Topics under Miscellaneous operations

The following table presents the various topics discussed by Bhāskara to solve commonly encountered basic arithmetic problems, in the section titled *Prakīrṇaka*.

क्रमः	विषयः	अनुवादः
1	व्यस्तविधिः	Inverting mathematical processes
2	इष्टकर्म	Operations with assumed numbers
3	सङ्क्रमणं विषमकर्म च	Operations with sums and differences
4	वर्गकर्म	Operations with squares of numbers
5	गुणकर्म	Dealing with univariate quadratic equations
6	त्रैराशिकादिः	Rules of three etc.
7	भाण्डप्रतिभाण्डकम्	Barter of commodities

वर्गकर्म —Operations with 'squares' of *rāśis* resulting in squares!

- Having dealt with problems involving one unknown variable (referred to as *rāśi*), Bhāskara proceeds to discuss problems involving two unknown variables.
- Here is an interesting problem that involves the determination of two unknown *rāśis*, the sum as well as difference of whose squares when reduced by one (unity), also results in squares. This has been nicely articulated as follows:

  ययोः द्वयोः अज्ञातराश्योः वर्गयोः योगः अन्तरं च रूपविहीनं वर्गौ एव भवतः तयोः साधनं वर्गकर्म भवति।

- In other words, Bhāskara seeks to determine two quantities a and b such that

$$a^2 + b^2 - 1 = S_1 \tag{1}$$
$$b^2 - a^2 - 1 = S_2, \tag{2}$$

where S_1 and S_2 are squares, with of course $b > a$.

इष्टकृतिरष्टगुणिता व्येका दलिता विभाजितेष्टेन ।
एकः स्यादस्य कृतिः दलिता सैकापरो राशिः ॥ ६० ॥
रूपं द्विगुणेष्टहृतं सेष्टं प्रथमोऽथवापरो रूपम् ।
कृतियुतिवियुती व्येके वर्गौ स्यातां ययो राश्योः ॥ ६१ ॥ । आर्या ।

The square of an assumed number multiplied by eight, deducted by one, [the result] halved, and divided by the assumed [number] would be one [quantity]. The square of this [quantity] halved, [added] with one would be the other quantity.[These methods determine two quantities], the sum and difference of whose squares decreased by one would be square numbers.

Solution 1:

$$a = \frac{8x^2 - 1}{2x} \qquad b = \frac{a^2}{2} + 1 = \frac{1}{2} \times \left(\frac{8x^2 - 1}{2x}\right)^2 + 1. \tag{3}$$

रूपं द्विगुणेष्टहृतं सेष्टं प्रथमोऽथवापरो रूपम् ।
कृतियुतिवियुती व्येके वर्गौ स्यातां ययो राश्योः ॥ ६१॥ । आर्या ।

Or, unity divided by twice an assumed number, [added] with the assumed [number] would the first [quantity]. Unity is the other [quantity]. [These methods determine two quantities], the sum and difference of whose squares decreased by one would be square numbers.

Solution 2:

$$a = \frac{1}{2x} + x = \frac{2x^2+1}{2x} \qquad b = 1. \tag{4}$$

Verification that the expressions given above lead to square numbers when substituted in (1) and (2) is straightforward.

Demonstration that the first set satisfies the condition

It can be easily seen that the expression results in a square number when substituted in (1).

$$a^2 + b^2 - 1 = \left[\frac{8x^2-1}{2x}\right]^2 + \left[\left(\frac{8x^2-1}{2x}\right)^2 \times \frac{1}{2} + 1\right]^2 - 1$$

$$= \frac{1}{4} \times \left[\frac{8x^2-1}{2x}\right]^4 + 2 \times \left[\frac{8x^2-1}{2x}\right]^2 + 1 - 1$$

$$= \left[\frac{8x^2-1}{2x}\right]^2 \times \left[\frac{1}{4} \times \left(\frac{8x^2-1}{2x}\right)^2 + 2\right]$$

$$= \left[\frac{8x^2-1}{2x}\right]^2 \times \left[\frac{64x^4+16x^2+1}{16x^2}\right]$$

$$= \left[\frac{8x^2-1}{2x}\right]^2 \times \left[\frac{8x^2+1}{4x}\right]^2$$

$$= \left[\frac{64x^4-1}{8x^2}\right]^2.$$

Demonstration that the first set satisfies the condition

Similarly,

$$
\begin{aligned}
b^2 - a^2 - 1 &= \left[\frac{1}{2} \times \left(\frac{8x^2-1}{2x}\right)^2 + 1\right]^2 - \left[\frac{8x^2-1}{2x}\right]^2 - 1 \\
&= \frac{1}{4} \times \left[\frac{8x^2-1}{2x}\right]^4 + \left[\frac{8x^2-1}{2x}\right]^2 - \left[\frac{8x^2-1}{2x}\right]^2 + 1 - 1 \\
&= \left[\frac{1}{2} \times \left(\frac{8x^2-1}{2x}\right)^2\right]^2 .
\end{aligned}
$$

Thus we see that the expression results in a square number when substituted in (2). The resulting number need not be an integer. All that Bhāskara mentions that it has to be a *varga*, which indeed is satisfied.

राश्योर्ययोः कृतिवियोगयुती निरेके
मूलप्रदे प्रवद तौ मम मित्र यत्र ।
क्लिश्यन्ति बीजगणिते पटवोऽपि मूढाः
षोढोक्तबीजगणितं परिभावयन्तः ॥६२॥ *[वसन्ततिलका]*

My friend! Tell those two numbers, the sum and difference of whose squares reduced by one result in square numbers, wherein even experts in algebra who know the cryptic mathematical techniques stated in six ways, being bewildered, face difficulty [in solving this problem].

x	a	b	S_1	S_2
$\frac{1}{2}$	1	$\frac{3}{2}$	$\left(\frac{3}{2}\right)^2$	$\left(\frac{1}{2}\right)^2$
1	$\frac{7}{2}$	$\frac{57}{8}$	$\left(\frac{63}{8}\right)^2$	$\left(\frac{49}{8}\right)^2$
2	$\frac{31}{4}$	$\frac{993}{32}$	$\left(\frac{255}{32}\right)^2$	$\left(\frac{961}{32}\right)^2$

x	a	b	S_1	S_2
1	$\frac{3}{2}$	1	$\left(\frac{3}{2}\right)^2$	$\left(\frac{1}{2}\right)^2$
2	$\frac{9}{4}$	1	$\left(\frac{9}{4}\right)^2$	$\left(\frac{7}{4}\right)^2$
3	$\frac{19}{6}$	1	$\left(\frac{19}{6}\right)^2$	$\left(\frac{17}{6}\right)^2$

Qualifications required to undertake study of astronomy

It may be worth recalling a couple of verses in this context.

- Bhāskara, in his *gola* chapter of the *Siddhāntasiromaṇi*, commenting on eligiblity criteria to study astronomy observes:

  यो वेद वेदवदनं सदनं हि सम्यक्
  ब्राह्म्याः स वेदमपि वेद किमन्यशास्त्रम् ।
  यस्मादतः प्रथममेतदधीत्य धीमान्
  शास्त्रान्तरस्य भवति श्रवणेऽधिकारी ।। ४ ।।

- Nityānanda goes one step beyond this and explicitly and derisively states that one who does not possess this skill set doesn't qualify to be counted as a competent astronomer:

  पाटीकुट्टकबीजगोलनिपुणो वेदाङ्गपारङ्गमः
  काव्यालङ्कृतिशिल्पशास्त्रकुशलो यो न्यायशास्त्रादिवित् ।
  सिद्धान्तेऽत्र समस्तवस्तुसहिते तस्याधिकारो भवेत्
  चेदेवं न यथाकथञ्चिदभिधामात्रं प्रसिद्धिं नयेत् ।।

इष्टस्य वर्गवर्गो घनश्च तावष्टसंगुणौ प्रथमः ।
सैको राशी स्यातामेवं व्यक्तेऽथवाव्यक्ते ॥६३॥ [आर्या]

The square of the square, as well as the cube of an assumed number, are both multiplied by eight, and the former is increased by one. Thus would be the two quantities in arithmetic or algebra.

Here are the solutions given above along with demonstration of the result:

$$a = 8x^4 + 1 \qquad b = 8x^3. \tag{5}$$

$$\begin{aligned} a^2 + b^2 - 1 &= (8x^4+1)^2 + (8x^3)^2 - 1 \\ &= 64x^8 + 64x^6 + 16x^4 + 1 - 1 \\ &= 16x^4 \times (4x^4 + 4x^2 + 1) \\ &= 16x^4 \times (2x^2+1)^2 \\ &= [4x^2 \times (2x^2+1)]^2. \end{aligned}$$

$$\begin{aligned} a^2 - b^2 - 1 &= (8x^4+1)^2 - (8x^3)^2 - 1 \\ &= 64x^8 - 64x^6 + 16x^4 + 1 - 1 \\ &= 16x^4 \times (4x^4 - 4x^2 + 1) \\ &= 16x^4 \times (2x^2-1)^2 \\ &= [4x^2 \times (2x^2-1)]^2. \end{aligned}$$

पाटीसूत्रोपमं बीजं गूढमित्यवभासते ।
नास्ति गूढममूढानां नैव षोढेत्यनेकधा ॥६४-१॥
अस्ति त्रैराशिकं पाटी बीजं च विमला मतिः ।
किमज्ञातं सुबुद्धीनां अतो मन्दार्थमुच्यते ॥६४-२॥

Algebraic rules [which are] similar to [well understood] arithmetic ones, appear cryptic. [However], for the intelligent, there is nothing which is hidden (unknown), nor are they merely six-fold, but can be manifold. ...What is unknown to the intelligent, therefore this is stated for the dim-witted.

What is noteworthy here is:

- the relative difficulty of algebra compared to arithmetic
- mention of arithmetic, rule of three, and pure intellect
- the intelligent apply algebra far beyond these six primary operations.

- Quadratic equations are univariate second order equations, usually represented in the form

$$ax^2 + bx + c = 0$$

- These equations have been studied in India at least, from the time of Āryabhaṭa, with Brahmagupta prominently addressing them in his *Brahmasphuṭasiddhānta*.
- Bhāskara too addresses problems of this type, but in contrast to the given equation, he frames his equation in the form

$$x^2 \mp bx = c,$$

  where x^2 is referred to as *rāśi*, b is called *guṇa* or *mūlaguṇa*, and c is *dṛṣṭa* or *dṛśya*.
- Bhāskara arrives at the solution to these kinds of problems by modifying the equation by adding the square of half the *guṇa* to both sides. Therefore, he refers to this operation as *guṇakarma*.

गुणघ्नमूलोनयुतस्य राशेः
दृष्टस्य युक्तस्य गुणार्धकृत्या ।
मूलं गुणार्धेन युतं विहीनं
वर्गीकृतं प्रष्टुरभीष्टराशिः ॥६५॥ ।उपजातिः ।

The square root of—[either] the *rāśi* minus or plus the product of the *guṇa* and the square root of the *rāśi*, [or] the *dṛṣṭa*, added by the square of half the *guṇa*—added or subtracted by half the *guṇa*, and [the result] squared, becomes the desired quantity of the questioner.

$$x^2 \mp bx + \left(\frac{b}{2}\right)^2 = c + \left(\frac{b}{2}\right)^2$$

$$\left(x \mp \frac{b}{2}\right)^2 = c + \left(\frac{b}{2}\right)^2$$

$$x \mp \frac{b}{2} = \sqrt{c + \left(\frac{b}{2}\right)^2}$$

$$x = \sqrt{c + \left(\frac{b}{2}\right)^2} \pm \frac{b}{2}$$

$$\therefore \quad x^2 = \left[\sqrt{c + \left(\frac{b}{2}\right)^2} \pm \frac{b}{2}\right]^2.$$

In modern mathematical texts, the roots of a quadratic equation of the form $ax^2 + bx + c = 0$ are usually represented as

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Bhāskara's expression can also be simplified and re-written as follows:

$$x = \frac{\sqrt{b^2 + 4c} \pm b}{2}.$$

यदा लवैश्चोनयुतः स राशिः
एकेन भागोनयुतेन भक्त्वा ।
दृश्यं तथा मूलगुणं च ताभ्यां
साध्यस्ततः प्रोक्तवदेव राशिः ॥६६॥ ।उपजातिः ।

When that *raśi* is subtracted or added by [its] parts, then having divided the *dṛśya* and the *mūlaguṇa* by one decreased or increased by [those] parts, thereafter, using those [modified *dṛśya* and *guṇa*], the *rāśi* should be calculated as stated [earlier].

In this verse, Bhāskara presents how to solve quadratic equations having coefficient of x^2 other than one

$$\left(1 \mp \frac{p}{q}\right) x^2 \mp bx = c,$$

then, dividing the entire equation by the coefficient of x^2, we have

$$x^2 \mp b'x = c',$$

where $b' = \dfrac{b}{1 \mp \frac{p}{q}}$ and $c' = \dfrac{c}{1 \mp \frac{p}{q}}$

Problem involving solution of a quadratic equation

Example based on flora and fauna – an enticing description!

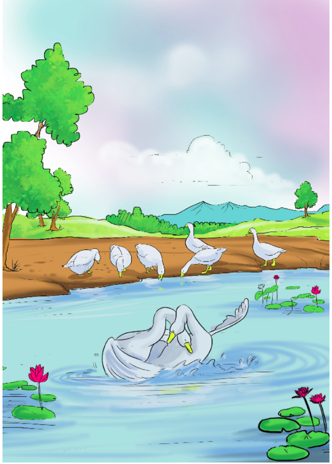
बाले मरालकुलमूलदलानि सप्त
तीरे विलासभरमन्थरगाण्यपश्यम् ।
कुर्वच्च केलिकलहं कलहंसयुग्मं
शेषं जले वद मरालकुलप्रमाणम् ॥६७॥ ।वसन्ततिलका ।

The given problem can be represented by the following quadratic equation, where x^2 is the total number of geese:

$$x^2 - 7\frac{x}{2} = 2.$$

In the above equation, the *guṇa* (b) is equal to $\frac{7}{2}$, and the *dṛṣṭa* (c) is 2. So,

$$x^2 = \left[\sqrt{2 + \left(\frac{1}{2} \times \frac{7}{2}\right)^2} + \frac{1}{2} \times \frac{7}{2}\right]^2 = 16.$$



स्वपदैर्नवभिर्युक्तः स्याच्चत्वारिंशताऽधिकम् ।
शतं द्वादशकं विद्वन् कः स राशिर्निगद्यताम् ॥६८॥ ।अनुष्टुभ् ।

O scholar! Let it be said, what is that number which added by nine times its square root would be twelve hundred more by forty.

The verse poses the following problem:

$$x^2 + 9x = 1240.$$

Substituting $b = 9$, and $c = 1240$, we have

$$x^2 = \left[\sqrt{1240 + \left(\frac{9}{2}\right)^2} - \frac{9}{2}\right]^2 = 961.$$

Problem involving गुणकर्म – Quadratic equations

यातं हंसकुलस्य मूलदशकं मेघागमे मानसं
प्रोड्डीय स्थलपद्मिनीवनमगात् अष्टांशकोऽम्भस्तटात् ।
बाले बालमृणालशालिनि जले केलिक्रियालालसं
दृष्टं हंसयुगत्रयं च सकलां यूथस्य[a] सङ्ख्यां वद ॥६९॥

।शार्दूलविक्रीडितम् ।

[a]यूथं = सजातीयसमूहः

Of a herd of swans (at a water body), ten times the square root went (migrated) to Mānasa (lake) on the approach of clouds (rainy season). One-eighth having flown, went to a forest of *sthalapadminī* (*Hibiscus mutabilis*) from the shore of the water. Three pairs of swans were observed to be absorbed in sporting in water having delicate stalks of lotuses. O girl! Tell the total number [of swans] the group has.

The given problem is as follows:

$$x^2 = \frac{x^2}{8} + 10x + 6$$

$$x^2\left(1-\frac{1}{8}\right) - 10x = 6$$

$$x^2 - \frac{10}{1-\frac{1}{8}}x = \frac{6}{1-\frac{1}{8}}$$

$$x^2 - \frac{80}{7}x = \frac{48}{7}$$

Applying the solution for quadratic equation, we have

$$x^2 = \left[\sqrt{\frac{48}{7} + \left(\frac{1}{2}\times\frac{80}{7}\right)^2} + \frac{1}{2}\times\frac{80}{7}\right]^2 = 144$$

Problem involving गुणकर्म – Quadratic equations

पार्थः कर्णवधाय मार्गणगणं[a] क्रुद्धो रणे सन्दधे
तस्यार्धेन निवार्य तच्छरगणं मूलैश्चतुर्भिर्हयान् ।
शल्यं षड्भिः अथेषुभिस्त्रिभिरपि छत्रं ध्वजं कार्मुकं
चिच्छेदास्य शिरः शरेण कति ते यानर्जुनः सन्दधे ॥७०॥

।शार्दूलविक्रीडितम् ।

[a]मार्गणः = मार्गयति लक्ष्यमिति ; शर इत्यर्थः।

An enraged Pārtha (Arjuna), in order to slay Karṇa in battle, shot a succession of arrows [on his bow]. Having countered his (Karṇa's) series of arrows with half of that (Arjuna's own), [he stopped] the horses by four times the square root [of his total arrows], [immobilised] Śalya by six arrows, then [destroyed] the umbrella, flag and bow by three arrows, [and] cut his head by one arrow. How many were those [arrows], which Arjuna nocked [on his bow]?

$$x^2 = \frac{x^2}{2} + 4x + 6 + 3 + 1.$$

Thanks!

धन्यवादाः!