श्रीमद्भास्कराचार्यविरचिता लीलावती

व्याख्यानम् 13: प्रकीर्णकम् (गुणकर्म त्रैराशिकादिश्च)

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प्रकीर्णकान्तर्गताः विषयाः — Topics under Miscellaneous operations

The following table presents the various topics discussed by Bhāskara to solve commonly encountered basic arithmetic problems, in the section titled *Prakīrṇaka*.

क्रमः	विषयः	अनुवादः
1	व्यस्तविधिः	Inverting mathematical processes
2	इष्टकर्म	Operations with assumed numbers
3	सङ्क्रमणं विषमकर्म च	Operations with sums and differences
4	वर्गकर्म	Operations with squares of numbers
5	गुणकर्म	Dealing with univariate quadratic equations
6	त्रैराशिकादिः	Rules of three etc.
7	भाण्डप्रतिभाण्डकम्	Barter of commodities

त्रैराशिकम् – Rule of three

प्रमाणमिच्छा च समानजाती आद्यन्तयोस्स्तः फलमन्यजातिः ।
मध्ये तदिच्छाहतमाद्यहृत् स्यात् इच्छाफलं व्यस्तविधिर्विलोमे ॥७३॥ ।उपजातिः ।

Pramāṇa and *icchā* [which] are of the same kind (units), are [placed] in the first and last positions. The result (*pramāṇaphala*) [which is] of another kind is [placed] in the middle. That (*pramāṇaphala*), multiplied by the *icchā*, and divided by the *ādya* (*pramāṇa*) would be *icchāphala*. In [case of] inverse proportion, reverse process [is employed].

The relation given in the verse to determine the *icchāphala* in terms of *ādi* and *anta* can be interpreted as follows:

$$\textit{icchāphala} = \frac{\textit{pramāṇaphala} \times \textit{icchā}}{\textit{pramāṇa}}$$

व्यस्तत्रैराशिकम् – Inverse proportions

इच्छावृद्धौ फले ह्रासः ह्रासे वृद्धिः फलस्य तु ।
व्यस्तं त्रैराशिकं तत्र ज्ञेयं गणितकोविदैः ॥७७॥ ।अनुष्टुभ् ।

Wherein reduction happens in [*icchā*] *phala* when *icchā* increases and increment happens [in *phala*] when *icchā* decreases, there inverse proportion should be considered by experts of mathematics.

$$\textit{icchāphala} \propto \frac{1}{\textit{icchā}} \qquad\qquad \textit{pramāṇaphala} \propto \frac{1}{\textit{pramāṇa}}$$

And so,

$$\textit{icchāphala} \times \textit{icchā} = \textit{pramāṇaphala} \times \textit{pramāṇa}$$

$$\text{or,} \qquad \textit{icchāphala} = \frac{\textit{pramāṇaphala} \times \textit{pramāṇa}}{\textit{icchā}}$$

जीवानां वयसो मौल्ये तौल्ये वर्णस्य हैमने ।
भागहारे च राशीनां व्यस्तत्रैराशिकं भवेत् ॥७८॥ ।अनुष्टुभ् ।

In valuation of living beings with respect to age, in weighing of gold with respect to quality (*varṇa* or purity), and in division of heaps [of grains with respect to volume], inverse proportion should be [applied].

- Two common situations one could think of during Bhāskara's time:
  - trading of livestock (say bulls). Here, $\text{price to be paid} \propto \dfrac{1}{\text{age of livestock}}$
  - hiring of lass for amorous activity. Here again, $\text{price} \propto \dfrac{1}{\text{age of the lass}}$
- Another common situation involves the purchase of gold. The colour (=quality) and weight of gold are inversely proportional to one another.
- A third scenario describes the division of grain into heaps. The number of divisions and the volume of each division are inversely proportional. Bhāskara poses problems involving each of these scenarios below.

प्राप्नोति चेत् षोडशवत्सरा स्त्री द्वात्रिंशतं विंशतिवत्सरा किम् ।
द्विधूर्वहो निष्कचतुष्कमुक्षा प्राप्नोतिधूःषट्कवहस्तदा किम् ॥७९॥ ।उपजातिः ।

If a sixteen year old woman gets thirty-two [*niṣkas*] [as *kaṇyāśulka*], then what would a twenty year old get? [If] an ox yoked and driven for two years gets four *niṣkas*, then what would one harnessed for six [years] get?

Solving the first problem we get,

$$icch\bar{a}phala = \frac{32 \times 16}{20} = \frac{128}{5}\ drammas.$$

The result of the second problem is,

$$icch\bar{a}phala = \frac{4 \times 2}{6} = \frac{4}{3}\ ni\d{s}kas.$$

Problems related to व्यस्तत्रैराशिकम् – Inverse proportions

दशवर्णसुवर्णं चेत् गद्याणकमवाप्यते ।
निष्केण तिथिवर्णं तु तदा वद कियन्मितम् ॥८०॥ ।अनुष्टुभ् ।

One *gadyāṇaka* of gold is obtained for one *niṣka*, if it were of ten *varṇas* [purity]. Then tell how much [gold can be obtained for one *niṣka*] if it were of fifteen *varṇas* [purity]?

The solution is,

$$icch\bar{a}phala = \frac{1 \times 10}{15} = \frac{2}{3}\ gady\bar{a}\d{n}akas.$$

- For a fixed price, the weight and purity of gold are inversely proportional. It is stated that 1 *gadayāṇaka* (*pramāṇaphala*) of gold of 10 *varṇas* (*pramāṇa*) can be obtained for one *niṣka*.
- The higher the *varṇa* the higher the price of the gold! (carat $\equiv$ *varṇa*??)
- Converting the weight of gold, the above result is equal to 1 *dharaṇa*, 2 *vallas*, and 2 *guñjas*.

Problems related to व्यस्तत्रैराशिकम् – Inverse proportions

सप्ताढकेन मानेन राशौ सस्यस्य मापिते ।
यदि मानशतं जातं तदा पञ्चाढकेन किम् ॥८१॥ ।अनुष्टुभ् ।

If a measure of hundred [heaps] is obtained when a certain quantity of grains are measured by a [volume] scale of seven *āḍhakas*, then what [number of heaps would be obtained] by using [a scale of] five *āḍhakas*?

We need to determine the number of heaps obtained (*icchāphala*), when each heap has a volume of 5 *āḍhakas* (*icchā*). The solution is,

$$icch\bar{a}phala = \frac{100 \times 7}{5} = 140 \text{ heaps.}$$

Principle: number of heaps $\propto \dfrac{1}{\text{volume of each heap}}$

1 घनहस्तः	=	1 खारी
1 खारी	=	16 द्रोणाः
1 द्रोणः	=	4 आढकाः
1 आढकः	=	4 प्रस्थाः
1 प्रस्थः	=	4 कुडवाः

पञ्चराशिकादिकम् – Compound proportions

- Simple proportions (*trairāśika*) have three known values and one unknown value, and involve only two different units (for example, price and weight).
- In contrast, compound proportions consist of more than two units (for example, principal, duration and interest), and more than three known values.
- In the following verses, Bhāskara deals with compound proportions having five, seven, nine and eleven known values.
- These are respectively known as *pañcarāśika*, *saptarāśika*, *navarāśika*, and *ekādaśarāśika*.
- However, in all these cases, only one unknown value is to be determined, just as in the case of *trairāśika*.

पञ्चराशिकादिकम् – Compound proportions

पञ्चसप्तनवराशिकादिकेऽन्योन्यपक्षनयनं फलच्छिदाम् ।
संविधाय बहुराशिजे वधे स्वल्पराशिवधभाजिते फलम् ॥८२॥ ।रथोद्धता ।

While dealing with *pañcarāśika, saptarāśika, navarāśika* or [compound proportions of] more [quantities], having done the exchange of the results and denominators [of any fraction] to mutual sides, when the product of [quantities in the side having] more quantities, is divided by the product of the [quantities in the side having] lesser quantities, the result is [obtained].

- This verse is more involved and needs more careful attention to be understood.
- Let us denote the five quantities involved in the problem of *pañcarāśika* as *pramāṇa*$_1$, *pramāṇa*$_2$, and *pramāṇaphala*, and their corresponding (same units) *icchā* values as *icchā*$_1$, and *icchā*$_2$.
- *Icchāphala*, which corresponds to the *pramāṇaphala*, is unknown and is to be determined.

The given method requires listing the known quantities in two columns, labelled as the *pramāṇapakṣa* and *icchāpakṣa*, the former containing the known *pramāṇa* quantities, and the latter the known *icchā* quantities.

Unit	***Pramāṇapakṣa***	***Icchāpakṣa***
Unit 1	$pramāṇa_1$	$icchā_1$
Unit 2	$pramāṇa_2$	$icchā_2$
Unit 3	*pramāṇaphala*	

Then the *pramāṇaphala* is transferred to to the other column as follows:

Pramāṇapakṣa	***Icchāpakṣa***
$pramāṇa_1$	$icchā_1$
$pramāṇa_2$	$icchā_2$
	pramāṇaphala
(*svalparāśi*)	(*bahurāśi*)

Then, to arrive at the result (*icchāphala*), the product of the quantities in the *bahurāśi* is divided by the product of the quantities in the *svalparāśi*:

$$icch\bar{a}phala = \frac{icch\bar{a}_1 \times icch\bar{a}_2 \times pram\bar{a}\d{n}aphala}{pram\bar{a}\d{n}a_1 \times pram\bar{a}\d{n}a_2}.$$

For a compound proportion having n known values (n is odd), the above expression can be generalised as follows:

$$icch\bar{a}phala = \frac{icch\bar{a}_1 \times icch\bar{a}_2 \times \cdots \times icch\bar{a}_{\frac{n-1}{2}} \times pram\bar{a}\d{n}aphala}{pram\bar{a}\d{n}a_1 \times pram\bar{a}\d{n}a_2 \times \cdots \times pram\bar{a}na_{\frac{n-1}{2}}}.$$

An optional intermediary step before carrying out the above operation is to transpose the denominators (*chid*) of fractional values to the opposite sides. In the above example, let $pramāṇa_1 = \frac{p}{q}$ and $icchā_1 = \frac{a}{b}$. Then, after transposing the *pramāṇaphala*, we have:

Pramāṇapakṣa	***Icchāpakṣa***
$\frac{p}{q}$	$\frac{a}{b}$
$pramāṇa_2$	$icchā_2$
	pramāṇaphala
(*svalparāśi*)	(*bahurāśi*)

After determining the *bahurāśi* and *svalparāśi*, the denominators of fractional quantities are transposed as follows:

Pramāṇapakṣa	***Icchāpakṣa***
p	a
b	q
$pramāṇa_2$	$icchā_2$
	pramāṇaphala
(*svalparāśi*)	(*bahurāśi*)

मासे शतस्य यदि पञ्च कलान्तरं स्यात्
वर्षे गते भवति किं वद षोडशानाम् ।
कालं तथा कथय मूलकलान्तराभ्यां
मूलं धनं गणक कालफले विदित्वा ॥८३॥ ।वसन्ततिलका ।

If the interest for hundred [units of a certain currency] in a month be five, [then] tell what [is the interest] for sixteen [units of same currency] after a year passes. Similarly O mathematician! Tell the duration from the principal and interest, [and also] the principal amount having known duration and interest.

Unit	*Pramāṇapakṣa*	*Icchāpakṣa*
Principal invested	100	16
Duration of investment (months)	1	12
Interest earned	5	

Solution to the first part by repeated application of त्रैराशिकम्

The above problem can be solved with the repeated application of the rule of three. As the first step, given that an investment of 100 earns an interest of 5 in one month, we determine how much an investment of 16 will earn in the same period. With respect to the rule of three, the *pramāṇa* and *pramāṇaphala* equal 100 and 5 respectively, while the *icchā* equals 16. Applying this we have

$$\textit{icchāphala} = \frac{5 \times 16}{100} = \frac{4}{5}$$

Now knowing that an investment of 16 earns an interest of $\frac{4}{5}$ in one month, we seek to determine, how much the same amount would earn in twelve months. Therefore, we apply *trairāśika* again, where the *pramāṇa* and the *pramāṇaphala* in this case equal 1 and $\frac{4}{5}$, while the *icchā* equals 12. Therefore, we have

$$\textit{icchāphala} = \frac{\frac{4}{5} \times 12}{1} = \frac{48}{5}$$

Illustrative example related to पञ्चराशिकादिकम् – Textual method (I part)

In the first step, the *pramāṇaphala* is transposed as follows:

Unit	*Pramāṇapakṣa*	*Icchāpakṣa*
Principal invested	100	16
Duration of investment (months)	1	12
Interest earned		5
	(*svalparāśi*)	(*bahurāśi*)

As the *icchāpakṣa* has more quantities, it is designated as *bahurāśi*, while the *pramāṇapakṣa* is designated *svalparāśi*. Dividing the product of the quantities in the *bahurāśi* with the product of the quantities in the *svalparāśi*, we get

$$icch\bar{a}phala = \frac{16 \times 12 \times 5}{100 \times 1} = \frac{48}{5},$$

which is the same as the solution obtained by the repeated application of the rule of three.

Illustrative example related to पञ्चराशिकादिकम् – II part of the problem

मासे शतस्य यदि पञ्च कलान्तरं स्यात्
वर्षे गते भवति किं वद षोडशानाम् ।
कालं तथा कथय मूलकलान्तराभ्यां ← *II part*
मूलं धनं गणक कालफले विदित्वा ॥८३॥ ।वसन्ततिलका ।

In the second part of the problem, we need to solve for the duration, taking the principal and interest as known. The problem can be stated as follows:

Unit	*Pramāṇapakṣa*	*Icchāpakṣa*
Principal invested	100	16
Duration of investment (months)	1	
Interest earned	5	$\frac{48}{5}$

Recall: In the formula $I = \frac{P \times T \times R}{100}$, assuming a fixed interest rate R, we have $P \propto I, \frac{1}{T}$ and $T \propto I, \frac{1}{P}$.

Solution to the second part by repeated application of त्रैराशिकम्

While duration is directly proportional to interest, it is inversely proportional to principal. Therefore, the given problem can be solved by applying a direct proportion, and then an inverse proportion. In this first step, given that an investment of 100 earns an interest of 5 (*pramāṇa* in 1 month (*pramāṇaphala*), we seek to determine the time required for the same principal to earn an interest of $\frac{48}{5}$ (*icchā*). Through direct proportion, we have

$$\textit{icchāphala} = \frac{1 \times \frac{48}{5}}{5} = \frac{48}{25} \text{ months.}$$

In the second step, given that an investment of 100 (*pramāṇa*) earns an interest of $\frac{48}{5}$ in $\frac{48}{25}$ months (*pramāṇaphala*), we seek to determine the time required for an investment of 16 (*icchā*) to earn the same interest. It is obvious that a smaller principal requires a longer period to earn the same interest. Therefore, applying inverse proportions, we have

$$\textit{icchāphala} = \frac{\frac{48}{25} \times 100}{16} = 12 \text{ months.}$$

In the second part of the problem, we need to solve for the duration, taking the principal and interest as known. First, transposing the *pramāṇaphala* and *icchāphala*, we determine the *bahurāśi* and *svalparāśi*:

Pramāṇapakṣa	*Icchāpakṣa*
100	16
1	
$\frac{48}{5}$	5
(*bahurāśi*)	(*svalparāśi*)

Pramāṇapakṣa	*Icchāpakṣa*
100	16
1	
48	5
	5
(*bahurāśi*)	(*svalparāśi*)

Then, dividing the product of the quantities in the *bahurāśi*, with the product of the quantities in the *svalparāśi*, we have

$$\text{Duration} = \frac{100 \times 1 \times 48}{16 \times 5 \times 5} = 12 \text{ months.}$$

Illustrative example related to पञ्चराशिकादिकम् – Textual method (III part)

In the final part of the problem, we need to determine the principal, from the known duration and interest. The problem is stated as follows:

Unit	*Pramāṇapakṣa*	*Icchāpakṣa*
Principal invested	100	
Duration of investment (months)	1	12
Interest earned	5	$\frac{48}{5}$

Transposing the *phalas*, we have:

Pramāṇapakṣa	*Icchāpakṣa*
100	
1	12
$\frac{48}{5}$	5
(*bahurāśi*)	(*svalparāśi*)

Skipping the optional step of transposing the denominators, and directly dividing the product of the quantities in the *bahurāśi* with the product of the quantities in the *svalparāśi*, we have

$$\text{Principal} = \frac{100 \times 1 \times \frac{48}{5}}{12 \times 5} = 16.$$

Problems related to पञ्चराशिकादिकम् – Compound proportions

सत्र्यंशमासेन शतस्य चेत् स्यात् कलान्तरं पञ्च सपञ्चमांशाः ।
मासैस्त्रिभिः पञ्चलवाधिकैस्तैः सार्धद्विषष्टेः फलमुच्यतां किम् ॥८४॥ ।उपजातिः ।

If the interest for hundred [units of a certain currency] in one and one-third months be five and one-fifth, then let it be told what would be that result for sixty-two and a half [units of same currency] with [the duration of] three months more by one-fifth part.

The problem can be stated as follows:

Unit	*Pramāṇapakṣa*	*Icchāpakṣa*
Principal invested	100	$\frac{125}{2}$
Duration of investment (months)	$\frac{4}{3}$	$\frac{16}{5}$
Interest earned	$\frac{26}{5}$	

Problems related to पञ्चराशिकादिकम् – Compound proportions

Transposing the *phala* (interest), we have:

Pramāṇapakṣa	*Icchāpakṣa*
100	$\frac{125}{2}$
$\frac{4}{3}$	$\frac{16}{5}$
	$\frac{26}{5}$
(*svalparāśi*)	(*bahurāśi*)

Now, further transposing the denominators of the fractions, we have:

Pramāṇapakṣa	*Icchāpakṣa*
100	125
2	3
4	16
5	26
5	
(*svalparāśi*)	(*bahurāśi*)

$$\text{Interest} = \frac{125 \times 3 \times 16 \times 26}{100 \times 2 \times 4 \times 5 \times 5} = \frac{39}{5}.$$

Illustrative example of सप्तराशिकम्

विस्तारे त्रिकराः कराष्टकमिता दैर्घ्ये विचित्राश्च चेत्
रूपैरुत्कटपट्टसूत्रपटिका अष्टौ लभन्ते शतम् । (पट्ट = कौशेय = कृमिकोषादिजातेवस्त्रे)
दैर्घ्ये सार्धकरत्रयाऽपरपटी हस्तार्धविस्तारिणी
तादृक् किं लभते द्रुतं वद वणिक् वाणिज्यकं वेत्सि चेत् ॥८५॥ ।शार्दूलविक्रीडितम् ।

If eight multi-coloured superior quality woven silk cloths which are of three *hastas* in breadth and eight *hastas* in length fetch a hundred [*niṣkas*], then O merchant! If you know trade, tell quickly how much [money] would another cloth which is of three and a half *karas* (*hastas*) in length and of half *hasta* breadth fetch.

Unit	*Pramāṇapakṣa*	*Icchāpakṣa*
Length	8	$\frac{7}{2}$
Breadth	3	$\frac{1}{2}$
Number of pieces	8	1
Price	100	

Illustrative example of सप्तराशिकम्

Here the *phala* is the price as it is directly proportional to the other units. Transposing the *phala* as well as the *chid*, we have:

Pramāṇapakṣa	*Icchāpakṣa*
8	7
2	1
3	1
2	100
8	
(*svalparāśi*)	(*bahurāśi*)

Then, we have

$$\text{Price} = \frac{7 \times 1 \times 1 \times 100}{8 \times 2 \times 3 \times 2 \times 8} = \frac{175}{192} \textit{ niṣkas}.$$

The above quantity is equal to 14 *drammas*, 9 *paṇas*, 1 *kākiṇī*, and $6\frac{2}{3}$ *varāṭakas*.

Thanks!

धन्यवादाः!