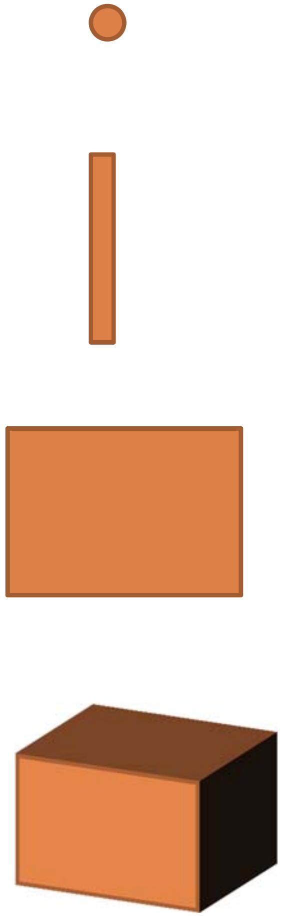
TENSOR ALGEBRA

Continuum Mechanics Course (MMC) - ETSECCPB - UPC

Introduction to Tensors

Tensor Algebra

Introduction

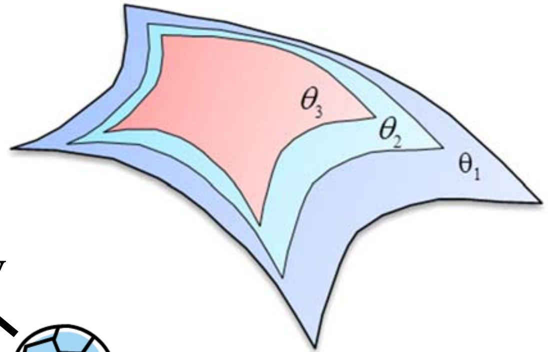


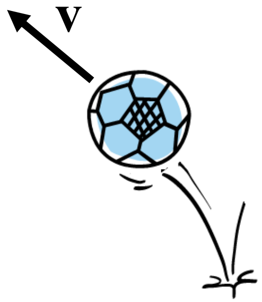
SCALAR — $\rho,\ \theta,\ \ldots$

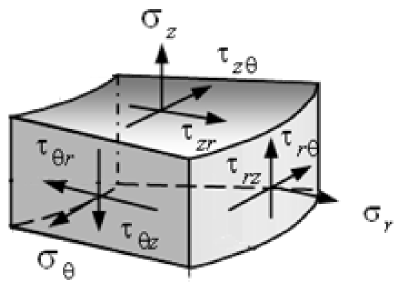
VECTOR — $\mathbf{v},\ \mathbf{f},\ \ldots$

MATRIX — $\boldsymbol{\sigma},\ \boldsymbol{\varepsilon},\ \ldots$

? — $\mathbb{C},\ \ldots$







Concept of Tensor

- A **TENSOR** is an algebraic entity with various components which **generalizes the concepts** of scalar, vector and matrix.
  - Many physical quantities are mathematically represented as tensors.
  - **Tensors** are **independent of any reference system** but, by need, are commonly represented in one by means of their “component matrices”.
  - The **components** of a tensor **will depend on the reference system** chosen and will vary with it.

Order of a Tensor

- The order of a tensor is given by the number of indexes needed to specify without ambiguity a component of a tensor.

a ■ **Scalar:** zero dimension $\quad \alpha = 3.14$

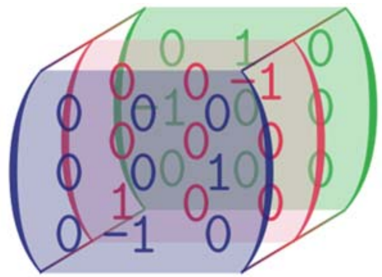
$\underline{a}$, $\mathbf{a}$ ■ **Vector:** 1 dimension $\quad \mathrm{v}_i = \begin{pmatrix} 1.2 \\ 0.3 \\ 0.8 \end{pmatrix}$

$\underline{\underline{A}}$, $\mathbf{A}$ ■ **2nd order:** 2 dimensions $\quad E_{ij} = \begin{pmatrix} 0.1 & 0 & 1.3 \\ 0 & 2.4 & 0.5 \\ 1.3 & 0.5 & 5.8 \end{pmatrix}$

$\underline{\underline{\underline{\mathcal{A}}}}$, $\mathcal{A}$ ■ **3rd order:** 3 dimensions

$\underline{\underline{\underline{\underline{\mathbb{A}}}}}$, $\mathbb{A}$ ■ **4th order** …

$\epsilon_{ijk} =$



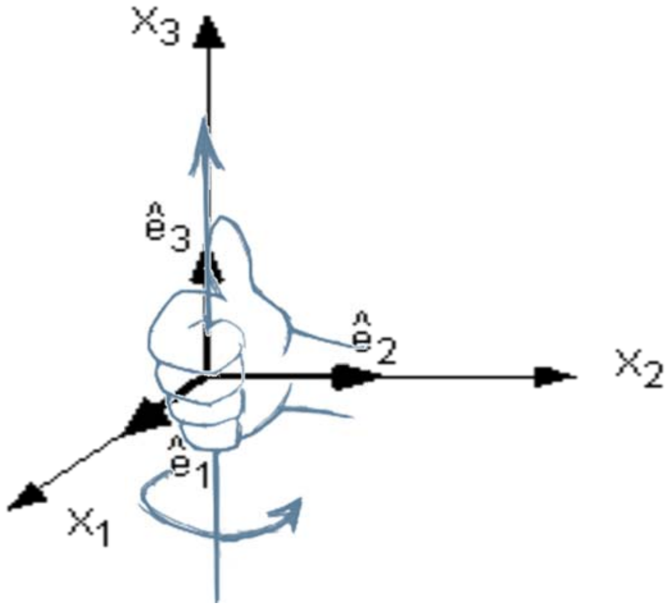
Cartesian Coordinate System

- Given an **orthonormal basis** formed by three mutually perpendicular unit vectors:

$$\hat{\mathbf{e}}_1 \perp \hat{\mathbf{e}}_2 \quad , \qquad \hat{\mathbf{e}}_2 \perp \hat{\mathbf{e}}_3 \quad , \qquad \hat{\mathbf{e}}_3 \perp \hat{\mathbf{e}}_1$$

Where:

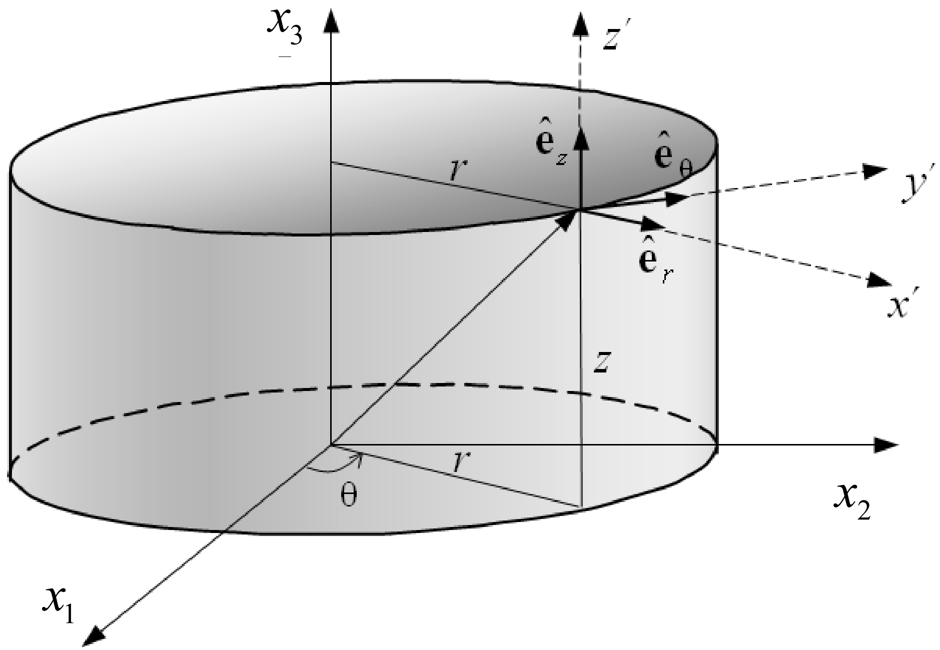
$$|\hat{\mathbf{e}}_1| = 1 \quad , \qquad |\hat{\mathbf{e}}_2| = 1 \quad , \qquad |\hat{\mathbf{e}}_3| = 1$$



- Note that

$$\boxed{\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \begin{Bmatrix} 1 & \text{if} & i = j \\ 0 & \text{if} & i \neq j \end{Bmatrix} = \delta_{ij}}$$

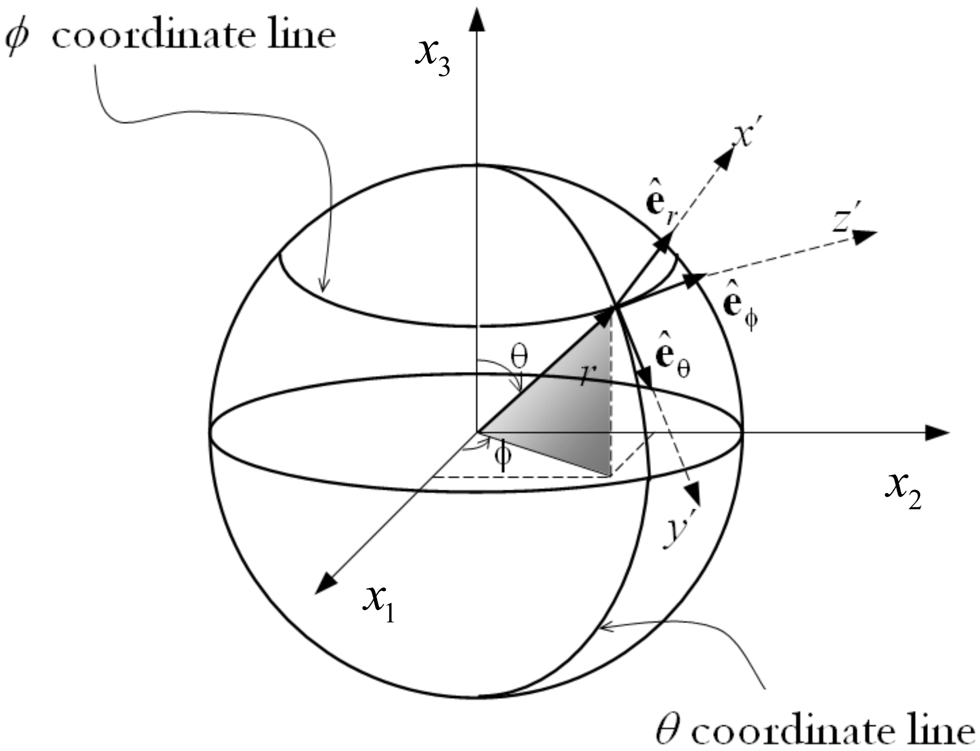
Cylindrical Coordinate System



$$\mathbf{x}(r,\theta,z) \equiv \begin{cases} x_1 = r \ \cos\theta \\ x_2 = r \ \sin\theta \\ x_3 = z \end{cases}$$

$$\hat{\mathbf{e}}_r = \cos\theta \ \hat{\mathbf{e}}_1 + \sin\theta \ \hat{\mathbf{e}}_2$$
$$\hat{\mathbf{e}}_\theta = -\sin\theta \ \hat{\mathbf{e}}_1 + \cos\theta \ \hat{\mathbf{e}}_2$$
$$\hat{\mathbf{e}}_z = \hat{\mathbf{e}}_3$$

Spherical Coordinate System



$$\mathbf{x}(r,\theta,\varphi) \equiv \begin{cases} x_1 = r\sin\theta\ \cos\varphi \\ x_2 = r\ \sin\theta\ \sin\varphi \\ x_3 = r\ \cos\theta \end{cases}$$

$$\hat{\mathbf{e}}_r = \sin\theta\sin\varphi\ \hat{\mathbf{e}}_1 + \sin\theta\cos\varphi\ \hat{\mathbf{e}}_2 + \cos\theta\ \hat{\mathbf{e}}_3$$
$$\hat{\mathbf{e}}_\theta = \cos\varphi\ \hat{\mathbf{e}}_1 - \sin\varphi\ \hat{\mathbf{e}}_2$$
$$\hat{\mathbf{e}}_\varphi = \cos\theta\sin\varphi\ \hat{\mathbf{e}}_1 + \cos\theta\cos\varphi\ \hat{\mathbf{e}}_2 - \sin\theta\ \hat{\mathbf{e}}_3$$

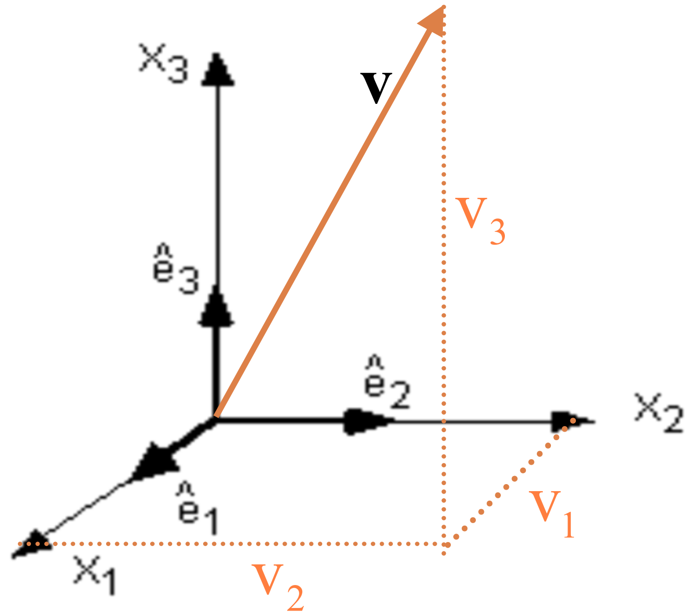
Indicial or (Index) Notation

Tensor Algebra

Tensor Bases – VECTOR

- A vector $\mathbf{v}$ can be written as a unique linear combination of the **three** vector basis $\hat{\mathbf{e}}_i$ for $i \in \{1,2,3\}$.

$$\boxed{\mathbf{v} = \mathrm{v}_1\hat{\mathbf{e}}_1 + \mathrm{v}_2\hat{\mathbf{e}}_2 + \mathrm{v}_3\hat{\mathbf{e}}_3}$$



- In matrix notation:

$$[\mathbf{v}] = \begin{bmatrix} \mathrm{v}_1 \\ \mathrm{v}_2 \\ \mathrm{v}_3 \end{bmatrix}$$

- In index notation:

$$\mathbf{v} = \sum_i \mathrm{v}_i \hat{\mathbf{e}}_i$$ tensor as a physical entity

$$[\mathbf{v}]_i = \mathrm{v}_i$$ component i of the tensor in the given basis $i \in \{1,2,3\}$

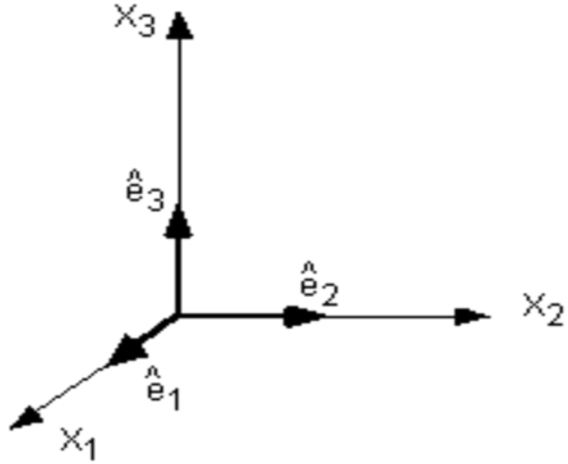
Tensor Bases – 2nd ORDER TENSOR

- A 2nd order tensor $\mathbf{A}$ can be written as a unique linear combination of the **nine** dyads $\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \equiv \hat{\mathbf{e}}_i \hat{\mathbf{e}}_j$ for $i, j \in \{1,2,3\}$.

$$\boxed{\begin{aligned}\mathbf{A} = {} & A_{11}\left(\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_1\right) + A_{12}\left(\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_2\right) + A_{13}\left(\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_3\right) + \\ & + A_{21}\left(\hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_1\right) + A_{22}\left(\hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_2\right) + A_{23}\left(\hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_3\right) + \\ & + A_{31}\left(\hat{\mathbf{e}}_3 \otimes \hat{\mathbf{e}}_1\right) + A_{32}\left(\hat{\mathbf{e}}_3 \otimes \hat{\mathbf{e}}_2\right) + A_{33}\left(\hat{\mathbf{e}}_3 \otimes \hat{\mathbf{e}}_3\right)\end{aligned}}$$

Alternatively, this could have been written as:

$$\begin{aligned}\mathbf{A} = {} & A_{11}\,\hat{\mathbf{e}}_1\hat{\mathbf{e}}_1 + A_{12}\,\hat{\mathbf{e}}_1\hat{\mathbf{e}}_2 + A_{13}\,\hat{\mathbf{e}}_1\hat{\mathbf{e}}_3 + \\ & + A_{21}\,\hat{\mathbf{e}}_2\hat{\mathbf{e}}_1 + A_{22}\,\hat{\mathbf{e}}_2\hat{\mathbf{e}}_2 + A_{23}\,\hat{\mathbf{e}}_2\hat{\mathbf{e}}_3 + \\ & + A_{31}\,\hat{\mathbf{e}}_3\hat{\mathbf{e}}_1 + A_{32}\,\hat{\mathbf{e}}_3\hat{\mathbf{e}}_2 + A_{33}\,\hat{\mathbf{e}}_3\hat{\mathbf{e}}_3\end{aligned}$$



Tensor Bases – 2nd ORDER TENSOR

$$\boxed{\begin{aligned}\mathbf{A} = {} & A_{11}\left(\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_1\right) + A_{12}\left(\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_2\right) + A_{13}\left(\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_3\right) + \\ & + A_{21}\left(\hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_1\right) + A_{22}\left(\hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_2\right) + A_{23}\left(\hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_3\right) + \\ & + A_{31}\left(\hat{\mathbf{e}}_3 \otimes \hat{\mathbf{e}}_1\right) + A_{32}\left(\hat{\mathbf{e}}_3 \otimes \hat{\mathbf{e}}_2\right) + A_{33}\left(\hat{\mathbf{e}}_3 \otimes \hat{\mathbf{e}}_3\right)\end{aligned}}$$

- In matrix notation:

$$[\mathbf{A}] = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

- In index notation:

$$\mathbf{A} = \sum_{ij} \mathrm{A}_{ij}\left(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j\right)$$

tensor as a physical entity

$$[\mathbf{A}]_{ij} = A_{ij}$$

component ij of the tensor in the given basis $i, j \in \{1, 2, 3\}$

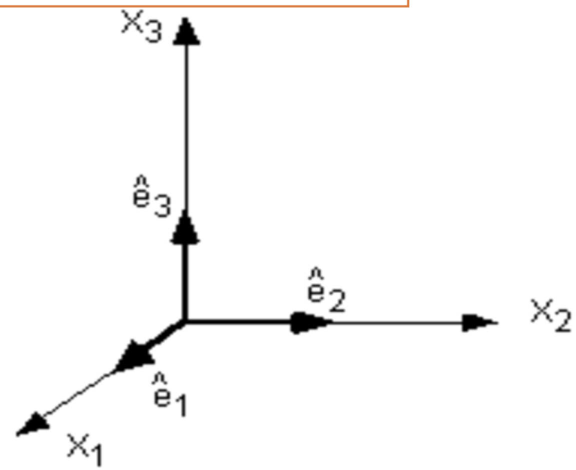
Tensor Bases – 3rd ORDER TENSOR

- A 3rd order tensor $\mathcal{A}$ can be written as a unique linear combination of the 27 tryads $\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k \equiv \hat{\mathbf{e}}_i \hat{\mathbf{e}}_j \hat{\mathbf{e}}_k$ for $i, j, k \in \{1, 2, 3\}$.

$$\boxed{\begin{aligned}\mathcal{A} = {} & \mathcal{A}_{111}\left(\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_1\right) + \mathcal{A}_{121}\left(\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_1\right) + \mathcal{A}_{131}\left(\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_3 \otimes \hat{\mathbf{e}}_1\right) + \\ & + \mathcal{A}_{211}\left(\hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_1\right) + \mathcal{A}_{221}\left(\hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_1\right) + \mathcal{A}_{231}\left(\hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_3 \otimes \hat{\mathbf{e}}_1\right) + \\ & + \mathcal{A}_{311}\left(\hat{\mathbf{e}}_3 \otimes \hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_1\right) + \mathcal{A}_{321}\left(\hat{\mathbf{e}}_3 \otimes \hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_1\right) + \mathcal{A}_{331}\left(\hat{\mathbf{e}}_3 \otimes \hat{\mathbf{e}}_3 \otimes \hat{\mathbf{e}}_1\right) + \\ & + \mathcal{A}_{112}\left(\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_2\right) + \mathcal{A}_{122}\left(\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_2\right) + \ldots\end{aligned}}$$

Alternatively, this could have been written as:

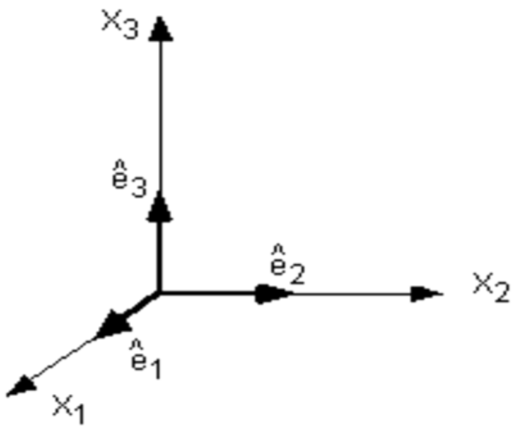
$$\begin{aligned}\mathcal{A} = {} & \mathcal{A}_{111}\,\hat{\mathbf{e}}_1\hat{\mathbf{e}}_1\hat{\mathbf{e}}_1 + \mathcal{A}_{121}\,\hat{\mathbf{e}}_1\hat{\mathbf{e}}_2\hat{\mathbf{e}}_1 + \mathcal{A}_{131}\,\hat{\mathbf{e}}_1\hat{\mathbf{e}}_3\hat{\mathbf{e}}_1 + \\ & + \mathcal{A}_{211}\,\hat{\mathbf{e}}_2\hat{\mathbf{e}}_1\hat{\mathbf{e}}_1 + \mathcal{A}_{221}\,\hat{\mathbf{e}}_2\hat{\mathbf{e}}_2\hat{\mathbf{e}}_1 + \mathcal{A}_{231}\,\hat{\mathbf{e}}_2\hat{\mathbf{e}}_3\hat{\mathbf{e}}_1 + \\ & + \mathcal{A}_{311}\,\hat{\mathbf{e}}_3\hat{\mathbf{e}}_1\hat{\mathbf{e}}_1 + \mathcal{A}_{321}\,\hat{\mathbf{e}}_3\hat{\mathbf{e}}_2\hat{\mathbf{e}}_1 + \mathcal{A}_{331}\,\hat{\mathbf{e}}_3\hat{\mathbf{e}}_3\hat{\mathbf{e}}_1 + \\ & + \mathcal{A}_{112}\,\hat{\mathbf{e}}_1\hat{\mathbf{e}}_1\hat{\mathbf{e}}_2 + \mathcal{A}_{122}\,\hat{\mathbf{e}}_1\hat{\mathbf{e}}_2\hat{\mathbf{e}}_2 + \ldots\end{aligned}$$



Tensor Bases – 3rd ORDER TENSOR

$$
\begin{aligned}
\mathcal{A} = {} & \mathcal{A}_{111}(\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_1) + \mathcal{A}_{121}(\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_1) + \mathcal{A}_{131}(\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_3 \otimes \hat{\mathbf{e}}_1) + \\
& + \mathcal{A}_{211}(\hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_1) + \mathcal{A}_{221}(\hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_1) + \mathcal{A}_{231}(\hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_3 \otimes \hat{\mathbf{e}}_1) + \\
& + \mathcal{A}_{311}(\hat{\mathbf{e}}_3 \otimes \hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_1) + \mathcal{A}_{321}(\hat{\mathbf{e}}_3 \otimes \hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_1) + \mathcal{A}_{331}(\hat{\mathbf{e}}_3 \otimes \hat{\mathbf{e}}_3 \otimes \hat{\mathbf{e}}_1) + \\
& + \mathcal{A}_{112}(\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_2) + \mathcal{A}_{122}(\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_2) + \ldots
\end{aligned}
$$

- In matrix notation:



$$
[\mathcal{A}] = \begin{bmatrix} \mathcal{A}_{111} & \mathcal{A}_{121} & \mathcal{A}_{131} \\ \mathcal{A}_{211} & \mathcal{A}_{221} & \mathcal{A}_{231} \\ \mathcal{A}_{311} & \mathcal{A}_{321} & \mathcal{A}_{331} \end{bmatrix}
\begin{bmatrix} \mathcal{A}_{112} & \mathcal{A}_{122} & \mathcal{A}_{132} \\ \mathcal{A}_{212} & \mathcal{A}_{222} & \mathcal{A}_{232} \\ \mathcal{A}_{312} & \mathcal{A}_{322} & \mathcal{A}_{332} \end{bmatrix}
\begin{bmatrix} \mathcal{A}_{113} & \mathcal{A}_{123} & \mathcal{A}_{133} \\ \mathcal{A}_{213} & \mathcal{A}_{223} & \mathcal{A}_{233} \\ \mathcal{A}_{313} & \mathcal{A}_{323} & \mathcal{A}_{333} \end{bmatrix}
$$

Tensor Bases – 3rd ORDER TENSOR

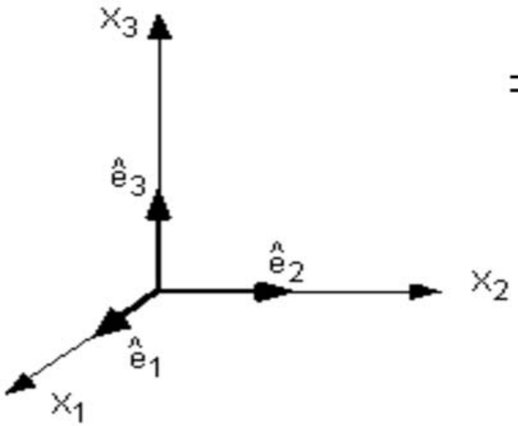
$$\begin{aligned}\mathcal{A} = {} & \mathcal{A}_{111}\left(\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_1\right) + \mathcal{A}_{121}\left(\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_1\right) + \mathcal{A}_{131}\left(\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_3 \otimes \hat{\mathbf{e}}_1\right) \\ & + \mathcal{A}_{211}\left(\hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_1\right) + \mathcal{A}_{221}\left(\hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_1\right) + \mathcal{A}_{231}\left(\hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_3 \otimes \hat{\mathbf{e}}_1\right) + \\ & + \mathcal{A}_{311}\left(\hat{\mathbf{e}}_3 \otimes \hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_1\right) + \mathcal{A}_{321}\left(\hat{\mathbf{e}}_3 \otimes \hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_1\right) + \mathcal{A}_{331}\left(\hat{\mathbf{e}}_3 \otimes \hat{\mathbf{e}}_3 \otimes \hat{\mathbf{e}}_1\right) + \\ & + \mathcal{A}_{112}\left(\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_2\right) + \mathcal{A}_{122}\left(\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_2\right) + \ldots\end{aligned}$$

- In index notation:

$$\mathcal{A} = \sum_{ijk} \mathcal{A}_{ijk}\left(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k\right) =$$

$$= \mathcal{A}_{ijk}\left(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k\right) \equiv \mathcal{A}_{ijk}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j\hat{\mathbf{e}}_k$$

tensor as a physical entity



$$\left[\mathcal{A}\right]_{ijk} = \mathcal{A}_{ijk}$$

component ijk of the tensor in the given basis $i, j, k \in \{1, 2, 3\}$

Higher Order Tensors

- A tensor of order n is expressed as:

$$\mathbf{A} = A_{i_1, i_2 \ldots i_n} \hat{\mathbf{e}}_{i_1} \otimes \hat{\mathbf{e}}_{i_2} \otimes \hat{\mathbf{e}}_{i_3} \otimes \ldots \otimes \hat{\mathbf{e}}_{i_n}$$

where $i_1, i_2 \ldots i_n \in \{1, 2, 3\}$

- The number of components in a tensor of order n is 3^n.

Repeated-index (or Einstein's) Notation

- The **Einstein Summation Convention:** repeated Roman indices are summed over.

i is a mute index

$$a_i b_i = \sum_{i=1}^{3} a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3$$

i is a talking index and *j* is a mute index

$$A_{ij} b_j = \sum_{j=1}^{3} A_{ij} b_j = A_{i1} b_1 + A_{i2} b_2 + A_{i3} b_3$$

  - A **"MUTE" (or DUMMY) INDEX** is an index that does not appear in a monomial after the summation is carried out (it can be arbitrarily changed of "name").
  - A **"TALKING" INDEX** is an index that is not repeated in the same monomial and is transmitted outside of it (it cannot be arbitrarily changed of "name").

REMARK

An index can only appear up to two times in a monomial.

Repeated-index (or Einstein's) Notation

Rules of this notation:

1. Sum over all repeated indices.
2. Increment all unique indices fully at least once, covering all combinations.
3. Increment repeated indices first.
4. A comma indicates differentiation, with respect to coordinate x_i.

$$u_{i,i} = \frac{\partial u_i}{\partial x_i} = \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i} \qquad u_{i,jj} = \frac{\partial^2 u_i}{\partial x_j \partial x_j} = \sum_{j=1}^{3} \frac{\partial^2 u_i}{\partial x_j^2} \qquad A_{ij,j} = \frac{\partial A_{ij}}{\partial x_j} = \sum_{j=1}^{3} \frac{\partial A_{ij}}{\partial x_j}$$

5. The number of talking indices indicates the order of the tensor result

Kronecker Delta δ

- The Kronecker delta δ_{ij} is defined as:

$$\boxed{\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}}$$

  - Both i and j may take on any value in $\{1,2,3\}$
  - Only for the three possible cases where $i = j$ is δ_{ij} non-zero.

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \quad (\delta_{11} = \delta_{22} = \delta_{33} = 1) \\ 0 & \text{if } i \neq j \quad (\delta_{12} = \delta_{13} = \delta_{21} \ldots = 0) \end{cases}$$

  - $$\delta_{ij} = \delta_{ji}$$

REMARK

Following Einsten's notation: $\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$

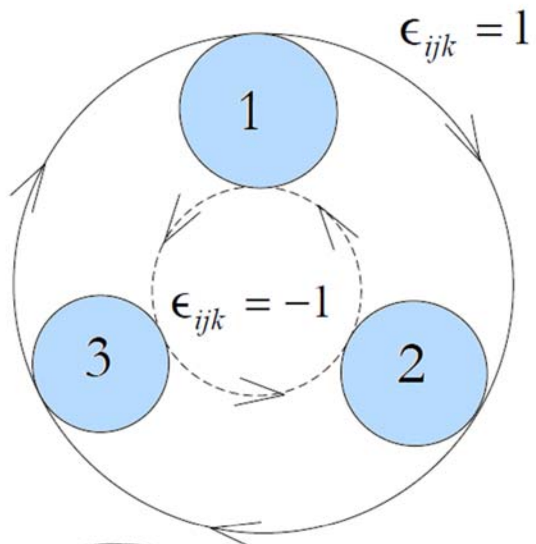
Kronecker delta serves as a replacement operator:

$$\delta_{ij} u_j = u_i \quad , \quad \delta_{ij} A_{jk} = A_{ik}$$

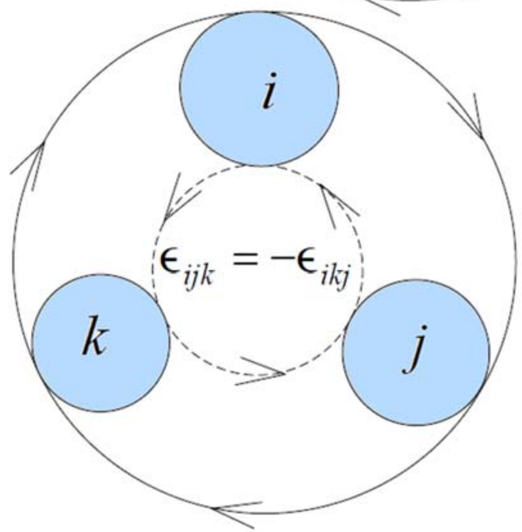
Levi-Civita Epsilon (permutation) ϵ

- The Levi-Civita epsilon ϵ_{ijk} is defined as:

$$\epsilon_{ijk} = \begin{cases} 0 & \text{if} \quad \text{there is a repeated index} \\ +1 & \text{if} \quad ijk = 123, 231 \text{ or } 312 \\ -1 & \text{if} \quad ijk = 213, 132 \text{ or } 321 \end{cases}$$



- 3 indices ⇨ 27 possible combinations.
- $\epsilon_{ijk} = -\epsilon_{ikj}$



REMARK

The Levi-Civita symbol is also named permutation or alternating symbol.

Relation between δ and ε

$$\epsilon_{ijk} = \det\begin{bmatrix} \delta_{i1} & \delta_{i2} & \delta_{i3} \\ \delta_{j1} & \delta_{j2} & \delta_{j3} \\ \delta_{k1} & \delta_{k2} & \delta_{k3} \end{bmatrix}$$

$$\epsilon_{ijk}\epsilon_{pqr} = \det\begin{bmatrix} \delta_{ip} & \delta_{iq} & \delta_{ir} \\ \delta_{jp} & \delta_{jq} & \delta_{jr} \\ \delta_{kp} & \delta_{kq} & \delta_{kr} \end{bmatrix}$$

$$\epsilon_{ijk}\epsilon_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}$$

$$\epsilon_{ijk}\epsilon_{pjk} = 2\delta_{pi}$$

$$\epsilon_{ijk}\epsilon_{ijk} = 6$$

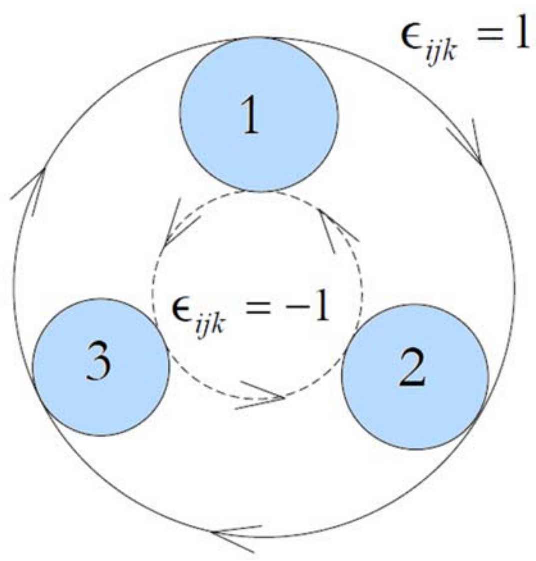
Example

- Prove the following expression is true:

$$\epsilon_{ijk}\epsilon_{ijk} = 6$$

Example - Solution

$$
\begin{array}{lcccl}
 & k=1 & k=2 & k=3 & \\
\epsilon_{ijk}\epsilon_{ijk} = & \cancel{\epsilon_{111}\epsilon_{111}} + & \cancel{\epsilon_{112}\epsilon_{112}} + & \cancel{\epsilon_{113}\epsilon_{113}} + & j=1 \\
i=1 \quad & + \cancel{\epsilon_{121}\epsilon_{121}} + & \cancel{\epsilon_{122}\epsilon_{122}} + & \overset{=1}{\epsilon_{123}}\epsilon_{123} + & j=2 \\
 & + \cancel{\epsilon_{131}\epsilon_{131}} + & \overset{=-1}{\epsilon_{132}}\epsilon_{132} + & \cancel{\epsilon_{133}\epsilon_{133}} + & j=3 \\
\hline
 & + \cancel{\epsilon_{211}\epsilon_{211}} + & \cancel{\epsilon_{212}\epsilon_{212}} + & \overset{=-1}{\epsilon_{213}}\epsilon_{213} + & \\
i=2 \quad & + \cancel{\epsilon_{221}\epsilon_{221}} + & \cancel{\epsilon_{222}\epsilon_{222}} + & \cancel{\epsilon_{223}\epsilon_{223}} + & \\
 & + \overset{=1}{\epsilon_{231}}\epsilon_{231} + & \cancel{\epsilon_{232}\epsilon_{232}} + & \cancel{\epsilon_{233}\epsilon_{233}} + & \\
\hline
 & + \cancel{\epsilon_{311}\epsilon_{311}} + & \overset{=1}{\epsilon_{312}}\epsilon_{312} + & \cancel{\epsilon_{313}\epsilon_{313}} + & \\
i=3 \quad & + \overset{=-1}{\epsilon_{321}}\epsilon_{321} + & \cancel{\epsilon_{322}\epsilon_{322}} + & \cancel{\epsilon_{323}\epsilon_{323}} + & \\
 & + \cancel{\epsilon_{331}\epsilon_{331}} + & \cancel{\epsilon_{332}\epsilon_{332}} + & \cancel{\epsilon_{333}\epsilon_{333}} & = 6
\end{array}
$$



Vector Operations

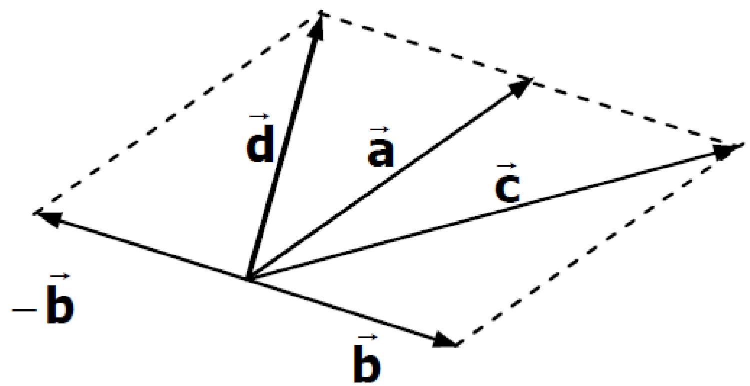
Tensor Algebra

Vector Operations

- **Sum and Subtraction. Parallelogram law.**

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a} = \mathbf{c} \quad \Rightarrow \quad c_i = a_i + b_i$$

$$\mathbf{a} - \mathbf{b} = \mathbf{d} \quad \Rightarrow \quad d_i = a_i - b_i$$



- **Scalar multiplication**

$$\alpha\mathbf{a} = \mathbf{b} = \alpha a_1\hat{\mathbf{e}}_1 + \alpha a_2\hat{\mathbf{e}}_2 + \alpha a_3\hat{\mathbf{e}}_3 \quad \Rightarrow \quad b_i = \alpha a_i$$

Vector Operations

- **Scalar or dot product** yields a scalar

$$\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta$$

where θ is the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$

- In index notation:

$$\mathbf{u}\cdot\mathbf{v} = \underbrace{u_i\hat{\mathbf{e}}_i}_{\mathbf{u}} \cdot \underbrace{\mathrm{v}_j\hat{\mathbf{e}}_j}_{\mathbf{v}} = u_i \mathrm{v}_j \underbrace{\hat{\mathbf{e}}_i\cdot\hat{\mathbf{e}}_j}_{\delta_{ij}} = u_i\mathrm{v}_j\underbrace{\delta_{ij}}_{\begin{cases} =0\,(i\neq j) \\ =1\,(j=i)\end{cases}} = u_i\mathrm{v}_i \left(= \sum_{i=1}^{i=3} u_i \mathrm{v}_i \right) = [\mathbf{u}]^T[\mathbf{v}]$$

- Norm of a vector

$$\|\mathbf{u}\| = (\mathbf{u}\cdot\mathbf{u})^{1/2} = (u_i u_i)^{1/2}$$

$$\Rightarrow \quad \|\mathbf{u}\|^2 = \mathbf{u}\cdot\mathbf{u} = u_i\hat{\mathbf{e}}_i\cdot u_j\hat{\mathbf{e}}_j = u_i u_j \delta_{ij} = u_i u_i$$

Vector Operations

- Some **properties** of the **scalar or dot product**

$$\mathbf{u}\cdot\mathbf{v}=\mathbf{v}\cdot\mathbf{u}$$

$$\mathbf{u}\cdot\mathbf{0}=0$$

$$\mathbf{u}\cdot(\alpha\mathbf{v}+\beta\mathbf{w})=\alpha(\mathbf{u}\cdot\mathbf{v})+\beta(\mathbf{u}\cdot\mathbf{w}) \Rightarrow \text{Linear operator}$$

$$\mathbf{u}\cdot\mathbf{u}>0 \iff \mathbf{u}\neq\mathbf{0}$$

$$\mathbf{u}\cdot\mathbf{u}=0 \iff \mathbf{u}=\mathbf{0}$$

$$\mathbf{u}\cdot\mathbf{v}=0,\quad \mathbf{u}\neq\mathbf{0},\quad \mathbf{v}\neq\mathbf{0} \iff \mathbf{u}\perp\mathbf{v}$$

Vector Operations

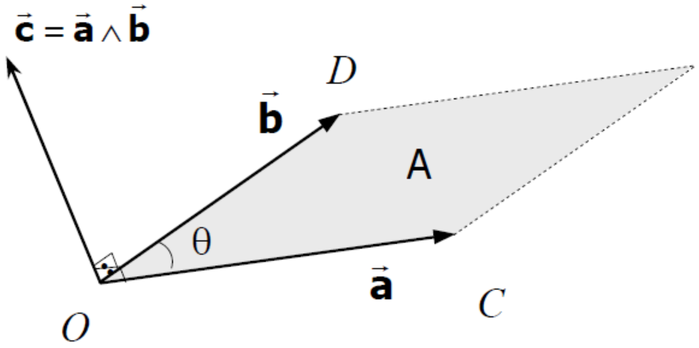
- **Vector product (or cross product)** yields another vector

$$\mathbf{c} = \mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$$

$$|\mathbf{c}| = |\mathbf{a}||\mathbf{b}|\sin\theta$$

where θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$

$$\boxed{0 \le \theta \le \pi}$$



- In index notation:

$$\mathbf{c} = c_i \hat{\mathbf{e}}_i = \epsilon_{ijk} a_j b_k \hat{\mathbf{e}}_i \quad \Rightarrow \quad c_i = \epsilon_{ijk} a_j b_k \quad i \in \{1,2,3\}$$

$$\mathbf{c} = \overbrace{\underbrace{(a_2 b_3 - a_3 b_2)}_{\underbrace{\epsilon_{123}}_{=1} a_2 b_3 + \underbrace{\epsilon_{132}}_{=-1} a_3 b_2}}^{i=1} \hat{\mathbf{e}}_1 + \overbrace{\underbrace{(a_3 b_1 - a_1 b_3)}_{\underbrace{\epsilon_{231}}_{=1} a_3 b_1 + \underbrace{\epsilon_{213}}_{=-1} a_1 b_3}}^{i=2} \hat{\mathbf{e}}_2 + \overbrace{\underbrace{(a_1 b_2 - a_2 b_1)}_{\underbrace{\epsilon_{312}}_{=1} a_1 b_2 + \underbrace{\epsilon_{321}}_{=-1} a_2 b_1}}^{i=3} \hat{\mathbf{e}}_3 \overset{symb}{=} \det \begin{bmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

Vector Operations

- Some **properties** of the **vector or cross product**

$$\mathbf{u}\times\mathbf{v} = -\left(\mathbf{v}\times\mathbf{u}\right)$$

$$\mathbf{u}\times\mathbf{v} = \mathbf{0}, \quad \mathbf{u}\neq\mathbf{0}, \mathbf{v}\neq\mathbf{0} \quad \Longleftrightarrow \quad \mathbf{u} \parallel \mathbf{v}$$

$$\mathbf{u}\times\left(a\mathbf{v}+b\mathbf{w}\right) = a\mathbf{u}\times\mathbf{v} + b\mathbf{u}\times\mathbf{w} \quad \Rightarrow$$ Linear operator

Vector Operations

- **Tensor product (or open or dyadic product) of two vectors:**

  $$\mathbf{A} = \mathbf{u} \otimes \mathbf{v} \equiv \mathbf{uv}$$

  Also known as the **dyad** of the vectors $\mathbf{u}$ and $\mathbf{v}$, which results in a 2nd order tensor $\mathbf{A}$.

  - Deriving the tensor product along an orthonormal basis $\{\hat{\mathbf{e}}_i\}$:

    $$\mathbf{A} = (\mathbf{u} \otimes \mathbf{v}) = (u_i \hat{\mathbf{e}}_i) \otimes (\mathrm{v}_j \hat{\mathbf{e}}_j) = u_i \mathrm{v}_j (\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j) = A_{ij} (\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j)$$

$$\boxed{[\mathbf{A}]_{ij} = A_{ij} = [\mathbf{u} \otimes \mathbf{v}]_{ij} = u_i \mathrm{v}_j \quad i, j \in \{1,2,3\}}$$

  - In matrix notation:

    $$[\mathbf{u} \otimes \mathbf{v}] = [\mathbf{u}][\mathbf{v}]^T = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \begin{bmatrix} \mathrm{v}_1 & \mathrm{v}_2 & \mathrm{v}_3 \end{bmatrix} = \begin{bmatrix} u_1\mathrm{v}_1 & u_1\mathrm{v}_2 & u_1\mathrm{v}_3 \\ u_2\mathrm{v}_1 & u_2\mathrm{v}_2 & u_2\mathrm{v}_3 \\ u_3\mathrm{v}_1 & u_3\mathrm{v}_2 & u_3\mathrm{v}_3 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

Vector Operations

- Some **properties** of the **open product:**

$$(\mathbf{u}\otimes\mathbf{v})\neq(\mathbf{v}\otimes\mathbf{u})$$

$$(\mathbf{u}\otimes\mathbf{v})\cdot\mathbf{w}=\mathbf{u}\otimes(\mathbf{v}\cdot\mathbf{w})=\mathbf{u}(\mathbf{v}\cdot\mathbf{w})=(\mathbf{v}\cdot\mathbf{w})\mathbf{u}$$

$$\mathbf{u}\otimes(\alpha\mathbf{v}+\beta\mathbf{w})=\alpha\mathbf{u}\otimes\mathbf{v}+\beta\mathbf{u}\otimes\mathbf{w} \Rightarrow$$ Linear operator

$$(\mathbf{u}\otimes\mathbf{v})(\mathbf{w}\otimes\mathbf{x})=(\mathbf{u}\otimes\mathbf{x})(\mathbf{v}\cdot\mathbf{w})$$

$$\mathbf{u}\cdot(\mathbf{v}\otimes\mathbf{w})=(\mathbf{u}\cdot\mathbf{v})\otimes\mathbf{w}=(\mathbf{u}\cdot\mathbf{v})\mathbf{w}=\mathbf{w}(\mathbf{u}\cdot\mathbf{v})$$

Example

- Prove the following property of the tensor product is true:

$$\mathbf{u}\cdot\left(\mathbf{v}\otimes\mathbf{w}\right)=\left(\mathbf{u}\cdot\mathbf{v}\right)\otimes\mathbf{w}$$

Example - Solution

$$\underbrace{\underbrace{\mathbf{u}}_{\text{vector}} \cdot \underbrace{(\mathbf{v} \otimes \mathbf{w})}_{\text{2nd order tensor (matrix)}}}_{\text{1st order tensor (vector)}} = \underbrace{\underbrace{(\mathbf{u} \cdot \mathbf{v})}_{\text{scalar}} \otimes \underbrace{\mathbf{w}}_{\text{vector}}}_{\text{1st order tensor (vector)}} \Rightarrow \mathbf{c}$$

$$c_k = \left[\mathbf{u} \cdot (\mathbf{v} \otimes \mathbf{w})\right]_k = [\mathbf{u}]_i \left[(\mathbf{v} \otimes \mathbf{w})\right]_{ik} = u_i \left(\mathrm{v}_i \mathrm{w}_k\right) = u_i \mathrm{v}_i \mathrm{w}_k \Rightarrow$$

k-component of vector $\mathbf{c}$

$$c_k = \left[(\mathbf{u} \cdot \mathbf{v}) \otimes \mathbf{w}\right]_k = u_i \mathrm{v}_i \otimes [\mathbf{w}]_k = u_i \mathrm{v}_i \mathrm{w}_k \Rightarrow$$

k-component of vector $\mathbf{c}$

$$\mathbf{c} = u_i \mathrm{v}_j \mathrm{w}_k \hat{\mathbf{e}}_k = \left[\mathbf{u} \cdot (\mathbf{v} \otimes \mathbf{w})\right]_k \hat{\mathbf{e}}_k = \left[(\mathbf{u} \cdot \mathbf{v}) \otimes \mathbf{w}\right]_k \hat{\mathbf{e}}_k$$

Example – Solution

$$\mathbf{u}\cdot(\mathbf{v}\otimes\mathbf{w})=(\mathbf{u}\cdot\mathbf{v})\otimes\mathbf{w}$$

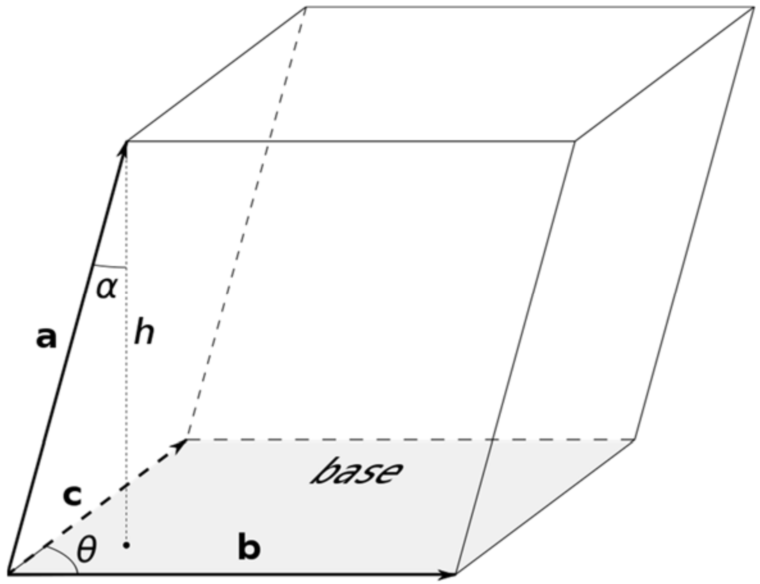
$$\mathbf{u}\cdot(\mathbf{v}\otimes\mathbf{w})=u_i\hat{\mathbf{e}}_i\cdot(\mathrm{v}_j\hat{\mathbf{e}}_j\otimes\mathrm{w}_k\hat{\mathbf{e}}_k)=u_i\hat{\mathbf{e}}_i\cdot(\mathrm{v}_j\mathrm{w}_k\hat{\mathbf{e}}_j\otimes\hat{\mathbf{e}}_k)=u_i\mathrm{v}_j\mathrm{w}_k(\hat{\mathbf{e}}_i\cdot\hat{\mathbf{e}}_j\otimes\hat{\mathbf{e}}_k)$$

$$(\mathbf{u}\cdot\mathbf{v})\otimes\mathbf{w}=(u_i\hat{\mathbf{e}}_i\cdot\mathrm{v}_j\hat{\mathbf{e}}_j)\otimes\mathrm{w}_k\hat{\mathbf{e}}_k=(u_i\mathrm{v}_j\hat{\mathbf{e}}_i\cdot\hat{\mathbf{e}}_j)\otimes\mathrm{w}_k\hat{\mathbf{e}}_k=u_i\mathrm{v}_j\mathrm{w}_k(\hat{\mathbf{e}}_i\cdot\hat{\mathbf{e}}_j\otimes\hat{\mathbf{e}}_k)$$

Vector Operations

- **Triple scalar or box product**

$$
\begin{aligned}
V &= \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) \\
&= \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) \\
&= |\mathbf{a}|\cos\alpha\,|\mathbf{b}\times\mathbf{c}| \\
&= \underbrace{|\mathbf{a}|\cos\alpha}_{\text{height}}\ \underbrace{|\mathbf{b}||\mathbf{c}|\sin\theta}_{\text{base area}}
\end{aligned}
$$



- In index notation:

$$V = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \epsilon_{ijk}a_i b_j c_k \quad \Rightarrow \quad V = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \det\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

Vector Operations

- **Triple vector product**

$$\mathbf{u}\times(\mathbf{v}\times\mathbf{w})=(\mathbf{u}\cdot\mathbf{w})\mathbf{v}-(\mathbf{u}\cdot\mathbf{v})\mathbf{w}$$

  - In index notation:

$$\mathbf{u}\times(\mathbf{v}\times\mathbf{w})=u_j\hat{\mathbf{e}}_j\times(\epsilon_{klm}\mathrm{v}_l\mathrm{w}_m\hat{\mathbf{e}}_k)=\epsilon_{ijk}u_j(\epsilon_{klm}\mathrm{v}_l\mathrm{w}_m)\hat{\mathbf{e}}_i=$$

$$=\epsilon_{ijk}\epsilon_{lmk}u_j\mathrm{v}_l\mathrm{w}_m\hat{\mathbf{e}}_i=(\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})u_j\mathrm{v}_l\mathrm{w}_m\hat{\mathbf{e}}_i=$$

$$=u_m\mathrm{v}_i\mathrm{w}_m\hat{\mathbf{e}}_i-u_l\mathrm{v}_l\mathrm{w}_i\hat{\mathbf{e}}_i$$

REMARK

$$\epsilon_{ijk}\epsilon_{pqk}=\delta_{ip}\delta_{jq}-\delta_{iq}\delta_{jp}$$

Summary

- **Scalar or dot product** yields a scalar

$$\mathbf{u}\cdot\mathbf{v} = [\mathbf{u}]^T[\mathbf{v}] = |\mathbf{u}||\mathbf{v}|\cos\theta \qquad \mathbf{u}\cdot\mathbf{v} = u_i \mathrm{v}_i$$

- **Vector or cross product** yields another vector

$$\mathbf{c} = \mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a} \qquad [\mathbf{a}\times\mathbf{b}]_i = c_i = \epsilon_{ijk} a_j b_k$$

- **Triple scalar or box product** yields a scalar

$$V = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = |\mathbf{a}|\cos\alpha\ |\mathbf{b}||\mathbf{c}|\sin\theta \qquad \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \epsilon_{ijk} a_i b_j c_k$$

- **Triple vector product** yields another vector

$$\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\mathbf{w} \qquad [\mathbf{u}\times(\mathbf{v}\times\mathbf{w})]_i = u_k \mathrm{w}_k \mathrm{v}_i - u_k \mathrm{v}_k \mathrm{w}_i$$

Tensor Operations

Tensor Algebra

Tensor Operations

- **Summation (only for equal order tensors)**

$$\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}=\mathbf{C} \quad \Rightarrow \quad C_{ij}=A_{ij}+B_{ij}$$

- **Scalar multiplication (scalar times tensor)**

$$\alpha\mathbf{A}=\mathbf{C} \quad \Rightarrow \quad C_{ij}=\alpha A_{ij}$$

Tensor Operations

- **Dot product (.) or single index contraction product**

$$\underbrace{\mathbf{A}}_{\substack{2^{nd}\\ \text{order}}} \cdot \underbrace{\mathbf{b}}_{\substack{1^{st}\\ \text{order}}} = \underbrace{\mathbf{c}}_{\substack{1^{st}\\ \text{order}}} \Rightarrow c_i = A_{ij} b_j \Rightarrow$$

Index "j" disappears (index contraction)

$$\underbrace{\mathcal{A}}_{\substack{3^{rd}\\ \text{order}}} \cdot \underbrace{\mathbf{b}}_{\substack{1^{st}\\ \text{order}}} = \underbrace{\mathbf{C}}_{\substack{2^{nd}\\ \text{order}}} \Rightarrow C_{ij} = \mathcal{A}_{ijk} b_k \Rightarrow$$

Index "k" disappears (index contraction)

$$\underbrace{\mathbf{A}}_{\substack{2^{nd}\\ \text{order}}} \cdot \underbrace{\mathbf{B}}_{\substack{2^{nd}\\ \text{order}}} = \underbrace{\mathbf{C}}_{\substack{2^{nd}\\ \text{order}}} \Rightarrow C_{ik} = A_{ij} B_{jk} \Rightarrow$$

Index "j" disappears (index contraction)

$$\mathbf{A} \cdot \mathbf{B} \neq \mathbf{B} \cdot \mathbf{A}$$

REMARK

$$\mathbf{A} \cdot \mathbf{A} = \mathbf{A}^2$$

Tensor Operations

- Some **properties:**

$$\mathbf{A}\cdot(\alpha\mathbf{b}+\beta\mathbf{c})=\alpha\mathbf{A}\cdot\mathbf{b}+\beta\mathbf{A}\cdot\mathbf{c} \quad \Rightarrow \quad \text{Linear operator}$$

- **2nd order unit** (or **identity**) **tensor**

$$\mathbf{1}\cdot\mathbf{u}=\mathbf{u}\cdot\mathbf{1}=\mathbf{u}$$

$$\begin{cases}\mathbf{1}=\delta_{ij}\mathbf{e}_j\otimes\mathbf{e}_i=\mathbf{e}_i\otimes\mathbf{e}_i\\ [1]_{ij}=\delta_{ij}\end{cases} \qquad [\mathbf{1}]=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

2nd Order Tensor Operations

- Some **properties:**

$$\mathbf{1}\cdot\mathbf{A}=\mathbf{A}=\mathbf{A}\cdot\mathbf{1}$$

$$\mathbf{A}\cdot\left(\mathbf{B}+\mathbf{C}\right)=\mathbf{A}\cdot\mathbf{B}+\mathbf{A}\cdot\mathbf{C}$$

$$\mathbf{A}\cdot\left(\mathbf{B}\cdot\mathbf{C}\right)=\left(\mathbf{A}\cdot\mathbf{B}\right)\cdot\mathbf{C}=\mathbf{A}\cdot\mathbf{B}\cdot\mathbf{C}$$

$$\mathbf{A}\cdot\mathbf{B}\neq\mathbf{B}\cdot\mathbf{A}$$

Example

- When does the relation $\mathbf{n} \cdot \mathbf{T} = \mathbf{T} \cdot \mathbf{n}$ hold true ?

Example - Solution

$$\underbrace{\underbrace{\mathbf{n}}_{\text{vector}} \cdot \underbrace{\mathbf{T}}_{\text{2nd order tensor}}}_{\text{vector}} = \mathbf{T} \cdot \mathbf{n} \quad \Rightarrow \quad \mathbf{c}$$

$$\left.\begin{array}{ll} \mathbf{c} = \mathbf{n} \cdot \mathbf{T} \quad \Rightarrow & c_k = n_i T_{ik} \\ \mathbf{c}^* = \mathbf{T} \cdot \mathbf{n} \quad \Rightarrow & c_k^* = T_{ki} n_i = n_i T_{ki} \end{array}\right\} \quad c_k = c_k^* \quad \text{if} \quad T_{ik} = T_{ki}$$

$$\mathbf{c} = c_k \hat{\mathbf{e}}_k = [\mathbf{n} \cdot \mathbf{T}]_k \, \hat{\mathbf{e}}_k = n_i T_{ik} \hat{\mathbf{e}}_k$$

$$\mathbf{c}^* = c_k \hat{\mathbf{e}}_k = [\mathbf{T} \cdot \mathbf{n}]_k \, \hat{\mathbf{e}}_k = n_i T_{ki} \hat{\mathbf{e}}_k$$

Example - Solution

$$\mathbf{n}\cdot\mathbf{T}=\mathbf{T}\cdot\mathbf{n}$$ ⇨ COMPACT NOTATION

$$n_i T_{ik} = T_{ki} n_i \qquad k\in\{1,2,3\}$$ ⇨ INDEX NOTATION

$$\underbrace{\left[\underbrace{[\mathbf{n}]^T[\mathbf{T}]}_{\mathbf{c}^T}\right]^T}_{\mathbf{c}} = \underbrace{[\mathbf{T}][\mathbf{n}]}_{\mathbf{c}}$$ ⇨ MATRIX NOTATION

Example - Solution

$$\underbrace{\left[\underbrace{[\mathbf{n}]^T\,[\mathbf{T}]}_{\mathbf{c}^T}\right]^T}_{\mathbf{c}} = \underbrace{[\mathbf{T}][\mathbf{n}]}_{\mathbf{c}}$$ ⇨ MATRIX NOTATION

$$\left[\begin{bmatrix} n_1, & n_2, & n_3 \end{bmatrix}\begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}\right]^T = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

$$\begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix}^T = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

2^{nd} Order Tensor Operations

- **Transpose**

$$[\mathbf{A}] = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \Rightarrow [\mathbf{A}^T] = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

$$\begin{cases} \left(\mathbf{A}^T\right)^T = \mathbf{A} \\ \left(\mathbf{A}\cdot\mathbf{B}\right)^T = \mathbf{B}^T\cdot\mathbf{A}^T \\ \left(\mathbf{u}\otimes\mathbf{v}\right)^T = \mathbf{v}\otimes\mathbf{u} \\ \left(\alpha\mathbf{A}+\beta\mathbf{B}\right)^T = \alpha\mathbf{A}^T+\beta\mathbf{B}^T \end{cases}$$

$$[\mathbf{A}^T]_{ij} = \mathbf{A}_{ji}$$

- **Trace** yields a scalar

$$\boxed{Tr\left(\mathbf{A}\right) = A_{ii} (= A_{11} + A_{22} + A_{33})}$$

$$Tr\left(\mathbf{a}\otimes\mathbf{b}\right) = Tr\left[a_i b_j\right] = a_i b_i = \mathbf{a}\cdot\mathbf{b}$$

- Some **properties:**

$$Tr\left(\mathbf{A}^T\right) = Tr\,\mathbf{A} \qquad Tr\left(\mathbf{A}+\mathbf{B}\right) = Tr\mathbf{A} + Tr\mathbf{B}$$

$$Tr\left(\alpha\mathbf{A}\right) = \alpha\, Tr\mathbf{A} \qquad Tr\left(\mathbf{A}\cdot\mathbf{B}\right) = Tr\left(\mathbf{B}\cdot\mathbf{A}\right)$$

2^{nd} Order Tensor Operations

- **Double index contraction** or **double (vertical) dot product** (:)

$$\underbrace{\mathbf{A}}_{\text{2nd order}} : \underbrace{\mathbf{B}}_{\text{2nd order}} = \underbrace{c}_{\substack{\text{zero order}\\ \text{(scalar)}}} \Rightarrow c = A_{ij} B_{ij} \Rightarrow$$

Indices "i,j" disappear (double index contraction)

$$\underbrace{\mathcal{A}}_{\text{3rd order}} : \underbrace{\mathbf{B}}_{\text{2nd order}} = \underbrace{\mathbf{c}}_{\text{1 st order}} \Rightarrow \mathbf{c}_{i} = \mathcal{A}_{ijk} B_{jk} \Rightarrow$$

Indices "j,k" disappear (double index contraction)

$$\underbrace{\mathbb{A}}_{\text{4th order}} : \underbrace{\mathbf{B}}_{\text{2nd order}} = \underbrace{\mathbf{C}}_{\text{2nd order}} \Rightarrow C_{ij} = {}_{ijkl} B_{kl} \Rightarrow$$

Indices "k,l" disappear (double index contraction)

  - **Indices contiguous to the double-dot** (:) operator get vertically repeated (contraction) and they **disappear** in the resulting tensor (4 order reduction of the sum of orders).

2nd Order Tensor Operations

- **Some properties**

$$\mathbf{A}:\mathbf{B} = Tr\left(\mathbf{A}^T\cdot\mathbf{B}\right) = Tr\left(\mathbf{B}^T\cdot\mathbf{A}\right) = Tr\left(\mathbf{A}\cdot\mathbf{B}^T\right) = Tr\left(\mathbf{B}\cdot\mathbf{A}^T\right) = \mathbf{B}:\mathbf{A}$$

$$\mathbf{1}:\mathbf{A} = Tr\mathbf{A} = \mathbf{A}:\mathbf{1}$$

$$\mathbf{A}:\left(\mathbf{B}\cdot\mathbf{C}\right) = \left(\mathbf{B}^T\cdot\mathbf{A}\right):\mathbf{C} = \left(\mathbf{A}\cdot\mathbf{C}^T\right):\mathbf{B}$$

$$\mathbf{A}:\left(\mathbf{u}\otimes\mathbf{v}\right) = \mathbf{u}\cdot\left(\mathbf{A}\cdot\mathbf{v}\right)$$

$$\left(\mathbf{u}\otimes\mathbf{v}\right):\left(\mathbf{w}\otimes\mathbf{x}\right) = \left(\mathbf{u}\cdot\mathbf{w}\right)\cdot\left(\mathbf{v}\cdot\mathbf{x}\right)$$

REMARK

$$\mathbf{A}:\mathbf{B} = \mathbf{C}:\mathbf{B} \not\Rightarrow \mathbf{A} = \mathbf{C}$$

2^{nd} Order Tensor Operations

- **Double index contraction** or **double (horizontal) dot product** $(\cdot\cdot)$

$$\underbrace{\mathbf{A}}_{2^{nd}\ \text{order}} \cdot\cdot \underbrace{\mathbf{B}}_{2^{nd}\ \text{order}} = \underbrace{c}_{\substack{\text{zero}\\ \text{order}\\ \text{(scalar)}}} \Rightarrow c = A_{ij} B_{ji} \Rightarrow$$

Indices "i,j" disappear (double index contraction)

$$\underbrace{\mathcal{A}}_{3^{rd}\ \text{order}} \cdot\cdot \underbrace{\mathbf{B}}_{2^{nd}\ \text{order}} = \underbrace{\mathbf{c}}_{1^{st}\ \text{order}} \Rightarrow \mathbf{c}_i = \mathcal{A}_{ijk} B_{kj} \Rightarrow$$

Indices "j,k" disappear (double index contraction)

$$\underbrace{\mathbb{A}}_{4^{th}\ \text{order}} \cdot\cdot \underbrace{\mathbf{B}}_{2^{nd}\ \text{order}} = \underbrace{\mathbf{C}}_{2^{nd}\ \text{order}} \Rightarrow C_{ij} = \quad {}_{ijkl} B_{lk} \Rightarrow$$

Indices "k,l" disappear (double index contraction)

  - **Indices contiguous to the double-dot** $(\cdot\cdot)$ operator get horizontally repeated (contraction) and they **disappear** in the resulting tensor (4 orders reduction of the sum of orders).

Tensor Operations

- **Norm** of a tensor is a non-negative real number defined by

$$\|\mathbf{A}\| = \left(\mathbf{A} : \mathbf{A}\right)^{1/2} = \left(A_{ij} A_{ij}\right)^{1/2} \geq 0$$

$$\mathbf{A} \cdot\cdot\, \mathbf{B} = Tr\left(\mathbf{A} \cdot \mathbf{B}\right) = Tr\left(\mathbf{B} \cdot \mathbf{A}\right) = \mathbf{B} \cdot\cdot\, \mathbf{A}$$

$$\mathbf{1} \cdot\cdot\, \mathbf{A} = Tr\mathbf{A} = \mathbf{A} \cdot\cdot\, \mathbf{1}$$

REMARK

$$\mathbf{A} : \mathbf{B} \neq \mathbf{A} \cdot\cdot\, \mathbf{B}$$

Unless one of the two tensors is symmetric.

Example

- Prove that:

$$\mathbf{A}:\mathbf{B} = Tr\left(\mathbf{A}^T \cdot \mathbf{B}\right)$$

$$\mathbf{A}\cdot\cdot\mathbf{B} = Tr\left(\mathbf{A}\cdot\mathbf{B}\right)$$

Example - Solution

$$c = Tr\left(\mathbf{A}^T \cdot \mathbf{B}\right) = \left[\mathbf{A}^T \cdot \mathbf{B}\right]_{kk} = \left[\mathbf{A}^T\right]_{ki} [\mathbf{B}]_{ik} = \underbrace{\mathrm{A}_{ik}\mathrm{B}_{ik}}_{k \to j} = \mathrm{A}_{ij}\mathrm{B}_{ij}$$

$$c = Tr\left(\mathbf{A} \cdot \mathbf{B}\right) = \left[\mathbf{A} \cdot \mathbf{B}\right]_{kk} = [\mathbf{A}]_{ki} [\mathbf{B}]_{ik} = \underbrace{\mathrm{A}_{ki}\mathrm{B}_{ik}}_{\substack{i \to j \\ k \to i}} = \mathrm{A}_{ij}\mathrm{B}_{ji}$$

2nd Order Tensor Operations

- **Determinant** yields a scalar

$$\det \mathbf{A} = \det[\mathbf{A}] = \det\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \epsilon_{ijk} A_{1i} A_{2j} A_{3k} = \frac{1}{6}\epsilon_{ijk}\epsilon_{pqr} A_{pi} A_{qj} A_{rk}$$

  - Some **properties:**

$$\det(\mathbf{A}\cdot\mathbf{B}) = \det \mathbf{A} \cdot \det \mathbf{B}$$

$$\det \mathbf{A}^T = \det \mathbf{A}$$

$$\det(\alpha \mathbf{A}) = \alpha^3 \det \mathbf{A}$$

REMARK

The tensor $\mathbf{A}$ is **SINGULAR** *if and only if* $\det \mathbf{A} = 0$.

$\mathbf{A}$ is **NONSINGULAR** *if* $\det \mathbf{A} \neq 0$.

- **Inverse**

There exists a **unique inverse** $\mathbf{A}^{-1}$ of $\mathbf{A}$ when $\mathbf{A}$ is nonsingular, which satisfies the reciprocal relation:

$$\begin{cases} \mathbf{A}\cdot\mathbf{A}^{-1} = \mathbf{1} = \mathbf{A}^{-1}\cdot\mathbf{A} \\ A_{ik} A^{-1}_{kj} = A^{-1}_{ik} A_{kj} = \delta_{ij} \qquad i,j,k \in \{1,2,3\} \end{cases}$$

2^{nd} Order Tensor Operations

- If $\mathbf{A}$ and $\mathbf{B}$ are **invertible,** the following **properties** apply:

$$(\mathbf{A}\cdot\mathbf{B})^{-1} = \mathbf{B}^{-1}\cdot\mathbf{A}^{-1}$$

$$\left(\mathbf{A}^{-1}\right)^{-1} = \mathbf{A}$$

$$(\alpha\mathbf{A})^{-1} = \frac{1}{\alpha}\mathbf{A}^{-1}$$

$$\left(\mathbf{A}^{-1}\right)^{T} = \left(\mathbf{A}^{T}\right)^{-1} = \mathbf{A}^{-T}$$

$$\mathbf{A}^{-2} = \mathbf{A}^{-1}\cdot\mathbf{A}^{-1}$$

$$\det\left(\mathbf{A}^{-1}\right) = (\det\mathbf{A})^{-1} = 1/\det\mathbf{A}$$

Example

- Prove that $\det \mathbf{A} = \epsilon_{ijk} A_{1i} A_{2j} A_{3k}$

Example - Solution

$$\det \mathbf{A} = \det \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

$$= A_{11}A_{22}A_{33} + A_{21}A_{32}A_{13} + A_{31}A_{12}A_{23} - A_{13}A_{22}A_{31} - A_{23}A_{32}A_{11} - A_{33}A_{12}A_{21}$$

Example - Solution

$$k=1 \qquad k=2 \qquad k=3$$

$$\epsilon_{ijk}A_{1i}A_{2j}A_{3k} = \cancel{\epsilon_{111}A_{11}A_{21}A_{31}} + \cancel{\epsilon_{112}A_{11}A_{21}A_{32}} + \cancel{\epsilon_{113}A_{11}A_{21}A_{33}} + \qquad j=1$$

$$i=1 \qquad + \cancel{\epsilon_{121}A_{11}A_{22}A_{31}} + \cancel{\epsilon_{122}A_{11}A_{22}A_{32}} + \overset{=1}{\epsilon_{123}}A_{11}A_{22}A_{33} + \qquad j=2$$

$$+ \cancel{\epsilon_{131}A_{11}A_{23}A_{31}} + \overset{=-1}{\epsilon_{132}}A_{11}A_{23}A_{32} + \cancel{\epsilon_{133}A_{11}A_{23}A_{33}} + \qquad j=3$$

$$+ \cancel{{}_{211}A_{12}A_{21}A_{31}} + \cancel{\epsilon_{212}A_{12}A_{21}A_{32}} + \overset{=-1}{\epsilon_{213}}A_{12}A_{21}A_{33} +$$

$$i=2 \qquad + \cancel{\epsilon_{221}A_{12}A_{22}A_{31}} + \cancel{\epsilon_{222}A_{12}A_{22}A_{32}} + \cancel{\epsilon_{223}A_{12}A_{22}A_{33}} +$$

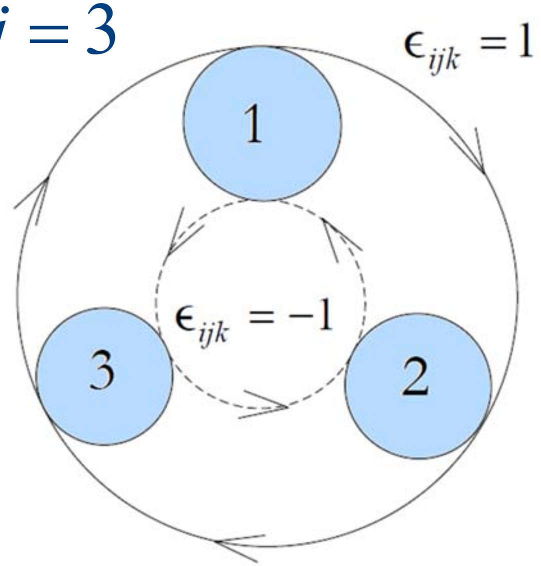
$$+ \overset{=1}{\epsilon_{231}}A_{12}A_{23}A_{31} + \cancel{\epsilon_{232}A_{12}A_{23}A_{32}} + \cancel{\epsilon_{233}A_{12}A_{23}A_{33}} +$$

$$+ \cancel{\epsilon_{311}A_{13}A_{21}A_{31}} + \overset{=1}{\epsilon_{312}}A_{13}A_{21}A_{32} + \cancel{\epsilon_{313}A_{13}A_{21}A_{33}} +$$

$$i=3 \qquad + \overset{=-1}{\epsilon_{321}}A_{13}A_{22}A_{31} + \cancel{\epsilon_{322}A_{13}A_{22}A_{32}} + \cancel{\epsilon_{323}A_{13}A_{22}A_{33}} +$$

$$+ \cancel{\epsilon_{331}A_{13}A_{23}A_{31}} + \cancel{\epsilon_{332}A_{13}A_{23}A_{32}} + \cancel{\epsilon_{333}A_{13}A_{23}A_{33}} =$$

$$= A_{11}A_{22}A_{33} + A_{12}A_{23}A_{31} + A_{13}A_{21}A_{32} - A_{13}A_{22}A_{31} - A_{12}A_{21}A_{33} - A_{11}A_{23}A_{32}$$



Summary - Tensor Operations

- **Dot product** – contraction of one index:

$$\boxed{\begin{aligned} &c = \mathbf{u}\cdot\mathbf{v} = u_i \mathrm{v}_i \\ &[\mathbf{C}]_{ij} = [\mathbf{A}\cdot\mathbf{B}]_{ij} = A_{ik}B_{kj} = C_{ij} \end{aligned}}$$

$$[\mathbf{D}]_{ij} = [\mathbf{B}\cdot\mathbf{A}]_{ij} = B_{ik}A_{kj} = D_{ij}$$

$$[\mathbf{c}]_j = [\mathbf{u}\cdot\mathbf{A}]_j = u_i A_{ij} = c_j$$

$$[\mathbf{d}]_i = [\mathbf{A}\cdot\mathbf{u}]_i = A_{ij}u_j = d_i$$

$$[\mathfrak{C}]_{ijk} = [\mathbf{A}\cdot\mathfrak{B}]_{ijk} = A_{im}\mathcal{B}_{mjk} = \mathcal{C}_{ijk}$$

$$[\mathfrak{D}]_{ijk} = [\mathfrak{B}\cdot\mathbf{A}]_{ijk} = \mathcal{B}_{ijm}A_{mk} = \mathcal{D}_{ijk}$$

$$[\mathbb{C}]_{ijkl} = [\mathcal{A}\cdot\mathfrak{B}]_{ijkl} = \mathcal{A}_{ijm}\mathcal{B}_{mkl} = \mathcal{C}_{ijkl}$$

$$[\mathbb{D}]_{ijkl} = [\mathfrak{B}\cdot\mathcal{A}]_{ijkl} = \mathcal{B}_{ijm}\mathcal{A}_{mkl} = \mathbb{D}_{ijkl}$$

Summary - Tensor Operations

- **Double dot product** – contraction of two indices:

$$\boxed{c = \mathbf{A} : \mathbf{B} = A_{ij} B_{ij}}$$

$$[\mathbf{C}]_{ij} = \left[\mathcal{A} : \mathcal{B} \right]_{ij} = \mathcal{A}_{ikm} \mathcal{B}_{kmj} = C_{ij}$$

$$[\mathbf{D}]_{ij} = \left[\mathcal{B} : \mathcal{A} \right]_{ij} = \mathcal{B}_{ikm} \mathcal{A}_{kmj} = D_{ij}$$

$$[\mathbf{c}]_k = [\mathbf{A} : \mathcal{B}]_k = A_{ij} \mathcal{B}_{ijk} = c_k$$

$$[\mathbf{d}]_i = [\mathcal{B} : \mathbf{A}]_i = \mathcal{B}_{ijk} A_{jk} = d_i$$

$$[\mathbb{C}]_{ijkl} = [\mathbb{A} : \mathbb{B}]_{ijkl} = \mathbb{A}_{ijmp} \mathbb{B}_{mpkl} = \mathbb{C}_{ijkl}$$

$$[\mathbb{D}]_{ijkl} = [\mathbb{B} : \mathbb{A}]_{ijkl} = \mathbb{B}_{ijmp} \mathbb{A}_{mpij} = \mathbb{D}_{ijkl}$$

$$[\mathfrak{C}]_{ijk} = [\mathbb{A} : \mathfrak{B}]_{ijk} = \mathbb{A}_{ijlm} \mathcal{B}_{lmk} = \mathcal{C}_{ijk}$$

$$[\mathfrak{D}]_{ijk} = [\mathfrak{B} : \mathbb{A}]_{ijk} = \mathcal{B}_{ilm} \mathbb{A}_{lmjk} = \mathcal{D}_{ijk}$$

Summary - Tensor Operations

- **Transposed double dot product** – contraction of two indexes:

$$\boxed{c = \mathbf{A}\cdot\cdot\mathbf{B} = A_{ij}B_{ji}}$$

$$[\mathbf{C}]_{ij} = [\mathcal{A}\cdot\cdot\mathcal{B}]_{ij} = \mathcal{A}_{ikm}\mathcal{B}_{mkj} = C_{ij}$$

$$[\mathbf{D}]_{ij} = [\mathcal{B}\cdot\cdot\mathcal{A}]_{ij} = \mathcal{B}_{ikm}\mathcal{A}_{mkj} = D_{ij}$$

$$[\mathbf{c}]_{k} = [\mathbf{A}\cdot\cdot\mathcal{B}]_{k} = A_{ij}\mathcal{B}_{jik} = c_{k}$$

$$[\mathbf{d}]_{i} = [\mathcal{B}\cdot\cdot\mathbf{A}]_{i} = \mathcal{B}_{ijk}A_{kj} = d_{i}$$

$$[\mathbb{C}]_{ijkl} = [\mathbb{A}\cdot\cdot\mathbb{B}]_{ijkl} = \mathbb{A}_{ijmp}\mathbb{B}_{pmkl} = \mathbb{C}_{ijkl}$$

$$[\mathbb{D}]_{ijkl} = [\mathbb{B}\cdot\cdot\mathbb{A}]_{ijkl} = \mathbb{B}_{ijmp}\mathbb{A}_{pmij} = \mathbb{D}_{ijkl}$$

$$[\mathfrak{C}]_{ijk} = [\mathbb{A}\cdot\cdot\mathcal{B}]_{ijk} = \mathbb{A}_{ijlm}\mathcal{B}_{mlk} = \mathcal{C}_{ijk}$$

$$[\mathfrak{D}]_{ijk} = [\mathcal{B}\cdot\cdot\mathbb{A}]_{ijk} = \mathcal{B}_{ilm}\mathbb{A}_{mljk} = \mathcal{D}_{ijk}$$

Summary - Tensor Operations

- **Open product** – expansion of indexes:

$$\boxed{\begin{aligned}[\mathbf{A}]_{ij} &= [\mathbf{u}\otimes\mathbf{v}]_{ij} = u_i \mathrm{v}_j = A_{ij}\\ [\mathbb{C}]_{ijkl} &= [\mathbf{A}\otimes\mathbf{B}]_{ijkl} = A_{ij}B_{kl} = \mathcal{C}_{ijkl}\end{aligned}}$$

$$[\mathbf{B}]_{ij} = [\mathbf{v}\otimes\mathbf{u}]_{ij} = \mathrm{v}_i u_j = B_{ij}$$

$$[\mathbb{D}]_{ijkl} = [\mathbf{B}\otimes\mathbf{A}]_{ijkl} = B_{ij}A_{kl} = \mathbb{D}_{ijkl}$$

$$[\mathfrak{C}]_{ijk} = [\mathbf{u}\otimes\mathbf{A}]_{ijk} = u_i A_{jk} = \mathcal{C}_{ijk}$$

$$[\mathfrak{D}]_{ijk} = [\mathbf{A}\otimes\mathbf{u}]_{ijk} = A_{ij}u_k = \mathcal{D}_{ijk}$$

$$[\mathfrak{C}]_{ijklm} = [\mathbf{A}\otimes\mathcal{B}]_{ijklm} = A_{ij}\mathcal{B}_{klm} = \mathfrak{C}_{ijklm}$$

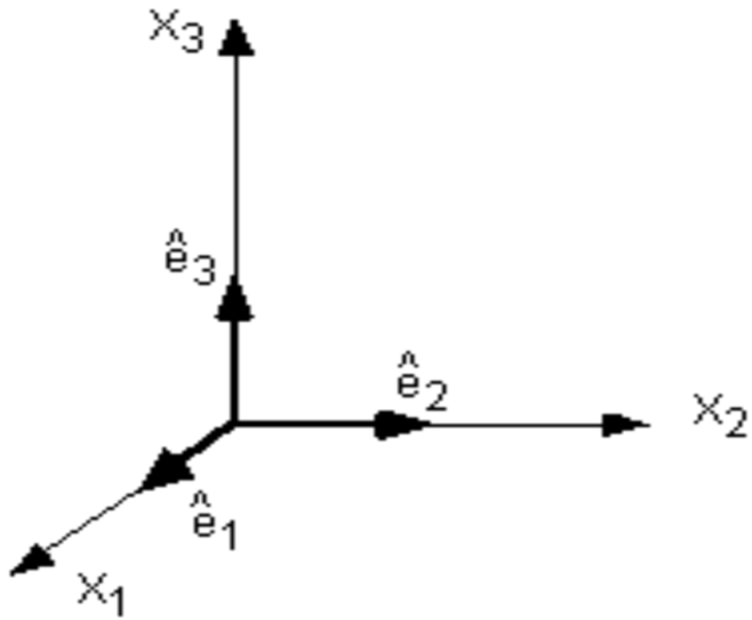
$$[\mathfrak{D}]_{ijklm} = [\mathcal{B}\otimes\mathbf{A}]_{ijklm} = \mathcal{B}_{ijk}A_{lm} = \mathfrak{D}_{ijklm}$$

Differential Operators

Tensor Algebra

Differential Operators

- A **differential operator** is a mapping that transforms a field $\mathbf{v}(\mathbf{x}), \mathbf{A}(\mathbf{x})\ldots$ into another field by means of partial derivatives.
  - The mapping is typically understood to be linear.
  - Examples:
    - Nabla operator
    - Gradient
    - Divergence
    - Rotation
    - …



Nabla Operator

- The **Nabla operator** ∇ is a differential operator "symbolically" defined as:

$$\boxed{\nabla \overset{symbolic}{=} \frac{\partial}{\partial \mathbf{x}} \overset{symb.}{=} \frac{\partial}{\partial x_i} \hat{\mathbf{e}}_i}$$

- In Cartesian coordinates, it can be used as a (symbolic) vector on its own:

$$[\nabla] \overset{symb.}{=} \begin{bmatrix} \dfrac{\partial}{\partial x_1} \\ \dfrac{\partial}{\partial x_2} \\ \dfrac{\partial}{\partial x_3} \end{bmatrix}$$

Gradient

- The **gradient** (or **open product of Nabla)** is a differential operator defined as:
  - Gradient of a scalar field $\Phi(\mathbf{x})$:
    - Yields a vector

$$\begin{cases} \left[\nabla\Phi\right]_i = \left[\nabla\otimes\Phi\right]_i = \left[\nabla\right]_i \Phi \overset{symb.}{=} \dfrac{\partial}{\partial x_i}\Phi = \dfrac{\partial\Phi}{\partial x_i} \qquad i\in\{1,2,3\} \\ \nabla\Phi = \left[\nabla\Phi\right]_i \hat{\mathbf{e}}_i = \dfrac{\partial\Phi}{\partial x_i}\hat{\mathbf{e}}_i \end{cases}$$

$$\boxed{\nabla\Phi = \frac{\partial\Phi}{\partial x_i}\hat{\mathbf{e}}_i}$$

  - Gradient of a vector field $\mathbf{v}(\mathbf{x})$:
    - Yields a 2nd order tensor

$$\begin{cases} \left[\nabla\otimes\mathbf{v}\right]_{ij} = \left[\nabla\right]_i \left[\mathbf{v}\right]_j \overset{symb.}{=} \dfrac{\partial}{\partial x_i}\mathrm{v}_j = \dfrac{\partial \mathrm{v}_j}{\partial x_i} \qquad i,j\in\{1,2,3\} \\ \nabla\mathbf{v} = \nabla\otimes\mathbf{v} = \left[\nabla\otimes\mathbf{v}\right]_{ij} \hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j = \dfrac{\partial \mathrm{v}_j}{\partial x_i}\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j \end{cases}$$

$$\boxed{\nabla\mathbf{v} = \frac{\partial \mathrm{v}_j}{\partial x_i}\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j}$$

Gradient

- Gradient of a 2nd order tensor field $\mathbf{A}(\mathbf{x})$:
  - Yields a 3rd order tensor

$$\begin{cases} [\nabla \mathbf{A}]_{ijk} = [\nabla \otimes \mathbf{A}]_{ijk} = [\nabla]_i [\mathbf{A}]_{jk} \overset{symb.}{=} \dfrac{\partial}{\partial x_i} \mathrm{A}_{jk} = \dfrac{\partial \mathrm{A}_{jk}}{\partial x_i} \qquad i,j,k \in \{1,2,3\} \\ \nabla \mathbf{A} = \nabla \otimes \mathbf{A} = [\nabla \otimes \mathbf{A}]_{ijk} \, \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k = \dfrac{\partial \mathrm{A}_{jk}}{\partial x_i} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k \end{cases}$$

$$\boxed{\nabla \mathbf{A} = \frac{\partial \mathrm{A}_{jk}}{\partial x_i} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k}$$

Divergence

- The **divergence** (or **dot product of Nabla)** is a differential operator defined as :
  - Divergence of a vector field $\mathbf{v}(\mathbf{x})$:
    - Yields a scalar

$$\nabla \cdot \mathbf{v} = [\nabla]_i [\mathbf{v}]_i \overset{symb.}{=} \frac{\partial}{\partial x_i} \mathrm{v}_i = \frac{\partial \mathrm{v}_i}{\partial x_i}$$

$$\boxed{\nabla \cdot \mathbf{v} = \frac{\partial \mathrm{v}_i}{\partial x_i}}$$

  - Divergence of a 2nd order tensor $\mathbf{A}(\mathbf{x})$:
    - Yields a vector

$$\begin{cases} [\nabla \cdot \mathbf{A}]_j = [\nabla]_i [\mathbf{A}]_{ij} \overset{symb.}{=} \dfrac{\partial}{\partial x_i} \mathrm{A}_{ij} = \dfrac{\partial \mathrm{A}_{ij}}{\partial x_i} \qquad j \in \{1,2,3\} \\ \nabla \cdot \mathbf{A} = [\nabla \cdot \mathbf{A}]_j \, \hat{\mathbf{e}}_j = \dfrac{\partial \mathrm{A}_{ij}}{\partial x_i} \hat{\mathbf{e}}_j \end{cases}$$

$$\boxed{\nabla \cdot \mathbf{A} = \frac{\partial \mathrm{A}_{ij}}{\partial x_i} \hat{\mathbf{e}}_j}$$

Divergence

- The divergence can only be performed on tensors of order 1 or higher.

- If $\nabla \cdot \mathbf{v} = 0$, the vector field $\mathbf{v}(\mathbf{x})$ is said to be **solenoid** (or **divergence-free**).

Rotation

- The **rotation** or **curl** (or **vector product of Nabla**) is a differential operator defined as:
  - Rotation of a vector field $\mathbf{v}(\mathbf{x})$:
    - Yields a vector

$$\begin{cases} [\nabla\times\mathbf{v}]_i \overset{symb.}{=} \epsilon_{ijk}[\nabla]_j[\mathbf{v}]_k \overset{symb.}{=} \epsilon_{ijk}\dfrac{\partial}{\partial x_j}\mathrm{v}_k = \epsilon_{ijk}\dfrac{\partial \mathrm{v}_k}{\partial x_j} & i\in\{1,2,3\} \\ \nabla\times\mathbf{v} = [\nabla\times\mathbf{v}]_i\,\hat{\mathbf{e}}_i = \epsilon_{ijk}\dfrac{\partial \mathrm{v}_k}{\partial x_j}\hat{\mathbf{e}}_i \end{cases}$$

$$\boxed{\nabla\times\mathbf{v} = \epsilon_{ijk}\frac{\partial \mathrm{v}_k}{\partial x_j}\hat{\mathbf{e}}_i}$$

  - Rotation of a 2nd order tensor $\mathbf{A}(\mathbf{x})$:
    - Yields a 2nd order tensor

$$\begin{cases} [\nabla\times\mathbf{A}]_{il} \overset{symb.}{=} \epsilon_{ijk}\dfrac{\partial}{\partial x_j}\mathrm{A}_{kl} = \epsilon_{ijk}\dfrac{\partial \mathrm{A}_{kl}}{\partial x_j} & i,j,k\in\{1,2,3\} \\ \nabla\times\mathbf{A} = [\nabla\times\mathbf{A}]_{il}\,\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_l = \epsilon_{ijk}\dfrac{\partial \mathrm{A}_{kl}}{\partial x_j}\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_l \end{cases}$$

$$\boxed{\nabla\times\mathbf{A} = \epsilon_{ijk}\frac{\partial \mathrm{A}_{kl}}{\partial x_j}\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_l}$$

Rotation

- The rotation can only be performed on tensors of order 1 or higher.

- If $\nabla \times \mathbf{v} = 0$, the vector field $\mathbf{v}(\mathbf{x})$ is said to be **irrotational** (or **curl-free**).

Differential Operators - Summary

	scalar field $\Phi(\mathbf{x})$	vector field $\mathbf{v}(\mathbf{x})$	2^{nd} order tensor $\mathbf{A}(\mathbf{x})$
GRADIENT	$[\nabla \otimes \Phi]_i = [\nabla \Phi]_i = \dfrac{\partial \Phi}{\partial x_i}$	$[\nabla \otimes \mathbf{v}]_{ij} = [\nabla \mathbf{v}]_{ij} = \dfrac{\partial v_j}{\partial x_i}$	$[\nabla \otimes \mathbf{A}]_{ijk} = [\nabla \mathbf{A}]_{ijk} = \dfrac{\partial A_{jk}}{\partial x_i}$
DIVERGENCE		$\nabla \cdot \mathbf{v} = \dfrac{\partial v_i}{\partial x_i}$	$[\nabla \cdot \mathbf{A}]_j = \dfrac{\partial A_{ij}}{\partial x_i}$
ROTATION		$[\nabla \times \mathbf{v}]_i = \epsilon_{ijk} \dfrac{\partial v_k}{\partial x_j}$	$[\nabla \times \mathbf{A}]_{il} = \epsilon_{ijk} \dfrac{\partial A_{kl}}{\partial x_j}$

Example

- Given the vector $\mathbf{v} = \mathbf{v}(\mathbf{x}) = x_1 x_2 x_3 \hat{\mathbf{e}}_1 + x_1 x_2 \hat{\mathbf{e}}_2 + x_1 \hat{\mathbf{e}}_3$ determine $\nabla \cdot \mathbf{v}, \quad \nabla \times \mathbf{v}, \quad \nabla \mathbf{v}.$

Example - Solution

$$\mathbf{v} = \mathbf{v}(\mathbf{x}) = x_1 x_2 x_3 \hat{\mathbf{e}}_1 + x_1 x_2 \hat{\mathbf{e}}_2 + x_1 \hat{\mathbf{e}}_3 \Rightarrow [\mathbf{v}] = \begin{bmatrix} x_1 x_2 x_3 \\ x_1 x_2 \\ x_1 \end{bmatrix}$$

- Divergence:

$$\boxed{\nabla \cdot \mathbf{v} = \frac{\partial \mathrm{v}_i}{\partial x_i}}$$

$$\nabla \cdot \mathbf{v} = \frac{\partial \mathrm{v}_i}{\partial x_i} = \frac{\partial \mathrm{v}_1}{\partial x_1} + \frac{\partial \mathrm{v}_2}{\partial x_2} + \frac{\partial \mathrm{v}_3}{\partial x_3} = x_2 x_3 + x_1$$

Example - Solution

- Divergence:

$$\boxed{\nabla \cdot \mathbf{v} = \frac{\partial \mathrm{v}_i}{\partial x_i}}$$

$$\boxed{[\mathbf{v}] = \begin{bmatrix} x_1 x_2 x_3 \\ x_1 x_2 \\ x_1 \end{bmatrix}}$$

  - In matrix notation:

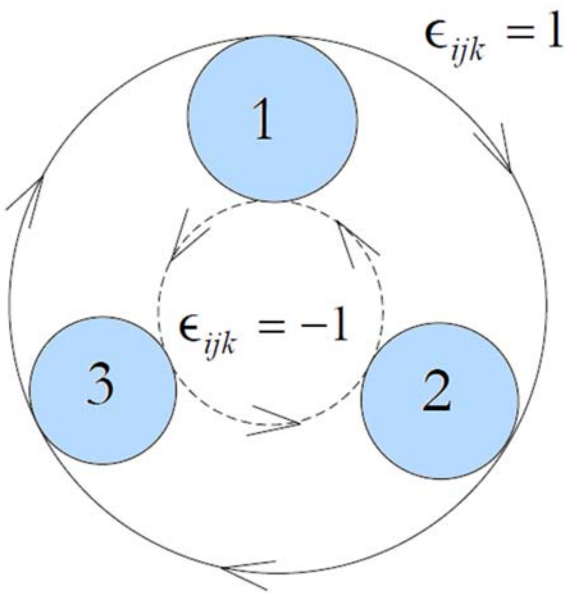
$$\nabla \cdot \mathbf{v} \overset{symb}{=} [\nabla]^T [\mathbf{v}] \overset{symb}{=} \underbrace{\begin{bmatrix} \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \end{bmatrix}^T}_{1\times 3} \underbrace{\begin{bmatrix} x_1 x_2 x_3 \\ x_1 x_2 \\ x_1 \end{bmatrix}}_{3\times 1} \overset{symb}{=}$$

$$\overset{symb}{=} \underbrace{\frac{\partial}{\partial x_1} x_1 x_2 x_3 + \frac{\partial}{\partial x_2} x_1 x_2 + \frac{\partial}{\partial x_3} x_1}_{1\times 1} = \frac{\partial (x_1 x_2 x_3)}{\partial x_1} + \frac{\partial (x_1 x_2)}{\partial x_2} + \frac{\partial x_1}{\partial x_3} = x_2 x_3 + x_1$$

Example - Solution

- Rotation:

$$\boxed{[\nabla \times \mathbf{v}]_i = \epsilon_{ijk} \frac{\partial \mathrm{v}_k}{\partial x_j}}$$



$$\boxed{[\mathbf{v}] = \begin{bmatrix} x_1 x_2 x_3 \\ x_1 x_2 \\ x_1 \end{bmatrix}}$$

  - In index notation:

$$[\nabla \times \mathbf{v}]_i = \epsilon_{ijk} \frac{\partial \mathrm{v}_k}{\partial x_j} =$$

$$= \epsilon_{i12} \frac{\partial \mathrm{v}_2}{\partial x_1} + \epsilon_{i13} \frac{\partial \mathrm{v}_3}{\partial x_1} + \epsilon_{i21} \frac{\partial \mathrm{v}_1}{\partial x_2} + \epsilon_{i23} \frac{\partial \mathrm{v}_3}{\partial x_2} + \epsilon_{i31} \frac{\partial \mathrm{v}_1}{\partial x_3} + \epsilon_{i32} \frac{\partial \mathrm{v}_2}{\partial x_3} =$$

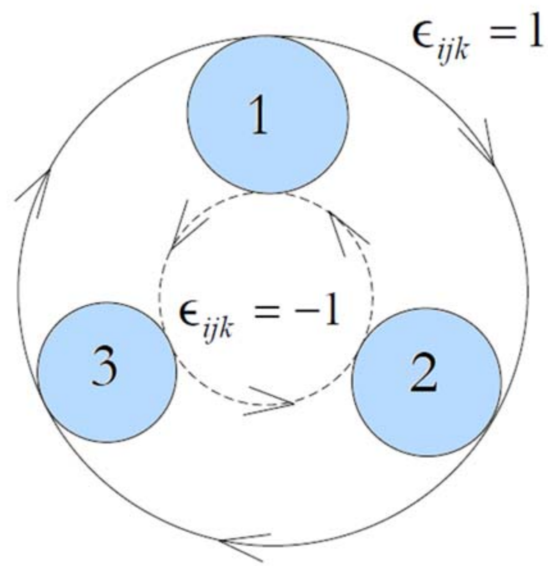
Example - Solution

- Rotation: $$[\nabla\times\mathbf{v}]_i = \epsilon_{i12}\frac{\partial \mathrm{v}_2}{\partial x_1} + \epsilon_{i13}\frac{\partial \mathrm{v}_3}{\partial x_1} + \epsilon_{i21}\frac{\partial \mathrm{v}_1}{\partial x_2} + \epsilon_{i23}\frac{\partial \mathrm{v}_3}{\partial x_2} + \epsilon_{i31}\frac{\partial \mathrm{v}_1}{\partial x_3} + \epsilon_{i32}\frac{\partial \mathrm{v}_2}{\partial x_3}$$

$$[\mathbf{v}] = \begin{bmatrix} x_1x_2x_3 \\ x_1x_2 \\ x_1 \end{bmatrix}$$

  - In matrix notation

$$[\nabla\times\mathbf{v}] = \begin{bmatrix} \underset{=1}{\epsilon_{123}}\dfrac{\partial \mathrm{v}_3}{\partial x_2} + \underset{=-1}{\epsilon_{132}}\dfrac{\partial \mathrm{v}_2}{\partial x_3} \\ \underset{=-1}{\epsilon_{213}}\dfrac{\partial \mathrm{v}_3}{\partial x_1} + \underset{=1}{\epsilon_{231}}\dfrac{\partial \mathrm{v}_1}{\partial x_3} \\ \underset{=1}{\epsilon_{312}}\dfrac{\partial \mathrm{v}_2}{\partial x_1} + \underset{=-1}{\epsilon_{321}}\dfrac{\partial \mathrm{v}_1}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 \\ x_1x_2 - 1 \\ x_2 - x_1x_3 \end{bmatrix}$$



  - In compact notation: $\nabla\times\mathbf{v} = (x_1x_2 - 1)\hat{\mathbf{e}}_2 + (x_2 - x_1x_3)\hat{\mathbf{e}}_3$

Example - Solution

- Rotation:
  - Calculated directly in matrix notation:

$$\mathrm{v}_1 = x_1 x_2 x_3$$
$$\mathrm{v}_2 = x_1 x_2$$
$$\mathrm{v}_3 = x_1$$

$$\nabla \times \mathbf{v} \overset{symb}{=} \begin{bmatrix} \dfrac{\partial}{\partial x_1} \\ \dfrac{\partial}{\partial x_2} \\ \dfrac{\partial}{\partial x_3} \end{bmatrix} \times \begin{bmatrix} \mathrm{v}_1 \\ \mathrm{v}_2 \\ \mathrm{v}_3 \end{bmatrix} = \det \begin{bmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ \mathrm{v}_1 & \mathrm{v}_2 & \mathrm{v}_3 \end{bmatrix} =$$

$$= \left(\frac{\partial \mathrm{v}_3}{\partial x_2} - \frac{\partial \mathrm{v}_2}{\partial x_3} \right) \hat{\mathbf{e}}_1 + \left(\frac{\partial \mathrm{v}_1}{\partial x_3} - \frac{\partial \mathrm{v}_3}{\partial x_1} \right) \hat{\mathbf{e}}_2 + \left(\frac{\partial \mathrm{v}_2}{\partial x_1} - \frac{\partial \mathrm{v}_1}{\partial x_2} \right) \hat{\mathbf{e}}_3 =$$

$$= \left(x_1 x_2 - 1 \right) \hat{\mathbf{e}}_2 + \left(x_2 - x_1 x_3 \right) \hat{\mathbf{e}}_3$$

Example - Solution

- Gradient:

$$\boxed{[\nabla \otimes \mathbf{v}]_{ij} = [\nabla \mathbf{v}]_{ij} = \frac{\partial v_j}{\partial x_i}}$$

$$\boxed{\begin{aligned} v_1 &= x_1 x_2 x_3 \\ v_2 &= x_1 x_2 \\ v_3 &= x_1 \end{aligned}}$$

- In matrix notation

$$[\nabla \mathbf{v}] \overset{symb}{=} [\nabla \otimes \mathbf{v}] \overset{symb}{=} [\nabla][\mathbf{v}]^T \overset{symb}{=} \underbrace{\begin{bmatrix} \dfrac{\partial}{\partial x_1} \\ \dfrac{\partial}{\partial x_2} \\ \dfrac{\partial}{\partial x_3} \end{bmatrix}}_{3\times 1} \underbrace{\begin{bmatrix} x_1 x_2 x_3 \ , & x_1 x_2 \ , & x_1 \end{bmatrix}}_{1\times 3} = \underbrace{\begin{bmatrix} x_2 x_3 & x_2 & 1 \\ x_1 x_3 & x_1 & 0 \\ x_1 x_2 & 0 & 0 \end{bmatrix}}_{3\times 3}$$

- In compact notation:

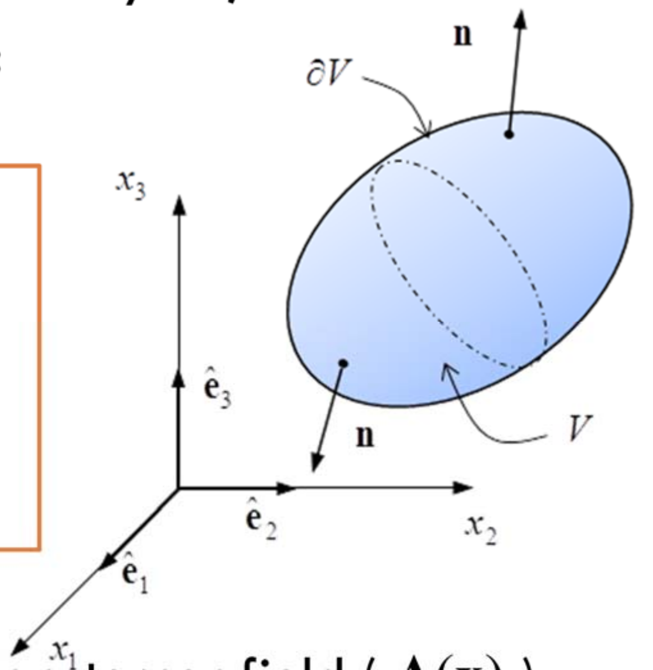
$$\nabla \mathbf{v} = x_2 x_3 \, \hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_1 + x_2 \hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_3 + x_1 x_3 \, \hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_1 + x_1 \, \hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_2 + x_1 x_2 \, \hat{\mathbf{e}}_3 \otimes \hat{\mathbf{e}}_1$$

Integral Theorems

Tensor Algebra

Divergence or Gauss Theorem

- Given a field $\mathcal{A}$ in a volume V with closed boundary surface ∂V and unit **outward** normal to the boundary $\mathbf{n}$, the **Divergence** (or **Gauss**) **Theorem** states:



$$\int_V \nabla \cdot \mathcal{A}\; dV = \int_{\partial V} \mathbf{n} \cdot \mathcal{A}\, dS$$

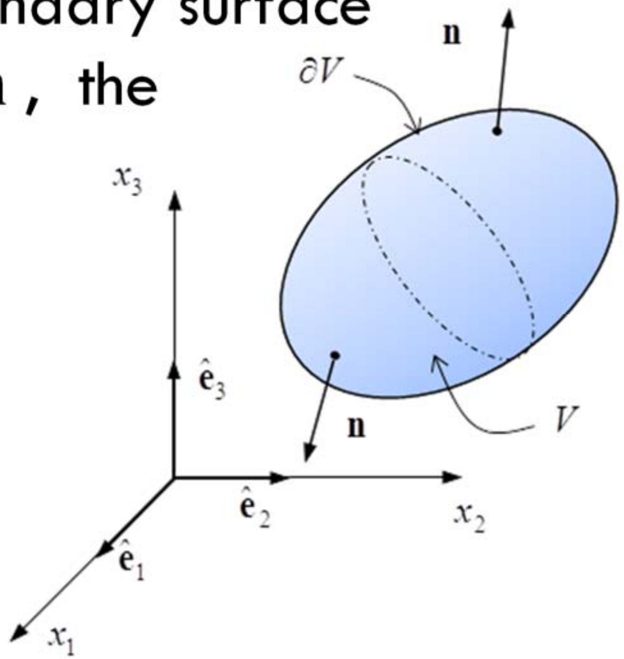
$$\int_V \mathcal{A} \cdot \nabla\; dV = \int_{\partial V} \mathcal{A} \cdot \mathbf{n}\, dS$$

Where:

- $\mathcal{A}$ represents either a vector field ($\mathbf{v}(\mathbf{x})$) or a tensor field ($\mathbf{A}(\mathbf{x})$).

Generalized Divergence Theorem

- Given a field $\mathcal{A}$ in a volume V with closed boundary surface ∂V and unit outward normal to the boundary $\mathbf{n}$, the **Generalized Divergence Theorem** states:



$$\boxed{\begin{aligned} \int_V \nabla * \mathcal{A}\, dV &= \int_{\partial V} \mathbf{n} * \mathcal{A}\, dS \\ \int_V \mathcal{A} * \nabla\, dV &= \int_{\partial V} \mathcal{A} * \mathbf{n}\, dS \end{aligned}}$$

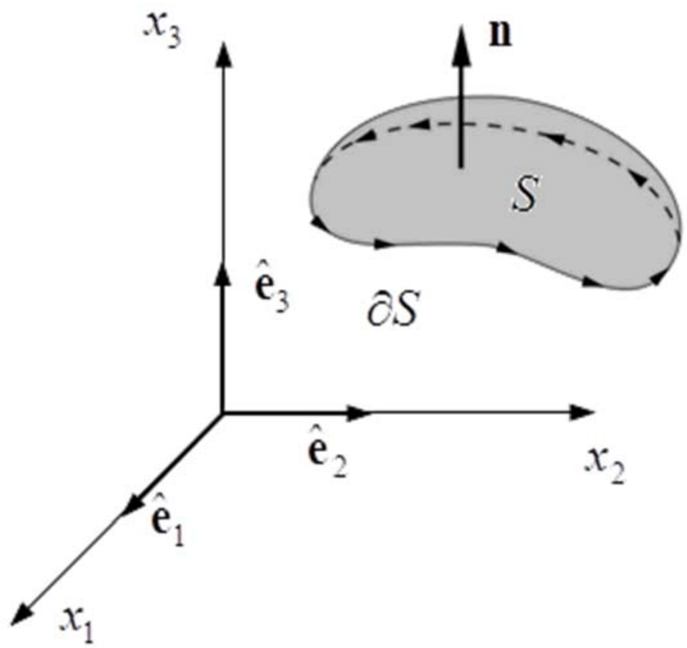
Where:

- $*$ represents either the dot product ($\cdot$), the cross product ($\times$) or the tensor product ($\otimes$).
- $\mathcal{A}$ represents either a scalar field ($\phi(\mathbf{x})$), a vector field ($\mathbf{v}(\mathbf{x})$) or a tensor field ($\mathbf{A}(\mathbf{x})$).

Curl or Stokes Theorem

- Given a vector field $\mathbf{u}$ in a surface S with closed boundary surface ∂S and unit outward normal to the boundary $\mathbf{n}$, the **Curl** (or **Stokes**) **Theorem** states:

$$\int_S (\nabla \times \mathbf{u}) \cdot \mathbf{n}\, dS = \int_{\partial S} \mathbf{u} \cdot d\mathbf{r}$$



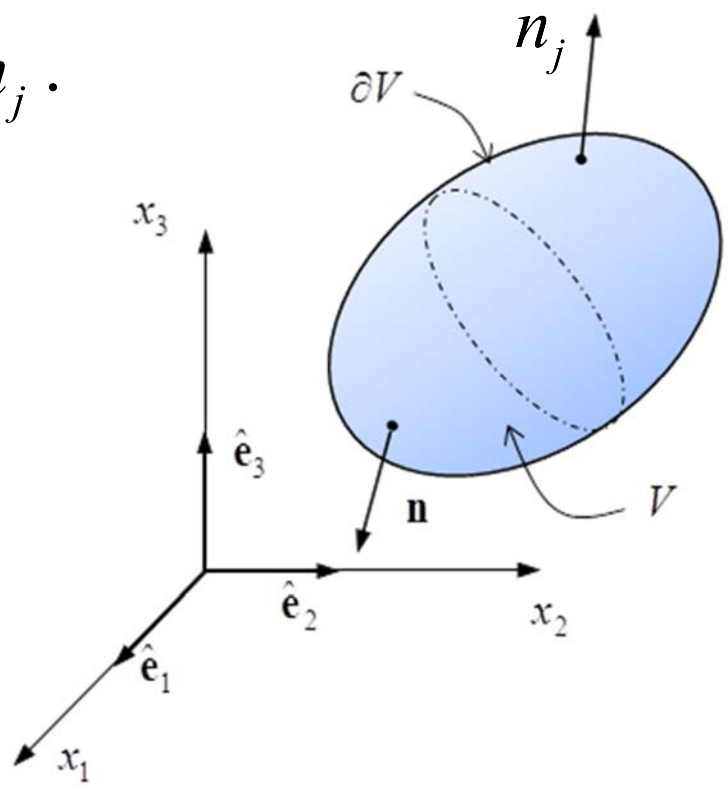
where the curve of the line integral must have positive orientation, such that $d\mathbf{r}$ points counter-clockwise when the unit normal points to the viewer, following the right-hand rule.

Example

- Use the Generalized Divergence Theorem to show that

$$\int_S x_i n_j \, dS = V\delta_{ij}$$

where x_i is the position vector of n_j .



$$\boxed{\int_{\partial V} \mathcal{A} * \mathbf{n} \, dS = \int_V \mathcal{A} * \nabla \, dV}$$

Example - Solution

$$\boxed{\int_S x_i n_j \, dS = V\delta_{ij}}$$

$$\int_S x_i n_j \, dS \Rightarrow \int_S \mathbf{x} \otimes \mathbf{n} \, dS$$

- Applying the Generalized Divergence Theorem:

$$\boxed{\int_{\partial V} \mathbf{x} \otimes \mathbf{n} \, dS = \int_V \mathbf{x} \otimes \nabla \; dV}$$

- Applying the definition of gradient of a vector:

$$[\nabla \mathbf{x}]_{ij} = \frac{\partial x_j}{\partial x_i} \Rightarrow [\mathbf{x}\nabla]_{ij} = \frac{\partial x_i}{\partial x_j}$$

Example - Solution

$$\int_S x_i n_j \, dS = V\delta_{ij}$$

- The Generalized Divergence Theorem in index notation:

$$\int_S x_i n_j \, dS = \int_V \frac{\partial x_i}{\partial x_j} \, dV$$

- Then,

$$\int_S x_i n_j \, dS = \int_V \frac{\partial x_i}{\partial x_j} \, dV = \int_V \delta_{ij} \, dV = \delta_{ij} V$$

References

Tensor Algebra

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