Linear Transformations

DEFINITION (Linear Transformation): A transformation (or mapping) T from a vector space V_1 to a vector space V_2, $T : V_1 \to V_2$ is a *linear transformation* (or a *linear operator*, a *linear map*, etc.), if:

(i) $T(\vec{\mathbf{u}} + \vec{\mathbf{v}}) = T\vec{\mathbf{u}} + T\vec{\mathbf{v}}$ for all vectors $\vec{\mathbf{u}}, \vec{\mathbf{v}}$ in V_1; and

(ii) $T(c\vec{\mathbf{u}}) = cT\vec{\mathbf{u}}$ for all vectors $\vec{\mathbf{u}}$ in V_1 and all scalars c.

EQUIVALENT DEFINITION (Linear Transformation): A transformation $T : V_1 \to V_2$ is a *linear transformation* if:

$T(a\vec{\mathbf{u}} + b\vec{\mathbf{v}}) = aT\vec{\mathbf{u}} + bT\vec{\mathbf{v}}$ for all vectors $\vec{\mathbf{u}}, \vec{\mathbf{v}}$ in V_1 and all scalars a, b.

BASIC FACTS:

- If T is a linear transformation, then $T\mathbf{0}$ must be $\mathbf{0}$. (So if you find $T\mathbf{0} \neq \mathbf{0}$, that means your T is not a linear transformation.)
- Any linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ can be given by a matrix A of type $m \times n$, $T(\vec{\mathbf{u}}) = A\vec{\mathbf{u}}$ for vectors $\vec{\mathbf{u}}$ in $\mathbb{R}^n$.

EXAMPLES: The following are linear transformations.

- $T : \mathbb{R}^5 \to \mathbb{R}^2$ defined by

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 - 5x_3 + 7x_4 + 6x_5 \\ -3x_1 + 4x_2 + 8x_3 - x_4 + x_5 \end{bmatrix},$$

  or equivalently,

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 & 2 & -5 & 7 & 6 \\ -3 & 4 & 8 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}.$$

- Scaling (expansion by factor 5): $T : \mathbb{R}^3 \to \mathbb{R}^3$ with matrix $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$.

- Scaling (contraction by factor 1/3): $T : \mathbb{R}^2 \to \mathbb{R}^2$ with matrix $\begin{bmatrix} 1/3 & 0 \\ 0 & 1/3 \end{bmatrix}$.

- Scaling in certain direction: $T : \mathbb{R}^2 \to \mathbb{R}^2$ with matrix $\begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$.

- Rotation in $\mathbb{R}^2$ about the origin by angle $\pi/3$ counterclockwise: $T : \mathbb{R}^2 \to \mathbb{R}^2$ with matrix $\begin{bmatrix} \cos(\pi/3) & -\sin(\pi/3) \\ \sin(\pi/3) & \cos(\pi/3) \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$.

- Reflection in $\mathbb{R}^2$ through the x_2 axis: matrix $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

- Orthogonal projection in $\mathbb{R}^3$ onto the x_1x_3-plane: matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

- Orthogonal projection in $\mathbb{R}^3$ onto the plane $x_1 - 2x_2 + 2x_3 = 0$: matrix $\frac{1}{9}\begin{bmatrix} 8 & 2 & -2 \\ 2 & 5 & 4 \\ -2 & 4 & -5 \end{bmatrix}$.

- A horizontal shear in $\mathbb{R}^2$: matrix $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$.

- $T : C[-\pi, \pi] \to \mathbb{R}$, defined by $Tf = \int_{-\pi}^{\pi} f(x)\sin(3x)dx$.

- $T : C^1[0, 1] \to \mathbb{R}$, defined by $Tf = f'(1) + 3f(1)$.

- $T : C^2[0, 3] \to C[0, 3]$, defined by $Tf(x) = -x^2 f''(x) - 3xf'(x) + e^x f(x)$.

EXAMPLES: The following are NOT linear transformations.

- $T : \mathbb{R}^5 \to \mathbb{R}^2$ defined by

$$T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 - 5x_3 + 7x_4 + 6x_5 + 777 \\ -3x_1 + 4x_2 + 8x_3 - x_4 + x_5 \end{bmatrix}.$$

- $T : \mathbb{R} \to \mathbb{R}$, $T(x) = x^2$.

- $T : \mathbb{R}^3 \to \mathbb{R}^2$, $T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + 2\sin(x_2) - 4x_3 \\ x_2 + 2x_3 \end{bmatrix}$.

- $T : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{cases} 2x_1 - 3x_2 & \text{if } x_1 \geq 0, \\ 4x_1 - 3x_2 & \text{if } x_1 < 0. \end{cases}$$

  (Check whether or not $T \begin{bmatrix} -3 \\ 0 \end{bmatrix} = -3T \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. If not satisfied, this T is not linear.)

- $T : C^2[0, 1] \to C[0, 1]$, $Tf = f'' + 1 - f$.

- $T : C^2[0, 1] \to C[0, 1]$, $Tf = f'' + f - e^f$.

EXERCISES

Are the following mappings linear transformations?

1. $T : \mathbb{R} \to \mathbb{R}, T(x) = 3x.$
2. $T : \mathbb{R} \to \mathbb{R}, T(x) = 3x - 2.$
3. $T : \mathbb{R} \to \mathbb{R}, T(x) = x^3.$
4. $T : \mathbb{R} \to \mathbb{R}, T(x) = (x+3)^2 - x^2 - 9.$
5. $T : \mathbb{R}^2 \to \mathbb{R}^3,\ T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$
6. $T : \mathbb{R}^2 \to \mathbb{R}^3,\ T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$
7. $T : \mathbb{R}^2 \to \mathbb{R}^3,\ T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \sin(x_1 + x_2) \\ 0 \\ 0 \end{bmatrix}.$
8. $T : \mathbb{R}^3 \to \mathbb{R}^3,\ T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 + x_3 \\ x_2 + 5x_3 \\ x_1 \end{bmatrix}.$
9. $T : C[0,1] \to C[0,1],\ Tf(x) = \int_0^1 f(y)\sin(x-y)dy.$
10. $T : C[0,1] \to \mathbb{R},\ Tf = \int_0^1 f^2(y)dy.$
11. $T : C^1[0,1] \to \mathbb{R},\ Tf = f'(0)f(0).$
12. $T : C^1[0,1] \to \mathbb{R},\ Tf = f'(0) - f(0).$
13. $T : C^4[0,1] \to C[0,1],\ Tf(x) = -\frac{d^4 f}{dx^4}(x) + \frac{d^2 f}{dx^2}(x) + f(x).$
14. $T : C^4[0,1] \to C[0,1],\ Tf(x) = -\frac{d^4 f}{dx^4}(x) + \frac{d^2 f}{dx^2}(x) + (1+x^2)f(x).$
15. $T : C^4[0,1] \to C[0,1],\ Tf(x) = -\frac{d^4 f}{dx^4}(x) + \frac{d^2 f}{dx^2}(x) + 1 + f(x)^2.$

ANSWERS: Y,N,N,Y,Y,N,N,Y,Y,N,N,Y,Y,Y,N