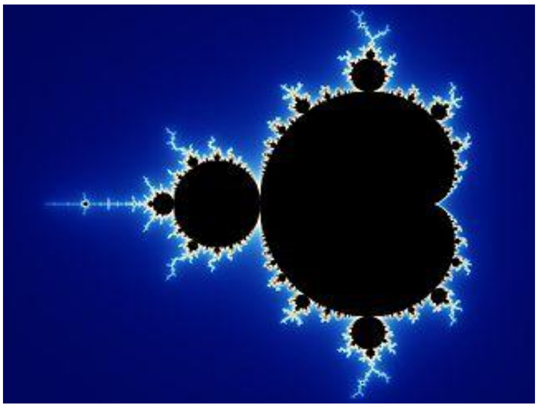
Beauty in Mathematics

Often, when reading a good maths book, the author will get to the end of an explanation of a particularly complicated proof, theorem, or idea, and mention the 'beauty' of the maths involved. I always wonder what, exactly, this means. Did I miss a particularly neat diagram? Or, as seems to be the case, is mathematical 'beauty' something buried deep: something that, perhaps, I need a PhD to get to grips with?

I used to think it was the latter. Maybe one day, after years of studying maths at its highest level, I'd suddenly gain a glimpse of some incomprehensibly deep truth and realise the incredible beauty of things which now seem boring and trivial.



Sometimes the correct answer is the former…

Well, actually, I think you can get a glimpse of what Mathematicians mean by 'beauty' without too much effort at all. That's what I'm going to try and convince you of in the rest of this article.

Sometimes, I think that Mathematics is a bit like a dense, never-ending jungle. It can feel like you're hacking away and away at it and never getting anywhere, but if you stop and look around yourself every once in a while there are huge numbers of incredible, exotic plants and animals to marvel at – and every so often huge new swathes of jungle are found to explore.

The particular thing that I want to introduce you to, that I think is so beautiful, is something that was mentioned in passing on a television programme I was watching. I hardly knew what it meant, and I certainly had no idea how it came about, but I knew I had to find out more.

I am talking about 'Euler's identity', $e^{i\pi} + 1 = 0$

So now you probably think I'm crazy. 'What's beautiful about that?' Well, I ought to warn you, I'm not alone – *Mathematical Intelligencer*'s readers voted the identity the 'most beautiful theorem in mathematics'. The physicist Richard Feynman called the formula it is derived from "one of the most remarkable, almost astounding, formulas in all of mathematics".

Leonard Euler; one of the greatest mathematicians ever.

But what is so special about it? Well, first I ought to explain what the symbols actually mean.

It's likely you're familiar with π, which is the number you multiply a circle's diameter by to get its circumference.

'*e*' is also a constant, and you may be vaguely familiar with it. It crops up in lots of different places. To 20 decimal places, *e* = 2.71828182845904523536. Both π and *e* are irrational numbers – they have an infinite number of decimal places and you can't write them down as one integer divided by another.

$e = 1 + 1/(1!) + 1/(2!) + 1/(3!) + 1/(4!) + \ldots$

An approximation of *e*. 'n!' is 'n factorial', and means n x (n-1) x (n-2) … x 2 x 1, so '4!' = 4 x 3 x 2 x 1 = 24

Probably the strangest of these unfamiliar numbers is 'i'. 'i' is the square root of -1, so that $i^2 = -1$. 'i' is an 'imaginary number'; it isn't found anywhere along the normal number line. If you add together 'i' and a normal (or 'real') number, you get a complex number, normally denoted by the letter 'z'.

Are you starting to get an idea of the beauty of Euler's identity yet? If you take the constant '*e*' to the power of (the constant 'π' multiplied by the very strange number 'i') then take away one, you get to zero. Isn't it a little odd how three very strange numbers which are not connected in any evident way combine to give such a normal, familiar number?

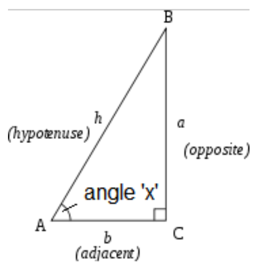
So, why does this happen? You might think that it is down to some really complex idea – how, anyway, do we take a number to the power of 'i'? Well, actually, it isn't too difficult to see how Euler's identity comes about – that is one thing that makes the identity so wonderful! But first you have to see Euler's formula, which leads to his beautiful identity, in full:

$$e^{ix} = \cos x + i \sin x$$

Doesn't look quite as nice and neat now, does it? Don't be put off. All you need to know is that the angle 'x' is in *radians*, which is just another way of measuring angles like degrees. Instead of splitting a circle into 360 'degrees', we are splitting it into 2π radians – π radians is the same as 180°. Cos and sin, of course, are the trigonometric functions.

I found the following proof of Euler's formula (and the images from it), which I am going to try and explain to you, on wikipedia.org.

To understand why the formula comes about, we need something called 'Taylor Series'. These are just a way of expressing functions such as sin x or cos x as infinite sums. They were invented by the Mathematician Brook Taylor, who was also part of the committee which adjudicated the claims of Isaac Newton and Gottfried Leibniz.



sin x = opposite/hypotenuse
cos x = adjacent/hypotenuse

The Taylor Series for the three important functions in Euler's formula are as follows:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots .$$

Now if we replace 'x' with the variable for complex numbers, 'z', multiplied by i, we get

$$e^{iz} = 1 + iz + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \frac{(iz)^4}{4!} + \frac{(iz)^5}{5!} + \frac{(iz)^6}{6!} + \frac{(iz)^7}{7!} + \frac{(iz)^8}{8!} + \cdots$$

But certain powers of 'i' can be simplified – for example, i^2 is equal to -1 by definition, and so $i^3 = -i$ and $i^4 = +1$. So we can simplify the above to

$$e^{iz} = 1 + iz - \frac{z^2}{2!} - \frac{iz^3}{3!} + \frac{z^4}{4!} + \frac{iz^5}{5!} - \frac{z^6}{6!} - \frac{iz^7}{7!} + \frac{z^8}{8!} + \cdots$$

But then we can gather the 'i' terms together to give

$$= \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \cdots\right) + i\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right)$$

You might notice that these series are the same as the series for sin x and cos x from earlier, so we can substitute these in to get…

$$e^{iz} = \cos z + i \sin z$$

i.e. Euler's formula!

All we have to do now is substitute in $z = \pi$. $\sin \pi = 0$ and $\cos \pi = -1$ so we get

$$e^{i\pi} + 1 = 0 \quad .$$


The mathematical jungle?

So, you see, after a long sequence of fairly complex mathematics we arrive back where we started – at the (seemingly) simple ideas of '1' and '0'. That is what I think is so beautiful about this identity: it links very strange numbers with very ordinary and fundamental ones. Seeing why it works feels a bit like treading a little-known path through the mathematical jungle to reach a secret destination somewhere in the thick undergrowth.

Bibliography:

These links were used for either the open source images in the article, for general research purposes, or for the proof of Euler's identity/formula. They were accessed on 29th March 2009.

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