

Econ 202A Macroeconomics: Section 1

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October 23, 25, 2024

Syllabus

Section Overview

- (1) Numerical solutions (2) using a finite difference method (3) for continuous time (4) heterogeneous agent models

- (1) Numerical solutions (2) using a finite difference method (3) for continuous time (4) heterogeneous agent models
- Useful references:
 1. **Benjamin Moll's website**
 2. Online Appendix for Achdou et al. (2022)
 3. LeVeque, 2007, *"Finite Difference Methods for Ordinary and Partial Differential Equations: Steady-State and Time-dependent Problems."*
 4. Candler, 1999, *"Finite Difference Methods for Continuous Time Dynamic Programming."*

Schedule (Subject to Change)

1. **October 23, 25:** Discrete and Continuous-Time Dynamics & Introduction to Finite Difference Method
2. **October 30, November 1:** Neoclassical Growth Model
3. **November 6, 8:** Huggett Model (Partial Equilibrium)
4. **November 13, 15:** Huggett Model (Kolmogorov Forward Equation and General Equilibrium)
5. **November 20, 22:** Coding: Huggett Model
6. **November 27, 29: No Section: Thanksgiving Break**
7. **December 4, 6:** TBD

- Office hours
 - **Fridays from 11am to 1pm** in Evans 546
 - If this time doesn't work for you, email me to arrange an alternative meeting.
 - Please email your questions in advance (especially if they involve coding—please attach the relevant code) so that I can review them beforehand.
- Email
 - Include [ECON 202A] at the beginning of the subject line.
 - Please allow up to two business days for a response.

Problem Sets

- Submission
 - Late assignments are not accepted.
 - For coding problems:
 1. Copy the code into a text file (e.g., .txt, LaTeX, Word, etc.)
 2. Submit a PDF version.
- Programming Language
 - All in-class demos will use MATLAB.
 - Some assignments may require you to adapt pre-written MATLAB code, which will not be available in other languages.
 - Berkeley offers free access to MATLAB. (<https://software.berkeley.edu/matlab>.)
 - You are welcome to use other languages, but support will only be provided for MATLAB.

Section 1

1. Dynamics in Discrete and Continuous Time
 - Discrete Time: Difference Equations
 - Continuous Time: Differential Equations
2. Solving Ordinary Differential Equations (ODEs)
 - Analytical Solution: Integrating Factor Method
 - Numerical Solution: **Finite Difference Method**
3. Exercises: Solving and Comparing Solutions
 - Capital Accumulation
 - Consumption Euler Equation (Assignment)

Section 1-1: Dynamics in Discrete and Continuous Time

- In macroeconomics, many key questions revolve around how variables evolve over time.

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- Two broad categories of dynamic models:
 1. Discrete Time: $K_{t+1} = (1 - \delta)K_t + I_t$
 2. Continuous Time: $\dot{K}_t = I_t - \delta K_t$

- In macroeconomics, many key questions revolve around how variables evolve over time.
- Two broad categories of dynamic models:
 1. Discrete Time: $K_{t+1} = (1 - \delta)K_t + I_t$
 2. Continuous Time: $\dot{K}_t = I_t - \delta K_t$
- Models can be translated from discrete time to continuous time, and vice versa.

Difference Equations

The **first-order linear difference equation** is defined as:

$$x_{t+1} = bx_t + cz_t \quad (1)$$

where $\{z_t\}$ is an exogenously given, bounded sequence.

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Exercise: Solve the First-Order Difference Equation

Solve the first-order linear autonomous difference equation:

$$x_{t+1} = bx_t + 1$$

Solving a First-Order Difference Equation

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Solve the first-order linear autonomous difference equation, **given an initial value** x_0 :

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General Solution Steps

1. Solve the homogeneous equation: $x_{t+1} = bx_t \Rightarrow x_t = Ab^t$
2. Find the stationary point or steady state: $x = \frac{1}{1-b}$ when $b \neq 1$
3. Solve for the general solution as the sum of the homogeneous solution and the particular solution: $x_t = Ab^t + x$
4. Determine A using the initial condition: $x_0 = Ab^0 + x \Rightarrow A = x_0 - x$
5. The general solution for the autonomous equation: $x_t = (x_0 - \frac{1}{1-b})b^t + \frac{1}{1-b}$

Difference Equations in Macroeconomics

Capital Accumulation Equation:

$$K_{t+1} = (1 - \delta)K_t + I_t$$

- This is a first-order linear difference equation.
- It requires one boundary condition.
- This is a **forward equation** since it describes how the system evolves forward in time.
- Therefore, it requires an **initial condition** K_0 .
- If $I_t = 0 \forall t$ and $0 < \delta < 1$, then $K_t \rightarrow 0$.
- If $I_t = c > 0 \forall t$, then K_t converges to $\frac{c}{\delta}$.

Difference Equations in Macroeconomics

Consumption Euler Equation:

$$\frac{1}{C_t} = \beta R_t \frac{1}{C_{t+1}}$$

- This is also a first-order linear difference equation.
- It requires one boundary condition.
- Unlike the capital accumulation equation, this is a **backward equation**, determining optimal consumption today based on future consumption.
- Therefore, it requires either a **terminal condition** or a transversality condition.
- In a finite-horizon model, the consumption level at a specific time $t = T$ must be given.
- In an infinite-horizon model, a transversality condition is commonly imposed to ensure that C_t converges to a specific value over time.

The **first-order linear ordinary differential equation (ODE)** is defined as:

$$\dot{X}(t) = a(t)X(t) + b(t) \quad (2)$$

- If $b(t) = 0$, equation (2) is a homogeneous equation.
- If $a(t)$ and $b(t)$ are non-zero constants, we say equation (2) has constant coefficients.

From Discrete to Continuous Time

Derivation of Continuous-Time Capital Accumulation

Discrete time with unit time steps:

$$K_{t+1} = (1 - \delta)K_t + I_t$$

With an arbitrary Δ time step:

$$K_{t+\Delta} = (1 - \Delta\delta)K_t + \Delta I_t$$

Rearranging:

$$\frac{K_{t+\Delta} - K_t}{\Delta} = I_t - \delta K_t$$

Taking the limit as $\Delta \rightarrow 0$:

$$\lim_{\Delta \rightarrow 0} \frac{K_{t+\Delta} - K_t}{\Delta} = \lim_{\Delta \rightarrow 0} (I_t - \delta K_t)$$

$$\therefore \dot{K}_t = I_t - \delta K_t$$

Exercise: Discrete to continuous-Time Transformation

Transform the discrete-time consumption Euler equation into continuous-time counterpart:

$$u'(C_t) = \beta R_t u'(C_{t+1}), \quad \text{where} \quad u(C_t) = \log(C_t)$$

Denote:

$$\beta = 1 - \rho, \quad R_t = 1 + r_t$$

Derivation of Continuous-Time Consumption Euler Equation

With an arbitrary Δ time step:

$$u'(C_t) = (1 - \Delta\rho)(1 + \Delta r_t)u'(C_{t+\Delta})$$

First approximation of $u'(C_{t+\Delta})$ around C_t :

$$u'(C_{t+\Delta}) = u'(C_t + (C_{t+\Delta} - C_t)) \simeq u'(C_t) + u''(C_t)(C_{t+\Delta} - C_t)$$

Plugging in and rearranging:

$$u'(C_t) = (1 - \Delta\rho)(1 + \Delta r_t) [u'(C_t) + u''(C_t)(C_{t+\Delta} - C_t)]$$

$$1 = (1 - \Delta\rho)(1 + \Delta r_t) \left[1 + \frac{u''(C_t)}{u'(C_t)}(C_{t+\Delta} - C_t) \right]$$

From Discrete to Continuous Time

Substitute:

$$\frac{u''(C_t)}{u'(C_t)} = -\frac{1}{C_t}$$

Rearranging:

$$1 = (1 - \Delta\rho + \Delta r_t - \Delta^2 \rho r_t) \left[1 - \frac{C_{t+\Delta} - C_t}{C_t} \right]$$

Dividing by Δ and taking the limit as $\Delta \rightarrow 0$:

$$\lim_{\Delta \rightarrow 0} \left[(1 - \Delta\rho + \Delta r_t - \Delta^2 \rho r_t) \frac{1}{C_t} \frac{C_{t+\Delta} - C_t}{\Delta} \right] = \lim_{\Delta \rightarrow 0} \left(\frac{-\Delta\rho + \Delta r_t - \Delta^2 \rho r_t}{\Delta} \right)$$

$$\therefore \frac{\dot{C}_t}{C_t} = r_t - \rho$$

Capital Accumulation Equation:

$$\dot{K}_t = I_t - \delta K_t$$

- This is a first-order linear ordinary differential equation.
- It requires one boundary condition.
- This is a forward equation since it describes how the system evolves forward in time.
- Therefore, it requires an initial condition K_0 .
- If $I_t = 0 \forall t$ and $0 < \delta < 1$, then $K_t \rightarrow 0$.
- If $I_t = c > 0 \forall t$, then K_t converges to $\frac{c}{\delta}$.

Consumption Euler Equation:

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- It requires one boundary condition.
- Unlike the capital accumulation equation, this is a backward equation, determining optimal consumption today based on future consumption.
- Therefore, it requires either a terminal condition or a transversality condition.
- In a finite-horizon model, the consumption level at a specific time $t = T$ must be given.
- In an infinite-horizon model, a transversality condition is commonly imposed to ensure that C_t converges to a specific value over time.

Section 1-2: Solving Ordinary Differential Equations (ODE)

Solving Ordinary Differential Equations (ODEs)

- Two approaches to solving ODEs:
 1. Analytical: Integrating Factor Method
 2. Numerical: Finite Difference Method

Solving Ordinary Differential Equations (ODEs)

- Two approaches to solving ODEs:
 1. Analytical: Integrating Factor Method
 2. Numerical: Finite Difference Method
- Continuous-time heterogeneous agent models are particularly analytically intractable, so they are typically solved numerically using the finite difference method.

Analytical Solution: Integrating Factor Method

1. Write the ODE in Standard Form

The first step is to ensure the ODE is in its standard form:

$$\dot{y}(t) + p(t)y(t) = q(t),$$

which represents a first-order linear ODE. If the equation is not in this form, manipulate it so that it matches this structure.

2. Find the Integrating Factor

The key idea is to find a function, known as the integrating factor, which simplifies the ODE. The integrating factor, denoted by $\mu(t)$, is chosen to make the left-hand side easier to integrate. It is defined as:

$$\mu(t) = e^{\int p(t) dt}.$$

This function will be used to multiply through the entire ODE.

Analytical Solution: Integrating Factor Method

3. Multiply the ODE by the Integrating Factor

Multiply the entire ODE by $\mu(t)$:

$$\mu(t)\dot{y}(t) + \mu(t)p(t)y(t) = \mu(t)q(t).$$

Since $\mu(t) = e^{\int p(t) dt}$, this transforms the left-hand side into the derivative of a product:

$$\frac{d}{dt} (\mu(t)y(t)) = \mu(t)q(t).$$

This step is crucial, as it significantly simplifies the equation.

4. Integrate Both Sides

Integrate both sides of the equation with respect to t :

$$\int \frac{d}{dt} (\mu(t)y(t)) dt = \int \mu(t)q(t) dt.$$

Analytical Solution: Integrating Factor Method

This yields:

$$\mu(t)y(t) = \int \mu(t)q(t) dt + C,$$

where C is the constant of integration.

5. **Solve for $y(t)$**

Finally, solve for $y(t)$ by dividing both sides by $\mu(t)$:

$$y(t) = \frac{1}{\mu(t)} \left(\int \mu(t)q(t) dt + C \right).$$

6. **Determine the Constant of Integration**

Solve for C using given boundary conditions.

This provides the general solution to the first-order linear ODE.

Exercise: Analytically Solve the Continuous-Time Capital Accumulation Equation

Solve the capital accumulation equation:

$$\dot{K}_t = I_t - \delta K_t$$

where I_t is the exogenously given investment and δ is the depreciation rate, using the initial condition K_0 .

Analytical Solution of Continuous-Time Capital Accumulation

Step 1: Write the ODE in Standard Form

The first step is to rewrite the equation in its standard form:

$$\dot{K}_t + \delta K_t = I_t$$

which corresponds to the form $\dot{y}(t) + p(t)y(t) = q(t)$, where $p(t) = \delta$ and $q(t) = I_t$.

Step 2: Find the Integrating Factor

The integrating factor $\mu(t)$ is defined as:

$$\mu(t) = e^{\int \delta dt} = e^{\delta t}$$

Solving Capital Accumulation

Step 3: Multiply the ODE by the Integrating Factor

Multiply both sides of the equation by $\mu(t) = e^{\delta t}$:

$$e^{\delta t} \dot{K}_t + e^{\delta t} \delta K_t = e^{\delta t} I_t$$

The left-hand side simplifies to the derivative of a product:

$$\frac{d}{dt} (e^{\delta t} K_t) = e^{\delta t} I_t$$

Step 4: Integrate Both Sides

Integrate both sides with respect to t :

$$\int \frac{d}{dt} (e^{\delta t} K_t) dt = \int e^{\delta t} I_t dt$$

Solving Capital Accumulation

This gives:

$$e^{\delta t} K_t = \int e^{\delta t} I_t dt + C$$

where C is the constant of integration.

Step 5: Solve for K_t

Solve for K_t by dividing both sides by $e^{\delta t}$:

$$K_t = e^{-\delta t} \left(\int_0^t e^{\delta s} I_s ds + C \right)$$

Step 6: Determine the Constant of Integration

Use the initial condition K_0 to find C :

$$K_0 = e^{-\delta \cdot 0} \left(\int_0^0 e^{\delta s} I_s ds + C \right) = C$$

Analytical Solution

The analytical solution to the capital accumulation equation is:

$$K_t = e^{-\delta t} \left(\int_0^t e^{\delta s} I_s ds + K_0 \right)$$

If I_t is constant at I

The solution to the capital accumulation equation is:

$$K_t = \frac{I}{\delta} (1 - e^{-\delta t}) + e^{-\delta t} K_0$$

Numerical Solution: Finite Difference Method

- The finite difference method **approximates derivatives using finite differences**—calculating changes between function values at discrete time steps.

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- The key idea involves:
 1. Discretizing the domain (i.e., time) into a finite number of intervals.
 2. Approximating the first-order derivative by transitioning from continuous to discrete time, as shown below:

$$\dot{X}_t = \frac{dX_t}{dt} \approx \frac{X_{t+\Delta} - X_t}{\Delta}$$

where Δ is the small time step.

Numerical Solution: Finite Difference Method

- The finite difference method **approximates derivatives using finite differences**—calculating changes between function values at discrete time steps.
- The key idea involves:
 1. Discretizing the domain (i.e., time) into a finite number of intervals.
 2. Approximating the first-order derivative by transitioning from continuous to discrete time, as shown below:

$$\dot{X}_t = \frac{dX_t}{dt} \approx \frac{X_{t+\Delta} - X_t}{\Delta}$$

where Δ is the small time step.

- Finite difference methods transform differential equations, which may be nonlinear, into systems of linear equations that can be solved using matrix algebra techniques.

Finite Difference Approximation

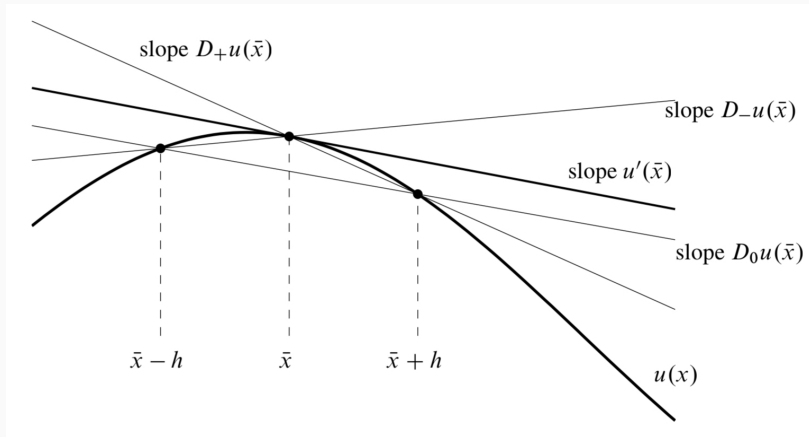


Figure 1: Various approximations to $u'(\bar{x})$ interpreted as the slope of secant lines.

Finite Difference Approximation

- One-sided approximations:

- Forward difference:

$$D_+ u(\bar{x}) \equiv \frac{u(\bar{x} + h) - u(\bar{x})}{h}$$

- Backward difference:

$$D_- u(\bar{x}) \equiv \frac{u(\bar{x}) - u(\bar{x} - h)}{h}$$

- Both are first-order accurate approximations.

- Centered approximation (central difference):

$$D_0 u(\bar{x}) \equiv \frac{u(\bar{x} + h) - u(\bar{x} - h)}{2h} = \frac{1}{2} [D_+ u(\bar{x}) + D_- u(\bar{x})]$$

- This is a second-order accurate approximation.

Section 1-3: Exercises: Solving and Comparing Solutions

Solving the Continuous-Time Capital Accumulation Equation

Exercise: Numerically Solve the Continuous-Time Capital Accumulation Equation

Solve the capital accumulation equation:

$$\dot{K}_t = I_t - \delta K_t$$

where $I_t = 5 \ \forall t$ is the exogenously given investment and $\delta = 0.05$ is the depreciation rate, with the initial condition $K_0 = 10$.

Update Rule for Forward Difference Method

Since we are given an initial boundary condition, we will use the **forward difference method** to solve this equation. The update rule is:

$$K(t + \Delta t) = K(t) + \Delta t (I(t) - \delta K(t))$$

where Δt is the time step size. You will iterate this equation forward in time, starting from $t = 0$ and progressing to $t = T$.

Finite Difference Approximations

We **discretize the time domain** into uniformly spaced grid points, where $\Delta t = t(i+1) - t(i)$ represents the distance (time step) between consecutive grid points.

Using the shorthand notations $K_i = K(t_i)$, the **finite difference approximation** of the capital accumulation equation is expressed as:

$$K(i+1) = K(i) + \Delta t (I(i) - \delta K(i))$$

Numerical (FDM) vs Analytical Solution

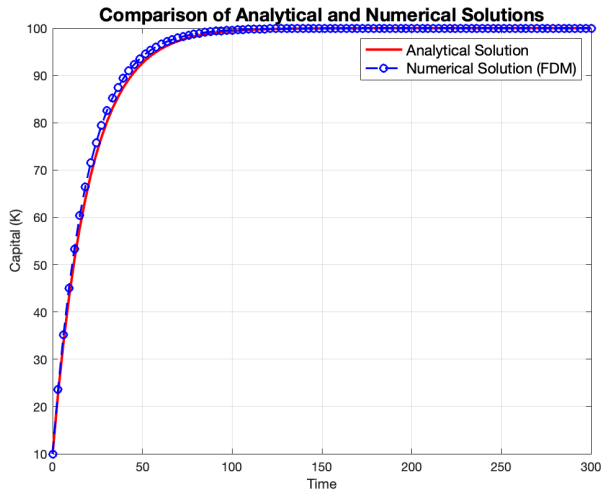


Figure 2: Numerical vs Analytical Solution for $I = 5$ with 100 Time Steps

Numerical (FDM) vs Analytical Solution

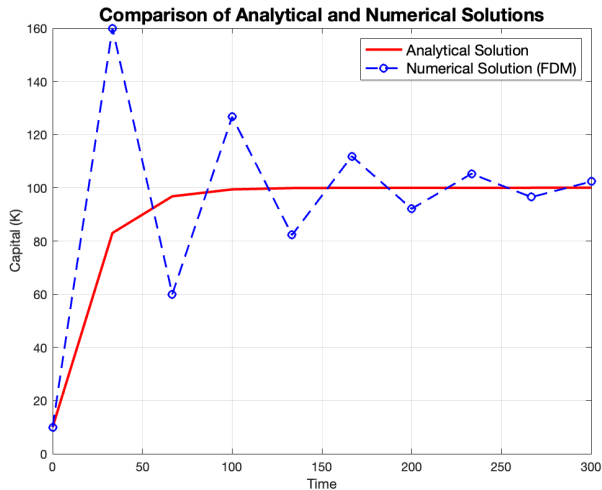


Figure 3: Numerical vs Analytical Solution for $l = 5$ with 10 Time Steps

Numerical (FDM) vs Analytical Solution

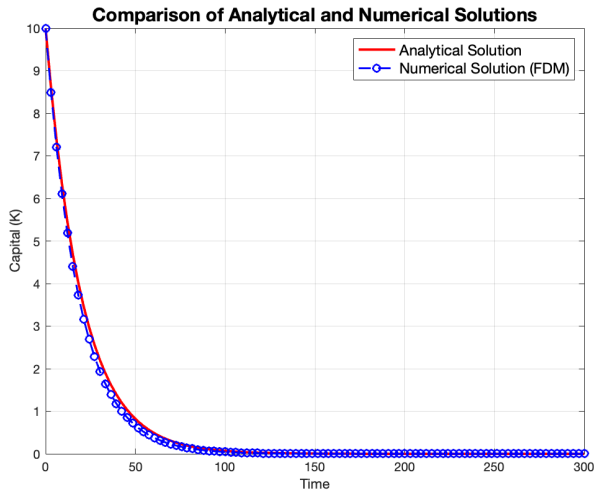


Figure 4: Numerical vs Analytical Solution for $I = 0$ with 100 Time Steps

Numerical (FD) vs Numerical (ODE45) Solution

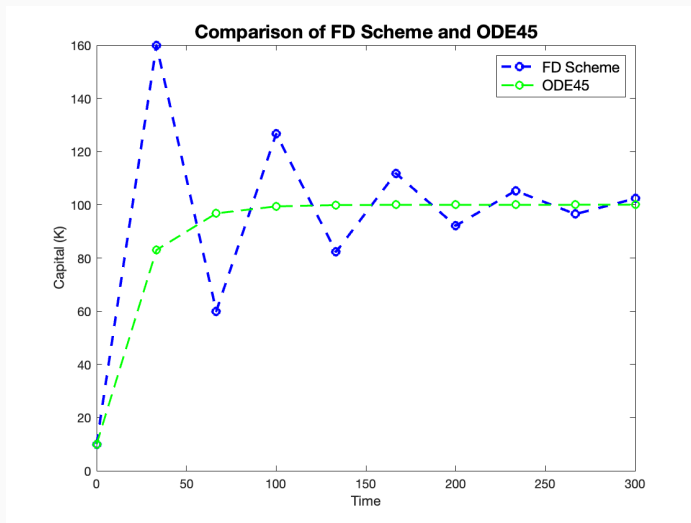


Figure 5: FD Scheme vs ODE45 Solution for $I = 5$ with 10 Time Steps

Numerical (FDM) vs Numerical (ODE45) Solution

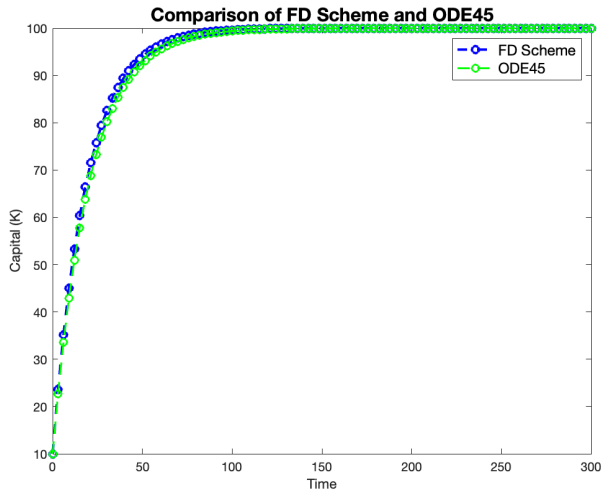


Figure 6: FD Scheme vs ODE45 Solution for $I = 5$ with 100 Time Steps

References

- Achdou, Y., J. Han, J.-M. Lasry, P.-L. Lions, and B. Moll (2022). Income and wealth distribution in macroeconomics: A continuous-time approach. The review of economic studies 89(1), 45–86.
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- LeVeque, R. J. (2007). Finite Difference Methods for Ordinary and Partial Differential Equations: Steady-State and Time-dependent Problems. Philadelphia, PA: Society for Industrial and Applied Mathematics.