Dynamic Programming and Applications

Deterministic Dynamic Programming in Continuous Time

Lectures 3 – 4

Andreas Schaab

Outline

Part 1: Differential equations

1. The continuous time limit
2. Ordinary differential equations (ODEs)
3. Boundary conditions
4. Linear first-order ODEs
5. Examples of ODEs in macro
6. Application: solving the Solow growth model
7. Partial differential equations (PDEs)

Outline

Part 2: Optimization with deterministic dynamics

1. Neoclassical growth model in continuous time
2. Calculus of variations
3. Optimal control theory
4. Simple example
5. Hamilton-Jacobi-Bellman (HJB) equation
6. First-order condition for consumption
7. Envelope condition and Euler equation
8. Connection between calculus of variations / optimal control and HJBs
9. Boundary conditions: no-borrowing in the wealth / capital dimension

Outline

Part 3: Applications

1. Labor: search and matching
2. Urban / trade / dynamic spatial: migration
3. Macro: sticky prices
4. IO: duopoly
5. Public finance: tax competition

Part 1: Differential Equations

1. Continuous time limit

- Consider the two key difference equations:

$$K_{t+1} = I_t + (1-\delta)K_t$$

  and

$$\frac{1}{C_t} = \beta R_t \frac{1}{C_{t+1}}$$

- On the board: (i) generalized discrete time step Δ and (ii) continuous time limit

2. Ordinary differential equations

- Consider the "discrete-time" equation

$$X_{t+\Delta t} - X_t = G(X_t, t, \Delta t)$$

- *Continuous-time limit*: consider the limit as $\Delta t \to 0$

$$\dot{X}_t \equiv \frac{dX}{dt} \equiv \lim_{\Delta t \to 0} \frac{X_{t+\Delta t} - X_t}{\Delta t} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} G(X_t, t, \Delta t) \equiv g(X_t, t)$$

- $\dot{X}_t = g(X_t)$ is *autonomous* and dropping subscripts: $\dot{X} = g(X)$

- This is a *first-order (ordinary) differential equation*, second-order equations are:

$$\frac{d^2 X_t}{dt^2} = g\left(\frac{dX_t}{dt}, X_t, t\right)$$

- We often consider ODEs in the *time dimension* but ODEs can be defined on any state space (e.g., space dimensions)

3. Boundary conditions

- Boundary conditions are critical for characterizing differential equations
- Consider an ODE on the time interval $t \in [0, 1]$. We call $[0, 1]$ the *state space*. $(0, 1)$ is the *interior of the state space* and $\{0, 1\}$ is the *boundary*
- The way to think about it: differential equations are defined on the interior of the state space but not on the boundary
- To characterize the function that satisfies the ODE on the interior on the *full* state space, we need a set of boundary conditions to also characterize the behavior on the boundary
- Heuristically: we need as many boundary conditions as the order of the differential equation

- Similar to discrete-time difference equations: forward equations have initial conditions, backward equations have terminal conditions

- For ODEs, you will often see the terminology:
  - *Initial value problems* specify a differential equation for X_t with some *initial condition* X_0
  - *Terminal value problems* instead specify X_T

- More broadly: We need sufficient information to characterize the function of interest along the boundary

- Types of boundary conditions: Dirichlet ($X_0 = c$), von-Neumann ($\frac{dX_0}{dt} = c$), reflecting boundaries, ...

- Boundary conditions are very important and can be very subtle (especially for PDEs)

4. Linear first-order ODEs

- Consider the equation:

$$\dot{X}(t) = a(t)X(t) + b(t) \tag{1}$$

- If $b(t) = 0$, (1) is a *homogeneous* equation, if $a(t) = a$ and $b(t) = b$ we say (1) has *constant coefficients*

- Start with $\dot{X}(t) = aX(t)$, divide by $X(t)$ and integrate with respect to t

$$\int \frac{\dot{X}(t)}{X(t)} dt = \int a dt$$

$$\log X(t) + c_0 = at + c_1$$

$$X(t) = Ce^{at}$$

where $C = e^{c_1 - c_0}$

- Pin down constant C by using the boundary condition (we need 1)

- Consider time-varying coefficient with $\dot{X}(t) = a(t)X(t)$ with initial condition $X(0) = \bar{x}$
- Dividing by $X(t)$, integrating, and exponentiating yields

$$X(t) = Ce^{\int_0^t a(s)ds}$$

- Constant of integration again pinned down by boundary condition: $C = \bar{x}$
- Finally, for $\dot{X}(t) = aX(t) + b$, we find

$$X(t) = -\frac{b}{a} + Ce^{at}$$

  after using change of variables $Y(t) = X(t) + \frac{b}{a}$

- Many results for systems of linear differential equations: $\dot{\boldsymbol{X}}(t) = \boldsymbol{A}\boldsymbol{X}(t)$

5. Examples of differential equations in macro

Capital accumulation:

$$\dot{K}_t = I_t - \delta K_t$$

- We can always map back and forth between DT and CT
- In discrete time with *unit* time steps, $K_{t+1} = I_t + (1 - \delta)K_t$
- With arbitrary Δ time step, $K_{t+\Delta} = K_t + \Delta(I_t - \delta K_t)$
- Continuous-time limit:

$$\begin{aligned}
K_{t+\Delta} &= K_t + \Delta(I_t - \delta K_t) \\
\frac{K_{t+\Delta} - K_t}{\Delta} &= I_t - \delta K_t \\
\dot{K}_t &= I_t - \delta K_t
\end{aligned}$$

- Suppose $\{I_t\}_{t\geq 0}$ exogenously given
- Solving this *inhomogeneous equation*, we use *integrating factor*:

$$\dot{K}_t + \delta K_t = I_t$$

$$e^{\int_0^t \delta ds}\dot{K}_t + e^{\int_0^t \delta ds}\delta K_t = e^{\int_0^t \delta ds} I_t$$

- Notice that $\int_0^t \delta ds = \delta \int_0^t ds = \delta[s]_0^t = \delta(t-0) = \delta t$, so

$$e^{\delta t}\dot{K}_t + e^{\delta t}\delta K_t = e^{\delta t} I_t$$

- We have $e^{\delta t}\dot{K}_t + e^{\delta t}\delta K_t = \frac{d}{dt}(K_t e^{\delta t})$, integrating:

$$K_t e^{\delta t} = \tilde{C} + \int_0^t e^{\delta s} I_s ds$$

$$K_t = C + \int_0^t e^{-\delta(t-s)} I_s ds$$

- Integrating constant solves initial condition: $C = K_0$

Wealth dynamics (*very important equation in this course*):

$$\dot{a}_t = r_t a_t + y_t - c_t$$

- r_t is the real rate of return on wealth, y_t is income, and c_t is consumption
- Structure of the equation similar to capital accumulation equation

Consumption Euler equation:

$$\frac{\dot{C}_t}{C_t} = r_t - \rho$$

- The Euler equation typically takes the form of a *backward equation* and comes with a terminal condition (C_T) or transversality condition ($\lim_{T\to\infty} C_T$)
- Stationary point only if $r_t = \rho$
- Suppose we are at $r_t = r = \rho$ and a shock is realized. $r_0 > r$ what happens? $r_0 < r$ what happens?

6. Example: Solow growth model

- As before, $Y_t = C_t + I_t$ and
$$\dot{K}_t = Y_t - C_t - \delta K_t$$
- Representative firms operates neoclassical production function
$$Y_t = F(K_t, L_t, A_t)$$
- Normalize labor to $L_t = 1$ and hold TFP constant $A_t = A$
- We again assume constant savings rate: $Y_t - C_t = I_t = sY_t$
- Assume Cobb-Douglas $Y_t = AK_t^\alpha$ so equilibrium allocation
$$\dot{K}_t = sAK_t^\alpha - \delta K_t$$

- Steady state is given by

$$K_{ss} = \left(\frac{sA}{\delta}\right)^{\frac{1}{1-\alpha}}$$

- Key equilibrium condition in $\dot{K}_t$ is *non-linear* — how to proceed?

- Let $X_t = K_t^{1-\alpha}$, then

$$\begin{aligned}
\dot{X}_t &= (1-\alpha)K_t^{-\alpha}\dot{K}_t \\
&= (1-\alpha)K_t^{-\alpha}(sAK_t^{\alpha} - \delta K_t) \\
&= (1-\alpha)sA - (1-\alpha)K_t^{1-\alpha}\delta \\
&= (1-\alpha)sA - (1-\alpha)\delta X_t
\end{aligned}$$

- Solution with initial condition X_0 (work this out):

$$X_t = X_{ss} + e^{-(1-\alpha)\delta t}\left[X_0 - X_{ss}\right], \quad \text{where } X_{ss} = \frac{sA}{\delta}$$

- Transition dynamics (rate of convergence) governed by $-(1-\alpha)\delta$

7. What are partial differential equations?

- Partial differential equations (PDEs) generalize ODEs to higher-dimensional state spaces
- PDEs are at the heart of (i) continuous-time **dynamic programming** and (ii) heterogeneous-agent models in macro
- PDEs have long been a core tool in physics, applied math, ...
  $\Longrightarrow$ increasingly used in economics

- Consider a function $u(x_1, x_2, \ldots, x_n)$ where $x_1, \ldots, x_n$ are coordinates in $\mathbb{R}^n$
- Partial derivatives of $u(\cdot)$

$$\frac{\partial u}{\partial x_i} \equiv \partial_{x_i} u \qquad \text{and} \qquad \frac{\partial^2 u}{\partial x_i \partial x_j} = \partial_{x_i x_j} u$$

- A PDE is an equation in u and its partial derivatives — fully generally:

$$0 = G(u, \partial_{x_1} u, \ldots, \partial_{x_n} u, \partial_{x_1 x_1} u, \ldots)$$

- The *order* of the PDE, is the order of the highest partial derivative
- Examples from physics
  - Heat equation: $\partial_t u = \partial_{xx} u$ (second-order, linear, homogeneous)
  - Wave equation: $\partial_{tt} u = \partial_{xx} u$ (second-order, linear, homogeneous)
  - Transport equation: $\partial_t u = \partial_x u$ (first-order, linear, homogeneous)
- Income distribution “solves heat equation”, wealth dynamics “solve transport equations”, dynamic programming often transport + heat

Part 2: Optimization with Deterministic Dynamics

1. Neoclassical growth model in continuous time

- The lifetime value of the representative household is

$$v(k_0) = \max_{\{c_t\}_{t\geq 0}} \int_0^\infty e^{-\rho t} u(c_t) dt$$

subject to

$$\dot{k}_t = F(k_t) - \delta k_t - c_t$$
$$k_0 \text{ given },$$

where $\dot{x}_t = \frac{d}{dt} x_t$, ρ is the discount rate, c_t is the rate of consumption, $u(\cdot)$ is instantaneous utility flow, and $\dot{k}_t$ is the rate of (net) capital accumulation

- No uncertainty for now

- This is the **sequence problem** in continuous time

2. Calculus of variations

- Resources:
  - LeVeque: Finite Difference Methods for Ordinary and Partial Differential Equations
  - Kamien and Schwartz: Dynamic Optimization
  - Gelfand and Fomin: Calculus of Variations
- This dynamic optimization problem is associated with the Lagrangian

$$L = \int_0^\infty e^{-\rho t} \left[u(c_t) + \mu_t \left(F(k_t) - \delta k_t - c_t - \dot{k}_t \right) \right] dt$$

- μ_t is the Lagrange multiplier on the capital accumulation ODE
- What do we do with $\dot{k}_t$??

- Integrate by parts:

$$\int_0^\infty e^{-\rho t}\mu_t \dot{k}_t dt = e^{-\rho t}\mu_t k_t \Big|_0^\infty - \int_0^\infty \frac{d}{dt}\left(e^{-\rho t}\mu_t\right) k_t dt$$
$$= -\mu_0 k_0 + \int_0^\infty e^{-\rho t}\rho \mu_t k_t dt - \int_0^\infty e^{-\rho t}\dot{\mu}_t k_t dt$$

- Plugging into Lagrangian:

$$L = \int_0^\infty e^{-\rho t}\left[u(c_t) + \mu_t\left(F(k_t) - \delta k_t - c_t\right) - \rho\mu_t k_t + \dot{\mu}_t k_t\right] dt + \mu_0 k_0$$

- What have we accomplished?

- Notice $\mu_0 k_0$, this is crucial. What's intuition?

$$L = \int_0^\infty e^{-\rho t} \Big[u(c_t) + \mu_t \Big(F(k_t) - \delta k_t - c_t \Big) - \rho \mu_t k_t + \dot{\mu}_t k_t \Big] dt + \mu_0 k_0$$

- The planner optimizes over paths $\{c_t\}$ and $\{k_t\}$
- At an optimum, there cannot be *any* small perturbation in these paths that the planner finds preferable
- Let $\{c_t\}$ and $\{k_t\}$ be *candidate* optimal paths. Consider $\hat{c}_t = c_t + \alpha h_t^c$ and $\hat{k}_t = k_t + \alpha h_t^k$ for arbitrary functions h_t^c and h_t^k

$$L(\alpha) = \int_0^\infty e^{-\rho t} \Big[u(c_t + \alpha h_t^c) + \mu_t \Big(F(k_t + \alpha h_t^k) - \delta k_t - \delta \alpha h_t^k - c_t - \alpha h_t^c \Big) \\ - \rho \mu_t (k_t + \alpha h_t^k) + \dot{\mu}_t (k_t + \alpha h_t^k) \Big] dt + \mu_0 (k_0 + \alpha h_0^k)$$

- What about *boundary conditions*? At $t = 0$, capital stock is fixed (k_0 given) while consumption is free. So must have: $h_0^k = 0$ while h_0^c is free

Necessary condition for optimality: $\frac{d}{d\alpha}L(0) = 0$

$$L(\alpha) = \int_0^\infty e^{-\rho t}\bigg[u(c_t + \alpha h_t^c) + \mu_t\Big(F(k_t + \alpha h_t^k) - \delta k_t - \delta\alpha h_t^k - c_t - \alpha h_t^c\Big)$$
$$- \rho\mu_t(k_t + \alpha h_t^k) + \dot{\mu}_t(k_t + \alpha h_t^k)\bigg]dt + \mu_0(k_0 + \alpha h_0^k)$$

Work this out yourselves (many times, in many applications!)

$$\frac{d}{d\alpha}L(0) = \int_0^\infty e^{-\rho t}\bigg[u'(c_t)h_t^c + \mu_t\Big(F'(k_t)h_t^k - \delta h_t^k - h_t^c\Big)$$
$$- \rho\mu_t h_t^k + \dot{\mu}_t h_t^k\bigg]dt + \mu_0 h_0^k$$

where $h_0^k = 0$ because k_0 is fixed

Group terms:

$$\frac{d}{d\alpha}L(0) = \int_0^\infty e^{-\rho t}\left[\Big(u'(c_t) - \mu_t\Big)h_t^c + \Big(\mu_t\Big(F'(k_t) - \delta\Big) - \rho\mu_t + \dot{\mu}_t\Big)h_t^k\right]dt$$

Fundamental Theorem of the Calculus of Variations: Since h_t^c and h_t^k were arbitrary, we must have *pointwise*

$$\begin{aligned} 0 &= u'(c_t) - \mu_t \\ 0 &= \mu_t\Big(F'(k_t) - \delta\Big) - \rho\mu_t + \dot{\mu}_t \end{aligned}$$

Proposition. (Euler equation for marginal utility)

$$\frac{\dot{\mu}_t}{\mu_t} = \frac{\dot{u}_{c,t}}{u_{c,t}} = \rho - F'(k_t) + \delta = \rho - r_t$$

- We have now solved the neoclassical growth model in continuous time. Its solution is given by a system of two ODEs.

- Suppose $u(c) = \log(c)$ and $F(k) = k^\alpha$, then:

$$\frac{\dot{c}_t}{c_t} = \alpha k_t^{\alpha-1} - \delta - \rho$$
$$\dot{k}_t = k_t^\alpha - \delta k_t - c_t$$

  with k_0 given

- Derive the consumption Euler equation yourselves!

- What are the boundary conditions? (Always ask about BCs!)
  - Initial condition on capital: k_0 given
  - Terminal condition on consumption : $\lim_{T\to\infty} c_T = c_{\text{ss}}$

3. Optimal control theory

- Optimal control theory emerged from the calculus of variations

- Applies to dynamic optimization problems in continuous time that feature (ordinary) differential equations as constraints

- Again the neoclassical growth model:

$$v(k_0) = \max_{\{c_t\}_{t\geq 0}} \int_0^\infty e^{-\rho t} u(c_t) dt$$

  subject to

$$\dot{k}_t = F(k_t) - \delta k_t - c_t, \quad k_0 \text{ given}$$

- Three new terms:
  - **State variable**: k_t
  - **Control variable**: c_t
  - **Hamiltonian**: $H(c_t, k_t, \mu_t) = u(c_t) + \mu_t \left[F(k_t) - \delta k_t - c_t \right]$

- With Hamiltonian in hand, *copy-paste* formula that we can always use:
  - **Optimality condition**: $\frac{\partial}{\partial c} H = 0$
  - **Multiplier condition**: $\rho \mu_t - \dot{\mu}_t = \frac{\partial}{\partial k} H$
  - **State condition**: $\dot{k}_t = \frac{\partial}{\partial \mu} H$

- This gives us the same equations that we derived using calc of variations:

$$\begin{aligned} u'(c_t) &= \mu_t \\ \rho \mu_t - \dot{\mu}_t &= \mu_t (F'(k_t) - \delta) \\ \dot{k}_t &= F(k_t) - \delta k_t - c_t \end{aligned}$$

- We again get system of Euler equation and capital accumulation:

$$\begin{aligned} \dot{c}_t &= \frac{u'(c_t)}{u''(c_t)} \Big(\rho - F'(k_t) + \delta \Big) \\ \dot{k}_t &= F(k_t) - \delta k_t - c_t \end{aligned}$$

4. Simple example [*skip*]

- Credit: Kamien-Schwartz p. 129
- Simple problem: not much intuition, but illustrates mechanics

$$\max \int_0^1 (x+u)dt$$

subject to $\dot{x} = 1 - u^2$ and initial condition $x_0 = 1$

- Step 1: form Hamiltonian $H(t,x,u,\lambda) = x + u + \lambda(1-u^2)$
- Step 2: necessary conditions (note: no discounting here)

$$\begin{aligned} 0 &= H_u = 1 - 2\lambda u \\ -\dot{\lambda} &= H_x = 1 \end{aligned}$$

and terminal condition $\lambda_1 = 0$ (because u_1 is *free*)

- Step 3: manipulate necessary conditions:

$$\lambda = 1 - t$$
$$u = \frac{1}{2\lambda}$$

and therefore: $u = \frac{1}{2}(1 - t)$

- Finally: solve for all paths (control, state, multiplier)

$$x_t = t - \frac{1}{4}(1 - t) + \frac{5}{4}$$
$$\lambda_t = 1 - t$$
$$u_t = \frac{1}{2}(1 - t)$$

5. Hamilton-Jacobi-Bellman equation

- Recall the neoclassical growth model in continuous time

$$v(k_0) = \max_{\{c_t\}_{t\geq 0}} \int_0^{\infty} e^{-\rho t} u(c_t) dt$$

subject to

$$\dot{k}_t = F(k_t) - \delta k_t - c_t$$
$$k_0 \text{ given },$$

where $\dot{x}_t = \frac{d}{dt} x_t$, ρ is the discount rate, c_t is the rate of consumption, $u(\cdot)$ is instantaneous utility flow, and $\dot{k}_t$ is the rate of (net) capital accumulation

- No uncertainty for now
- This is the infinite-horizon sequence problem, $t \in [0, \infty)$
- A function $v(\cdot)$ that solves this problem is a solution to the neoclassical growth model

- We will now work towards a recursive representation (good reference: Stokey textbook)
- The discrete-time Bellman equation would be

$$v(k_t) = \max_c \left\{ u(c)\Delta + \frac{1}{1+\rho\Delta} v(k_{t+\Delta}) \right\}$$

where $\beta = \frac{1}{1+\rho\Delta}$

- Next: multiply by $1 + \rho\Delta$

$$\begin{aligned}
(1+\rho\Delta)v(k_t) &= \max_c \left\{ (1+\rho\Delta)u(c)\Delta + v(k_{t+\Delta}) \right\} \\
\rho\Delta v(k_t) &= \max_c \left\{ u(c)\Delta + \rho u(c)\Delta^2 + v(k_{t+\Delta}) - v(k_t) \right\} \\
\rho v(k_t) &= \max_c \left\{ u(c) + \rho u(c)\Delta + \frac{v(k_{t+\Delta}) - v(k_t)}{\Delta} \right\}
\end{aligned}$$

$$\rho v(k_t) = \max_c \left\{ u(c) + \rho u(c)\Delta + \frac{v(k_{t+\Delta}) - v(k_t)}{\Delta} \right\}$$

- Now we take limit $\Delta \to 0$
- Notice $\rho u(c)\Delta \to 0$ and also

$$\lim_{\Delta \to 0} \frac{v(k_{t+\Delta}) - v(k_t)}{\Delta} = \lim_{\Delta \to 0} \frac{v(k(t+\Delta)) - v(k(t))}{\Delta} = \frac{dv(k(t))}{dt}$$

- Therefore we arrive at

$$\rho v(k(t)) = \max_c \left\{ u(c) + \frac{dv(k(t))}{dt} \right\}$$

- Only step left for us to do: What is $\frac{dv(k(t))}{dt}$?
- Simply use Chain rule! $\frac{dv(k(t))}{dt} = \frac{dv}{dk}\frac{dk}{dt}$ and recall $\frac{dk}{dt} = F(k) - \delta k - c$
- Therefore, we arrive at the **Hamilton-Jacobi-Bellman equation**:

$$\rho v(k) = \max_c \left\{ u(c) + \Big(F(k) - \delta k - c \Big) v'(k) \right\}$$

- We drop t subscripts for clarity: this equation holds for all possible k
- Notice: We conjectured a stationary value function (what does this mean?)

6. First-order condition for consumption

- HJB still has "max" operator:

$$\rho v(k) = \max_c \left\{ u(c) + \Big(F(k) - \delta k - c \Big) v'(k) \right\}$$

- To get rid of this, we have to resolve optimal consumption choice
- First-order condition:

$$u'(c(k)) = v'(k)$$

- This defines the **consumption policy function**
- We can now plug back in, obtaining an ODE in $v'(k)$

$$\rho v(k) = u(c(k)) + \Big(F(k) - \delta k - c(k) \Big) v'(k)$$

- Why is this a "stationary" value function and ODE? What would a time-dependent ODE look like? When would we get one?

7. Envelope condition and Euler equation

- We now derive the Euler equation in continuous time
- We start with the **HJB envelope condition**. Differentiating in k:

$$\rho v'(k) = u'(c(k))c'(k) + \Big(F'(k) - \delta - c'(k)\Big)v'(k) + \Big(F(k) - \delta k - c(k)\Big)v''(k)$$

$$\rho v'(k) = \Big(\underbrace{F'(k) - \delta}_{\text{interest rate } r}\Big)v'(k) + \Big(F(k) - \delta k - c(k)\Big)v''(k)$$

$$(\rho - r)v'(k) = \Big(F(k) - \delta k - c(k)\Big)v''(k)$$

- Next, we characterize *process* $dv'(k)$. Using Ito's lemma (even though no uncertainty):

$$\begin{aligned} dv'(k) &= v''(k)dk \\ &= v''(k)(F(k) - \delta k - c(k))dt \\ &= (\rho - r)v'(k)dt. \end{aligned}$$

- Recall first-order condition $u'(c(k)) = v'(k)$.
- The **Euler equation for marginal utility** is given by

$$\frac{du'(c)}{u'(c)} = (\rho - r)dt.$$

- To go from marginal utility to consumption, we use CRRA utility: $u(c) = \frac{1}{1-\gamma}c^{1-\gamma}$. $u'(c) = c^{-\gamma}$ is a function of *process* c, so by Ito's lemma:

$$\begin{aligned} du'(c) &= -\gamma c^{-\gamma-1} dc \\ &= -\gamma u'(c)\frac{dc}{c} \end{aligned}$$

- Plugging in yields **Euler equation for consumption** in continuous time:

$$\frac{dc}{c} = \frac{r-\rho}{\gamma}dt$$

or (you'll often see this notation when no uncertainty): $\frac{\dot{c}}{c} = \frac{r-\rho}{\gamma}$

Connection between calculus of variations and HJB:

- What is the connection between costate / multiplier μ_t and marginal value of wealth $V'(k)$?
- What is the connection between multiplier equation and envelope condition?

9. Boundary conditions

- Important: everything we have done so far is only valid in the **interior of the state space**
- What's the state space of a model?
- For the neoclassical growth model without uncertainty, state space is $k \in [0, \infty)$, or

  $$\left\{k \mid k \in [0, \bar{k}]\right\}$$

  where we impose an upper boundary $\bar{k}$ (think about putting this on the computer)
- This is like the domain of the function $v(k)$ that will be valid
- We say $\{0, \bar{k}\}$ is the **boundary** of the state space and $(0, \bar{k})$ is the **interior**
- As is the case **for all differential equations**, the HJB holds on the interior and we need **boundary conditions** to characterize $v(k)$ along the boundary

- What differential equation is HJB in this model? How many boundary conditions do we need?
- In terms of the economics, what is correct economic behavior at the boundary $k \in \{0, \bar{k}\}$?

- What differential equation is HJB in this model? How many boundary conditions do we need?
- In terms of the economics, what is correct economic behavior at the boundary $k \in \{0, \bar{k}\}$?
- We want households to not leave the state space, so we impose that they do not dissave / borrow as $k \to 0$ and save as $k \to \bar{k}$
- Nice intuition: 2 boundary inequalities do same job as 1 boundary equality
- This implies: (why?)

$$
\begin{aligned}
u'(c(0)) &\geq v'(0) \\
u'(c(\bar{k})) &\leq v'(\bar{k})
\end{aligned}
$$

- In neoclassical growth model, Inada conditions take care of this (you never reach boundary)

Part 3: Applications

1. Labor: search and matching

- One of most important ideas in labor: frictional search and matching
  Diamond-Mortensen-Pissarides (DMP) model

- This is just a simple application of dynamic programming

Firms: Can post vacancies at cost (rate) c, vacancy filled at rate q. Value of a vacancy is given by HJB

$$rV = -c + q(J - V).$$

Assume firms post until $V = 0$. Once matched, workers produce revenue at rate p and cost wage w. Match separates at rate s. Value of job is given by HJB

$$rJ = p - w - sJ$$

Labor demand: In equilibrium, labor demand schedule given by

$$p - w = (r + s)\frac{c}{q}$$

Profit $p - w$ equalized with amortized cost of search / posting vacancies

Workers: When worker is unemployed, gets benefit b and can search at intensity λ, which costs $\psi(\lambda)$. When employed, gets wage w but separates at rate s. Let U be value of unemployment and E value of employment:

$$
\begin{aligned}
rU &= \max_{\lambda} \left\{ b - \psi(\lambda) + \lambda(E - U) \right\} \\
rE &= w + s(U - E)
\end{aligned}
$$

Labor supply schedule characterized by FOC for search intensity

$$\psi'(\lambda) = E - U$$

where E and U solve the coupled system of HJBs

2. Urban / trade / dynamic spatial: migration

- One of most important themes in urban, trade, international and dynamic spatial literatures: people move (migrate) in response to shocks

  For example: To what extent do households migrate in response to China trade shock or climate change?

- Turns out: state-of-the-art dynamic migration model (Caliendo-Dvorkin-Parro) is a simple application of our tools

Households: There are N regions indexed by j. Consider a household i and denote her region $j_{i,t}$. Lifetime utility is

$$V_{i,0} = \max \mathbb{E} \int_0^\infty e^{-\rho t} u(c_{i,t}) dt$$

- Household inelastically supplies 1 unit of labor, earns wage $w_{j_{i,t}}$
- For simplicity: They cannot save or borrow, so $c_{i,t} = w_{j_{i,t}}$ (“hand-to-mouth”)
- Problem will be stationary because w_j are time-invariant

Migration: discrete-choice optimal stopping problem

- Households face fixed cost κ_{jk} to move from j to k
- **Key trick:** At rate μ, household draws opportunity and **extreme-value taste shock** ϵ_k with shape parameter θ for possible destinations k

Recursive representation:

$$\rho V_j = u(w_j) + \mu\Big(\mathbb{E}\Big[\max_k V_k - \kappa_{jk} + \epsilon_k\Big] - V_j\Big)$$

- Using extreme-value taste shocks $\rightarrow$ nice expression for migration flows
- Incredibly easy to solve on computer using tools you're learning in Section

3. Macro: sticky prices

- In the data, firms do not adjust prices instantly
- Price stickiness (nominal rigidities) is at the heart of the New Keynesian model
- We will derive the New Keynesian Phillips Curve as simple application of our tools
  $\Longrightarrow$ much easier to derive in continuous time!!
- Consider a continuum of firms indexed by j that compete monopolistically
- Firm j faces demand function

$$Y_{j,t} = \left(\frac{P_{j,t}}{P_t}\right)^{-\epsilon} Y_t$$

  where Y_t and P_t are aggregate (industry) demand and price index

- Firms produce intermediate varieties with the linear production function

$$Y_{j,t} = A_t N_{j,t}$$

- A_t is aggregate productivity and $N_{j,t}$ firm j's labor demand
- Firm j sells at price $P_{j,t}$, profit = revenue net of operating expenses

$$\Pi_{j,t} = P_{j,t} Y_{j,t} - W_t N_{j,t}$$

- Firms maximize NPV of future profit streams, discounted at interest rate r

- Firms set prices optimally over time by choosing inflation $\dot{P}_{j,t} = P_{j,t}\pi_{j,t}$
- Firms pay quadratic adjustment cost $\frac{\delta}{2}\pi_{j,t}^2 P_t Y_t$ to adjust nominal price
- Firm problem:

$$\max_{\{\pi_{j,t}, N_{j,t}\}_{t\geq 0}} \int_0^\infty e^{-rds}\left(P_{j,t}Y_{j,t} - W_t N_{j,t} - \frac{\delta}{2}\pi_{j,t}^2 P_t\right)dt,$$

- Firms are small and take as given $\{W_t, Y_t, P_t\}_{t\geq 0}$ and initial condition $P_{j,0}$
- Any two firms j and j' with same initial price $P_{j,0} = P_{j',0}$ adopt identical inflation and production policies $\implies$ we get back to representative firm

- Hamiltonian (state: $P_{j,t}$, control: $\pi_{j,t}$, multiplier: $\eta_{j,t}$):

$$\mathcal{H}_t(P_{j,t}, \pi_{j,t}, \eta_{j,t}) = P_{j,t}^{1-\epsilon} P_t^{\epsilon} Y_t - \frac{W_t}{A_t} P_{j,t}^{-\epsilon} P_t^{\epsilon} Y_t - \frac{\delta}{2} \pi_{j,t}^2 P_t Y_t + \eta_{j,t} P_{j,t} \pi_{j,t}$$

- Conditions for optimum:

$$i_t \eta_{j,t} - \dot{\eta}_{j,t} = (1-\epsilon) P_{j,t}^{-\epsilon} P_t^{\epsilon} Y_t + \epsilon \frac{W_t}{A_t} P_{j,t}^{-\epsilon-1} P_t^{\epsilon} Y_t + \eta_{j,t} \pi_{j,t}$$

$$0 = -\delta \pi_{j,t} P_t Y_t + \eta_{j,t} P_{j,t},$$

as well as the initial condition for the multiplier $\eta_{j,0} = 0$

- Now we can impose symmetric equilibrium: $P_{j,t} = P_t$ for all j

$$i_t \eta_t - \dot{\eta}_t = (1-\epsilon) P_t^{-\epsilon} P_t^{\epsilon} Y_t + \epsilon \frac{W_t}{A_t} P_t^{-\epsilon-1} P_t^{\epsilon} Y_t + \eta_t \pi_t$$
$$0 = -\delta \pi_t P_t Y_t + \eta_t P_t$$

- Or simply:

$$i_t \eta_t - \dot{\eta}_t = (1-\epsilon) Y_t + \epsilon \frac{w_t}{A_t} Y_t + \eta_t \pi_t$$
$$\eta_t = \delta \pi_t Y_t$$

- Differentiating eq. 2 ($\dot{\eta}_t = \delta \dot{\pi}_t Y_t + \delta \pi_t \dot{Y}_t$) yields:

$$\dot{\pi}_t = \pi_t \left(i_t - \pi_t - \frac{\dot{Y}_t}{Y_t} \right) - \frac{\epsilon}{\delta} \left(\frac{w_t}{A_t} - \frac{\epsilon - 1}{\epsilon} \right)$$

- From previous slide:

$$\dot{\pi}_t = \pi_t\left(i_t - \pi_t - \frac{\dot{Y}_t}{Y_t}\right) - \frac{\epsilon}{\delta}\left(\frac{w_t}{A_t} - \frac{\epsilon - 1}{\epsilon}\right)$$

- Last step: recall Euler equation of the representative household

$$\frac{\dot{C}_t}{C_t} = r_t - \rho$$

and use goods market clearing

$$Y_t = C_t$$

- **NKPC:**

$$\dot{\pi}_t = \rho\pi_t - \frac{\epsilon}{\delta}\left(\frac{w_t}{A_t} - \frac{\epsilon - 1}{\epsilon}\right)$$

4. IO: duopoly

- Consider continuous-time variant of Ericson-Pakes (1995) quality-ladder model

- Duopolistic competition: 2 firms $i \in \{A, B\}$ produce good with quality q_t^i and maximize NPV of profits: $\max \int_0^\infty e^{-rt} \pi_t^i dt$. They compete over investments ι_t^i:

$$\dot{q}^i = \iota_t^i - \delta q_t^i$$

- Profits π_t^i depend on both firms' product qualities $\rightarrow$ state variables for recursive representation are $\omega \equiv (\omega^A, \omega^B)$

- Best-response of firm A to firm B characterized by HJB

$$rV^A(\omega) = \pi^A(\omega) + \max_{\iota} \left\{ (\iota - \delta\omega^A) V^A_{\omega^A}(\omega) - \Phi(\iota) \right\} + (\iota^B - \delta\omega^B) V^A_{\omega^B}(\omega)$$

where $\Phi(\cdot)$ is cost of investment, and best-response takes ι^B as given

5. Public finance: tax competition

- Two countries, $i \in \{A, B\}$, setting corporate tax rates τ_t^i on firms operating / headquartered in country i

- Mass of multinational firms j, with μ_t denoting % in country A at time t

- Firms relocate activity / headquarters at rate θ towards low-tax country:

$$d\mu_t = \theta \mu_t (\tau_t^B - \tau_t^A)^\gamma dt$$

- Country A maximizes tax revenue: $\max \int_0^\infty e^{-\rho t} \tau_t^A \mu_t dt$. Countries compete over taxes $\{\tau_{it}\}$

- Dynamic Nash: country A sets τ_t^A as best response taking τ_t^B as given

- Recursive representation: the only state variable is μ_t

$$\rho V^A(\mu) = \max_{\tau^A} \left\{ \tau^A \mu + \theta \mu \left(\tau^B(\mu) - \tau^A \right)^\gamma \partial_\mu V^A(\mu) \right\}$$

  Best response strategies: $0 = \mu + \gamma \theta \mu (\tau^B(\mu) - \tau^A)^{\gamma - 1} V_\mu^A(\mu)$