Dynamic Programming and Applications

Heterogeneous Agents and Inequality

Lecture 11

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Outline

Part 2: Huggett (1993) in continuous time (Bewley-Huggett-Aiyagari)

This is the textbook heterogeneous agent model / standard incomplete markets model

1. Households, firms and market clearing
2. Competitive equilibrium in sequence form
3. Recursive representation using dynamic programing
4. Competitive equilibrium in recursive form
5. Stationary competitive equilibrium in recursive form

I learned all this from Benjamin Moll!

Income and Wealth Distribution in Macroeconomics: A Continuous-Time Approach

See also Ben's teaching material: `https://benjaminmoll.com/lectures/`

Part 2: Huggett (1993)

Model overview

- Time is continuous, $t \in [0, \infty)$
- No aggregate uncertainty $\implies$ focus on one-time, unanticipated ("MIT") shocks (perfect foresight wrt. macroeconomic aggregates)
- Two types of agents: continuum of households (measure 1) + representative firm
- Households face "uninsurable idiosyncratic income risk"
- There is a single riskfree asset in zero net supply ($\sim$ government bond)

Plan:

1. Present model in sequence form focusing exposition on individual i
2. Recursive representation of competitive equilibrium

Households

Preferences. The individual lifetime utility of a household $i \in [0,1]$ is

$$V_{i,0} = \max_{\{c_{i,t}\}_{t\geq 0}} \mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_{i,t}) dt$$

Budget and **borrowing constraints**:

$$\dot{a}_{i,t} = w_t y_{i,t} + r_t a_{i,t} - c_{i,t}$$
$$a_{i,t} \geq \underline{a}$$

Idiosyncratic income risk: each $y_{i,t}$ follows a Markov chain (later: diffusion)

$$y_{i,t} \in \{y_1, y_2\} \quad \text{Poisson with intensities } \lambda_1, \lambda_2$$

Definition. The problem of household i (in sequence form) is

$$\max_{\{c_{i,t}\}_{t\geq 0}} \mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_{i,t}) dt \qquad \text{s.t.}$$

$$\dot{a}_{i,t} = w_t y_{i,t} + r_t a_{i,t} - c_{i,t}$$

$$y_{i,t} \in \{y_1, y_2\} \text{ Poisson with intensities } \lambda_1, \lambda_2$$

$$a_{i,t} \geq \underline{a}$$

taking as given initial $(a_{i,0}, y_{i,0})$

A **solution** to the household problem is a stochastic process $\{c_{i,t}, a_{i,t}\}_{t\geq 0}$

Firms

- A representative firm produces the (homogeneous) final consumption good using technology
$$Y_t = A_t \ell_t$$
- Firms are small and perfectly competitive $\implies$ firm problem is
$$\max_{\ell_t} Y_t - w_t \ell_t$$
- Firm problem is **static** $\implies$ otherwise we would have to think hard about ownership

Markets

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Goods market:

$$Y_t = \int_0^1 c_{i,t} di$$

Labor market:

$$\ell_t = \int_0^1 y_{i,t} di$$

Asset market:

$$0 = \int_0^1 a_{i,t} di$$

Competitive Equilibrium

Definition. (Competitive equilibrium: sequence form) *Taking as given an initial distribution of assets and individual labor productivities* $\{a_{i,0}, y_{i,0}\}_i$ *as well as an exogenous path for TFP* $\{A_t\}$*, a competitive equilibrium comprises an allocation* $\{Y_t, \ell_t, c_{i,t}, a_{i,t}\}$ *and prices* $\{r_t, w_t\}$ *such that: (i) households optimize, (ii) firms optimize, and (iii) markets clear.*

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- Always show definition of your equilibrium!
- Why is this a definition of competitive equilibrium “in sequence form”?
- How many parts are there to this definition?

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  Recursive representation = functions over state variables
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  What's the difference between PE and GE?

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- What about the household problem in PE? What are the state variables? Firms?
  What's the difference between PE and GE?
- What about markets and GE? What does $\int_0^1 c_{i,t} di$ mean?

Recall household problem: Given $(a_{i,0}, y_{i,0})$,

$$\max_{\{c_{i,t}\}_{t\geq 0}} \mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_{i,t}) dt \qquad \text{s.t.}$$

$$\dot{a}_{i,t} = w_t y_{i,t} + r_t a_{i,t} - c_{i,t}$$

$$y_{i,t} \in \{y_1, y_2\} \text{ Poisson with intensities } \lambda_1, \lambda_2$$

$$a_{i,t} \geq \underline{a}$$

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Recursive representation:

$$\rho V_t(a, y) = \max_c \left\{ u(c) + \mathbb{E}_t \frac{dV_t(a, y)}{dt} \right\}$$

Two **Q**s: What is $\mathbb{E}_t \frac{dV_t(a,y)}{dt}$? And what about borrowing constraint?

From previous slide:

$$\rho V_t(a,y) = \max_c \left\{ u(c) + \mathbb{E}_t \frac{dV_t(a,y)}{dt} \right\}$$

Q1: What is continuation value?

From previous slide:

$$\rho V_t(a,y) = \max_c \left\{ u(c) + \mathbb{E}_t \frac{dV_t(a,y)}{dt} \right\}$$

Q1: What is continuation value?

$$\rho V_t(a,y_j) = \max_c \left\{ u(c) + (r_t a + w_t y_j - c)\partial_a V_t(a,y_j) + \lambda_j (V_t(a,y_{-j}) - V_t(a,y_j)) + \partial_t V_t(a,y_j) \right\}$$

Resolving max operator gives FOC, which defines **consumption policy function**:

$$u'(c_t(a,y_j)) = \partial_a V_t(a,y_j)$$

for all a and j. Define **savings policy function** as $s_t(a,y_j) = r_t a + w_t y_j - c_t(a,y_j)$

Q2: Where is the borrowing constraint $a_{i,t} \geq \underline{a}$ in the HJB?

Answer: in the boundary condition!

- Borrowing constraint gives rise to **state constraint boundary condition**

$$\partial_a V_t(\underline{a}, y_j) \geq u'(r_t\underline{a} + w_t y_j)$$

- Economic intuition: value of saving must be weakly larger than value of consuming
- Heuristic derivation: the FOC still holds at the borrowing constraint

$$u'(c_t(\underline{a}, j)) = \partial_a V_t(a, y_j)$$

- But borrowing constraint requires that

$$s_t(\underline{a}, y_j) = r_t\underline{a} + w_t y_j - c_t(\underline{a}, y_j) \geq 0$$

- Borrowing constraint showing up as boundary condition = major advantage of continuous time!

Summary: A solution to the household problem in **recursive form** is a set of two functions $V_t(a, y)$ and $c_t(a, y)$ that satisfy

$$\rho V_t(a, y_j) = u(c_t(a, y_j)) + (r_t a + w_t y_j - c_t(a, y_j))\partial_a V_t(a, y_j) + \lambda_j(V_t(a, y_{-j}) - V_t(a, y_j)) + \partial_t V_t(a$$

$$u'(c_t(a, y_j)) = \partial_a V_t(a, y_j)$$

with HJB boundary condition

$$\partial_a V_t(\underline{a}, y_j) \geq u'(r_t \underline{a} + w_t y_j)$$

To save space, will now use savings policy function $s_t(a, y_j) \equiv r_t a + w_t y_j - c_t(a, y_j)$ as shorthand

Income and Wealth Distribution

- We now have recursive representations of the household (and firm) problem
- But how do we do GE? We have conditions like $Y_t = \int_0^1 c_{i,t} di$
- Instead, we want to get **aggregate consumption** by integrating over $c_t(a, y)$
- Issue: there may be many households i in state (a, y)! Remember (important): household i is **uniquely** identified by her states (a_i, y_i) in this model
- Solution: integrate against the joint density $\sim$ income-wealth distribution $g_t(a, y)$:

$$Y_t = \iint c_t(a, y) g_t(a, y)\, da\, dy \equiv \sum_j \int c_t(a, y_j) g_t(a, y_j) da$$

Kolmogorov Forward Equation

- If we knew $g_t(a, y)$, we'd be done! Why?

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- Where does this equilibrium condition come from?

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- Notice: $g_t(a,y)$ is a new object that wasn't part of the eq. definition in sequence form
  $\Longrightarrow$ Need new equilibrium condition!
- Where does this equilibrium condition come from?
  $\Longrightarrow$ $g_t(a,y)$ must be consistent with household behavior
- Turns out: $g_t(a,y)$ solves a **Kolmogorov forward equation**
- Another major advantage of continuous time!

Result: the joint density $g_t(a,y)$ solves the Kolmogorov forward (KF) equation

$$\partial_t g_t(a,y_j) = -\partial_a \Big[(r_t a + w_t y_j - c_t(a,y_j)) g_t(a,y_j) \Big] - \lambda_j g_t(a,y_j) + \lambda_{-j} g_t(a,y_{-j})$$

Proof: Define aggregate consumption as cross-sectional average consumption

$$C_t = \mathbb{E}_i(c_{i,t}) = \mathbb{E}_g(c_t(a,y)) \equiv \iint c_t(a,y) g_t(a,y)\, da\, dy$$

By "Ito's lemma", we have

$$dc_t(a,y_j) = \partial_t c_t(a,y_j)dt + s_t(a,y_j)\partial_a c_t(a,y_j)dt + \lambda_j(c_t(a,y_{-j}) - c_t(a,y_j))dt$$

On the **one hand**, we have

$$dC_t = \mathbb{E}_g(dc_t(a,y)) = \iint dc_t(a,y)g_t(a,y)\,da\,dy$$

and so plugging in

$$= \sum_j \int \Big[\partial_t c_t(a,y_j) + s_t(a,y_j)\partial_a c_t(a,y_j) + \lambda_j(c_t(a,y_{-j}) - c_t(a,y_j))\Big] g_t(a,y_j)\,da\,dt$$

On the **other hand**, we have

$$dC_t = d\iint c_t(a,y)g_t(a,y)\,da\,dy = \sum_j \int \Big[g_t(a,y_j)\partial_t c_t(a,y_j) + c_t(a,y_j)\partial_t g_t(a,y_j)\Big]\,dadt$$

We now **equate the two** (next slide)

$$0 = \sum_j \int \left\{ \Big[s_t(a, y_j) \partial_a c_t(a, y_j) + \lambda_j (c_t(a, y_{-j}) - c_t(a, y_j)) \Big] g_t(a, y_j) - \Big[c_t(a, y_j) \partial_t g_t(a, y_j) \Big] \right\} da\, d$$

Integrating the first term by parts:

$$\sum_j \int \Big[- \partial_a [s_t(a, y_j) g_t(a, y_j] - \lambda_j g_t(a, y_j) + \lambda_{-j} g_t(a, y_{-j}) \Big] c_t(a, y)\, da\, dt$$

(plus boundary conditions: abstract from those for now)

Finally arrive at:

$$0 = \sum_j \int \Big[- \partial_t g_t(a, y_j) - \partial_a [s_t(a, y_j) g_t(a, y_j] - \lambda_j g_t(a, y_j) + \lambda_{-j} g_t(a, y_{-j}) \Big] c_t(a, y_j)\, da\, dt$$

Concluding: this must hold "for all" $c_t(a, y)$, so term in brackets $= 0$

Competitive Equilibrium: Recursive Form

Definition. Taking as given an initial joint density $g_0(a,y)$ and an exogenous path of TFP $\{A_t\}$, a competitive equilibrium (in recursive form) comprises **functions**

$$\Big\{V_t(a,y), c_t(a,y), g_t(a,y)\Big\} \quad \text{and} \quad \Big\{Y_t, \ell_t, r_t, w_t\Big\}$$

such that (i) households optimize, (ii) firms optimize, (iii) markets clear, and (iv) the joint density evolves consistently with household behavior.

HJB and **FOC**:

$$\rho V_t(a,y_j) = u(c_t(a,y_j)) + s_t(a,y_j)\partial_a V_t(a,y_j) + \lambda_j(V_t(a,y_{-j}) - V_t(a,y_j)) + \partial_t V_t(a,y_j)$$

$$\partial_a V_t(\underline{a},y_j) \geq u'(r_t\underline{a} + A_t y_j)$$

$$u'(c_t(a,y_j)) = \partial_a V_t(a,y_j)$$

KF: $$\partial_t g_t(a,y_j) = -\partial_a \Big[s_t(a,y_j) g_t(a,y_j) \Big] - \lambda_j g_t(a,y_j) + \lambda_{-j} g_t(a,y_{-j})$$

Bond market: $$0 = \sum_j \int a g_t(a,y_j) da$$

(We plugged in for $w_t = A_t$ and dropped goods market clearing by Walras' law)

In **continuous time**: HA models = system of 2 coupled PDEs!

Stationary Competitive Equilibrium

Definition. With $A_t = A$, a **stationary competitive equilibrium** comprises **functions**

$$\Big\{V(a,y), c(a,y), g(a,y)\Big\} \quad \text{and} \quad \Big\{Y, \ell, r, w\Big\}$$

such that (i) households optimize, (ii) firms optimize, (iii) markets clear, and (iv) the joint density evolves consistently with household behavior.

- Natural extension of “steady state” concept to HA economies
- Macroeconomic aggregates are constant. Distribution $g(a,y)$ is constant but households still move around as they draw idiosyncratic income shocks
- Usual notion of “steady” is: “if you start there, you stay there”

Stationary Competitive Equilibrium Conditions

$$\rho V(a, y_j) = u(c(a, y_j)) + s(a, y_j)\partial_a V(a, y_j) + \lambda_j (V(a, y_{-j}) - V(a, y_j))$$

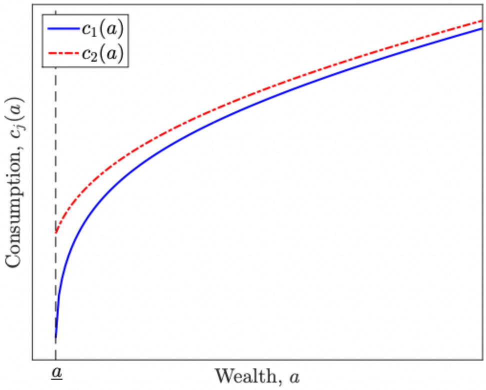
$$\partial_a V(\underline{a}, y_j) \geq u'(r\underline{a} + Ay_j)$$

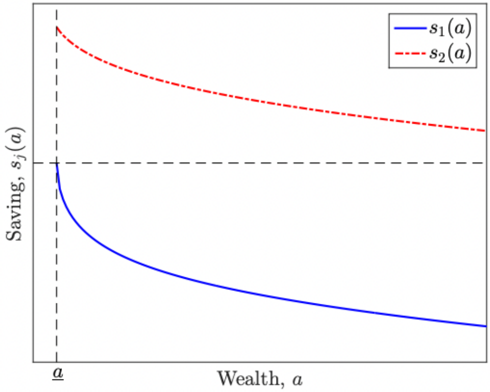
$$u'(c(a, y_j)) = \partial_a V(a, y_j)$$

$$0 = -\partial_a \Big[s(a, y_j) g(a, y_j) \Big] - \lambda_j g(a, y_j) + \lambda_{-j} g(a, y_{-j})$$

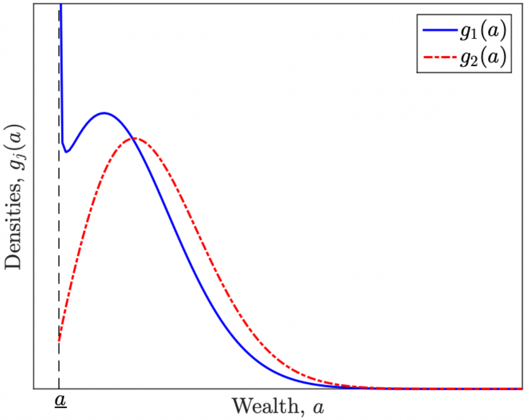
$$0 = \sum_j \int a g(a, y_j) da$$

Typical Consumption and Saving Policy Functions

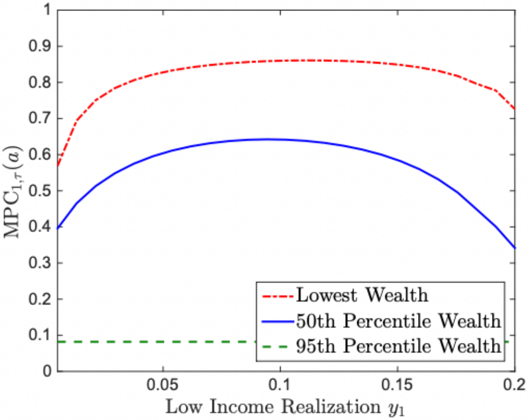




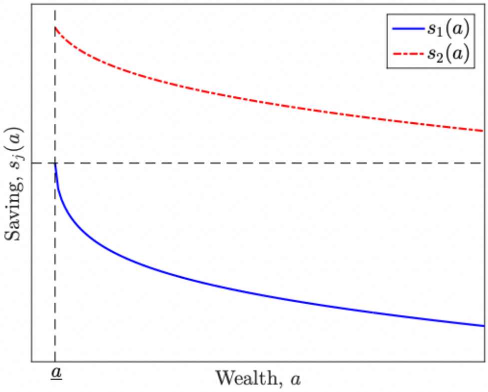
Typical Stationary Distribution

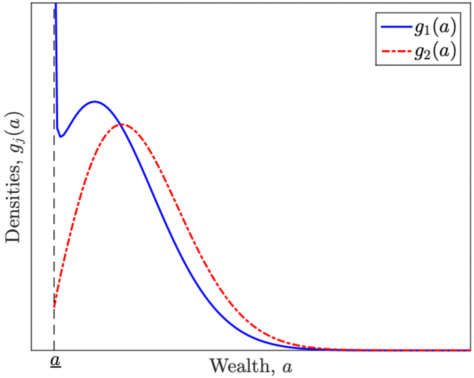


Household MPC

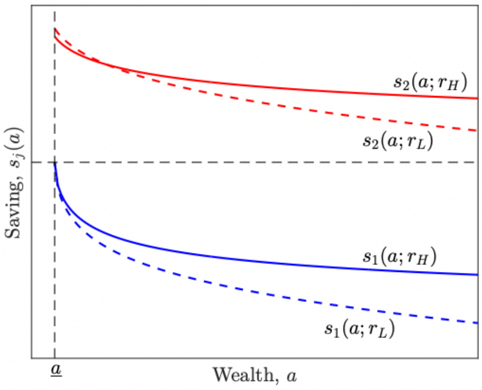


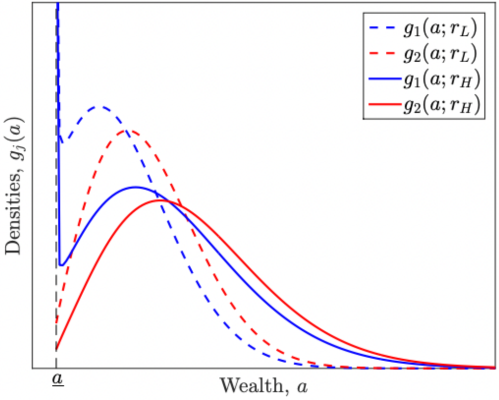
General Equilibrium: Existence and Uniqueness



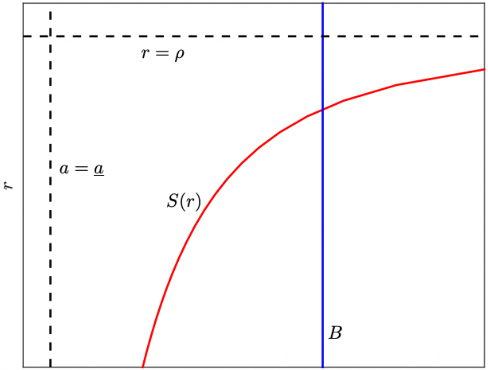


Increase in r from r_L to $r_H > r_L$



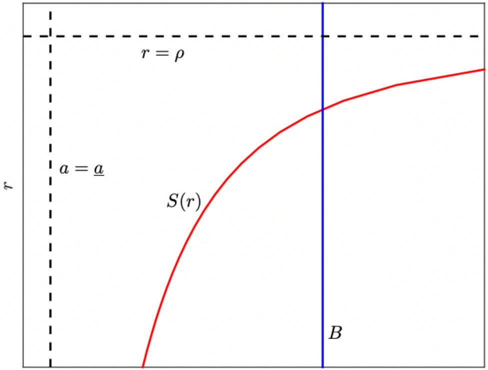


Stationary Equilibrium



Asset Supply $\quad S(r) = \int_{\underline{a}}^{\infty} a g_1(a;r)da + \int_{\underline{a}}^{\infty} a g_2(a;r)da$

Stationary Equilibrium



Asset Supply $\quad S(r) = \int_{\underline{a}}^{\infty} a g_1(a;r)da + \int_{\underline{a}}^{\infty} a g_2(a;r)da$

Proposition: a stationary equilibrium exists (also unique!)

Extension: Diffusion Income Process