202A: Dynamic Programming and Applications

Homework #3 Solutions

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Problem 1: Brownian Motion

This problem collects several exercises on Brownian motion and stochastic calculus. We denote standard Brownian motion by B_t.

(a) Show that $\mathbb{C}ov(B_s, B_t) = \min\{s, t\}$ for two times $0 \leq s < t$. Use the following tricks: Use the covariance formula $\mathbb{C}ov(A, B) = \mathbb{E}(AB) - \mathbb{E}(A)\mathbb{E}(B)$. Use $B_t \sim \mathcal{N}(0, t)$ as well as $B_t - B_s \sim \mathcal{N}(0, t-s)$. And use $B_t = B_s + (B_t - B_s)$.

First, we have

$$\mathbb{C}ov(B_s, B_t) = \mathbb{E}(B_s B_t) - \mathbb{E}(B_s)\mathbb{E}(B_t) = \mathbb{E}(B_s B_t)$$

Then

$$\mathbb{E}(B_s B_t) = \mathbb{E}(B_s(B_s + (B_t - B_s))) = \mathbb{E}(B_s^2) + \mathbb{E}(B_s(B_t - B_s))$$

Notice that B_s and $(B_t - B_s)$ are two independent random variables, hence

$$\mathbb{C}ov(B_s, B_t) = s + \mathbb{E}(B_s)\mathbb{E}(B_t - B_s) = s = \min\{s, t\}$$

(b) Let $X_t = B_t^2$. What is $\mathbb{E}X_t$? What is $\mathbb{C}ov(X_t, X_s)$?

Since $X_t = B_t^2$ and $B_t \sim \mathcal{N}(0, t)$, we know that B_t^2 follows a chi-square distribution with one degree of freedom scaled by t. Thus, $\mathbb{E}(B_t^2) = t$, so

$$\mathbb{E}(X_t) = \mathbb{E}(B_t^2) = t$$

Now, to compute $\mathbb{C}(X_t, X_s)$, we use

$$\mathbb{C}(X_t, X_s) = \mathbb{E}(B_t^2 B_s^2) - \mathbb{E}(B_t^2)\mathbb{E}(B_s^2)$$

Since $\mathbb{E}(B_t^2) = t$ and $\mathbb{E}(B_s^2) = s$, we have $\mathbb{E}(B_t^2)\mathbb{E}(B_s^2) = ts$.
Now, let's find $\mathbb{E}[B_t^2 B_s^2]$. Express $B_t = B_s + (B_t - B_s)$, then

$$B_t^2 = (B_s + (B_t - B_s))^2 = B_s^2 + 2B_s(B_t - B_s) + (B_t - B_s)^2$$

Thus,

$$B_t^2 B_s^2 = \left(B_s^2 + 2B_s(B_t - B_s) + (B_t - B_s)^2\right) B_s^2$$

Expanding this gives

$$B_t^2 B_s^2 = B_s^4 + 2B_s^3(B_t - B_s) + B_s^2(B_t - B_s)^2$$

Since B_s and $B_t - B_s$ are independent, the cross-term $\mathbb{E}[B_s^3(B_t - B_s)] = \mathbb{E}[B_s^3] \cdot \mathbb{E}[B_t - B_s] = 0$, as $\mathbb{E}[B_t - B_s] = 0$. So, we're left with

$$\mathbb{E}[B_t^2 B_s^2] = \mathbb{E}[B_s^4] + \mathbb{E}[B_s^2] \cdot \mathbb{E}[(B_t - B_s)^2]$$

For a Brownian motion:

$$\mathbb{E}[B_s^4] = 3s^2, \quad \mathbb{E}[B_s^2] = s, \quad \text{and} \quad \mathbb{E}[(B_t - B_s)^2] = t - s$$

Therefore,

$$\mathbb{E}[B_t^2 B_s^2] = 3s^2 + s(t - s)$$

Finally,

$$\mathbb{C}(X_t, X_s) = \mathbb{E}[B_t^2 B_s^2] - ts = \left(3s^2 + s(t - s)\right) - ts = 2s^2$$

(c) Let $X_t = B_{t+s} - B_s$ for some fixed $s > 0$. Let $Y_t = \frac{1}{\sqrt{\lambda}} B_{\lambda t}$. Show that both X_t and Y_t are standard Brownian motion; that is, show the following 5 properties:

(i) $X_0 = Y_0 = 0$

(ii) $X_t, Y_t \sim \mathcal{N}(0, t)$

(iii) Stationarity

(iv) Independent increments

(v) Continuous

(i) By definition, $X_0 = B_{0+s} - B_s = B_s - B_s = 0$ and $Y_0 = \frac{1}{\sqrt{\lambda}} B_{\lambda \cdot 0} = B_0 = 0$.

(ii) For X_t, note that $B_{t+s} - B_s \sim \mathcal{N}(0, t)$ because $B_{t+s} - B_s$ is the increment of a Brownian motion over an interval of length t. For Y_t, since $B_{\lambda t} \sim \mathcal{N}(0, \lambda t)$, it follows that $Y_t = \frac{1}{\sqrt{\lambda}} B_{\lambda t} \sim \mathcal{N}(0, t)$.

(iii) For X_t, $X_{t+h} - X_t = (B_{t+h+s} - B_s) - (B_{t+s} - B_s) = B_{t+h+s} - B_{t+s} \sim \mathcal{N}(0, h)$, which depends only on h, not on t. Similarly for Y_t, $Y_{t+h} - Y_t = \frac{1}{\sqrt{\lambda}}(B_{\lambda(t+h)} - B_{\lambda t}) \sim \mathcal{N}(0, h)$, which depends only on h, not on t.

(iv) Increments of X_t are independent because they are increments of Brownian motion B_t. For instance, $X_{t+h} - X_t$ and $X_t - X_{t-k}$ are independent for non-overlapping intervals. Similarly, increments of Y_t are also independent because B_t has independent increments, and scaling by $\frac{1}{\sqrt{\lambda}}$ preserves this property.

(v) Both X_t and Y_t are continuous processes because they are defined in terms of continuous Brownian motion.

(d) Geometric Brownian motion evolves as: $dX_t = \mu X_t dt + \sigma X_t dB_t$ given an initial value X_0. Show that $X_t = X_0 e^{\mu t - \frac{\sigma^2}{2}t + \sigma B_t}$. Also show that $\mathbb{E} = X_0 e^{\mu t}$.

To derive the solution it will be useful to apply Ito's lemma to the function $f = log(X_t)$. Recall with a function of stochastic process we cannot use standard calculus, instead we can use Ito's lemma as a stochastic version of the chain rule.
Applying Ito's lemma to f:

$$df = dlog(X_t) = \partial_t f dt + \partial_X f \mu X_t dt + \frac{1}{2}\partial_{XX} f \sigma^2 X_t^2 dt + \sigma X_t \partial_X f dB_t$$

The partial derivatives are: $\partial_t f = \frac{\partial log(X_t)}{\partial t} = 0$, $\partial_X f = \frac{1}{X_t}$, and $\partial_{XX} f = -\frac{1}{X_t^2}$.
Substituting:

$$dlog(X_t) = (\mu - \frac{\sigma^2}{2}) + \sigma dB_t$$

Integrating (and use $B_0 = 0$):

$$log(X_t) - log(X_0) = (\mu - \frac{\sigma^2}{2})t + \sigma B_t$$

$$e^{log(\frac{X_t}{X_0})} = e^{\mu t - \frac{\sigma^2}{2}t + \sigma B_t}$$

$$\Rightarrow X_t = X_0 e^{\mu t - \frac{\sigma^2}{2}t + \sigma B_t}$$

Taking expectations:

$$\mathbb{E}(X_t) = X_0 e^{\mu t - \frac{\sigma^2}{2}t}\mathbb{E}(e^{\sigma B_t})$$

Recall $B_t \sim \mathcal{N}(0, t)$, it is useful to substitute $\sigma B_t = \sigma\sqrt{t}Z$ where $Z \sim \mathcal{N}(0, 1)$.

$$\mathbb{E}(e^{\sigma B_t}) = \int e^{\sigma\sqrt{t}Z}\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}dz$$

For the term in the exponential we have:

$$\sigma\sqrt{t}Z - \frac{z^2}{2} = -\frac{1}{2}(z^2 - 2\sigma\sqrt{t}z) = -\frac{1}{2}(z - \sigma\sqrt{t})^2 + \frac{\sigma^2 t}{2}$$

Substitute back:

$$\mathbb{E}(e^{\sigma B_t}) = e^{\frac{\sigma^2 t}{2}}\int \frac{e^{-\frac{(z-\sigma\sqrt{t})^2}{2}}}{\sqrt{2\pi}}dz = e^{\frac{\sigma^2 t}{2}}$$

Because the second term is the density of a $\mathcal{N}(\sigma\sqrt{t}, 1)$ which integrates to one.
Substituting into the first equation we get the result.

(e) The Ornstein-Uhlenbeck (OU) process is like a continuous-time variant of the AR(1) process. It evolves as $dX_t = -\mu X_t dt + \sigma dB_t$ for drift parameter μ, diffusion parameter σ, and some X. Show that it solves $X_t = X_0 e^{-\mu t} + \sigma \int_0^t e^{-\mu(t-s)} dB_s$.

As before we derive the solution by applying the Ito's lemma to a properly chosen function. Let $f = e^{\mu t} X_t$, applying Ito's lemma:

$$df = de^{\mu t} X_t = \partial_t f dt + \partial_X f(-\mu X_t) dt + \frac{1}{2} \partial_{XX} f \sigma^2 dt + \sigma \partial_X f dB_t$$

Substitute the partial derivatives:

$$de^{\mu t} X_t = \mu e^{\mu t} X_t dt + -\mu X_t e^{\mu t} dt + 0 + \sigma e^{\mu t} dB_t$$

Integrate:

$$e^{\mu t} X_t - X_0 = \sigma \int_0^t e^{\mu s} dB_s$$

Multiplying by $e^{-\mu t}$ we get the result.

(f) Let $X_t = \int_0^t B_s ds$ and $Y_t = \int_0^t B_s^2 ds$. Compute $\mathbb{E}(X_t)$ and $\mathbb{E}(Y_t)$, as well as $\mathbb{V}ar(X_t)$ and $\mathbb{V}ar(Y_t)$.

Since $X_t = \int_0^t B_s \, ds$ and B_s is a standard Brownian motion with $\mathbb{E}(B_s) = 0$, we have:

$$\mathbb{E}(X_t) = \mathbb{E}\left(\int_0^t B_s \, ds\right) = \int_0^t \mathbb{E}(B_s) \, ds = \int_0^t 0 \, ds = 0$$

For $Y_t = \int_0^t B_s^2 \, ds$, since B_s^2 has mean s (since $B_s \sim \mathcal{N}(0, s)$):

$$\mathbb{E}(Y_t) = \mathbb{E}\left(\int_0^t B_s^2 \, ds\right) = \int_0^t \mathbb{E}(B_s^2) \, ds = \int_0^t s \, ds = \frac{t^2}{2}$$

To find $\mathbb{V}(X_t)$, we first compute $\mathbb{E}(X_t^2)$:

$$X_t^2 = \left(\int_0^t B_s \, ds\right)^2 = \int_0^t \int_0^t B_s B_u \, ds \, du$$

Taking expectations, we use $\mathbb{E}(B_s B_u) = \min(s, u)$:

$$\mathbb{E}(X_t^2) = \int_0^t \int_0^t \min(s, u) \, ds \, du = \frac{t^3}{3}$$

Thus,

$$\mathbb{V}(X_t) = \mathbb{E}(X_t^2) - (\mathbb{E}(X_t))^2 = \frac{t^3}{3}$$

To find $\mathbb{V}(Y_t)$, we first compute $\mathbb{E}(Y_t^2)$:

$$Y_t^2 = \int_0^t B_s^2 \, ds \int_0^t B_r^2 \, dr$$

Expanding and applying the expectation $\mathbb{E}(B_s^2 B_r^2) = 2\min(s^2, r^2) + sr$, we get:

$$\mathbb{E}(Y_t^2) = \int_0^t \int_0^t \left(2\min(s,r)^2 + sr\right)\, ds\, dr$$

Splitting the integral, we have:

$$\begin{aligned}\mathbb{E}(Y_t^2) &= 2\int_0^t \int_0^r s^2\, ds\, dr + 2\int_0^t \int_r^t r^2\, ds\, dr + \int_0^t \int_0^t sr\, ds\, dr \\ &= 2\int_0^t \frac{r^3}{3}\, dr + 2\int_0^t \left(r^2 t - r^3\right)\, dr + \int_0^t \frac{rt^2}{2}\, dr \\ &= \frac{1}{6}t^4 + 2(\frac{1}{3}t^4 - \frac{1}{4}t^4) + \frac{t^4}{4}\end{aligned}$$

Thus,

$$\mathbb{V}(Y_t) = \mathbb{E}(Y_t^2) - (\mathbb{E}(Y_t))^2 = \frac{1}{6}t^4 + \frac{2}{3}t^4 - \frac{1}{2}t^4 + \frac{t^4}{4} - \frac{t^4}{4} = \frac{t^4}{3}$$

Problem 2: Poisson Process

(a) Consider a two-state Markov chain in continuous time denoted Y_t, which can take on values in $\{Y^1, Y^2\}$. The transition rate is given by λ regardless of the current state. Suppose we are in state 1 at $t = 0$. Compute the expected time until the process switches to state 2.

In a continuous time Markov chain, the transition from one state to another is driven by an exponential waiting time. The expected waiting time $\mathbb{E}(T)$ for an exponential distribution with rate λ is simply:

$$\mathbb{E}(T) = \frac{1}{\lambda}$$

So, the expected time until the process switches to state Y^2 is $\frac{1}{\lambda}$.

(b) Now suppose that the transition rates differ depending on which state we are in. That is, if we're in state 1 we transition to rate 2 at rate λ_1, and vice versa at rate λ_2. Show that the fraction of time the process spends in states 1 and 2 converges in the long run to $\frac{\lambda_2}{\lambda_1+\lambda_2}$ and $\frac{\lambda_1}{\lambda_1+\lambda_2}$.

With transition rates λ_1 (from Y^1 to Y^2) and λ_2 (from Y^2 to Y^1), we can set up the steady-state probabilities. Let π_1 and π_2 represent the long-run fractions of time spent in states Y^1 and Y^2, respectively. The steady state represents a situation where the rate of flow into each state equals the rate of flow out of each state. The balance equations are:

$$\pi_1 \lambda_1 = \pi_2 \lambda_2$$

$$\pi_1 + \pi_2 = 1$$

From the first equation, we get $\pi_2 = \frac{\pi_1 \lambda_1}{\lambda_2}$. Substituting this into the second equation:

$$\pi_1 + \frac{\pi_1 \lambda_1}{\lambda_2} = 1$$

Solving for π_1:

$$\pi_1 \left(1 + \frac{\lambda_1}{\lambda_2}\right) = 1 \Rightarrow \pi_1 = \frac{\lambda_2}{\lambda_1 + \lambda_2}$$

Then $\pi_2 = 1 - \pi_1 = \frac{\lambda_1}{\lambda_1+\lambda_2}$.
Thus, the fraction of time spent in Y^1 is $\frac{\lambda_2}{\lambda_1+\lambda_2}$, and in Y^2 it is $\frac{\lambda_1}{\lambda_1+\lambda_2}$.

(c) What is $\mathbb{E}(Y_t \mid Y_0 = Y^1)$

To find $\mathbb{E}(Y_t \mid Y_0 = Y^1)$, we need to consider the expected value of Y_t at time t given that the process starts in state Y^1 at $t = 0$. Since Y_t is a two-state Markov chain with transition rates λ_1 from Y^1 to Y^2 and λ_2 from Y^2 to Y^1, the probability that the process is in state Y^1 at time t (denoted $P(Y_t = Y^1 \mid Y_0 = Y^1)$) evolves according to the exponential decay formula for a two-state Markov chain.
This probability is given by:

$$P(Y_t = Y^1 \mid Y_0 = Y^1) = \frac{\lambda_2}{\lambda_1 + \lambda_2} + \frac{\lambda_1}{\lambda_1 + \lambda_2} e^{-(\lambda_1+\lambda_2)t}.$$

Similarly, the probability of being in state Y^2 at time t is:

$$P(Y_t = Y^2 \mid Y_0 = Y^1) = \frac{\lambda_1}{\lambda_1 + \lambda_2} - \frac{\lambda_1}{\lambda_1 + \lambda_2} e^{-(\lambda_1+\lambda_2)t}.$$

Now, we can compute $\mathbb{E}(Y_t \mid Y_0 = Y^1)$ as a weighted average:

$$\mathbb{E}(Y_t \mid Y_0 = Y^1) = Y^1 \cdot P(Y_t = Y^1 \mid Y_0 = Y^1) + Y^2 \cdot P(Y_t = Y^2 \mid Y_0 = Y^1)$$

Substituting the probabilities:

$$\mathbb{E}(Y_t \mid Y_0 = Y^1) = Y^1 \left(\frac{\lambda_2}{\lambda_1 + \lambda_2} + \frac{\lambda_1}{\lambda_1 + \lambda_2} e^{-(\lambda_1+\lambda_2)t}\right) + Y^2 \left(\frac{\lambda_1}{\lambda_1 + \lambda_2} - \frac{\lambda_1}{\lambda_1 + \lambda_2} e^{-(\lambda_1+\lambda_2)t}\right)$$

Simplifying, we get:

$$\mathbb{E}(Y_t \mid Y_0 = Y^1) = \frac{\lambda_2 Y^1 + \lambda_1 Y^2}{\lambda_1 + \lambda_2} + (Y^1 - Y^2) \frac{\lambda_1}{\lambda_1 + \lambda_2} e^{-(\lambda_1+\lambda_2)t}$$

Thus, the expected value $\mathbb{E}(Y_t \mid Y_0 = Y^1)$ is:

$$\mathbb{E}(Y_t \mid Y_0 = Y^1) = \frac{\lambda_2 Y^1 + \lambda_1 Y^2}{\lambda_1 + \lambda_2} + (Y^1 - Y^2) \frac{\lambda_1}{\lambda_1 + \lambda_2} e^{-(\lambda_1+\lambda_2)t}$$

Problem 3: Ito's Lemma

(a) Let $dX_t = -\alpha X_t dt + \sigma dB_t$, and $f(t, X_t) = e^{\alpha t} X_t$. Show that $df = \sigma e^{\alpha t} dB$.

We apply Ito's lemma to $f(t, X_t) = e^{\alpha t} X_t$.
The partial derivatives are:

$$\frac{\partial f}{\partial t} = \alpha e^{\alpha t} X_t, \quad \frac{\partial f}{\partial X} = e^{\alpha t}, \quad \frac{\partial^2 f}{\partial X^2} = 0$$

Using Ito's lemma:

$$df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial X} dX_t + \frac{1}{2} \frac{\partial^2 f}{\partial X^2} (dX_t)^2$$

Substituting $dX_t = -\alpha X_t \, dt + \sigma \, dB_t$, we get:

$$df = \alpha e^{\alpha t} X_t \, dt + e^{\alpha t} (-\alpha X_t \, dt + \sigma \, dB_t)$$

Simplifying terms:

$$df = \sigma e^{\alpha t} \, dB_t$$

Therefore,

$$df = \sigma e^{\alpha t} \, dB_t$$

(b) Consider the capital accumulation equation $dK_t = (\iota - \delta) K_t dt + \sigma K_t dB_t$, where ι is the investment rate. Suppose our value function is $V(K_t)$. Use Ito's lemma to solve for dV_t.

We apply Ito's lemma to $V(K_t)$.
The partial derivatives are:

$$V_K = \frac{\partial V}{\partial K}, \quad V_{KK} = \frac{\partial^2 V}{\partial K^2}$$

Using Ito's lemma:

$$dV = V_K \, dK_t + \frac{1}{2} V_{KK} (dK_t)^2$$

Substituting for $dK_t = (\iota - \delta) K_t dt + \sigma K_t dB_t$, we get:

$$dV = V_K \left((\iota - \delta) K_t \, dt + \sigma K_t \, dB_t \right) + \frac{1}{2} V_{KK} \sigma^2 K_t^2 \, dt$$

Expanding terms:

$$dV = \left(V_K (\iota - \delta) K_t + \frac{1}{2} V_{KK} \sigma^2 K_t^2 \right) dt + V_K \sigma K_t \, dB_t$$

Thus,

$$dV_t = \left(V_K (\iota - \delta) K_t + \frac{1}{2} V_{KK} \sigma^2 K_t^2 \right) dt + V_K \sigma K_t \, dB_t$$

(c) Suppose $X_t = f(t, B_t)$ for some function f. In this problem, we will solve for f. The only information we have is that

$$dX_t = X_t dB_t$$

Use Ito's lemma to derive two conditions on the partial derivatives of $f(\cdot)$. (That is, group the dt and dB terms and reason via coefficient-matching.) Show that the function $f(t, x) = e^{x - \frac{1}{2}t}$ satisfies these conditions.

We apply Ito's lemma to $X_t = f(t, B_t)$.
Using Ito's lemma, we have:

$$dX_t = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dB_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} (dB_t)^2$$

Since $(dB_t)^2 = dt$, this simplifies to:

$$dX_t = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dB_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} dt = \frac{\partial f}{\partial x} dB_t + \left(\frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}\right) dt$$

We are given that $dX_t = X_t \, dB_t = f(t, B_t) \, dB_t$. For these expressions for dX_t to be equal, the terms involving dt and dB_t must match.
Matching the coefficients of dB_t terms:

$$\frac{\partial f}{\partial x} = f$$

This implies that $f(t, x)$ must satisfy the partial differential equation:

$$\frac{\partial f}{\partial x} = f$$

Solving this gives $f(t, x) = g(t)e^x$ for some function $g(t)$.
Matching the coefficients of dt terms:

$$\frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} = 0$$

Calculating the second derivative $\frac{\partial^2 f}{\partial x^2}$, we get:

$$\frac{\partial^2 f}{\partial x^2} = f$$

Substituting into the dt term condition, we have:

$$\frac{\partial f}{\partial t} + \frac{1}{2} f = 0$$

This implies $\frac{\partial f}{\partial t} = -\frac{1}{2} f$, which gives $f(t, x) = e^{x - \frac{1}{2}t}$ as a solution.

Problem 4: Generator

We defined the generator of a stochastic process in class. The generator is extremely useful when we want to quickly write down the HJB associated with a dynamic optimization problem.

(a) Consider the wealth and income processes

$$da_t = (ra_t + y_t - c_t)dt + \sigma dB_t$$

$$dy_t = \theta(\bar{y} - y_t)dt + \nu W_t$$

where B_t and W_t are independent standard Brownian motion. Write down the generator $\mathcal{A}$ for the two-dimensional process

$$\begin{pmatrix} da_t \\ dy_t \end{pmatrix}$$

That is, what is $\mathcal{A}V$ for a given (smooth) function $V(a_t, y_t)$?

The generator $\mathcal{A}$ of a two-dimensional process (a_t, y_t) applied to a smooth function $V(a_t, y_t)$ can be derived using the dynamics of da_t and dy_t
The partial derivatives of $V(a_t, y_t)$ are:

$$V_a = \frac{\partial V}{\partial a}, \quad V_y = \frac{\partial V}{\partial y}, \quad V_{aa} = \frac{\partial^2 V}{\partial a^2}, \quad V_{yy} = \frac{\partial^2 V}{\partial y^2}$$

Since $da_t = (ra_t + y_t - c_t)\,dt + \sigma\,dB_t$ and $dy_t = \theta(\bar{y} - y_t)\,dt + \nu\,dW_t$, the generator $\mathcal{A}$ applied to V is:

$$\mathcal{A}V = V_a(ra_t + y_t - c_t) + V_y\theta(\bar{y} - y_t) + \frac{1}{2}V_{aa}\sigma^2 + \frac{1}{2}V_{yy}\nu^2$$

(b) Consider the capital accumulation process

$$dk_t = (\iota - \delta)k_t dt.$$

Also suppose that firm technology A_t follows a two-state Markov chain (Poisson process) with transition rates λ. That is, A_t can take on values in $\{A^1, A^2\}$. Suppose that the enterprise value of the firm is given by some function $V(k_t, A_t)$. Characterize the generator of the process

$$\begin{pmatrix} dk_t \\ dA_t \end{pmatrix}$$

That is, what is $\mathcal{A}V$ for a given (smooth) function $V(k_t, A_t)$? (Recall that $\mathcal{A}V = \mathbb{E}[dV]$, so the expression you just solved for tells us how the firm's enterprise value evolves in expectation.)

The generator $\mathcal{A}$ of the process (k_t, A_t) applied to a smooth function $V(k_t, A_t)$ includes both the continuous dynamics of k_t and the jump dynamics of A_t.
For the continuous part, $dk_t = (\iota - \delta)k_t\,dt$, we have:

$$V_k = \frac{\partial V}{\partial k}, \quad V_{kk} = \frac{\partial^2 V}{\partial k^2}$$

The contribution to $\mathcal{A}V$ from k_t is:

$$V_k(\iota - \delta)k_t$$

For the Markov chain A_t, which jumps between states A^1 and A^2 at rate λ, the generator term accounts for transitions between these states:

$$\text{If } A_t = A^1: \quad \lambda(V(k, A^2) - V(k, A^1))$$

$$\text{If } A_t = A^2: \quad \lambda(V(k, A^1) - V(k, A^2))$$

Combining both parts, the generator $\mathcal{A}V$ is given by:

$$\mathcal{A}V(k, A^j) = V_k(\iota - \delta)k_t + \lambda(V(k, A^{-j}) - V(k, A^j)) \text{ for } j = 1, 2$$

Problem 5: Consumption with Income Uncertainty

Consider a household whose lifetime utility is given by

$$V_0 = \max_{\{c_t\}} \mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t) dt.$$

The household makes consumption-savings decisions facing the budget constraint

$$da_t = ra_t + y_t - c_t.$$

Here, y_t is the household's income process. We cannot perfectly forecast our future income, so we will assume that y_t is a stochastic process. Below, you will be asked to write down the HJB associated with this problem for the two canonical models of income uncertainty.

(a) Suppose that income follows the diffusion (Ornstein-Uhlenbeck / AR1) process

$$dy_t = \theta(\bar{y} - y_t)dt + \sigma dB_t.$$

Write down the HJB for this problem that characterizes the household value function $V(a, y)$.

$$\rho V(a,y) = \max_c \left\{ u(c) + [ra + y - c] \frac{\partial V}{\partial a}(a,y) + \theta(\bar{y} - y) \frac{\partial V}{\partial y}(a,y) + \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial y^2}(a,y) \right\}$$

(b) Now suppose that income follows a two-state Markov chain (Poisson process). Income can be high or low, $\in \{y^L, y^H\}$. The transition rate from high to low is λ^H and the transition rate from low to high is λ^L. Write down the HJB for this problem that characterizes the household value function $V(a,y)$.

$$\rho V(a,y^j) = \max_c \left\{ u(c) + \left[ra + y^j - c\right] \frac{\partial V}{\partial a}(a,y^j) + \lambda^j \left[V(a,y^{-j}) - V(a,y^j)\right] \right\} \quad j = L, H$$

(c) Suppose that the interest rate is not constant but varies over time. We know with certainty, however, how the interest rate evolves. So $\{r_t\}$ is a deterministic sequence that is exogenously given to us. (In other words, we are characterizing the household problem in partial equilibrium; the household takes the interest rate as given.) Write down the HJB for this problem that characterizes the household value function $V(t,a,y)$. Why is this value function no longer stationary?

When the income process follows the Ornstein-Uhlenbeck process:

$$\rho V(t,a,y) = \max_c \left\{ u(c) + \frac{\partial V}{\partial t}(t,a,y) + [r_t a + y - c] \frac{\partial V}{\partial a}(t,a,y) \right.$$
$$\left. + \theta(\bar{y} - y) \frac{\partial V}{\partial y}(t,a,y) + \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial y^2}(t,a,y) \right\}$$

When the income process follows a two-state Markov chain with states y^L and y^H, and the interest rate r_t varies over time, the HJB equations for $V(t,a,y)$ are:

$$\rho V(t,a,y^j) = \max_c \left\{ u(c) + \frac{\partial V}{\partial t}(t,a,y) + \left[r_t a + y^j - c\right] \frac{\partial V}{\partial a} V(t,a,y^j) \right.$$
$$\left. + \lambda^j \left[V(t,a,y^{-j}) - V(t,a,y^j)\right] \right\} \quad j = L, H$$

The value function is no longer stationary because the interest rate r_t changes over time, introducing an explicit dependence on time t. This means that the optimal consumption and savings decisions may vary at different points in time even if the state variables a and y are the same.

Problem 6: Consumption and Portfolio Choice

Consider a household that can trade two assets. The first asset is a stock. Stocks trade at price Q_t and they pay the holder dividends at rate D_t. That is, when you hold the stock, your "return" comprises both the dividend payouts and the change in price until you sell the stock, which may be positive or negative. Formally, the rate of return on the stock is

$$dR = \frac{Ddt + dQ}{Q},$$

where $\frac{D}{Q}$ is the dividend-price ratio and $\frac{dQ}{Q}$ is capital gains (change in price). We will now assume that the return process takes the form

$$dR = \mu dt + \sigma dB.$$

for some μ and σ, where B is standard Brownian motion.

The second asset the household can trade is a bond, which trades at price P. And we assume that the bond price evolves simply according to

$$\frac{dP}{P} = rdt,$$

which is the same as saying that holding the bond earns a riskfree rate of return rdt. Crucially, the household can both buy and sell these assets. Assume there are no borrowing (or short-sale) constraints.

The household's lifetime value is given by

$$V_0 = \max \mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t) dt.$$

Let's denote by b_t and k_t the household's bond and stock holdings. Then the household budget constraint is

$$Qdk + Pdb + cdt = Dkdt.$$

Why does the budget constraint take this form? On the RHS, households earn dividends at rate D for each unit of stock they hold. On the LHS, households can spend on consumption, or they can choose to purchase stocks or bonds. If $dk > 0$, the household purchases new stocks at price Q on top of the current stock holdings. If $dk < 0$, the household sells stocks.

We now define *net worth* n and the *risky portfolio share* θ implicity via

$$\theta n = Qk$$

$$(1-\theta)n = Pb$$

In other words, n is a notion of total wealth of the household. And we say that the household invests fraction θ of total wealth in stocks, and fraction $1-\theta$ in bonds.

(a) Derive the law of motion for household net worth and show that it satisfies:

$$dn = \Big[rn + \theta n\Big(\mu - r\Big) - c\Big]dt + \theta n\sigma dB$$

Using $\theta n = Qk$ and $(1-\theta)n = Pb$:

$$dn = \theta dn + (1-\theta)dn = (Qdk + kdQ) + (Pdb + bdP)$$

Using $Qdk + Pdb + cdt = Dkdt$ and $dP = Prdt$:

$$dn = kdQ + (Qdk + Pdb) + bdP = kdQ + (Dkdt - cdt) + bPrdt$$

Substitute $QdR = Ddt + dQ$ and $dR = \mu dt + \sigma dB$:

$$dn = k(dQ + Ddt) - cdt + bPrdt = kQdR - cdt + bPrdt = kQ(\mu dt + \sigma dB) - cdt + bPrdt$$

Using $\theta n = Qk$ and $(1-\theta)n = Pb$:

$$dn = \theta n(\mu dt + \sigma dB) - cdt + (1-\theta)nrdt$$

Simplify to:

$$dn = \big[rn + \theta n(\mu - r) - c\big]dt + \theta n\sigma dB$$

(b) Derive the recursive representation of the households optimization problem. That is, write down the HJB equation that characterizes the household value function $V(n)$. Why was it useful to rewrite the household problem in terms of net worth (rather than stock and bond holdings)? Why is this value function stationary?

The HJB equation for the value function $V(n)$ is:

$$\rho V(n) = \max_{c,\theta}\left\{u(c) + V'(n)\big[rn + \theta n(\mu - r) - c\big] + \frac{1}{2}V''(n)(\theta n\sigma)^2\right\}$$

Rewriting in terms of net worth makes the problem stationary because n fully encapsulates wealth, eliminating time-dependence caused by stock and bond holdings.

(c) Derive the first-order conditions for c and the portfolio share θ.

The FOC with respect to c:

$$u'(c) = V'(n)$$

The FOC with respect to θ:

$$n(\mu - r)V'(n) + \sigma^2\theta n^2 V''(n) = 0$$

or equivalently:

$$\theta = -\frac{(\mu - r)V'(n)}{\sigma^2 n V''(n)}$$

(d) *Optional and more difficult*: Derive the Euler equation for marginal utility and show that it satisfies

$$\frac{du_c}{u_c} = (\rho - r)dt - \frac{\mu - r}{\sigma}dB.$$

Show that with CRRA utility, this implies a consumption Euler equation

$$\frac{dc}{c} = \frac{r-\rho}{\gamma}dt + \frac{1+\gamma}{2}\left(\frac{\mu - r}{\gamma\sigma}\right)^2 dt + \frac{\mu - r}{\gamma\sigma}dB.$$

Lemma 1. *The household Euler equation for marginal utility is given by*

$$\frac{du_c}{u_c} = (\rho - r)dt - \frac{\mu - r}{\sigma}dB$$

Proof. The HJB envelope condition is given by

$$(\rho - r)V_n = V_{nn}\left[rn + \theta n(\mu - r) - c\right] + \frac{1}{2}V_{nnn}(\theta n\sigma)^2$$

Applying Ito's Lemma to V_n, we have:

$$dV_n = V_{nn}\,dn + \frac{1}{2}V_{nnn}(dn)^2$$

where the law of motion for n is:

$$dn = \left[rn + \theta n(\mu - r) - c\right]dt + \theta n\sigma dB$$

Substitute dn and $(dn)^2 = (\theta n\sigma)^2 dt$ into dV_n:

$$dV_n = V_{nn}\left[rn + \theta n(\mu - r) - c\right]dt + \frac{1}{2}V_{nnn}(\theta n\sigma)^2 dt + V_{nn}\theta n\sigma dB$$

Putting this together with the HJB condition:

$$(\rho - r)V_n = V_{nn}\left[rn + \theta n(\mu - r) - c\right] + \frac{1}{2}V_{nnn}(\theta n\sigma)^2$$

we have:

$$dV_n = (\rho - r)V_n dt + V_{nn}\theta n\sigma dB$$

The optimal portfolio share is given by:

$$\theta = -\frac{(\mu - r)V_n}{\sigma^2 n V_{nn}}$$

Substitute θ into the expressions for dV_n:

$$dV_n = (\rho - r)V_n dt - \frac{(\mu - r)V_n}{\sigma^2}\sigma dB$$

Using $u'(c) = V_n$, the marginal utility growth equation is:

$$\frac{du_c}{u_c} = (\rho - r)\,dt - \frac{\mu - r}{\sigma}dB \tag{1}$$

This establishes the household Euler equation for marginal utility. ■

Lemma 2. *The household Euler equation for consumption is given by*

$$\frac{dc}{c} = \frac{r-\rho}{\gamma}dt + \frac{1}{2}(1+\gamma)\left(\frac{\mu-r}{\gamma\sigma}\right)^2 + \frac{\mu-r}{\gamma\sigma}dB$$

Proof. Using Ito's lemma for $u_c(c)$, we have:

$$du_c = u_{cc}dc + \frac{1}{2}u_{ccc}(dc)^2$$

Plugging in (1):

$$\left[(\rho-r)\,dt - \frac{\mu-r}{\sigma}dB\right]u_c = u_{cc}dc + \frac{1}{2}u_{ccc}(dc)^2$$

Using the CRRA utility function $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$:

$$u_c(c) = c^{-\gamma}, \quad u_{cc}(c) = -\gamma c^{-\gamma-1}, \quad u_{ccc}(c) = \gamma(\gamma+1)c^{-\gamma-2}$$

Substitute $u_c = c^{-\gamma}$, $u_{cc} = -\gamma c^{-\gamma-1}$, and $u_{ccc} = \gamma(\gamma+1)c^{-\gamma-2}$ into :

$$\left[(\rho-r)\,dt - \frac{\mu-r}{\sigma}dB\right]c^{-\gamma} = -\gamma c^{-\gamma-1}dc + \frac{1}{2}\gamma(\gamma+1)c^{-\gamma-2}(dc)^2$$

Divide through by $c^{-\gamma}$:

$$(\rho-r)\,dt - \frac{\mu-r}{\sigma}dB = -\gamma\frac{dc}{c} + \frac{1}{2}\gamma(\gamma+1)\left(\frac{dc}{c}\right)^2$$

Now substitute $\frac{dc}{c} = \mu_c\,dt + \sigma_c\,dB$, where μ_c is the drift and σ_c is the volatility of consumption growth. Then:

$$\left(\frac{dc}{c}\right)^2 = \sigma_c^2\,dt$$

Using this, the equation becomes:

$$(\rho-r)\,dt - \frac{\mu-r}{\sigma}dB = -\gamma\left(\mu_c\,dt + \sigma_c\,dB\right) + \frac{1}{2}\gamma(\gamma+1)\sigma_c^2\,dt$$

Separate the dt and dB terms:

$$(\rho-r)\,dt = -\gamma\mu_c\,dt + \frac{1}{2}\gamma(\gamma+1)\sigma_c^2\,dt$$

$$-\frac{\mu-r}{\sigma}dB = -\gamma\sigma_c\,dB$$

From the dB term:

$$\sigma_c = \frac{\mu-r}{\gamma\sigma}$$

Substitute $\sigma_c^2 = \left(\frac{\mu-r}{\gamma\sigma}\right)^2$ into the dt term:

$$\mu_c = \frac{r-\rho}{\gamma} + \frac{1}{2}(\gamma+1)\left(\frac{\mu-r}{\gamma\sigma}\right)^2$$

Thus, the dynamics of consumption growth are:

$$\frac{dc}{c} = \frac{r-\rho}{\gamma}\,dt + \frac{1}{2}(1+\gamma)\left(\frac{\mu-r}{\gamma\sigma}\right)^2 dt + \frac{\mu-r}{\gamma\sigma}\,dB$$

■

(e) Guess and verify that

$$V(n) = \frac{1}{1-\gamma}\kappa^{-\gamma}n^{1-\gamma}$$

where $\kappa = \frac{1}{\gamma}\left[\rho - (1-\gamma)r - \frac{1-\gamma}{2\gamma}\left(\frac{\mu-r}{\sigma}\right)^2\right]$.

From the FOC with respect to consumption, $u'(c) = V'(n)$.
Using CRRA utility, $u'(c) = c^{-\gamma}$, we guess:

$$V'(n) = (\kappa n)^{-\gamma}$$

Integrating $V'(n)$ gives:

$$V(n) = \frac{1}{1-\gamma}\kappa^{-\gamma}n^{1-\gamma}$$

The HJB equation is:

$$\rho V(n) = \max_{c,\theta}\left\{u(c) + V'(n)\left[rn + \theta n(\mu-r) - c\right] + \frac{1}{2}V''(n)(\theta n\sigma)^2\right\}$$

Substitute $V(n) = \frac{1}{1-\gamma}\kappa^{-\gamma}n^{1-\gamma}$, $V'(n) = (\kappa n)^{-\gamma}$, and $V''(n) = -\gamma\kappa^{-\gamma}n^{-\gamma-1}$:

$$\rho\frac{\kappa^{-\gamma}n^{1-\gamma}}{1-\gamma} = \max_{c,\theta}\left\{\frac{c^{1-\gamma}}{1-\gamma} + (\kappa n)^{-\gamma}\left[rn + \theta n(\mu-r) - c\right] - \frac{1}{2}\gamma\kappa^{-\gamma}n^{-\gamma-1}(\theta n\sigma)^2\right\}$$

The FOC with respect to c is:

$$c^{-\gamma} = (\kappa n)^{-\gamma}$$

Solving for c:

$$c = \kappa n$$

The FOC with respect to θ is:

$$(\mu-r)(\kappa n)^{-\gamma} - \gamma\kappa^{-\gamma}n^{-\gamma-1}\theta n\sigma^2 = 0$$

Simplify and solve for θ:

$$\theta = \frac{\mu-r}{\gamma\sigma^2}$$

Substitute $c = \kappa n$ and $\theta = \frac{\mu-r}{\gamma\sigma^2}$:

$$rn + \theta n(\mu-r) - c = rn + \frac{(\mu-r)^2 n}{\gamma\sigma^2} - \kappa n$$

$$\frac{1}{2}\gamma\kappa^{-\gamma}n^{-\gamma-1}(\theta n\sigma)^2 = \frac{1}{2}\gamma\kappa^{-\gamma}n^{-\gamma-1}\left(\frac{(\mu-r)n}{\gamma\sigma}\right)^2$$

Combine terms:

$$\rho\frac{\kappa^{-\gamma}n^{1-\gamma}}{1-\gamma} = \frac{(\kappa n)^{1-\gamma}}{1-\gamma} + (\kappa n)^{-\gamma}\left[rn + \frac{(\mu-r)^2 n}{\gamma\sigma^2} - \kappa n\right] - \frac{1}{2}\gamma\kappa^{-\gamma}n^{-\gamma-1}\left(\frac{(\mu-r)n}{\gamma\sigma}\right)^2$$

Equating the coefficients of $n^{1-\gamma}$, solve for κ:

$$\kappa = \frac{1}{\gamma}\left[\rho - (1-\gamma)r - \frac{1-\gamma}{2\gamma}\left(\frac{\mu-r}{\sigma}\right)^2\right]$$

(f) Given the solution for $V(n)$, use the FOCs to also solve for $c(n)$ and $\theta(n)$. You have now solved the household portfolio choice problem in closed form!!

$$c = c(n) = \kappa n = \frac{1}{\gamma}\left[\rho - (1-\gamma)r - \frac{1-\gamma}{2\gamma}\left(\frac{\mu-r}{\sigma}\right)^2\right]n$$
$$\theta = \theta(n) = \frac{\mu-r}{\gamma\sigma^2}$$

(g) What does this model tell us? Consider the interesting special case of log utility with $\gamma = 1$. Show that consumption collapses to $c = \rho n$. This implies that you want to consume a constant fraction ρ of lifetime net worth. But how much should you invest in the stock market according to this model? Take the formula for θ and plug in log utility. Let's assume an equity premium of 6%, so $\mu - r = 0.0.06$. And let's assume volatility of stock returns of 16%, so $\sigma = 0.16$. Solve for the numeric value of θ, i.e., the fraction of your total wealth that this model tells you to invest in the stock market. Is this high or low? What if "total wealth" also includes a notion of your human capital?

When $\gamma = 1$:

$$\kappa = \frac{1}{\gamma}\left[\rho - (1-\gamma)r - \frac{1-\gamma}{2\gamma}\left(\frac{\mu-r}{\sigma}\right)^2\right] = \rho$$
$$\Rightarrow c = \kappa n = \rho n$$

For θ, substitute $\mu - r = 0.06$, $\sigma = 0.16$:

$$\theta = \frac{\mu-r}{\sigma^2} = \frac{0.06}{0.16^2} = 2.34$$

This implies investing 234% of wealth in stocks, which may be unrealistic unless human capital is included in total wealth.