202A: Dynamic Programming and Applications

Homework #1

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Problem 1: Discrete Time Markov Chains

Time is discrete, with $t \in \{0, 1, ...\}$. Consider the stochastic process $\{y_t\}$ which follows a Markov chain. Let y_t denote the employment / earnings state of an individual in periods t. Consider the state space $y_t \in Y = \{y^U, y^E\}$, where y^U corresponds to unemployment and y^E corresponds to employment. Let $\boldsymbol{y}$ denote the column vector $(y^U, y^E)'$ representing this state space (this is the grid you would construct on a computer). Suppose the employment dynamics of the individual are characterized by the invariant transition matrix

$$P = \begin{pmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{pmatrix}.$$

We interpret a time period as a quarter.

(a) Give economic interpretations of $\alpha = P_{11}$ and $\beta = P_{22}$

(b) Why do the rows of P sum to 1?

(c) Is there an absorbing state in this model?

(d) Compute the probability of being unemployed two quarters after being employed.

(e) Denote the *marginal (probability) distribution* of y_t at time t by ψ_t. $\psi_t(y^L)$ is the probability that process y_t is in state y^L at time t. It is easiest to think of ψ_t as a time-varying row vector. Use the law of total probability to decompose $y_{t+1} = y^L$, accounting for all the possible ways in which state y^L can be reached at time $t+1$.

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(f) Show that the resulting equation can be written as the vector-matrix product

$$\psi_{t+1} = \psi_t P.$$

Therefore: The evolution of the marginal distribution of a Markov chain is obtained by post-multiplying by the transition matrix.

(g) Show that

$$X_0 \sim \psi_0 \implies X_t \sim \psi_0 P^t,$$

where $\sim$ reads "is distributed according to".

(h) We call ψ^* a *stationary distribution* of the Markov chain if it satisfies

$$\psi^* = \psi^* P.$$

Compute the probability of being unemployed n quarters after being employed. Take $n \to \infty$ and find the stationary distribution of this Markov chain. Find the stationary distribution by alternatively plugging into the above equation for ψ^*.

(i) Suppose $y_0 = y^H$. Solve for $\mathbb{E}_0(y_t)$. Use the law of total / iterated expectation to relate expectation to probabilities. Then use the formulas for marginal (probability) distributions derived above.

Problem 2: Proof of the Contraction Mapping Theorem

In class, we defined the Bellman operator B, which operates on functions w, and is defined by

$$(Bw)(x) \equiv \max_{x' \in \Gamma(x)} \left\{ F(x, x') + \beta w(x') \right\}$$

for all $x \in \mathcal{X}$ in the state space, where $\Gamma(x)$ is some constraint set—in our case, this was the budget constraint. The definition is expressed pointwise, but it applies to all possible values in the state space. We call B an operator because it maps a function w to a new function Bw. So both w and Bw map $\mathcal{X}$ into $\mathbb{R}$. Operator B maps *functions* and is therefore called a functional operator. In class, we showed that the solution of the Bellman equation—the value function—is a fixed point of the Bellman operator.

What does it mean to *iterate* $B^n w$?

$$(Bw)(x) = \max_{x' \in \Gamma(x)} \left\{ F(x, x') + \beta w(x') \right\}$$

$$(B(Bw))(x) = \max_{x' \in \Gamma(x)} \left\{ F(x, x') + \beta (Bw)(x') \right\}$$

$$(B(B^2 w))(x) = \max_{x' \in \Gamma(x)} \left\{ F(x, x') + \beta (B^2 w)(x') \right\}$$

$$\vdots\vdots$$

$$(B(B^n w))(x) = \max_{x' \in \Gamma(x)} \left\{ F(x, x') + \beta (B^n w)(x') \right\}.$$

What does it mean for functions to converge to a limiting function? Let v_0 be some guess for the value function, then convergence would mean

$$\lim_{n \to \infty} B^n v_0 = v.$$

And why might $B^n w$ converge as $n \to \infty$? The answer is that B is a *contraction mapping*.

Definition 1. Let (S, d) be a metric space and $B : S \to S$ be a function that maps S into intself. B is a contraction mapping if for some $\beta \in (0, 1)$, $d(Bf, Bg) \leq \beta d(f, g)$, for any two functions f and g.[1]

Intuitively, B is a contraction mapping if applying the operator B to any two functions f and g (that are not the same) moves them strictly closer together. Bf and Bg are strictly closer together than f and g. We can now state the contraction mapping theorem.

Theorem 2. *If (S, d) is a complete metric space and $B : S \to S$ is a contraction mapping, then:*

(i) B has exactly one fixed point $v \in S$

(ii) For any $v_0 \in S$, $\lim_{n \to \infty} B^n v_0 = v$

(iii) $B^n v_0$ has an exponential convergence rate at least as great as $-\ln(\beta)$

In this problem, we will illustrate and prove the contraction mapping theorem.

(a) Consider the contraction mapping $(Bw)(x) \equiv h(x) + \alpha w(x)$ with $\alpha \in (0, 1)$. Iteratively apply the operator B and show that

$$\lim_{n \to \infty} (B^n f)(x) = \frac{h(x)}{1 - \alpha}$$

[1] A metric d is a way of representing the distance between two functions, or two members of (metric) space S. One example: the supremum pointwise gap.

Argue that this shows that the fixed point of this operator B is consequently the function $v(x) = \frac{1}{1-\alpha} h(x)$. Show that $(Bv)(x) = v(x)$.

(b) **Optional:** Now we will prove the contraction mapping theorem in 3 steps (we will not prove the convergence rate). Show that $\{B^n f_0\}_{n=0}^{\infty}$ is a Cauchy sequence. (Cauchy sequence definition: Fix any $\epsilon > 0$. Then there exists N such that $d(B^m f_0, B^n f_0) \leq \epsilon$ for all $m, n \leq N$.)

(c) **Optional:** Show that the limit point v is a fixed point of B.

(d) **Optional:** Show that only one fixed point exists.

Problem 3: Blackwell's Sufficiency Conditions

We now show that there are in fact sufficient conditions for an operator to be contraction mapping.

Theorem 3 (Blackwell's Sufficient Conditions)**.** *Let $X \subset \mathbb{R}^l$ and let $C(X)$ be a space of bounded functions $f : X \to \mathbb{R}$, with the sup-metric. Let $B : C(X) \to C(X)$ be an operator satisfying two conditions:*

1. *monotonicity: if $f, g \in C(X)$ and $f(x) \leq g(x)\ \forall x \in X$,*
   then $(Bf)(x) \leq (Bg)(x), \forall x \in X$

2. *discounting: there exists some $\delta \in (0,1)$ such that*

$$[B(f+a)](x) \leq (Bf)(x) + \delta a \ \ \forall\, f \in C(X),\ a \geq 0,\ x \in X.$$

Then, B is a contraction with modulus δ.

Note that a is a constant and $(f + a)$ is the function generated by adding a to the function f. Blackwell's conditions are sufficient but not necessary for B to be a contraction.

In this problem, we will prove these sufficient conditions.

(a) Let d be the sup-metric and show that, for any $f, g \in C(X)$, we have $f(x) \leq g(x) + d(f, g)$ for all x

(b) Use monotonicity and discounting to show that, for any $f, g \in C(X)$, we have $(Bf)(x) \leq (Bg)(x) + \delta d(f, g)$ and $(Bg)(x) \leq (Bf)(x) + \delta d(f, g)$.

(c) Combine these to show that $d(Bf, Bg) \leq \delta d(f, g)$.

Problem 4: Example of Blackwell's Conditions

We will now work out a simple example to illustrate these sufficient conditions. In particular, consider the Bellman operator in a consumption problem with stochastic asset returns, stochastic labor income, and a liquidity constraint:

$$(Bf)(x) = \sup_{c \in [0,x]} \left\{ u(c) + \delta \mathbb{E} f(\tilde{R}_{+1}(x - c) + \tilde{y}_{+1}) \right\} \quad \forall x$$

Notionally, $\tilde{R}$ and $\tilde{y}$ just underscore that these are random variables. The $_{+1}$ subscript underscores that these random variables are realized next period (in class, we used $'$ for this). The liquidity constraint is encoded in $c \in [0, x]$. (Why?)

(a) Interpret each term in the definition of this Bellman operator.

(b) Explicitly write out the budget constraint that is used here.

(c) Check the first of Blackwell's conditions: monotonicity.

(d) Check the second of Blackwell's conditions: discounting.

Problem 5: Neoclassical Growth Model with Log Utility

Recall the neoclassical growth model we discussed in class. Assuming log utility, full depreciation, and a decreasing-returns production function, preferences can be written as

$$V(k_0) = \max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \log(k_t^{\alpha} - k_{t+1})$$

where $0 < \alpha < 1$, subject to the constraint

$$k_{t+1} \in [0, k_t^{\alpha}] \equiv \Gamma(k_t).$$

Think of k_t^{α} as the resources you have available, and so the most you would be allowed to save is k_t^{α}. We represent this constraint by the *feasibility set* $\Gamma(k_t)$. (This is the more general notation you will find in Stokey-Lucas, for example.)

Consider the associated Bellman equation

$$V(k) = \max_{k' \in \Gamma(k)} \left\{ \ln(k^{\alpha} - k') + \beta V(k') \right\}.$$

(a) Try to solve the Bellman equation by *guessing* a solution. Specifically, start by guessing that the form of the solution is

$$V(k) = \psi + \phi \log k.$$

We will solve for the coefficients ψ and ϕ, and show that $V(k)$ solves the functional equation. Rewrite the functional equation substituting in $V(k) = \psi + \phi \log k$. Use the Envelope Theorem (ET) and the First Order Condition (FOC) to show

$$\phi = \frac{\alpha}{1 - \alpha\beta}.$$

Now use the FOC to show

$$k'(k) = \alpha\beta k^{\alpha}.$$

Finally, show that the functional equation is satisfied at all feasible values of k_0 if

$$\psi = \frac{1}{1-\beta}\left[\log(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\log(\alpha\beta)\right].$$

You have now solved the functional equation by using the guess and verify method.

(b) We have derived the policy function

$$k'(k) = g(k) = \alpha\beta k^{\alpha}.$$

Derive the optimal sequence of state variables $\{k_t^*\}_{t=0}^{\infty}$ which would be generated by this policy function. Show that

$$V(k_0) = \sum_{t=0}^{\infty} \beta^t \log\left((k_t^*)^{\alpha} - k_{t+1}^*\right),$$

thereby confirming that this policy function is optimal.

(c) Consider the Bellman (functional) operator B defined by

$$(Bf)(k) = \sup_{k' \in \Gamma(k)} \left\{ \log(k^{\alpha} - k') + \beta f(k') \right\}.$$

Let $\hat{V}(k) = \frac{\alpha \log k}{1-\alpha\beta}$. Show that

$$(B^n\hat{V})(k) = \frac{1-\beta^n}{1-\beta}\left[\log(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\log(\alpha\beta)\right] + \frac{\alpha \log k}{1-\alpha\beta}.$$

To prove this you'll need to show that $k' = \alpha\beta k^{\alpha}$, and substitute this expression into the functional operator. Let,

$$\lim_{n\to\infty}(B^n\hat{V})(k) = V(k)$$

Confirm that $V(k)$ is the same solution to the the functional equation that you derived in part (a). You have now solved the functional equation by iterating the operator T on a starting guess.

Problem 6: A Model with Equity

Assume that a consumer with only equity wealth must choose period by period consumption in a discrete-time dynamic optimization problem. Specifically, consider the sequence problem

$$V(x_0) = \max_{\{c_t\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} e^{-\rho t} u(c_t)$$

subject to the constraints

$$x_{t+1} = e^{r+\sigma u_t - \frac{\sigma^2}{2}}(x_t - c_t)$$

where u_t is iid and $u_t \sim \mathcal{N}(0,1)$. There is a feasibility constraint $c_t \in [0, x_t]$. And we assume an endowment $x_0 > 0$. Here, x_t represents equity wealth at period t and c_t is consumption in period t. The consumer has discount rate ρ. The consumer can only invest in a risky asset with expected return $e^r = \mathbb{E}e^{r+\sigma u_t - \frac{\sigma^2}{2}}$. And we assume CRRA preferences with $u(c) = \frac{1}{1-\gamma}c^{1-\gamma}$, with $\gamma \in [0, \infty]$. We call this *constant* relative risk aversion because the relative risk aversion coefficient

$$-\frac{cu''(c)}{u'(c)} = \gamma$$

is constant. Notice that CRRA consumption preferences are *homothetic*, and this is what allows us to analytically solve this problem.

The associated Bellman equation is

$$V(x) = \max_{x' \in [0,x]} \left\{ u(x - x') + \mathbb{E}e^{-\rho} V\left(e^{r+\sigma u - \frac{\sigma^2}{2}} x'\right) \right\}.$$

(a) Explain all terms in this Bellman equation. Why is u not a state variable, i.e., why don't we have $V(x, u)$? Make sure that this equation makes sense to you.

(b) Now guess that the value function takes the special form

$$V(x) = \phi \frac{x^{1-\gamma}}{1-\gamma}.$$

Note the close similarity between this functional form and the functional form of the utility function. Assuming that the value function guess is correct, use the Envelope Theorem to derive the consumption function:

$$c = \phi^{-\frac{1}{\gamma}} x.$$

Now verify that the Bellman Equation is satisfied for a particular value of ϕ. Do not solve for ϕ (it's a very nasty expression). Instead, show that

$$\log(1 - \phi^{-\frac{1}{\gamma}}) = \frac{1}{\gamma}\Big[(1-\gamma)r - \rho\Big] + \frac{1}{2}(\gamma - 1)\sigma^2.$$

(c) Now consider the log of the ratio of c_{t+1} and c_t. Show that

$$\mathbb{E} \log \left(\frac{c_{t+1}}{c_t} \right) = \frac{1}{\gamma}(r - \rho) + \frac{\gamma}{2}\sigma^2 - \sigma^2.$$

(d) Interpret the previous equation for the certainty case $\sigma = 0$. Note that $\log(\frac{c_{t+1}}{c_t}) = \log c_{t+1} - \log c_t$ is the growth rate of consumption. Explain why $\log c_{t+1} - \log c_t$ increases in r and decreases in ρ. Why does the coefficient of relative risk aversion γ appear in the denominator of the expression? Why does the coefficient of relative risk aversion regulate the consumer's willingness to substitute consumption between periods?