202A: Dynamic Programming and Applications

Homework #4 Solutions

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Problem 1: Building Intuition

Part 1: Consider the isoelastic utility function:

$$u(c) = \frac{c^{1-\frac{1}{\sigma}} - 1}{1 - \frac{1}{\sigma}},$$

where $\sigma > 0$.

(a) Prove that $\lim_{\sigma \to 1} u(c) = \ln(c)$. (Hint: use l'hopital's rule)

As $\sigma \to 1$, we can see that $\frac{c^{1-1/\sigma}-1}{1-1/\sigma} \to \frac{0}{0}$, which doesn't tell us much. To solve this, recall that L'Hopital's Rule says that $\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}$. Therefore, as $\sigma \to 1$, this utility function approaches $\frac{\ln(c) \cdot \frac{1}{\sigma^2}}{\frac{1}{\sigma^2}} = \ln(c)$.

(b) The coefficient of relative prudence is

$$-\frac{u'''(c)c}{u''(c)}$$

derive it. What is it related to?

We start by finding the second and third derivatives of the utility function $u(c)$ with respect to c:

$$u'(c) = c^{-\frac{1}{\sigma}}$$

$$u''(c) = -\frac{1}{\sigma} c^{-\frac{1}{\sigma}-1}$$

$$u'''(c) = \frac{1}{\sigma}\left(\frac{1}{\sigma} + 1\right) c^{-\frac{1}{\sigma}-2}$$

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\begin{aligned}
-\frac{u'''(c)c}{u''(c)} &= -\frac{\frac{1}{\sigma}\left(\frac{1}{\sigma}+1\right)c^{-\frac{1}{\sigma}-2}\cdot c}{-\frac{1}{\sigma}c^{-\frac{1}{\sigma}-1}} \\
&= \frac{\left(\frac{1}{\sigma}+1\right)c^{-\frac{1}{\sigma}-2}\cdot c}{c^{-\frac{1}{\sigma}-1}} \\
&= \frac{1}{\sigma}+1
\end{aligned}
$$

The coefficient of relative prudence is related to the agent's attitude towards risk and uncertainty, specifically in the context of precautionary savings. A higher value indicates greater sensitivity to uncertainty, leading to more savings as a buffer against future risks.

Part 2: Consider an agent who lives for two periods, $t = 0, 1$. The agent can freely borrow or lend at interest rate r. The agent has period preferences given by $u(c_t) = \frac{c_t^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}}$. In period 0, the agent discounts the utility of future consumption at rate β. The agent receives income y_0 in period 0 , but has uncertain income in period 1. Therefore, the agent maximizes expected utility subject to certain income y_0 and expected income $\mathbb{E}(y_1)$.

(a) Derive the consumption Euler equation.

$$
c_0^{-\frac{1}{\sigma}} = \beta(1+r)E\left[c_1^{-\frac{1}{\sigma}}\right]
$$

(b) Suppose $\sigma \to \infty$. What conditions are placed on $\beta(1+r)$ if the agent has positive consumption in period 0? Interpret your answer in light of question 1. Recall that we call σ the intertemporal elasticity of substitution.

As $\sigma \to \infty$, the utility function becomes linear in consumption, implying no curvature and no aversion to changes in consumption across time. In this case, the Euler equation requires $\beta(1+r) = 1$ for the agent to have positive consumption in both periods (an interior solution). If $\beta(1+r) > 1$, the agent would prefer to save all income in period 0, leading to a "corner solution" where consumption in period 0 is zero. Conversely, if $\beta(1+r) < 1$, the agent would consume all their income in period 0 and save nothing for period 1. Thus, the condition $\beta(1+r) = 1$ ensures that the market incentives to save (via the interest rate) are exactly offset by the agent's time preference (via the discount factor β). This balance makes the agent indifferent to consuming in period 0 or period 1, allowing for an interior solution where both periods feature positive consumption. This result highlights the role of the intertemporal elasticity of substitution ($\frac{1}{\sigma}$) in determining the agent's willingness to smooth consumption over time.

(c) Assume $y_1 \in \{y_L, y_H\}$, with $y_H > y_L$. Argue that this implies $c_1 \in \{c_L, c_H\}$, with $c_H > c_L$, for some unknown values c_L and c_H. Let b_0 denote period 0 savings. Then $c_1 = (1+r)b_0 + y_1$

Given that the agent receives uncertain income in period 1, with $y_1 \in \{y_L, y_H\}$ and $y_H > y_L$, their period 1 consumption will also take on two corresponding values: $c_1 \in \{c_L, c_H\}$, where $c_H > c_L$. The agent saves $b_0 = y_0 - c_0$ in period 0 and brings those savings into period 1, earning interest at rate r. In period 1, the agent will consume all their resources since it is the terminal period. Thus, in the high-income state, consumption will be:

$$c_H = (1+r)b_0 + y_H = (1+r)(y_0 - c_0) + y_H$$

Similarly, in the low-income state, consumption will be:

$$c_L = (1+r)b_0 + y_L = (1+r)(y_0 - c_0) + y_L$$

These are the values of c_H and c_L, which depend on the agent's savings decision in period 0. Since $y_H > y_L$, it follows that $c_H > c_L$. This demonstrates how period 1 consumption directly reflects period 0 savings and the realization of period 1 income.

Part 3: Consider the 2 period model with, for simplicity, $y_1 = 0$. It gives rise to the consumption function

$$c_0 = \frac{1}{1+\beta^\sigma(1+r)^{\sigma-1}} y_0.$$

(a) Differentiate c_0 with respect to $1+r$.

$$\begin{aligned}\frac{\partial c_0}{\partial(1+r)} &= -(\sigma-1)\beta^\sigma(1+r)^{\sigma-2}\left[1+\beta^\sigma(1+r)^{\sigma-1}\right]^{-2} y_0 \\ &= \frac{(1-\sigma)\beta^\sigma(1+r)^{\sigma-2} y_0}{\left(1+\beta^\sigma(1+r)^{\sigma-1}\right)^2}\end{aligned}$$

(b) Explain why your answer to (a) shows that period 0 consumption responds positively to a decrease in the real interest rate if and only if $\sigma > 1$.

The derivative $\frac{\partial c_0}{\partial(1+r)}$ is proportional to $(1-\sigma)$. All other terms in the expression are strictly positive. If $\sigma > 1$, then $(1-\sigma) < 0$, making the derivative negative. This implies that a decrease in $(1+r)$ (the real interest rate) leads to an increase in c_0, which is consistent with a higher preference for smoothing consumption over time when intertemporal elasticity of substitution is low. Conversely, if $\sigma < 1$, then $(1-\sigma) > 0$, making the derivative positive. In this case, a decrease in $(1+r)$ would reduce c_0, reflecting a preference for allocating more consumption to period 1 rather than smoothing. Therefore, period 0 consumption increases in response to a decrease in the real interest rate if and only if $\sigma > 1$, highlighting the role of σ as the reciprocal of the intertemporal elasticity of substitution in determining consumption behavior.

(c) Show that:

$$c_1 = \left[\frac{\beta^\sigma(1+r)^\sigma}{1+\beta^\sigma(1+r)^{\sigma-1}}\right] y_0.$$

Since $y_1 = 0$, the budget constraint is:

$$c_0 + \frac{c_1}{1+r} = y_0$$

Rearranging for c_1 gives:

$$c_1 = (1+r)\left(y_0 - c_0\right)$$

Substitute $c_0 = \frac{1}{1+\beta^\sigma(1+r)^{\sigma-1}} y_0$ into the equation for c_1:

$$\begin{aligned}
c_1 &= (1+r)\left[1 - \frac{1}{1+\beta^\sigma(1+r)^{\sigma-1}}\right] y_0 \\
&= (1+r)\left[\frac{\beta^\sigma(1+r)^{\sigma-1}}{1+\beta^\sigma(1+r)^{\sigma-1}}\right] y_0 \\
&= \left[\frac{\beta^\sigma(1+r)^\sigma}{1+\beta^\sigma(1+r)^{\sigma-1}}\right] y_0
\end{aligned}$$

(d) Show that if $\sigma > 0$, then $\frac{\partial c_1}{\partial(1+r)} > 0$.

We compute the derivative of c_1 with respect to $1+r$ using the quotient rule and chain rule:

$$\begin{aligned}
\frac{\partial c_1}{\partial(1+r)} &= y_0\left[\sigma\beta^\sigma(1+r)^{\sigma-1}\left(1+\beta^\sigma(1+r)^{\sigma-1}\right) - \beta^\sigma(1+r)^\sigma\left((\sigma-1)\beta^\sigma(1+r)^{\sigma-2}\right)\right] \\
&= y_0\beta^\sigma(1+r)^{\sigma-1}\left[\sigma\left(1+\beta^\sigma(1+r)^{\sigma-1}\right) - (\sigma-1)\beta^\sigma(1+r)^{\sigma-1}\right] \\
&= y_0\beta^\sigma(1+r)^{\sigma-1}\left[\sigma + \sigma\beta^\sigma(1+r)^{\sigma-1} - (\sigma-1)\beta^\sigma(1+r)^{\sigma-1}\right] \\
&= y_0\beta^\sigma(1+r)^{\sigma-1}\left[\sigma + \beta^\sigma(1+r)^{\sigma-1}\right]
\end{aligned}$$

Since $y_0 > 0$, $\beta^\sigma > 0$, and $(1+r)^{\sigma-1} > 0$ for $\sigma > 0$, all terms in the expression are positive, making $\frac{\partial c_1}{\partial(1+r)} > 0$. This indicates that period 1 consumption increases with the real interest rate when $\sigma > 0$.

(e) Why does the response of c_0 to $(1+r)$ depend on the value of σ, but the response of c_1 does not? (Hint: your answer should reference the direction of income and substitution effects for consumption in each period.)

Since the individual earns no income in period 1, they are a net saver. An increase in r generates both income and substitution effects:

— Income Effect: A higher r makes the individual richer overall because they earn more on their savings. The income effect suggests consuming more in both periods 0 and 1.

— Substitution Effect: A higher r increases the relative price of consuming in period 0 (since saving becomes more rewarding). The substitution effect encourages consuming less in period 0 and more in period 1.

For c_0, these two effects work in opposite directions:

— The income effect increases c_0.

— The substitution effect decreases c_0.

The net response of c_0 depends on the relative strength of these effects, which is determined by σ, the inverse of the intertemporal elasticity of substitution. When $\sigma > 1$, the substitution effect dominates, leading to a decrease in c_0 as r increases. Conversely, when $\sigma < 1$, the income effect dominates, increasing c_0. For c_1, however, both effects work in the same direction:

— The income effect increases c_1 because the individual is richer.

— The substitution effect also increases c_1 because a higher r incentivizes saving more today to consume more tomorrow.

As a result, c_1 always increases with r, regardless of the value of σ. The response of c_1 is unaffected by the relative strength of substitution versus income effects, unlike c_0.

Problem 2: Eat-the-pie in discrete time

Time is discrete and there is no uncertainty. Consider an agent that faces the sequence problem

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t),$$

subject to the budget constraints

$$W_{t+1} = R(W_t - c_t)$$

where R is the constant (gross) real interest rate and W_t is the agent's wealth at date t,

$$0 \leq c_t \leq W_t,$$

and taking as given an initial wealth level

$$W_0 > 0.$$

(a) Motivate the economic problem above. What are the implicit assumptions? What is economically sensible and what is not sensible about this modeling set-up? Why do you think this problem might be called the "eat-the-pie model"?

The consumer selects consumption in each period, given their current wealth, which equals savings in the previous period times the gross rate of return. The following are implicit assumptions in the model:

- The consumer knows the rate of return on savings with certainty, eliminating a motive for precautionary savings.
- The consumer is assumed to be able to sell claims to all their future labor income for their expected discounted value. Under this interpretation the assumption that no borrowing is allowed is natural (it follows from a no-Ponzi condition).
- The consumer discounts the future exponentially and has time-consistent preferences, even though there is some evidence indicating that consumers find it difficult to adhere to their previous consumption plans.
- The utility function is additively separable across time periods, excluding habit formation or state-dependent preferences.
- The decision problem assumes to face an infinite-horizon decision problem. This assumption be justified if consumers were altruistic toward future generations of their families.

The problem is called the "eat-the-pie model" because the agent must decide how to consume their wealth over time, similar to eating a finite pie while ensuring it lasts as desired.

(b) Explain why the Bellman equation for this problem is given by:

$$v(W) = \sup_{c\in[0,W]} \Big\{ u(c) + \beta v\Big(R(W-c)\Big)\Big\}, \qquad \forall W.$$

Why is there no expectation operator on the continuation value? Why is there no t subscript on $v(W)$?

The Bellman equation reflects the recursive structure of the problem. The current value of wealth $v(W)$ is equal to the flow payoff from current consumption, $u(c)$, plus the discounted value of future wealth $\beta v(R(W-c))$, where current consumption $c \in [0, W]$ is chosen to maximize the sum of the latter two terms.

The expectation operator is not needed because the consumer is not exposed to any uncertainty (the stochastic income stream was already sold for a constant, known amount of assets and the return rate on savings is known with certainty). The absence of a time subscript on $v(W)$ reflects the stationary nature of the problem, as the environment and preferences do not change over time.

(c) Using Blackwell's sufficiency conditions, prove that the Bellman operator B, defined by

$$(Bf)(W) = \sup_{c\in[0,W]} \Big\{ u(c) + \beta f\Big(R(W-c)\Big)\Big\}, \qquad \forall W.$$

is a contraction mapping. You should assume that u is a bounded function. (Why is this boundedness assumption necessary for the application of Blackwell's Theorem?) Explain what the contraction mapping property implies about iterative solution methods.

The Bellman operator B maps bounded functions into bounded functions. We can restrict our attention to the space of bounded functions because the fact that u is bounded implies that the value function for our sequence problem:

$$v^{SP}(W_0) = \max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t),$$

is also bounded. To see this, note that if $K > 0$ is an upper bound for $u(x)$ then $\frac{K}{1-\beta}$ is an upper bound for $v^{SP}(W_0)$, so that the latter is also bounded. Since our ultimate goal is to compute the solution to the sequence problem, we can restrict our attention to solutions $v(x)$ of the Bellman equation that are bounded. Also note that, if u is bounded, the Bellman operator B will always map a bounded function f to another bounded function Bf, since the latter is the sum of two bounded functions, $u(\cdot)$ and $\beta f(\cdot)$. Therefore, B maps the space of bounded functions to itself, $B : C(X) \to C(X)$, as Blackwell's Theorem requires.

Then, we can apply Blackwell's Theorem, which provides sufficient conditions for a mapping from the space of bounded functions into itself to be a contraction. We now show that our Bellman operator B satisfies Blackwell's sufficient conditions::

i. Monotonicity: For any functions f, g such that $f \leq g$, we have:

$$Bf(W) = \sup_{c\in[0,W]} \{u(c) + \beta f(R(W-c))\} \leq \sup_{c\in[0,W]} \{u(c) + \beta g(R(W-c))\} = Bg(W)$$

where the inequality is a straightforward consequence of the fact that $f(W) \leq g(W)$ for each W.

ii. Discounting: For any function f and constant $\alpha \geq 0$:

$$\begin{aligned} B(f+\alpha)(W) &= \sup_{c\in[0,W]} \{u(c) + \beta[f(R(W-c)) + \alpha]\} \\ &= \sup_{c\in[0,W]} \{u(c) + \beta f(R(W-c))\} + \beta\alpha \\ &= Bf(W) + \beta\alpha. \end{aligned}$$

Since B satisfies these conditions, B is a contraction mapping. The contraction mapping theorem ensures that the repeated iteration of B on a starting function W generates a sequence of functions that converges to the unique fixed point $v(W)$ of the Bellman operator, thus providing the solution to the Bellman equation.

(d) Now assume that,

$$u(c) = \left\{ \begin{array}{ll} \frac{c^{1-\gamma}}{1-\gamma} & \text{if } \gamma \in (0,\infty) \text{ and } \gamma \neq 1 \\ \ln c & \text{if } \gamma = 1 \end{array} \right\}.$$

(So u is no longer bounded.) Use the guess method to solve the Bellman equation. Specifically, guess the form of the solution:

$$v(W) = \left\{ \begin{array}{lll} \psi \frac{W^{1-\gamma}}{1-\gamma} & \text{if} & \gamma \in (0,\infty) \text{ and } \gamma \neq 1 \\ \phi + \psi \ln W & \text{if} & \gamma = 1 \end{array} \right\}.$$

Derive the optimal policy rule:

$$c = \psi^{-\frac{1}{\gamma}} W$$

$$\psi^{-\frac{1}{\gamma}} = 1 - (\beta R^{1-\gamma})^{\frac{1}{\gamma}}$$

Note that this rule applies for all values of γ. Confirm that this solution to the Bellman Equation works.

The first-order condition and the envelope theorem yield the following:

$$u'(c) = \beta R v'(R(W-c)) = v'(W)$$

Given the form of the utility function and the value function being assumed, we have $u'(c) = c^{-\gamma}$ and $v'(W) = \psi W^{-\gamma}$ which is valid for any $\gamma \in (0,\infty)$. Substituting $u'(c)$ and $v'(W)$ gives:

$$c^{-\gamma} = \beta R \psi [R(W-c)]^{-\gamma}$$

This simplifies to the policy rule:

$$c = \psi^{-\frac{1}{\gamma}} W$$

where $\psi^{-\frac{1}{\gamma}} = 1 - (\beta R^{1-\gamma})^{\frac{1}{\gamma}}$.
The Bellman equation is satisfied because:

$$\psi \frac{W^{1-\gamma}}{1-\gamma} = \sup_{c \in [0,W]} \left\{ \frac{c^{1-\gamma}}{1-\gamma} + \beta \psi \frac{[R(W-c)]^{1-\gamma}}{1-\gamma} \right\}$$

$$\implies \psi \frac{W^{1-\gamma}}{1-\gamma} = \frac{(\psi^{-\frac{1}{\gamma}} W)^{1-\gamma}}{1-\gamma} + \beta \psi \frac{\left[R\left(W - \psi^{-\frac{1}{\gamma}} W\right)\right]^{1-\gamma}}{1-\gamma}$$

$$\implies 1 = \psi^{-\frac{1}{\gamma}} + \beta \left[R\left(1 - \psi^{-\frac{1}{\gamma}}\right)\right]^{1-\gamma}$$

$$\implies \psi = \left(1 - \beta^{\frac{1}{\gamma}} R^{\frac{1-\gamma}{\gamma}}\right)^{-\gamma}$$

For $\gamma = 1$, we have $\psi^{-1} = 1 - \beta$ and $c = (1-\beta)W$; so that, the constant ϕ can be recovered as follows:

$$\phi + \psi \ln W = \sup_{c \in [0,W]} \left\{ \ln c + \beta \Big(\phi + \psi \ln[R(W-c)] \Big) \right\}$$

$$\implies \phi + \frac{1}{1-\beta}\ln W = \ln(1-\beta) + \ln W + \beta\Big(\phi + \frac{1}{1-\beta}[\ln(\beta R) + \ln W]\Big)$$

$$\implies \phi = \frac{1}{1-\beta}\Big[\ln(1-\beta) + \frac{\beta}{1-\beta}\ln(\beta R)\Big]$$

which confirms that the solution indeed satisfies the Bellman equation for $\gamma = 1$.

(e) When $\gamma = 1$ the consumption rule collapses to $c_t = (1-\beta)W_t$. Why does consumption no longer depend on the value of the interest rate (for a given W_t)? Hint: think about income effects and substitution effects.

When $\gamma = 1$, consumption does not depend *directly* on the gross rate of return R, because the income and substitution effects of an interest rate change on consumption cancel each other out.

Problem 3: Consumption-savings with deterministic income fluctuations

Our next problem is similar to the previous "eat-the-pie" problem, but we allow for labor income as well as a time-varying interest rate. We also switch to continuous time.

Consider a household with preferences

$$\max_{\{c_t\}} \int_0^\infty e^{-\rho t} u(c_t) dt.$$

There is no uncertainty. The household budget constraint is given by

$$da_t = r_t a_t + y_t - c_t,$$

given an initial wealth a_0. The household's labor income process $\{y_t\}$ as well as the interest rate path $\{r_t\}$ are deterministic. The household takes these as exogenously given. Consequently, the household does not face any uncertainty in this problem.

(a) We start by characterizing what's known as the "natural borrowing limit". In particular, assume that the household may accumulate debt, $a_t < 0$, as long as the household repays "in the long run". (Formally, imagine there is a lender that is risk-neutral with respect to when she is repaid by the household, as long as the household does not default outright.) Intuitively, argue that there must exist a lower bound $\underline{a}^{\text{nat}}$ such that the household must respect $a_t \geq \underline{a}^{\text{nat}}$. Derive this so-called natural borrowing limit in terms of $\{r_t\}$ and $\{y_t\}$.

To derive the natural borrowing limit, consider the household's intertemporal budget constraint, which must hold over an infinite horizon. The natural borrowing limit ensures that the household can eventually repay its debt, even when debt is allowed. When individuals face the natural borrowing limit, they are permitted to borrow up to the present discounted value of all their future income. This ensures that debt is sustainable and can be repaid over time. Mathematically, the natural borrowing limit $\underline{a}^{\text{nat}}$ is given by:

$$\underline{a}^{\text{nat}} = -\int_0^\infty e^{-\int_0^t r_s\, ds} y_t\, dt$$

This formula represents the maximum amount the household can borrow, which corresponds to the present value of its future income, discounted by the interest rate path $\{r_t\}$. The negative sign reflects the fact that this value is a lower bound on wealth, as it accounts for debt.

(b) Assuming that the household may accumulate debt as long as she respects her natural borrowing limit, derive the household's lifetime budget constraint. Denote the household's "initial lifetime wealth" by W and show that we can write it as a function $W = \mathcal{W}(a_0, \{r_t\}, \{y_t\})$. That is, initial lifetime wealth is a function of initial wealth a_0, as well as the paths of real interest rates $\{r_t\}$ and income $\{y_t\}$.

Lemma 1. *(Lifetime Budget Constraint) For any linear ODE*

$$\frac{dy}{dt} = r(t)y(t) + x(t)$$

we have the integration result

$$y(T) = y(0)e^{\int_0^T r(s)ds} + \int_0^T e^{\int_t^T r(s)ds} x(t)dt.$$

Proof. Consider any ODE:

$$\frac{dy}{dt} = r(t)y(t) + x(t)$$

Using an integrating factor approach, we have:

$$e^{\int -r(s)ds}\frac{dy}{dt} - e^{\int -r(s)ds}r(t)y(t) = e^{\int -r(s)ds}x(t)$$

The LHS can then be written as a product rule, so that :

$$\frac{d}{dt}\left(y(t)e^{\int -r(s)ds}\right) = \frac{dy}{dt}e^{\int -r(s)ds} + y(t)e^{\int -r(s)ds}\frac{d}{dt}\left(\int -r(s)ds\right) = e^{\int -r(s)ds}x(t)$$

The last derivative follows from the fundamental theorem of calculus for indefinite integrals.

Alternatively, since I know that I will work on the definite time horizon $t \in [0, T]$, I can choose a slightly different integrating factor: I can write $u(t) = e^{-\int_0^t r(s)ds}$, so:

$$e^{\int_0^t -r(s)ds}\frac{dy}{dt} - e^{\int_0^t -r(s)ds}r(t)y(t) = e^{\int_0^t -r(s)ds}x(t)$$

Using Leibniz rule, I have:

$$\begin{aligned}\frac{d}{dt}\left(y(t)e^{\int_0^t -r(s)ds}\right) &= \frac{dy}{dt}e^{\int_0^t -r(s)ds} + y(t)e^{\int_0^t -r(s)ds}\frac{d}{dt}\left(\int_0^t -r(s)ds\right) \\ &= \frac{dy}{dt}e^{\int_0^t -r(s)ds} - y(t)e^{\int_0^t -r(s)ds}r(t)\end{aligned}$$

Now, I have :

$$\frac{d}{dt}\left(y(t)u(t)\right) = u(t)x(t)$$

Finally, this implies:

$$y(T)u(T) - y(0)u(0) = \int_0^T u(t)x(t)$$

or noting $u(0) = 1$:

$$y(T)e^{-\int_0^T r(s)ds} = y(0) + \int_0^T e^{-\int_0^t r(s)ds}x(t)dt$$

Rearranging:

$$y(T) = y(0)e^{\int_0^T r(s)ds} + \int_0^T e^{\int_t^T r(s)ds} x(t)dt$$

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Using this formula for the household budget constraint, we have:

$$\lim_{T\to\infty} a(T) = a(0) \lim_{T\to\infty} e^{\int_0^T r(s)ds} - \lim_{T\to\infty} \int_0^T e^{\int_t^T r(s)ds} \left[c(t) - y_t\right] dt$$

Under the no-Ponzi assumption of $\lim_{T\to\infty} a(T) = 0$, this further simplifies to:

$$\int_0^\infty e^{-\int_0^t r(s)ds} c(t)dt = a(0) + \int_0^\infty e^{-\int_0^t r(s)ds} y(t)dt \equiv W$$

(c) It will be a useful exercise to characterize the response of lifetime wealth dW to a general perturbation of this economy. In other words, consider a perturbation that leads to changes in the households initial wealth, as well as interest rates and income $\{da_0, \{dr_t\}, \{dy_t\}\}$. Under such a perturbation, lifetime wealth changes

$$dW = \mathcal{W}_{a_0} da_0 + \int_0^\infty \mathcal{W}_{r_t} dr_t dt + \int_0^\infty \mathcal{W}_{y_t} dy_t dt,$$

where $\mathcal{W}_x = \frac{\partial \mathcal{W}}{\partial x}$. Work out the derivatives $\mathcal{W}_{a_0}$, $\mathcal{W}_{r_t}$ and $\mathcal{W}_{y_t}$ and interpret.

We analyze how lifetime wealth W responds to perturbations in initial wealth, interest rates, and income paths, denoted by $\{da_0, \{dr_t\}, \{dy_t\}\}$. The change in lifetime wealth can be expressed as:

$$dW = \mathcal{W}_{a_0} da_0 + \int_0^\infty \mathcal{W}_{r_t} dr_t \, dt + \int_0^\infty \mathcal{W}_{y_t} dy_t \, dt$$

where $\mathcal{W}_x = \frac{\partial \mathcal{W}}{\partial x}$. We derive the terms $\mathcal{W}_{a_0}$, $\mathcal{W}_{r_t}$, and $\mathcal{W}_{y_t}$ below.
The initial wealth a_0 directly contributes to lifetime wealth:

$$W = a_0 + \int_0^\infty e^{-\int_0^t r_s ds} y_t \, dt$$

Thus, the partial derivative with respect to a_0 is:

$$\mathcal{W}_{a_0} = 1$$

The income y_t affects lifetime wealth through its present discounted value. The derivative is:

$$\mathcal{W}_{y_t} = e^{-\int_0^t r_s ds}$$

This term represents the present value discount factor at time t.

The only challenging term is the effect of interest rates r_t. Differentiating W with respect to r_τ, we have:

$$
\begin{aligned}
W_{r_\tau} &= \frac{\partial}{\partial r_\tau}\left[\int_0^\infty e^{-\int_0^t r_s ds} w_t dt\right] \\
&= \int_0^\infty \frac{\partial}{\partial r_\tau}\left[e^{-\int_0^t r_s ds}\right] y_t dt \\
&= \int_\tau^\infty \frac{\partial}{\partial r_\tau}\left[e^{-\int_0^t r_s ds}\right] y_t dt \\
&= -\int_\tau^\infty e^{-\int_0^t r_s ds}\left[\frac{\partial}{\partial r_\tau}\int_0^t r_s ds\right] y_t dt \\
&= -\int_\tau^\infty e^{-\int_0^t r_s ds} y_t dt \\
&= -\int_\tau^\infty e^{-(\int_0^t r_s ds + \int_0^\tau r_s ds - \int_0^\tau r_s ds)} y_t dt \\
&= -e^{-\int_0^\tau r_s ds}\int_\tau^\infty e^{-\int_\tau^t r_s ds} y_t dt
\end{aligned}
$$

Let $R_{0,t} = e^{\int_0^t r_s ds}$ for ease of exposition. Rewriting, this yields:

$$
W_{r_\tau} = -\frac{1}{R_{0,\tau}}\int_\tau^\infty \frac{y_t}{R_{\tau,t}} dt
$$

This is the effect on lifetime wealth of a change in interest rates at time τ.

(d) Characterize a recursive representation of the household's problem using a as the only state variable. That is, write down an HJB for the value function $V(t, a)$. What does the dependence on calendar time t capture?

$$
\rho V(t,a) = u(c) + V_a\left[r_t a + w_t - c\right] + V_t
$$

The dependence on calendar time t captures the non-stationarity of the environment, such as time-varying interest rates r_t and labor income w_t, which directly affect the household's optimization problem.

(e) Use the HJB, the first-order condition, and the envelope condition to derive an Euler equation of the form

$$
\frac{du_{c,t}}{u_{c,t}} = (\rho - r_t)dt
$$

where $u_{c,t} \equiv u'(c_t)$.

The envelope condition is given by:

$$
(\rho - r_t)V_a = V_{at} + V_{aa}\left[r_t a + w_t - c\right]
$$

Since $c = c(t, a)$, it follows that $u_c = u_c(t, a)$. Using Ito's Lemma for V_a, we have:

$$dV_a = V_{at}dt + V_{aa}\Big[r_t a + w_t - c\Big]dt$$

Dividing through by V_a, this becomes:

$$\begin{aligned}\frac{dV_a}{V_a} &= \frac{V_{at}dt + V_{aa}\Big[r_t a + w_t - c\Big]dt}{V_a} \\ &= \frac{(\rho - r_t)V_a dt}{V_a} \\ &= (\rho - r_t)dt\end{aligned}$$

Since $u_c = V_a$ by the first-order condition, we can equivalently write:

$$\frac{du_c}{u_c} = (\rho - r_t)dt$$

Thus, the Euler equation for consumption is derived as:

$$\frac{du_{c,t}}{u_{c,t}} = (\rho - r_t)dt$$

(f) In this setting, where $\{r_t\}$ and $\{y_t\}$ are entirely deterministic, the Euler equation is of course also a deterministic equation. Let $R_{s,t} = e^{-\int_s^t r_s ds}$. Now show that the Euler equation implies a relationship between marginal utility at any two dates $t > s$ given by

$$u_c(c_s) = e^{-\rho(t-s)}R_{s,t}u_c(c_t).$$

Using separation of variables, we rearrange terms to integrate:

$$\int_s^t \frac{1}{u_c}du_c = \int_s^t (\rho - r_\tau)d\tau$$

Integrating both sides:

$$\ln(u_c(t)) - \ln(u_c(s)) = \int_s^t (\rho - r_\tau)d\tau$$

Exponentiating both sides to eliminate the logarithm:

$$u_c(t) = u_c(s)e^{\int_s^t (\rho - r_\tau)d\tau}$$

Separating the terms in the exponent:

$$u_c(t) = u_c(s)e^{\int_s^t \rho d\tau}e^{-\int_s^t r_\tau d\tau}$$

Recalling that $R_{s,t} = e^{-\int_s^t r_\tau d\tau}$, we can rewrite:

$$u_c(t) = u_c(s)e^{\rho(t-s)}R_{s,t}$$

Rearranging, we obtain the desired relationship:

$$u_c(c_s) = e^{-\rho(t-s)} R_{s,t} u_c(c_t)$$

(g) Show that with CRRA utility we can write consumption as

$$c_t = c_0 \left[e^{-\rho t} R_{0,t} \right]^{\frac{1}{\gamma}}.$$

Interpret this equation.

The CRRA utility function is given by:

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}$$

where γ is the coefficient of relative risk aversion. The marginal utility of consumption is:

$$u_c(c_t) = c_t^{-\gamma}$$

From (f):

$$u_c(c_t) = u_c(c_0) e^{\int_0^t (\rho - r_s) ds}$$

Substituting $u_c(c_t) = c_t^{-\gamma}$, we have:

$$c_t^{-\gamma} = c_0^{-\gamma} e^{\int_0^t (\rho - r_s) ds}$$

Simplify:

$$\begin{aligned}
c_t &= c_0 \left(e^{\int_0^t (\rho - r_s) ds} \right)^{-\frac{1}{\gamma}} \\
&= c_0 \left(e^{\int_0^t \rho \, ds} \cdot e^{- \int_0^t r_s \, ds} \right)^{-\frac{1}{\gamma}} \\
&= c_0 \left(e^{\rho t} \cdot R_{0,t} \right)^{-\frac{1}{\gamma}} \\
&= c_0 \left(e^{-\rho t} R_{0,t} \right)^{\frac{1}{\gamma}}
\end{aligned}$$

(h) Using the lifetime budget constraint and the consumption policy function, show that

$$W = c_0 \int_0^\infty e^{-\frac{\rho}{\gamma} t} R_{0,t}^{\frac{1-\gamma}{\gamma}} \, dt.$$

Interpret this condition.

The lifetime budget constraint implies:

$$\int_0^\infty e^{-\int_0^t r(s)ds} c(t)dt = \int_0^\infty \frac{c_t}{R_{0,t}} dt = W$$

Substituting in for c_t obtained in (g):

$$\begin{aligned} W &= \int_0^\infty \frac{c_0 \left[e^{-\rho t} R_{0,t}\right]^{\frac{1}{\gamma}}}{R_{0,t}} dt \\ &= c_0 \int_0^\infty e^{-\frac{\rho}{\gamma} t} R_{0,t}^{\frac{1-\gamma}{\gamma}} dt \end{aligned}$$

The condition ensures that total discounted consumption aligns with the household's lifetime wealth, where the discounting reflects the household's time preferences, risk aversion, and the influence of the interest rate path.

(i) Finally, we will characterize the household's marginal propensity to consume (MPC). For simplicity, consider the simple case with $r_t = r$ constant. Define

$$\text{MPC}_{s,t} = \frac{\partial c_t}{\partial y_s}.$$

This object characterizes the household's behavioral consumption response at date t to a marginal change in (unearned) income at date s. We refer to $\text{MPC}_{s,t}$ as the household's *intertemporal MPC* or iMPC. Notice that $R_{0,t} = e^{\int_0^t r ds} = e^{rt}$. Solve for $\text{MPC}_{s,t}$ and interpret, considering both cases $s < t$ and $t \geq s$.

Considering the simple case with $r_t = r$ constant gives:

$$W = \int_0^\infty e^{-\frac{1}{\gamma}\left[\rho - (1-\gamma)r\right]t} c_0 \, dt$$

Differentiate this equation with respect to W:

$$1 = \frac{\partial c_0}{\partial W} \int_0^\infty e^{-\frac{1}{\gamma}\left[\rho - (1-\gamma)r\right]t} dt$$

The integral simplifies using the exponential integral formula, for any $\kappa \neq 0$:

$$\int e^{\kappa x} dx = \frac{1}{\kappa} e^{\kappa x} + C$$

Let $\kappa = \frac{1}{\gamma}\left[\rho - (1-\gamma)r\right] > 0$. Then:

$$\frac{\partial c_0}{\partial W} = \left[\int_0^\infty e^{-\kappa t} dt\right]^{-1} = \left[-\frac{1}{\kappa} e^{-\kappa t} + C\Big|_0^\infty\right]^{-1} = \left[-\frac{1}{\kappa}(0-1)\right]^{-1} = \left[\frac{1}{\kappa}\right]^{-1} = \kappa$$

The consumption is given by:

$$c_t = c_0 \left[e^{-\rho t} R_{0,t}\right]^{\frac{1}{\gamma}} = c_0 e^{\frac{1}{\gamma}(r-\rho)t}$$

$$\frac{\partial c_t}{\partial W} = \frac{\partial c_t}{\partial c_0}\frac{\partial c_0}{\partial W} = e^{\frac{1}{\gamma}(r-\rho)t}\kappa$$

The income y_t affects lifetime wealth through its present discounted value. The derivative is:

$$\mathcal{W}_{y_t} = e^{-\int_0^t r_s ds} = e^{-rt}$$

This term represents the present value discount factor at time t.

$$\begin{aligned}
\text{MPC}_{s,t} &= \frac{\partial c_t}{\partial y_s} \\
&= \frac{\partial c_t}{\partial W}\frac{\partial W}{\partial y_s} \\
&= \kappa e^{\frac{1}{\gamma}(r-\rho)t} e^{-rs} \\
&= \kappa e^{\left[\frac{1}{\gamma}(r-\rho)t - rs\right]}
\end{aligned}$$

As s increases, the present value of income declines due to stronger discounting (e^{-rs}), which reduces $\frac{\partial W}{\partial y_s}$. Consequently, the impact of income received at s on consumption at t diminishes. As t increases, given $r > \rho$, savings grow faster than the household's impatience to consume, amplifying the effect of lifetime wealth on future consumption, as returns on savings dominate. Thus, $\frac{\partial c_t}{\partial c_0}$ increases with t.

Problem 4: Consumption-savings with uncertain wealth dynamics

We now introduce uncertainty and solve an analytically tractable variant of the household consumption-savings problem with stochastic returns on savings. The key to making this tractable is to assume that the household faces no borrowing constraint.

Time is continuous. The evolution of wealth is given by

$$da_t = (ra_t - c_t)dt + \sigma a_t dB,$$

where B_t is standard Brownian motion. We assume that the household is not subject to a borrowing constraint, so a_t can go negative.

(a) Write the generator for the stochastic process of wealth.

The generator for the stochastic process of wealth is:

$$\mathcal{A}f = (ra_t - c_t)f' + \frac{1}{2}(\sigma^2 a_t^2)f''$$

(b) Use it to derive the HJB:

$$\rho v(a) = \max_c \left\{ u(c) + v'(a)[ra - c] + \frac{\sigma^2}{2}a^2 v''(a) \right\}.$$

Why is there no t subscript on $v(a)$? What kind of differential equation is this? Why is it not a PDE?

The HJB is given by:

$$\begin{aligned}\rho v(a) &= \max_c \left\{ u(c) + \mathcal{A}v \right\} \\ &= \max_c \left\{ u(c) + v'(a)[ra - c] + \frac{\sigma^2}{2}a^2 v''(a) \right\}\end{aligned}$$

There is no t subscript on $v(a)$ because the problem is stationary—time does not explicitly affect preferences or constraints. This is an ordinary differential equation (ODE), not a partial differential equation (PDE), because there is only one state variable, a (wealth).

(c) Guess that the policy function is linear in wealth (because of log), in particular: $c(a) = \rho a$. And show:

$$v(a) = \frac{1}{\rho}\log(\rho a) + \frac{r - \rho}{\rho^2} - \frac{\sigma^2}{2\rho^2}.$$

Interpret this expression. What is the household's MPC in this model?

The first-order condition for consumption is given by:

$$v'(a) = \frac{1}{c(a)}$$

Using the guess $c(a) = \rho a$, we integrate to find the value function:

$$v(a) = \frac{1}{c'(a)} \ln[c(a)] + \kappa$$

where κ is a constant. Substituting this into the HJB equation, we get:

$$\begin{aligned}\frac{\rho}{c'(a)} \ln[c(a)] + \rho\kappa &= \ln[c(a)] + [ra - c(a)]\frac{1}{c(a)} - \frac{\sigma^2}{2}a^2\frac{c'(a)}{c(a)^2} \\ &= \ln[c(a)] + \frac{ra}{c(a)} - 1 - \frac{\sigma^2}{2}a^2\frac{c'(a)}{c(a)^2}\end{aligned}$$

Substituting $c(a) = \rho a$:

$$\ln(\rho a) + \rho\kappa = \ln(\rho a) + \frac{r-\rho}{\rho} - \frac{\sigma^2}{2\rho}$$

This implies:

$$\kappa = \frac{r-\rho}{\rho^2} - \frac{\sigma^2}{2\rho^2}$$

Thus, the value function is:

$$v(a) = \frac{1}{\rho}\log(\rho a) + \frac{r-\rho}{\rho^2} - \frac{\sigma^2}{2\rho^2}$$

The term $\frac{1}{\rho}\ln(\rho a)$ represents the utility derived from consumption at a given level of wealth. The terms $\frac{r-\rho}{\rho^2}$ and $-\frac{\sigma^2}{2\rho^2}$ adjust for the effects of the interest rate and uncertainty, respectively. The policy function $c(a) = \rho a$ implies that the MPC is constant and equal to ρ, as consumption is proportional to wealth in this model.

Problem 5: The equity premium

Consider a representative household. Time is discrete. We consider a set of assets that the household can trade and index these assets by j. For each asset j, optimal portfolio choice implies an Euler equation of the form

$$U'(C_t) = \beta E[R^j_{t+1} U'(C_{t+1})]$$

where R^j_{t+1} is the potentially stochastic return on asset j, and where $\beta = e^{-\rho}$. We can write

$$1 = e^{-\rho} E\left[R^j_{t+1} \frac{U'(C_{t+1})}{U'(C_t)}\right].$$

(a) Explain why the above Euler equation must hold for every asset j. Make sure you are comfortable with this logic. Start with a CRRA utility function $u(c) = \frac{1}{1-\gamma} c^{1-\gamma}$ and let $r^j_{t+1} = \ln R^j_{t+1}$. Show that

$$1 = E\left[e^{r^j_{t+1} - \rho - \gamma \Delta \ln C_{t+1}}\right],$$

where $\Delta \ln C_{t+1} \equiv \ln C_{t+1} - \ln C_t$.

$$1 = e^{-\rho} E\left[R^j_{t+1} \frac{C^{-\gamma}_{t+1}}{C^{-\gamma}_t}\right] = E\left[e^{-\rho} R^j_{t+1} e^{-\gamma \ln(C_{t+1}/C_t)}\right] = E\left[R^j_{t+1} e^{-\rho - \gamma \Delta \ln C_{t+1}}\right]$$

Finally, denote by $r^j_{t+1} = \ln R^j_{t+1}$ the log return of the asset, then we arrive at:

$$1 = E\left[e^{r^j_{t+1} - \rho - \gamma \Delta \ln C_{t+1}}\right]$$

Euler equation under log-normality. A log-normal RV is characterized via the representation

$$X = e^{\mu + \sigma Z},$$

where Z is a standard normal random variable, and (μ, σ) are the parameters of the log-normal. The mean of the log-normal is given by

$$E(X) = e^{\mu + \frac{1}{2}\sigma^2}$$

and its variance by

$$\mathrm{Var}(X) = [e^{\sigma^2} - 1] e^{2\mu + \sigma^2}.$$

The Euler equation can be further simplified when we assume

$$R^j_{t+1} = e^{r^j_{t+1} + \sigma^j \epsilon^j_{t+1} - \frac{1}{2}(\sigma^j)^2},$$

where $\epsilon^j_{t+1} \sim \mathcal{N}(0,1)$, so that

$$R^j_{t+1} \sim \log \mathcal{N}\left(r^j_{t+1} - \frac{1}{2}(\sigma^j)^2, \sigma^j\right).$$

Assume also that $\Delta \ln C_{t+1}$ is conditionally normal, with mean $\mu_{C,t}$ and variance $\sigma^2_{C,t}$. Furthermore assume that the two normals are also jointly, conditionally normal.

(b) Derive the asset pricing equation

$$1 = E_t[\exp(X_t)],$$

where

$$X_t = -\rho + r^j_{t+1} + \sigma^j \epsilon^j_{t+1} - \frac{1}{2}(\sigma^j)^2 - \gamma \Delta \ln C_{t+1}$$

so

$$X_t \sim -\rho + \mathcal{N}\left(r^j_{t+1} - \frac{1}{2}(\sigma^j)^2 - \gamma\mu_{C,t}, (\sigma^j)^2 + \gamma^2\sigma^2_{C,t} - 2\rho_{j,C}\gamma\sigma^j\sigma_{C,t}\right).$$

The Euler equation can be further simplified when we assume

$$R^j_{t+1} = e^{r^j_{t+1} + \sigma^j\epsilon^j_{t+1} - \frac{1}{2}(\sigma^j)^2}$$

where $\epsilon^j_{t+1} \sim \mathcal{N}(0, 1)$, so that

$$R^j_{t+1} \sim \log \mathcal{N}\left(r^j_{t+1} - \frac{1}{2}(\sigma^j)^2, \sigma^j\right)$$

Then, the Euler equation is given as:

$$1 = E_t\left[\exp\left(-\rho + r^j_{t+1} + \sigma^j\epsilon^j_{t+1} - \frac{1}{2}(\sigma^j)^2 - \gamma\Delta\ln C_{t+1}\right)\right]$$

Define X_t as:

$$X_t = -\rho + r^j_{t+1} + \sigma^j \epsilon^j_{t+1} - \frac{1}{2}(\sigma^j)^2 - \gamma \Delta \ln C_{t+1}$$

Assume that $\Delta \ln C_{t+1}$ is conditionally normal with mean $\mu_{C,t}$ and variance $\sigma^2_{C,t}$. Also assume that ϵ^j_{t+1} and $\Delta \ln C_{t+1}$ are jointly normal. Given the assumptions, X_t is normally distributed because it is a linear combination of jointly normal random variables. The mean of X_t is:

$$\mathbb{E}[X_t] = -\rho + \left(r^j_{t+1} - \frac{1}{2}(\sigma^j)^2\right) - \gamma\mu_{C,t}$$

The variance of X_t is:

$$\text{Var}(X_t) = (\sigma^j)^2 + \gamma^2\sigma^2_{C,t} - 2\rho_{j,C}\gamma\sigma^j\sigma_{C,t}$$

where $\rho_{j,C}$ is the correlation coefficient between ϵ^j_{t+1} and $\Delta \ln C_{t+1}$. Thus, $X_t \sim \mathcal{N}(\mu, \sigma^2_X)$, where:

$$\mu = -\rho + r^j_{t+1} - \frac{1}{2}(\sigma^j)^2 - \gamma\mu_{C,t}$$

and

$$\sigma^2_X = (\sigma^j)^2 + \gamma^2\sigma^2_{C,t} - 2\rho_{j,C}\gamma\sigma^j\sigma_{C,t}$$

(c) Taking expecations and logs show:

$$0 = -\rho + r^j_{t+1} - \frac{1}{2}(\sigma^j)^2 - \gamma E_t(\Delta \ln C_{t+1}) + \frac{1}{2}\text{Var}_t(\sigma^j\epsilon^j_{t+1} - \gamma\Delta\ln C_{t+1}) \qquad (1)$$

Since $\exp(X_t)$ is log-normal, we have:

$$1 = \exp\left\{ -\rho + r^j_{t+1} - \frac{1}{2}(\sigma^j)^2 - \gamma\mu_{C,t} + \frac{1}{2}(\sigma^j)^2 + \frac{1}{2}\gamma^2\sigma^2_{C,t} - \gamma\rho_{j,C}\sigma^j\sigma_{C,t} \right\}$$

Taking the log from both sides, we finally arrive at:

$$0 = -\rho + r^j_{t+1} - \gamma\mu_{C,t} + \frac{\gamma^2}{2}\sigma^2_{C,t} - \gamma\rho_{r,C}\sigma^j\sigma_{C,t}$$

More generally, we can leave this as:

$$0 = -\rho + r^j_{t+1} - \frac{1}{2}(\sigma^j)^2 - \gamma E_t(\Delta \ln C_{t+1}) + \frac{1}{2}\mathrm{Var}_t(\sigma^j\epsilon^j_{t+1} - \gamma\Delta \ln C_{t+1})$$

(d) Use this last formula to derive the risk-free rate r^f (Hint: for $j = f$ set $\sigma^f = 0$)

For the risk-free rate, simply plug in $\sigma^f = 0$. Then we have:

$$0 = -\rho + r^f_{t+1} - \gamma E_t(\Delta \ln C_{t+1}) + \frac{\gamma^2}{2}\mathrm{Var}_t(\Delta \ln C_{t+1})$$

(e) Consider a class of equities with risk σ^E, we define the equity premium as

$$\pi^E_{t+1} \equiv r^E_{t+1} - r^f_{t+1}$$

Show

$$\pi^E_{t+1} = \gamma\sigma_{C,E}$$

where $\sigma_{C,E}$ is the covariance between equity returns and log consumption growth.

For the asset class of equities, which we denote by the return R^E_{t+1}, we have:

$$\pi^E_{t+1} \equiv r^E_{t+1} - r^f_{t+1} = \frac{1}{2}(\sigma^E)^2 - \frac{1}{2}\mathrm{Var}_t(\sigma^E\epsilon_{t+1} - \gamma\Delta \ln C_{t+1}) + \frac{\gamma^2}{2}\mathrm{Var}_t(\Delta \ln C_{t+1})$$

Now use the formula:

$$\mathrm{Var}(aX - bY) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y) - 2ab\mathrm{Cov}(X, Y)$$

Thus:

$$\begin{aligned}\pi^E_{t+1} =& \frac{1}{2}(\sigma^E)^2 - \frac{1}{2}\Big(\mathrm{Var}_t(\sigma^E\epsilon_{t+1}) + \mathrm{Var}_t(\gamma\Delta \ln C_{t+1}) - 2\sigma^E\gamma\mathrm{Cov}_t(\epsilon_{t+1}, \Delta \ln C_{t+1})\Big) \\ &+ \frac{\gamma^2}{2}\mathrm{Var}_t(\Delta \ln C_{t+1})\end{aligned}$$

Rewriting, we arrive at:

$$\pi^E_{t+1} = \gamma\sigma_{C,E}$$

where $\sigma_{C,E}$ is the covariance between equity returns and log consumption growth.

Problem 6: Brunnermeier-Sannikov (2014)

This is the most challenging problem you will encounter in this course. Consider it optional, but I would encourage you to give it a try.

In this problem, you will work through a simple variant of the seminal Brunnermeier and Sannikov (2014, AER) paper. This problem brings together many of the tools we have learned and topics we have discussed. It combines a model of intertemporal consumption-savings with a model of investment (similar to Tobin's Q) and portfolio choice.

Consider an agent (household) that can consume, save and invest in a risky asset. Denote by $\{D_t\}_{t\geq 0}$ the *dividend stream* and by $\{Q_t\}_{t\geq 0}$ the price of the asset. Assume that the asset price evolves according to

$$\frac{dQ}{Q} = \mu_Q dt + \sigma_Q dB,$$

where you can interpret μ_Q and σ_Q as simple constants (alternatively, think of them as more complicated objects that would be determined in general equilibrium, which we abstract from here).

This is a model of two assets, capital and bonds. Bonds pay the riskfree rate of return r_t. Capital is accumulated and owned by the agent. Capital is traded at price Q_t and yields dividends at rate D_t.

The key interesting feature of this problem is that the agent faces both (idiosyncratic) earnings risk and (aggregate) asset price risk.

Households take as given all aggregate prices and behave according to preferences given by

$$\mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t) dt.$$

Households consume and save, investing their wealth into bonds and capital. Letting k denote a household's units of capital owned and b units of bonds, the budget constraint is characterized by

$$\begin{aligned} dk_t &= \Phi(\iota_t)k_t - \delta k_t \\ db_t &= r_t b_t + D_t k_t + w_t z_t - c_t - \iota_t k_t. \end{aligned}$$

The rate of investment is given by ι_t. Investment adjustment costs are captured by the concave technology Φ. Dividends are paid to households in units of the numeraire, thus entering the equation for db. The law of motion for household earnings are given by

$$dz = \mu_z dt + \sigma_z dW.$$

(a) Interpret all terms of the two budget constraints. Make sure these budget constraints make sense to you.

It is convenient to rewrite the household problem in terms of liquid net worth, defined by the equations

$$\theta n = Qk$$

$$(1-\theta)n = b,$$

so that total liquid net worth is $n = Qk + b$.

The household increases its capital stock k_t by investing at rate $\Phi(\iota_t)$. However, capital depreciates at rate δ. The household's liquid assets (bonds) increase through returns on bonds ($r_t b_t$), dividends from capital ($D_t k_t$), and labor income ($w_t z_t$). These are offset by consumption expenditures (c_t) and the cost of investment ($\iota_t k_t$).

(b) How would you refer to θ? Assume households' capital accumulation is non-stochastic. That is, there is no capital quality risk. Show that the liquid net worth evolves according to

$$dn = rn + \theta n\left[\frac{D-\iota}{Q} + \frac{dQ}{Q} + \Phi(\iota) - \delta - r\right] + wz - c.$$

We interpret θ as the fraction of total liquid net worth n invested in capital, while $1-\theta$ represents the fraction of wealth invested in bonds. This fraction reflects the household's portfolio allocation between the risky asset (capital) and the risk-free bond.

Differentiating n with respect to time, we have:

$$\begin{aligned} dn &= d(Qk) + db \\ &= kdQ + Qdk + (dk)(dQ) + db \end{aligned}$$

Substituting $(dk)(dQ) = 0$ because there is no risk, and the given budget constraints:

$$dk_t = (\Phi(\iota_t)k_t - \delta k_t)dt, \quad db_t = (r_t b_t + D_t k_t + w_t z_t - c_t - \iota_t k_t)dt$$

we write:

$$dn = k\,dQ + Q(\Phi(\iota_t)k_t - \delta k_t)dt + \left(r_t b_t + D_t k_t + w_t z_t - c_t - \iota_t k_t\right)dt$$

Express k and b in terms of n:

$$k_t = \frac{\theta n}{Q}, \quad b_t = (1-\theta)n$$

Substitute these into the equation for dn:

$$dn = \frac{\theta n}{Q}dQ + Q\left[\Phi(\iota_t)\frac{\theta n}{Q} - \delta\frac{\theta n}{Q}\right]dt + \left[r_t(1-\theta)n + D_t\frac{\theta n}{Q} + w_t z_t - c_t - \iota_t\frac{\theta n}{Q}\right]dt$$

Simplify:

$$dn = \left(rn + \theta n \left[\frac{D - \iota}{Q} + \Phi(\iota) - \delta - r \right] + wz - c \right) dt + \theta n \frac{dQ}{Q}$$

(c) Argue why the choice of ι is entirely static in this setting and show it is only a function of capital $\iota = \iota(Q)$

Households optimize investment to maximize the marginal value of installed capital, which is determined by the first-order condition of the household's problem:

$$\Phi'(\iota) = \frac{1}{Q}.$$

This equation relates the marginal adjustment cost of investment ($\Phi'(\iota)$) to the capital price (Q). Solving for ι, we find:

$$\iota = \iota(Q).$$

This expression implies that investment decisions are determined solely by the current capital price, given the adjustment cost function Φ.

Recall also that households take as given aggregate "prices" (r, w, D, Q). This will allow us to work with a simplified representation. Define

$$dR = \underbrace{\frac{D - \iota(Q)}{Q}}_{\text{Dividend yield}} dt + \underbrace{\left[\Phi(\iota(Q)) - \delta \right] dt + \frac{dQ}{Q}}_{\text{Capital gains}} \equiv \mu_R dt + \sigma_R dB$$

to be the effective rate of return on households' capital investments. And where

$$\mu_R = \frac{D - \iota(Q)}{Q} + \Phi(\iota(Q)) - \delta + \mu_Q$$

$$\sigma_R = \sigma_Q.$$

After solving for $\iota = \iota(Q)$, this return is exogenous from the perspective of the household: it depends on macro conditions and prices, but not on the particular portfolio composition of the household.

(d) Show that the law of motion of the household's liquid net worth satisfies the following equation. Why do you think using liquid net worth is useful? And why do we want this law of motion?

$$dn = rn + \theta n(\mu_R - r) + wz - c + \theta n \sigma_R dB.$$

$dn = rn + \theta n(dR - r) + wz - c$ and then substitute using the definitions of μ_R and σ_R. By focusing on liquid net worth n, the problem is simplified to a single state variable (plus z), abstracting away from the portfolio composition.

Recursive representation. We denote the agent's individual states by (n, z). But notice that the agent also faces time-varying prices (macroeconomic aggregates) like Q_t. To make our lives simple, we make the following assumption: Suppose there is a scalar stochastic process X_t that fully summarizes the aggregate state of the macroeconomy. This means that we can represent all other prices as functions of it, i.e.,

$$r_t = r(X_t), \quad D_t = D(X_t), \quad Q_t = Q(X_t).$$

We refer to X_t as the *aggregtate state of the economy*. And let's assume that it follows a simple diffusion process given by

$$dX = \mu dt + \sigma dB.$$

This now allows us to write the household problem recursively with X as an extra state variable. That is, our state variables are (n, z, X). Note that otherwise, we would need to keep track of all the prices separately.

(e) Show that the household problem satisfies the following HJB

$$\rho V(n, z, X) = \max_{c,\theta} \left\{ u(c) + V_n \Big[rn + \theta n(\mu_R - r) + wz - c \Big] + \frac{1}{2} V_{nn} (\theta n \sigma_R)^2 + V_z \mu_z + \frac{1}{2} V_{zz} \sigma_z^2 \right.$$
$$\left. + V_{nX} \theta n \sigma_R \sigma + V_X \mu + \frac{1}{2} \sigma^2 V_X \right\},$$

where you can assume that $\mathbb{E}(dWdB) = 0$. This means that households' earnings risk is uncorrelated with the aggregate state X. This assumption is at odds with the data! But it simplifies the HJB here. (Why?)

We have 3 state variables (n, z, X) and 2 controls (c, θ). The HJB with the generator is:

$$\rho V(n, z, X) = \max_{c,\theta} \left\{ u(c) + \mathcal{A}V \right\}$$

Remember also that we get the generator by taking expectations after applying Ito's Lemma. We do the two steps, apply Ito's lemma to dV (ignoring the dt):

$$\begin{aligned} dV =& V_n \Big[rn + \theta n(\mu_R - r) + wz - c \Big] + \frac{1}{2} V_{nn} (\theta n \sigma_R)^2 + V_n \theta n \sigma_R dB \\ & + V_z \mu_z + \frac{1}{2} V_{zz} \sigma_z^2 + V_{zz} \sigma_z dW + V_X \mu + \frac{1}{2} \sigma^2 V_{XX} + V_{XX} \sigma dB + V_{nX} (\theta n \sigma_R) \sigma \\ & + V_{nz} (\theta n \sigma_R) \sigma_z dW dB + V_{zX} \sigma \sigma_z dW dB \end{aligned}$$

Because there are 3 state variables, we also have to take into account the cross-partial derivatives (and correlations). For the term V_{nX} because n and X depend on the same stochastic process we have $dBdB = dt$. Taking expectations the terms with dB, dB and $dBdW$ disappear. Substituting back into the HJB we get the result.

(f) Derive the first-order conditions for consumption and portfolio choice.

$$u_c = V_n$$

$$\theta = -\left(\frac{V_n}{nV_{nn}}\frac{\mu_R - r}{\sigma_R^2} + \frac{V_{nX}}{nV_{nn}}\frac{\boldsymbol{\sigma}}{\sigma_R}\right)$$

(g) Difficult: Use the envelope condition and apply Ito's lemma to $V_n(n, z, X)$, and show that household marginal utiliy evolves according to

$$\frac{du_c}{u_c} = (\rho - r)dt - \frac{\mu_R - r}{\sigma_R}dB - X\frac{c_z}{c}\sigma_z dW.$$

The HJB envelope condition is given by:

$$(\rho - r)V_n = V_{nn}s + V_n\theta(\mu_R - r) + \frac{1}{2}V_{nnn}(\theta n\sigma_R)^2 + V_{nn}n(\theta\sigma_R)^2 + V_{ny}\mu_y + \frac{\sigma_y^2}{2}V_{yny}$$
$$+ V_{nnX}\theta n\sigma_R\boldsymbol{\sigma} + V_{nX}\theta\sigma_R\boldsymbol{\sigma} + V_{nX}\boldsymbol{\mu} + \frac{1}{2}\boldsymbol{\sigma}^2 V_{nXX}$$

Applying Ito's lemma to $V_n(n, y, X)$, we have:

$$dV_n = V_{nn}dn + \frac{1}{2}V_{nnn}(dn)^2 + V_{ny}dy + \frac{1}{2}V_{nyy}(dy)^2 + V_{nX}dX + \frac{1}{2}V_{nXX}(dX)^2 + V_{nnX}(dn)(dX)$$
$$= V_{nn}(s + \theta n\sigma_R dB) + \frac{1}{2}V_{nnn}(\theta n\sigma_R)^2 + V_{ny}(\mu_y + \sigma_y dW) + \frac{1}{2}V_{nyy}\sigma_y^2$$
$$+ V_{nX}(\boldsymbol{\mu} + \boldsymbol{\sigma}dB) + \frac{1}{2}\boldsymbol{\sigma}_X^T V_{nXX}\boldsymbol{\sigma} + V_{nnX}(\theta n\sigma_R)(\boldsymbol{\sigma}_X)$$

Putting this together with:

$$(\rho - r)V_n - V_n\theta(\mu_R - r) - V_{nn}n(\theta\sigma_R)^2 - V_{nX}\theta\sigma_R\boldsymbol{\sigma} = V_{nn}s + \frac{1}{2}V_{nnn}(\theta n\sigma_R)^2 + V_{ny}\mu_y + \frac{\sigma_y^2}{2}V_{yny}$$
$$+ V_{nnX}\theta n\sigma_R\boldsymbol{\sigma} + V_{nX}\boldsymbol{\mu} + \frac{1}{2}\boldsymbol{\sigma}^2 V_{nXX}$$

we have:

$$dV_n = V_{nn}\theta n\sigma_R dB + V_{ny}\sigma_y dW + V_{nX}\boldsymbol{\sigma}dB$$
$$+ (\rho - r)V_n - V_n\theta(\mu_R - r) - V_{nn}n(\theta\sigma_R)^2 - V_{nX}\theta\sigma_R\boldsymbol{\sigma}$$

Using:

$$\theta nV_{nn} = -\left(V_n\frac{\mu_R - r}{\sigma_R^2} + V_{nX}\frac{\boldsymbol{\sigma}}{\sigma_R}\right)$$

we have:

$$dV_n = -\left(V_n\frac{\mu_R - r}{\sigma_R^2} + V_{nX}\frac{\boldsymbol{\sigma}}{\sigma_R}\right)\sigma_R dB + V_{ny}\sigma_y dW + V_{nX}\boldsymbol{\sigma}dB$$
$$+ (\rho - r)V_n - V_n\theta(\mu_R - r) + \left(V_n\frac{\mu_R - r}{\sigma_R^2} + V_{nX}\frac{\boldsymbol{\sigma}}{\sigma_R}\right)\theta\sigma_R^2 - V_{nX}\theta\sigma_R\boldsymbol{\sigma}$$

Simplifying:

$$dV_n = -V_n \frac{\mu_R - r}{\sigma_R} dB - V_{nX}\boldsymbol{\sigma} dB + V_{ny}\sigma_y dW + V_{nX}\boldsymbol{\sigma} dB$$
$$+ (\rho - r)V_n - V_n\theta(\mu_R - r) + V_n(\mu_R - r)\theta + V_{nX}\boldsymbol{\sigma}\theta\sigma_R - V_{nX}\theta\sigma_R\boldsymbol{\sigma}$$

Combining terms yield the result.

(h) Difficult: Show that household consumption evolves according to

$$\frac{dc}{c} = \frac{r-\rho}{X}dt + \frac{1}{2}(1+X)\left[\left(\frac{\mu_R - r}{X\sigma_R}\right)^2 + \left(\frac{c_z}{c}\sigma_z\right)^2\right]dt + \frac{\mu_R - r}{X\sigma_R}dB + \frac{c_z}{c}\sigma_z dW.$$

This implies that

$$\mathbb{E}\left[\frac{dc}{c}\right] = \frac{r-\rho}{X}dt + \frac{1}{2}(1+X)\left[\left(\frac{\mu_R - r}{X\sigma_R}\right)^2 + \left(\frac{c_z}{c}\sigma_z\right)^2\right]dt.$$

Relate this expression to our discussion of *precautionary savings* in class.

The consumption policy function is given by $c = c(n, y, X)$. Thus:

$$dc = c_n dn + c_y dy + c_X dX + \frac{1}{2}c_{nn}(dn)^2 + \frac{1}{2}c_{yy}\sigma_y^2 + \frac{1}{2}\boldsymbol{\sigma}^T c_{XX}\boldsymbol{\sigma} + c_{nX}\theta n\sigma_R\boldsymbol{\sigma}$$

and so:

$$\begin{aligned}
(dc)^2 &= \Big(c_n\theta n\sigma_R dB + c_y\sigma_y dW + c_X\boldsymbol{\sigma} dB\Big)^2 \\
&= \Big(c_n\theta n\sigma_R dB + c_X\boldsymbol{\sigma} dB\Big)^2\Big(c_y\sigma_y dW\Big)^2 \\
&= \Big[(c_n\theta n\sigma_R)^2 + 2c_n c_X\theta n\sigma_R\boldsymbol{\sigma} + (c_X\boldsymbol{\sigma})^2 + (c_y\sigma_y)^2\Big]dt
\end{aligned}$$

Let's simplify this expression a bit. Notice that we have:

$$\begin{aligned}
(c_n n\sigma_R)^2\theta\theta + 2c_n c_X\theta n\sigma_R\boldsymbol{\sigma} &= -(c_n n\sigma_R)^2\theta\left(\frac{u_c}{nu_{cc}c_n}\frac{\mu_R - r}{\sigma_R^2} + \frac{u_{cc}c_X}{nu_{cc}c_n}\frac{\boldsymbol{\sigma}}{\sigma_R}\right) + 2c_n c_X\theta n\sigma_R\boldsymbol{\sigma} \\
&= -(c_n n\sigma_R)^2\theta\frac{1}{nc_n}\left(-c\frac{\mu_R - r}{X\sigma_R^2} + c_X\frac{\boldsymbol{\sigma}}{\sigma_R}\right) + 2c_n c_X\theta n\sigma_R\boldsymbol{\sigma} \\
&= (c_n n\sigma_R)^2\theta\frac{1}{nc_n}c\frac{\mu_R - r}{X\sigma_R^2} - (c_n n\sigma_R)\theta c_X\boldsymbol{\sigma} + 2c_n c_X\theta n\sigma_R\boldsymbol{\sigma} \\
&= \left(\frac{\mu_R - r}{X\sigma_R^2} - \frac{c_X}{c}\frac{\boldsymbol{\sigma}}{\sigma_R}\right)c^2\frac{\mu_R - r}{X} + c_n c_X\theta n\sigma_R\boldsymbol{\sigma} \\
&= c^2\left(\frac{\mu_R - r}{X\sigma_R}\right)^2 - c_X\boldsymbol{\sigma}c\frac{\mu_R - r}{X\sigma_R} + c_n c_X\theta n\sigma_R\boldsymbol{\sigma}
\end{aligned}$$

Therefore, we have:

$$(dc)^2 = \left[c^2\left(\frac{\mu_R - r}{X\sigma_R}\right)^2 - c_X\sigma c\frac{\mu_R - r}{X\sigma_R} + c_n c_X \theta n \sigma_R \sigma + (c_X\sigma)^2 + (c_y\sigma_y)^2\right]dt$$

We can take one last step here, noting that:

$$\begin{aligned}(dc)^2 &= \left[c^2\left(\frac{\mu_R - r}{X\sigma_R}\right)^2 - c_X\sigma c\frac{\mu_R - r}{X\sigma_R} + c_X\sigma c\frac{\mu_R - r}{X\sigma_R} - (c_X\sigma)^2 + (c_X\sigma)^2 + (c_y\sigma_y)^2\right]dt \\ &= \left[c^2\left(\frac{\mu_R - r}{X\sigma_R}\right)^2 + (c_y\sigma_y)^2\right]dt\end{aligned}$$

I haven't used this yet in the rest of the proof, but it's the same thing.
Using Ito's lemma for $u_c(c)$, we have

$$du_c = u_{cc}dc + \frac{1}{2}u_{ccc}(dc)^2$$

Plugging in and using CRRA, we have:

$$\begin{aligned}&(\rho - r)dt - \frac{\mu_R - r}{\sigma_R}dB - X\frac{c_y}{c}\sigma_y dW \\ &= \frac{u_{cc}}{u_c}dc + \frac{1}{2}\frac{u_{ccc}}{u_c}\left[c^2\left(\frac{\mu_R - r}{X\sigma_R}\right)^2 - c_X\sigma c\frac{\mu_R - r}{X\sigma_R} + c_n c_X \theta n \sigma_R \sigma + (c_X\sigma)^2 + (c_y\sigma_y)^2\right]dt\end{aligned}$$

Plugging in for CRRA:

$$\begin{aligned}&\frac{r - \rho}{X}dt + \frac{\mu_R - r}{X\sigma_R}dB + \frac{c_y}{c}\sigma_y dW \\ &= \frac{dc}{c} - \frac{1}{2}\frac{1 + X}{c^2}\left[c^2\left(\frac{\mu_R - r}{X\sigma_R}\right)^2 - c_X\sigma c\frac{\mu_R - r}{X\sigma_R} + c_n c_X \theta n \sigma_R \sigma + (c_X\sigma)^2 + (c_y\sigma_y)^2\right]dt\end{aligned}$$

Therefore, we have:

$$\begin{aligned}\frac{dc}{c} =& \frac{r - \rho}{X}dt + \frac{\mu_R - r}{X\sigma_R}dB + \frac{c_y}{c}\sigma_y dW \\ &+ \frac{1}{2}(1 + X)\left[\left(\frac{\mu_R - r}{X\sigma_R}\right)^2 - \frac{c_X}{c}\sigma\frac{\mu_R - r}{X\sigma_R} + \frac{1}{c^2}c_n c_X \theta n \sigma_R \sigma + \left(\frac{c_X}{c}\sigma\right)^2 + \left(\frac{c_y}{c}\sigma_y\right)^2\right]dt\end{aligned}$$

Lastly, I can substitute in for θ again, so that this term becomes:

$$\begin{aligned}\frac{1}{c^2}c_n c_X \theta n \sigma_R \sigma &= \frac{1}{c^2}c_n c_X n \sigma_R \sigma \frac{c}{n c_n}\left(\frac{\mu_R - r}{X\sigma_R^2} - \frac{c_X}{c}\frac{\sigma}{\sigma_R}\right) \\ &= \frac{1}{c}c_X \sigma_R \sigma\left(\frac{\mu_R - r}{X\sigma_R^2} - \frac{c_X}{c}\frac{\sigma}{\sigma_R}\right) \\ &= \frac{c_X}{c}\sigma\frac{\mu_R - r}{X\sigma_R} - \left(\frac{c_X}{c}\sigma\right)^2\end{aligned}$$

Therefore:

$$\frac{dc}{c} = \frac{r-\rho}{X}dt + \frac{\mu_R - r}{X\sigma_R}dB + \frac{c_y}{c}\sigma_y dW$$

$$+\frac{1}{2}(1+X)\left[\left(\frac{\mu_R - r}{X\sigma_R}\right)^2 - \frac{c_X}{c}\boldsymbol{\sigma}\frac{\mu_R - r}{X\sigma_R} + \frac{c_X}{c}\boldsymbol{\sigma}\frac{\mu_R - r}{X\sigma_R} - \left(\frac{c_X}{c}\boldsymbol{\sigma}\right)^2 + \left(\frac{c_X}{c}\boldsymbol{\sigma}\right)^2 + \left(\frac{c_y}{c}\sigma_y\right)^2\right]dt$$

Simplifying yields the result.