Dynamic Programming and Applications

Consumption

Lectures 7–8

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Outline

Part 1: Early theories of consumption

1. The old-Keynesian consumption function
2. The consumption Euler equation
3. The permanent income hypothesis (PIH)
4. The certainty equivalence model (Hall 1978)
5. Precautionary savings
6. Linearization of the Euler Equation

Outline

Part 2: Empirical regularities

1. Empirical tests of the consumption Euler equation
2. How does consumption move with income fluctuations?
3. What is the marginal propensity to consume (MPC) out of income?

Part 3: Tractable models of consumption and MPCs

1. Eat-the-pie
2. Unearned income
3. Labor supply
4. Precautionary savings

Part 4: The canonical consumption model with liquidity constraints

Part 1: Early Theories

1. The old-Keynesian consumption function

Keynes (1936, p. 96):

> *The fundamental psychological law, upon which we are entitled to depend with great confidence both* a priori *and from our knowledge of human nature and from detailed facts of experience, is that [people] are disposed, as a rule and on average, to increase their consumption as their income increases, but not by as much as the increase in their income.*

Keynes (1936, p. 93-94):

> *The usual type of short-period fluctuation in the rate of interest is not likely, however, to have much* direct *influence on spending either way. There are not many people who will alter their way of living because the rate of interest has fallen from 5 to 4 per cent, if their aggregate income is the same as before.*

Keynes' Consumption Function

$$C_t = \alpha + \gamma(Y_t - T_t)$$

- Consumption a function of after-tax income
- Marginal propensity to consume (γ) between zero and one
- Interest rates not important
- Future income not important

Three Landmark Empirical Studies

“dealt a fatal blow to this extraordinarily simple view of the savings process” (Modigliani 86)

- Simon Kuznetz (1946):
  - National Income and Product Accounts back to 1899
  - No rise in aggregate savings over time
- Dorothy Brady and Rose D. Friedman (1947):
  - Re-analyze budget study data
  - Consumption function shifts up over time as average income increases
- Margaret Reid (unpublished):
  - Re-analyzes budget study data
  - Introduces concept of “permanent component of income”

(See Burns (2022) for history of “Hidden Figures.”)

2. Canonical model of consumption

- Canonical model known as **consumption-savings** or **income-fluctuations** problem:

$$V(a_0) = \max_{\{c_t\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$a_{t+1} = R_{t+1}(a_t - c_t) + y_t$$

- a_t is wealth, R_{t+1} is deterministic (gross) interest rate and y_t is iid. income risk
- Assume $u(\cdot)$ concave ($u' > 0$ and $u'' < 0$ for all c) and $\lim_{c \to 0} u'(c) = \infty$
- What should we assume about borrowing capacity?
  - Natural borrowing limit: $a_t \geq a^n$
  - Ad-hoc borrowing limit: $a_t \geq \underline{a}$

- Bellman equation:

$$V_t(a) = \max_{c,a'} \left\{ u(c) + \beta \mathbb{E}_t V_{t+1}(a') \right\} \;\; s.t. \;\; a' = R_{t+1}(a - c) + y$$

- Bellman equation may not be stationary or time-independent: R_t
- If income process $\{y_t\}$ persistent, would need y as second state variable
- First-order conditions (with ad-hoc borrowing limit $\underline{a}$):

$$u'(c_t(a)) = \beta R_{t+1} \mathbb{E}_t \frac{\partial V_{t+1}(a')}{\partial a'} \qquad \text{if } a' > \underline{a}$$

$$u'(c_t(a)) \geq \beta R_{t+1} \mathbb{E}_t \frac{\partial V_{t+1}(a')}{\partial a'} \qquad \text{if } a' = \underline{a}$$

- Envelope theorem: $\frac{\partial V_t(a)}{\partial a} = u'(c_t(a))$
- So we again get a consumption Euler equation:

$$
\begin{aligned}
u'(c_t) &= \beta R_{t+1} \mathbb{E}_t u'(c_{t+1}) \quad \text{if } a' > \underline{a} \\
u'(c_t) &\geq \beta R_{t+1} \mathbb{E}_t u'(c_{t+1}) \quad \text{if } a' = \underline{a}
\end{aligned}
$$

Perturbation intuition:

- What is the cost of consuming ϵ dollars less today?

$$\text{Utility loss today} = \epsilon \cdot u'(c_t)$$

- What is the expected, discounted gain of consuming $\epsilon \cdot R_{t+1}$ dollars more tomorrow?

$$\text{Utility gain tomorrow} = \beta(\epsilon \cdot R_{t+1}) \mathbb{E}_t u'(c_{t+1})$$

3. Permanent Income Hypothesis (PIH)

- Originally developed independently by:
  - Modigiani and Brumberg (1954) (Life-Cycle Hypothesis)
  - Friedman (1957) (Permanent Income Hypothesis)
- Basic idea:
  - Utility maximization and perfect markets imply that current consumption is determined by net present value of life-time income
- Dramatically different from Keynesian consumption function

Life Cycle Hypothesis: Modigliani & Brumberg (1954)

- $R_t = R$ for all t, and $\beta R = 1$
- No assumption on (stochastic) income process $\{y_t\}$
- Perfect capital markets (i.e., no moral hazard), so that future income, y_t, can be exchanged for current capital. Let's assume that your counter-party is risk neutral

Bellman Equation:

$$v(x) = \max_{c \leq x} \left\{ u(c) + \beta v(x) \right\}$$

subject to

$$x' = R(x-c) \qquad \text{and} \qquad x_0 = \mathbb{E} \sum_{t=0}^{\infty} R^{-t} y_t$$

- Sometimes referred to as "eating a pie/cake problem"
- Euler Equation implies
$$u'(c_t) = \beta R u'(c_{t+1}) = u'(c_{t+1})$$
- Hence, consumption is constant: $c_t = c_{t-1}$ regardless of y_t

- Budget constraint:

$$\sum_{t=0}^{\infty} R^{-t} c_t = \mathbb{E}_0 \sum_{t=0}^{\infty} R^{-t} y_t$$

- Substitute Euler Equation, $c_0 = c_t$ to find

$$\sum_{t=0}^{\infty} R^{-t} c_0 = \mathbb{E}_0 \sum_{t=0}^{\infty} R^{-t} y_t$$

- So Euler Equation + budget constraint implies

$$c_0 = \left(1 - \frac{1}{R}\right) \mathbb{E}_0 \sum_{t=0}^{\infty} R^{-t} y_t$$

- Consumption is an annuity. Note that $(1 - \frac{1}{R}) \simeq (1 - \frac{1}{1}) + \frac{1}{R^2_{|R=1}}(R-1) = R - 1 = r$

4. Certainty Equivalence Model: Hall (1978)

- $R_t = R$ for all t and $\beta R = 1$
- Quadratic utility: $u(c) = \alpha c - \frac{\gamma}{2}c^2$
  - This admits negative consumption
  - And this does **not** imply $\lim_{c \downarrow 0} u'(c) = \infty$
- Now assume you can't sell claims to labor income
- No interest rate uncertainty, but household now bears exposure to stochastic income fluctuations

- Euler Equation

$$u'(c_t) = \beta R \mathbb{E}_t u'(c_{t+1})$$

implies,

$$\begin{aligned} -\gamma c_t &= -\gamma \mathbb{E}_t c_{t+1} \\ c_t &= \mathbb{E}_t c_{t+1} = \mathbb{E}_t c_{t+n}. \end{aligned}$$

So consumption is a random walk (martingale):

$$c_{t+1} = c_t + \eta_{t+1}.$$

- So $\Delta c_{t+1} \equiv c_{t+1} - c_t$ cannot be predicted by any information available at time t

- Budget constraint at date t is (along any proceeding history)

$$\sum_{s=0}^{\infty} R^{-s} c_{t+s} = x_t + \sum_{s=1}^{\infty} R^{-s} y_{t+s}$$

$$\implies \mathbb{E}_t \sum_{s=0}^{\infty} R^{-s} c_{t+s} = x_t + \mathbb{E}_t \sum_{s=1}^{\infty} R^{-s} y_{t+s}$$

- Substitute $c_t = \mathbb{E}_t c_{t+s}$:

$$\sum_{s=0}^{\infty} R^{-s} c_t = x_t + \mathbb{E}_t \sum_{s=1}^{\infty} R^{-s} y_{t+s}$$

- So Euler Equation + budget constraint implies

$$c_t = \left(1 - \frac{1}{R}\right)\left(x_t + \mathbb{E}_t \sum_{s=1}^{\infty} R^{-s} y_{t+s}\right)$$

5. Precautionary Savings

- Recall certainty equivalence model:

$$c_t = \mathbb{E}_t c_{t+1}$$

- Imagine there are two periods, interest rate is 0

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Scenario **A**:

- You get $200 today
- You get $100 tomorrow with certainty
- How much would you save today?

5. Precautionary Savings

- Recall certainty equivalence model:

$$c_t = \mathbb{E}_t c_{t+1}$$

- Imagine there are two periods, interest rate is 0

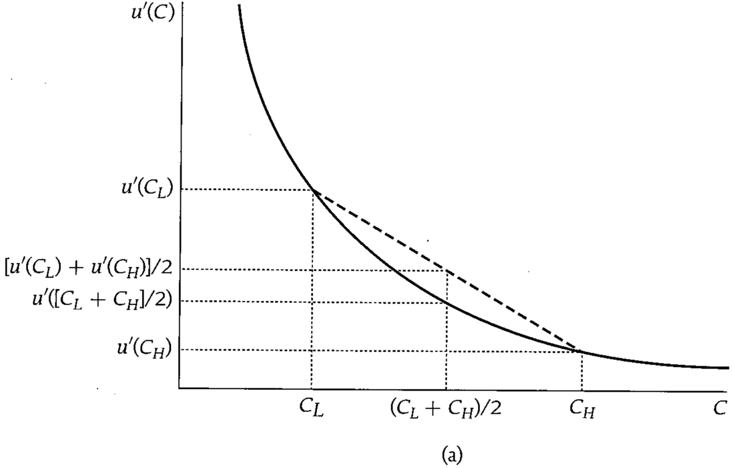
Scenario **B**:

- You get \$200 today as before
- You get \$200 tomorrow with probability $\frac{1}{2}$, and \$0 otherwise
- How much would you save today?

- Curvature of utility almost surely falls as consumption rises:

$$u'''(c_t) > 0$$

- This implies that marginal utility $u'(c_t)$ is convex
- Suppose you currently consume such that $c_t = \mathbb{E}_t c_{t+1}$. Is this optimal?
- Notice: $u'(c_t) = u'(\mathbb{E}_t c_{t+1}) < \mathbb{E}_t u'(c_{t+1})$
- Therefore: marginally raising c_t increases utility
- This extra saving today (higher c_t) relative to certainty equivalence case is called **precautionary savings**



Source: Romer (2019). 50-50 chance of C_H and C_L.

6. Linearizing the Euler Equation

- Recall Euler Equation:

$$u'(c_t) = \mathbb{E}_t \beta R_{t+1} u'(c_{t+1})$$

- Want to transform this equation so it is more amenable to empirical analysis

- Assume that R_{t+1} is known at time t

- Assume u is isoelastic (i.e., constant relative risk aversion) utility function,

$$u(c) = \frac{c^{1-\gamma} - 1}{1-\gamma}.$$

(Aside: $\lim_{\gamma \to 1} \frac{c^{1-\gamma}-1}{1-\gamma} = \ln c$. Important special case.)

- Recall $-\log\beta = \rho$ and $\log R_{t+1} = r_{t+1}$
- Noting that $u'(c) = c^{-\gamma}$, rewrite Euler equation as

$$
\begin{aligned}
c_t^{-\gamma} &= \mathbb{E}_t \beta R_{t+1} c_{t+1}^{-\gamma} \\
1 &= \mathbb{E}_t \exp\left[\log(\beta R_{t+1} c_{t+1}^{-\gamma} c_t^{\gamma})\right] \\
1 &= \mathbb{E}_t \exp\left[-\rho + r_{t+1} - \gamma \log(c_{t+1}/c_t)\right] \\
1 &= \mathbb{E}_t \exp\left[r_{t+1} - \rho - \gamma \Delta \log c_{t+1}\right]
\end{aligned}
$$

- Assume that $\Delta \log c_{t+1}$ is conditionally normally distributed. So,

$$1 = \exp\left[r_{t+1} - \rho - \gamma \mathbb{E}_t \Delta \log c_{t+1} + \frac{1}{2}\gamma^2 \mathbb{V}ar_t \Delta \log c_{t+1}\right]$$

- Taking logs of both sides

$$\mathbb{E}_t \Delta \log c_{t+1} = \frac{1}{\gamma}(r_{t+1} - \rho) + \frac{\gamma}{2}\mathbb{V}ar_t \Delta \log c_{t+1}.$$

- **Precautionary savings**: $\frac{\gamma}{2}\mathbb{V}ar_t \Delta \log c_{t+1} > 0$ implies positive expected consumption growth. Why?

Part 2: Empirical Regularities

1. Empirical tests of the Euler equation

- Recall the consumption Euler equation

$$u'(c_t) = \mathbb{E}_t \beta R_{t+1} u'(c_{t+1})$$

- Rewrite linearized Euler equation in regression form

$$\Delta \log c_{t+1} = \frac{1}{\gamma}(r_{t+1} - \rho) + \frac{\gamma}{2}\mathbb{V}ar_t \Delta \log c_{t+1} + \varepsilon_{t+1}$$

  where ε_{t+1} is orthogonal to any information known at date t

- The conditional variance term is often referred to as the “precautionary savings term” (more on this later)

- We sometimes (counterfactually) assume that $\mathbb{V}ar_t \Delta \log c_{t+1}$ is constant (i.e., independent of time). So the Euler equation reduces to:

$$\Delta \log c_{t+1} = \text{constant} + \frac{1}{\gamma}(r_{t+1} - \rho) + \varepsilon_{t+1}$$

- When we replace the precautionary term with a constant, we are effectively ignorning its effect (since it is no longer separately identified from the other constant term: $\frac{\rho}{\gamma}$)

- Hundreds of papers have estimated a linearized Euler equation:

$$\Delta \log c_{t+1} = \text{constant} + \frac{1}{\gamma} r_{t+1} + \beta X_t + \varepsilon_{t+1}$$

- The principal goals of these regressions are twofold:

1. Estimate $\frac{1}{\gamma}$, the elasticity of intertemporal substitution (EIS) $= \frac{\partial \Delta \log c_{t+1}}{\partial r_{t+1}}$. For example, see Hall (1988).
    - For this model, the EIS is the inverse of the CRRA

2. Test the orthogonality restriction:

$$\left\{\Omega_t \equiv \text{information set at date} t\right\} \perp \varepsilon_{t+1}$$

- In other words, test the restriction that information available at time t does **not** predict consumption growth in the following regression

$$\Delta \log c_{t+1} = \text{constant} + \frac{1}{\gamma} r_{t+1} + \beta X_t + \varepsilon_{t+1}$$

- For example, does the date t expectation of income growth, $\mathbb{E}_t \Delta \log y_{t+1}$ predict date $t+1$ consumption growth?

$$\Delta \log c_{t+1} = \text{constant} + \frac{1}{\gamma}\mathbb{E}_t r_{t+1} + \alpha \mathbb{E}_t \Delta \log y_{t+1} + \varepsilon_{t+1}$$

- Finding: $\hat{\alpha} \in [0.1, 0.35]$ so $\mathbb{E}_t \Delta \log y_{t+1}$ covaries with $\Delta \log c_{t+1}$

  E.g. Campbell and Mankiw 1989, Shea 1995, Shapiro 2005, Parker and Broda 2014, Gelman, Kariv, Shapiro, Silverman, and Tadelis 2015, Ganong and Noel 2018

- Orthogonality restriction is violated: information at date t predicts consumption growth from t to $t+1$

- In other words, the assumptions (1) the Euler equation is true, (2) the utility function is in the CRRA class, (3) the linearization is accurate, and (4) $\mathbb{V}ar_t \Delta \log c_{t+1}$ is constant, are jointly rejected

  Recall we also assumed households have perfect foresight over R_{t+1}

2. How does consumption respond to windfalls?

- Related literature estimates the marginal propensity to consume non-durables (MPC) out of wealth “windfalls” (or unearned income)

$$c_{t+1} = \text{constant} + \alpha w_{t+1} + \varepsilon_{t+1}$$

- $\hat{\alpha} \in [0.1, 0.35]$ so MPC is much higher than in classical models

  E.g. Havranek and Sokolova 2020, Ganong, Jones, Noel, Greig, Farrell, and Wheat 2020, Parker et al 2013, Kueng 2018, Fagereng et al 2019

- Note that marginal propensity for expenditure (MPX) is about three times as high as $\hat{\alpha}$ due to expenditures on durables (Laibson-Maxted-Moll 2022)

Why does $\mathbb{E}_t \Delta \log y_{t+1}$ predict $\Delta \log c_{t+1}$?

- Welfare costs of smoothing are second-order (Cochrane 1989, Pischke 1995, Browning and Crossley 2001, Kueng 2015, Gabaix2016)
- Leisure and consumption expenditure are substitutes (Heckman 1974, Ghez and Becker 1975, Aguiar and Hurst 2005, 2007, Stephens and Toohey 2019)
- Work-based expenses (see Ganong and Noel 2016)
- Households support lots of dependents in mid-life when income is highest (Browning 1992, Attanasio 1995, Seshadri et al 2006)

Why does $\mathbb{E}_t \Delta \log y_{t+1}$ predict $\Delta \log c_{t+1}$?

- Some consumers use rules of thumb: $c_{it} = \alpha y_{it}$ (Campbell and Mankiw 1989, Thaler and Shefrin 1981, Gabaix 2016)

- Markets are incomplete, households are impatient, and utiliy is CRRA (Deaton 1991, Carroll 1992): these assumptions make the variance term time-varying ($V_t \Delta \ln c_{t+1}$)

- Markets are incomplete, households are present-biased, and utility is CRRA (Laibson 1997, Harris and Laibson 2001, Shapiro 2005, Laibson, Maxted, Repetto, and Tobacman 2016): these assumptions violate the Euler equation altogether

Part 3: Tractable Models of Consumption

What's the goal in this part of the lecture?

- Start with simplest model of consumption and add ingredients one by one
- Each model admits closed-form solution $\rightarrow$ illustrates economic mechanism
  Closed-form solutions are rare and special. CT does a lot of work here!
- We will study 3 crucial extensions to simplest model:
  1. Labor supply
  2. Precautionary savings
  3. Incomplete markets and liquidity constraints
- By the end, we arrive at **the canonical consumption model**

1. Eat-the-pie: no risk, no income

- Already saw variant of this in DT $\rightarrow$ even more tractable with CT!
- Time is continuous, $t \in [0, \infty)$
- Infinitely-lived household's wealth evolves according to

$$\dot{a} = ra - c$$

  given initial wealth position a_0, constant interest rate r

- The HJB equation is given by

$$\rho V(a) = \frac{1}{1-\gamma} c(a)^{1-\gamma} + V_a(ra - c(a))$$

  where $c(a)$ solves first-order condition $c(a)^{-\gamma} = V_a$

- Consider Ansatz $V = Aa^B$, so that $V_a = ABa^{B-1}$ and $c = [ABa^{B-1}]^{-\frac{1}{\gamma}}$

- Then HJB becomes

$$\rho A a^B = \frac{1}{\gamma}\left([ABa^{B-1}]^{-\frac{1}{\gamma}}\right)^{1-\gamma} + ABa^{B-1}\left(ra - [ABa^{B-1}]^{-\frac{1}{\gamma}}\right)$$
$$\rho A a^B - rABa^B = \frac{\gamma}{1-\gamma}[ABa^{B-1}]^{\frac{\gamma-1}{\gamma}}$$

Plug in $B = 1-\gamma$ and solve for (do this yourself!)

$$A = \frac{1}{1-\gamma}\left[\frac{1}{\gamma}\rho - \frac{1-\gamma}{\gamma}r\right]^{-\gamma}$$

- Solution of this model:

$$V(a) = \frac{1}{1-\gamma}\kappa^{-\gamma}a^{1-\gamma} \quad \text{and} \quad V_a(a) = \kappa^{-\gamma}a^{-\gamma} \quad \text{and} \quad c(a) = \kappa a$$

- Marginal propensity to consume (MPC) is defined as

$$c'(a) = \kappa \equiv \frac{1}{\gamma}\rho - \frac{1-\gamma}{\gamma}r$$

- Consider $u(c) = \log(c)$, with $\gamma \to 1$ (income and substitution effects offset each other)

$$c'(a) = \kappa = \rho \approx 5\% \text{ annually}$$

- The standard CRRA calibration with $\gamma = 2$ yields

$$c'(a) = \kappa \equiv \frac{1}{2}(\rho + r).$$

In models with uninsurable risk and incomplete financial markets, $r < \rho$

What you should take away from this: In the simplest model of consumption

1. The consumption policy function is linear: $c(a) = \kappa a$
2. MCP is small, $\kappa \approx 5\%$
3. MPC does not vary with wealth or income

These predictions are strongly rejected in the data!

Q: Is there a better theory of consumption consistent with MPCs that (i) are large and (ii) vary with wealth / income?

2. What happens when we add (unearned) income?

- Household now earns exogenous income w and wealth evolves as

$$\dot{a} = ra + w - c$$

- HJB equation becomes

$$\rho V(a) = \frac{1}{1-\gamma} c(a)^{1-\gamma} + V_a(ra + w - c(a))$$

- Solution of this model (work this out yourself):

$$V(a) = \frac{1}{1-\gamma} \kappa^{-\gamma} \left(a + \frac{w}{r}\right)^{1-\gamma} \quad \text{and} \quad c(a) = \kappa \left(a + \frac{w}{r}\right)$$

  with MPC given by: $c'(a) = \kappa$ (as before)

- Intuition? Human capital affects lifetime wealth but not MPC

3. What happens with a labor supply decision?

- Households solve: $\max \int_0^\infty e^{-\rho t} u(c_t, h_t) dt$, with $u(c,h) = \frac{1}{1-\gamma}(c - \frac{h^{1+\eta}}{1+\eta})^{1-\gamma}$ (GHH)

$$\rho V(a) = \max_{c,h} \left\{ u(c,h) + (ra + wh - c)\partial_a V(a) \right\}$$

where FOCs are $u_c = V_a$ and $u_h = -wV_a$, so

$$\left(c - \frac{h^{1+\eta}}{1+\eta}\right)^{-\gamma} = V_a \quad \text{and} \quad \left(c - \frac{h^{1+\eta}}{1+\eta}\right)^{-\gamma} h^\eta = wV_a$$

- Putting FOCs together, $c = V_a^{-\frac{1}{\gamma}} + \frac{1}{1+\eta} w^{\frac{1+\eta}{\eta}}$ and $h = w^{\frac{1}{\eta}}$
- Intuition: under GHH there is no income effect on labor supply

- HJB becomes

$$
\begin{aligned}
\rho V &= \frac{1}{1-\gamma} V_a^{\frac{\gamma-1}{\gamma}} + V_a \left(ra + wh - c \right) \\
&= \frac{1}{\gamma} V_a^{\frac{\gamma-1}{\gamma}} + V_a \left(ra + \frac{\eta}{1+\eta} w^{\frac{1+\eta}{\eta}} - V_a^{-\frac{1}{\gamma}} \right)
\end{aligned}
$$

- Solution of this model given by (work this out yourself):

$$
V(a) = \frac{1}{1-\gamma} \kappa^{-\gamma} \left(a + \frac{\eta}{1+\eta} \frac{w^{\frac{1+\eta}{\eta}}}{r} \right)^{1-\gamma}
$$

implying

$$
\begin{aligned}
V'(a) &= \kappa^{-\gamma} \left(a + \frac{\eta}{1+\eta} \frac{w^{\frac{1+\eta}{\eta}}}{r} \right)^{-\gamma} \\
c(a) &= \kappa a + \left(\frac{\eta}{1+\eta} \frac{\tilde{\kappa}}{r} + \frac{1}{1+\eta} \right) w^{\frac{1+\eta}{\eta}}
\end{aligned}
$$

- The household's MPC (out of wealth) is still constant and given by $c'(a) = \text{MPC} = \kappa$ (same κ as before)
- Intuition: we correct for an *effective wage adjustment* in human capital / lifetime wealth but consumption is unaffected by labor supply under GHH
- This is because GHH shuts down income effects on labor supply

4. Precautionary savings I: return risk

- In the data, we see (a) much higher MPCs and (b) MPCs are higher at low income / wealth

- So far: we considered deterministic consumption-savings problems. They all yielded (roughly) MPC $\approx \rho \approx 5\%$ annually.

- We now start exploring theories of consumption that can break this and match the data much better

- Let's start with a simple example of return risk: You can only trade stocks (no bonds) and stocks trade at a stochastic price

$$Qdk = Dk - c$$

where D is the dividend

- Define net worth as $a = Qk$, so $da = kdQ + Qdk$ by Ito's product rule, noting that $(dk)(dQ) = 0$, so

$$da = \frac{D}{Q} - c + a\frac{dQ}{Q}$$

- Assume stock prices follow a diffusion process (geometric Brownian):

$$\frac{dQ}{Q} = \mu dt + \sigma dB$$

- Rewrite wealth as: $da = (\mu_R - c)dt + a\sigma dB$, where $\mu_R = \mu_Q + \frac{D}{Q}$ (dividend + capital gains yield)

- HJB:

$$\rho V(a) = u(c(a) + V_a(\mu_R a - c) + \frac{1}{2}(a\sigma)^2 V_{aa}$$

- Solution of this model is (work this out yourself):

$$V(a) = \frac{1}{1-\gamma}\tilde{\kappa}^{-\gamma}a^{1-\gamma}$$
$$c(a) = \tilde{\kappa}a,$$

where

$$\kappa \equiv \frac{1}{\gamma}\Big[\rho - (1-\gamma)\mu_R + \underbrace{\gamma(1-\gamma)\frac{\sigma^2}{2}}_{\text{Precautionary savings}}\Big]$$

- Households' MPC now has a precautionary savings term $\frac{1}{2}(1-\gamma)\sigma^2$

- The consumption Euler equation for this model (work this out yourself) is:

$$\mathbb{E}\left(\frac{dc}{c}\right) = \frac{r-\rho}{\gamma}dt - \frac{1-\gamma}{2}\sigma^2 dt$$

- For $\gamma = 2$, precautionary term is negative, so households tilt consumption profile towards the future (hence, precautionary savings)

Part 4: Liquidity Constraints

- We now work through arguably *the* benchmark model of consumption in macro
- Household faces (1) uninsurable income risk and (2) a borrowing constraint
- Capture income risk via stochastic process $\{z_t\}$
- Household can trade a bond, a_t, but subject to constraint $a_t \geq \underline{a}$

Sequence problem:

$$V_0 = \max_{\{c_t\}_{t \geq 0}} \mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t) dt$$

$$\begin{aligned} da_t &= ra_t + e^{z_t} - c_t \\ a_t &\geq \underline{a} \\ dz_t &= -\theta z_t dt + \sigma dB_t \end{aligned}$$

- Household states are (a, z), so recursive representation is

$$\rho V = u(c) + (ra + e^z - c)V_a - \theta z V_z + \frac{\sigma^2}{2} V_{zz}$$

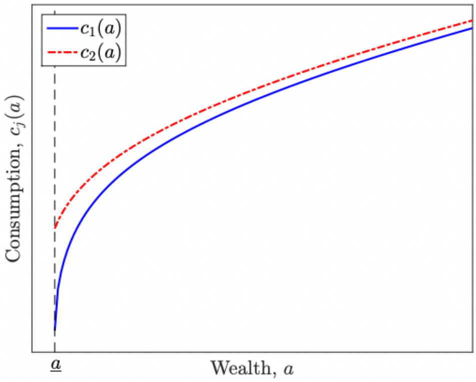
- Consumption Euler equation for CRRA (derivation on Homework):

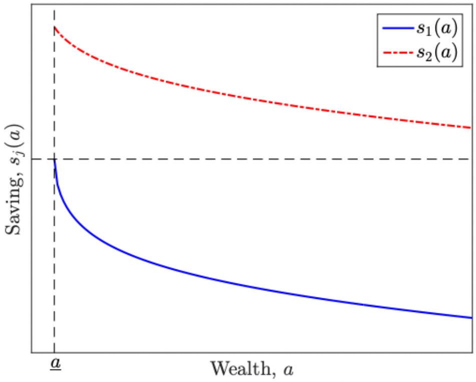
$$\frac{dc}{c} = \frac{r-\rho}{\gamma}dt + \frac{1+\gamma}{2}\left(\frac{\sigma c_z}{c}\right)^2 dt + \frac{c_z}{c}\sigma dB,$$

  where c_z is $\approx$ the marginal propensity out of income shocks

- The term $\frac{1+\gamma}{2}\left(\frac{\sigma c_z}{c}\right)^2$ captures a precautionary savings motive due to uncertainty about future income fluctuations (that are not insurable)

- Where does the borrowing constraint show up here??? $\rightarrow$ continuous time!


$c_1(a)$
$c_2(a)$
Consumption, $c_j(a)$
$\underline{a}$
Wealth, a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Predictions

- The consumption function is concave $\rightarrow$ MPC varies with wealth
- Consumption function becomes steep at the borrowing constraint $\rightarrow$ low-wealth households have large MPCs
- Consumption tracks income $\rightarrow \Delta \log y_{t+1}$ predicts $\Delta \log c_{t+1}$
- Households save to build up “buffer stock” to smooth income fluctuations