202A: Dynamic Programming and Applications

Homework #1 Solutions

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Problem 1: Discrete Time Markov Chains

Credit: QuantEcon https://python.quantecon.org/finite_markov.html#exercises

Time is discrete, with $t \in \{0, 1, ...\}$. Consider the stochastic process $\{y_t\}$ which follows a Markov chain. Let y_t denote the employment / earnings state of an individual in periods t. Consider the state space $y_t \in Y = \{y^U, y^E\}$, where y^U corresponds to unemployment and y^E corresponds to employment. Let $\boldsymbol{y}$ denote the column vector $(y^U, y^E)'$ representing this state space (this is the grid you would construct on a computer). Suppose the employment dynamics of the individual are characterized by the invariant transition matrix

$$P = \begin{pmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{pmatrix}.$$

We interpret a time period as a quarter.

(a) Give economic interpretations of $\alpha = P_{11}$ and $\beta = P_{22}$

$\alpha = P_{11}$: The probability that an unemployed individual does not find a job and remains unemployed at time t, i.e., $P(y_{t+1} = y^U \mid y_t = y^U)$.

$\beta = P_{22}$: The probability that an employed individual does not lose their job and remains employed at time t, i.e., $P(y_{t+1} = y^E \mid y_t = y^E)$.

(b) Why do the rows of P sum to 1?

This is because each row represents a probability distribution, and the total probability of transitioning to any state must sum to 1. Intuitively, in the model, an individual is either employed or unemployed.

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This course builds on the excellent teaching material developed, in particular, by David Laibson and Benjamin Moll. I would also like to thank Gabriel Chodorow-Reich, Pablo Kurlat, Jón Steinsson, and Gianluca Violante, as well as QuantEcon, on whose teaching material I have drawn.

(c) Is there an absorbing state in this model?

We say y is an absorbing state if

$$P(y_{t+1} = y | y_t = y) = 1$$

There are no absorbing states in this model if $\alpha, \beta \in (0, 1)$.

(d) Compute the probability of being unemployed two quarters after being employed.

$$\begin{aligned}
&P(y_{t+2} = y^U | y_t = y^E) \\
&= P(y_{t+2} = y^U, y_{t+1} = y^U | y_t = y^E) + P(y_{t+2} = y^U, y_{t+1} = y^E | y_t = y^E) \\
&= \alpha(1-\beta) + (1-\beta)\beta = (\alpha+\beta)(1-\beta)
\end{aligned}$$

(e) Denote the *marginal (probability) distribution* of y_t at time t by ψ_t. $\psi_t(y^L)$ is the probability that process y_t is in state y^L at time t. It is easiest to think of ψ_t as a time-varying row vector. Use the law of total probability to decompose $y_{t+1} = y^L$, accounting for all the possible ways in which state y^L can be reached at time $t+1$.

$$\begin{aligned}
\psi_{t+1}(y^L) &= P(y_{t+1} = y^L) \\
&= \psi_t(y^L) P(y_{t+1} = y^L | y_t = y^L) + (1 - \psi_t(y^L)) P(y_{t+1} = y^L | y_t \neq y^L)
\end{aligned}$$

(f) Show that the resulting equation can be written as the vector-matrix product

$$\psi_{t+1} = \psi_t P.$$

Therefore: The evolution of the marginal distribution of a Markov chain is obtained by post-multiplying by the transition matrix.

It is clear that the conditional probabilities from question (e) are obtained from one column of the transition matrix P, so we can write it as the vector-matrix product.

(g) Show that

$$X_0 \sim \psi_0 \implies X_t \sim \psi_0 P^t,$$

where $\sim$ reads "is distributed according to".

First, $\psi_1 = \psi_0 P$, and then $\psi_2 = \psi_1 P = \psi_0 P^2$. By iteration, $\psi_t = \psi_0 P^t$. Hence, $X_t \sim \psi_t = \psi_0 P^t$.

(h) We call ψ^* a *stationary distribution* of the Markov chain if it satisfies

$$\psi^* = \psi^* P.$$

Compute the probability of being unemployed n quarters after being employed. Take $n \to \infty$ and find the stationary distribution of this Markov chain. Find the stationary distribution by alternatively plugging into the above equation for ψ^*.

(Theorem) Every stochastic matrix P has at least one stationary distribution. Furthermore, this stationary distribution is unique as long as P is irreducible[a].

(Theorem) If P is both aperiodic and irreducible, then:

(a) P has exactly one stationary distribution ψ^*.

(b) For any initial marginal distribution ψ_0, we have $\|\psi_0 P^t - \psi^*\| \to 0$ as $t \to \infty$.

[a] A Markov Chain is irreducible if all states communicate, that is there is a positive probability of going from any y to any x (possibly in many steps).

To find the stationary distribution, we need to solve the equation above. Note that $\psi = 0$ is a solution, but it is not a valid probability distribution. Let $\psi^* = (\psi^*(y^U), 1 - \psi^*(y^U))$, then solving the equation gives us:

$$\psi^*(y^U) = \frac{1-\beta}{2-\alpha-\beta} = \frac{1-\beta}{(1-\alpha)+(1-\beta)}$$

Economic Intuition: The unemployment rate is higher (i.e., individuals spend more time unemployed) if the probability of finding a job (job-finding rate) is lower or if the probability of losing a job (separation rate) is higher.

(i) Suppose $y_0 = y^H$. Solve for $\mathbb{E}_0(y_t)$. Use the law of total / iterated expectation to relate expectation to probabilities. Then use the formulas for marginal (probability) distributions derived above.

$$\mathbb{E}_0(y_t) = P(y_t = y^U | y_0 = y^E) y^U + P(y_t = y^E | y_0 = y^E) y^E$$

where we get the transition probabilities from P^t.

Problem 2: Proof of the Contraction Mapping Theorem

Credit: David Laibson

In class, we defined the Bellman operator B, which operates on functions w, and is defined by

$$(Bw)(x) \equiv \max_{x' \in \Gamma(x)} \left\{ F(x, x') + \beta w(x') \right\}$$

for all $x \in \mathcal{X}$ in the state space, where $\Gamma(x)$ is some constraint set—in our case, this was the budget constraint. The definition is expressed pointwise, but it applies to all possible values in the state space. We call B an operator because it maps a function w to a new function Bw. So both w and Bw map $\mathcal{X}$ into $\mathbb{R}$. Operator B maps *functions* and is therefore called a functional operator. In class, we showed that the solution of the Bellman equation—the value function—is a fixed point of the Bellman operator.

What does it mean to *iterate* $B^n w$?

$$
\begin{aligned}
(Bw)(x) &= \max_{x' \in \Gamma(x)} \left\{ F(x,x') + \beta w(x') \right\} \\
(B(Bw))(x) &= \max_{x' \in \Gamma(x)} \left\{ F(x,x') + \beta (Bw)(x') \right\} \\
(B(B^2 w))(x) &= \max_{x' \in \Gamma(x)} \left\{ F(x,x') + \beta (B^2 w)(x') \right\} \\
&\vdots \\
(B(B^n w))(x) &= \max_{x' \in \Gamma(x)} \left\{ F(x,x') + \beta (B^n w)(x') \right\}.
\end{aligned}
$$

What does it mean for functions to converge to a limiting function? Let v_0 be some guess for the value function, then convergence would mean

$$\lim_{n \to \infty} B^n v_0 = v.$$

And why might $B^n w$ converge as $n \to \infty$? The answer is that B is a *contraction mapping*.

Definition 1. Let (S,d) be a metric space and $B : S \to S$ be a function that maps S into intself. B is a contraction mapping if for some $\beta \in (0,1)$, $d(Bf, Bg) \leq \beta d(f,g)$, for any two functions f and g.[1]

Intuitively, B is a contraction mapping if applying the operator B to any two functions f and g (that are not the same) moves them strictly closer together. Bf and Bg are strictly closer together than f and g. We can now state the contraction mapping theorem.

Theorem 2. *If (S,d) is a complete metric space and $B : S \to S$ is a contraction mapping, then:*

(i) B has exactly one fixed point $v \in S$

(ii) For any $v_0 \in S$, $\lim_{n\to\infty} B^n v_0 = v$

(iii) $B^n v_0$ has an exponential convergence rate at least as great as $-\ln(\beta)$

[1] A metric d is a way of representing the distance between two functions, or two members of (metric) space S. One example: the supremum pointwise gap.

In this problem, we will illustrate and prove the contraction mapping theorem.

(a) Consider the contraction mapping $(Bw)(x) \equiv h(x) + \alpha w(x)$ with $\alpha \in (0,1)$. Iteratively apply the operator B and show that

$$\lim_{n\to\infty}(B^n f)(x) = \frac{h(x)}{1-\alpha}$$

Argue that this shows that the fixed point of this operator B is consequently the function $v(x) = \frac{1}{1-\alpha}h(x)$. Show that $(Bv)(x) = v(x)$.

$$\begin{aligned}
(Bf)(x) &= h(x) + \alpha f(x) \\
(B(Bf))(x) &= h(x) + \alpha (Bf)(x) \\
&= h(x) + \alpha\left[h(x) + \alpha f(x)\right] \\
&= (1+\alpha)h(x) + \alpha^2 f(x) \\
\Rightarrow (B^n f)(x) &= \left(1 + \alpha + \cdots + \alpha^{n-1}\right)h(x) + \alpha^n f(x) \\
\therefore \lim_{n\to\infty}(B^n f)(x) &= \frac{h(x)}{1-\alpha}
\end{aligned}$$

$$\begin{aligned}
(Bv)(x) &= h(x) + \alpha v(x) \\
(Bv)(x) &= v(x) (\because \text{fixed point}) \\
\Rightarrow h(x) + \alpha v(x) &= v(x) \\
\therefore v(x) &= \frac{h(x)}{1-\alpha}
\end{aligned}$$

$$(Bv)(x) = h(x) + \alpha v(x) = h(x) + \alpha \cdot \frac{h(x)}{1-\alpha} = \frac{h(x)}{1-\alpha} = v(x)$$

(b) **Optional:** Now we will prove the contraction mapping theorem in 3 steps (we will not prove the convergence rate). Show that $\{B^n f_0\}_{n=0}^{\infty}$ is a Cauchy sequence. (Cauchy sequence definition: Fix any $\epsilon > 0$. Then there exists N such that $d(B^m f_0, B^n f_0) \leq \epsilon$ for all $m, n \leq N$.)

Choose some $f_0 \in S$. Let $f_n = B^n f_0$. Since B is a contraction

$$d(f_2, f_1) = d(Bf_1, Bf_0) \leq \delta d(f_1, f_0).$$

Continuing by induction,

$$d(f_{n+1}, f_n) \leq \delta^n d(f_1, f_0) \quad \forall n$$

We can now bound the distance between f_n and f_m when $m > n$.

$$\begin{aligned} d(f_m, f_n) &\leq d(f_m, f_{m-1}) + ... + d(f_{n+2}, f_{n+1}) + d(f_{n+1}, f_n) \\ &\leq \left[\delta^{m-1} + ... + \delta^{n+1} + \delta^n\right] d(f_1, f_0) \\ &= \delta^n \left[\delta^{m-n-1} + ... + \delta^1 + 1\right] d(f_1, f_0) \\ &< \frac{\delta^n}{1-\delta} d(f_1, f_0) \end{aligned}$$

So $\{f_n\}_{n=0}^{\infty}$ is Cauchy.

(c) **Optional:** Show that the limit point v is a fixed point of B.

Since S is complete, we have $f_n \to v \in S$.
Thus, v is a candidate for a fixed point.
To show that $Bv = v$, note that:

$$\begin{aligned} d(Bv, v) &\leq d(Bv, B^n f_0) + d(B^n f_0, v) \\ &\leq \delta d(v, B^{n-1} f_0) + d(B^n f_0, v). \end{aligned}$$

These inequalities must hold for all n.
As $n \to \infty$, both terms on the right-hand side converge to zero.
Thus, $d(Bv, v) = 0$, which implies that $Bv = v$.

(d) **Optional:** Show that only one fixed point exists.

Now we will show that the fixed point is unique.
Suppose there are two fixed points, $v \neq v^*$ toward contradiction.
Then, we have $Bv = v$ and $Bv^* = v^*$ since both are fixed points.
We also know that $d(Bv, Bv^*) < d(v, v^*)$ since B is a contraction.
Thus, $d(v, v^*) = d(Bv, Bv^*) < d(v, v^*)$, which is a contradiction.
Therefore, the fixed point must be unique.

Problem 3: Blackwell's Sufficiency Conditions

Credit: David Laibson

We now show that there are in fact sufficient conditions for an operator to be contraction mapping.

Theorem 3 (Blackwell's Sufficient Conditions)**.** *Let $X \subset \mathbb{R}^l$ and let $C(X)$ be a space of bounded functions $f : X \to \mathbb{R}$, with the sup-metric. Let $B : C(X) \to C(X)$ be an operator satisfying two conditions:*

1. *monotonicity: if* $f, g \in C(X)$ *and* $f(x) \leq g(x)\ \forall x \in X$,
   then $(Bf)(x) \leq (Bg)(x), \forall x \in X$

2. *discounting: there exists some* $\delta \in (0,1)$ *such that*

$$[B(f+a)](x) \leq (Bf)(x) + \delta a \ \ \forall\, f \in C(X),\ a \geq 0,\ x \in X.$$

Then, B is a contraction with modulus δ.

Note that a is a constant and $(f+a)$ is the function generated by adding a to the function f. Blackwell's conditions are sufficient but not necessary for B to be a contraction.

In this problem, we will prove these sufficient conditions.

(a) Let d be the sup-metric and show that, for any $f, g \in C(X)$, we have $f(x) \leq g(x) + d(f, g)$ for all x

$$d(f,g) = \sup_{x \in X} |f(x) - g(x)|$$

$$f(x) - g(x) \leq |f(x) - g(x)| \leq \sup_{x \in X} |f(x) - g(x)| = d(f,g)$$

$$\therefore f(x) \leq g(x) + d(f,g)$$

(b) Use monotonicity and discounting to show that, for any $f, g \in C(X)$, we have $(Bf)(x) \leq (Bg)(x) + \delta d(f, g)$ and $(Bg)(x) \leq (Bf)(x) + \delta d(f, g)$.

Using monotonicity and discounting, we have for all x

$$(Bf)\,(x) \leq [B(g + d(f,g))](x) \leq (Bg)\,(x) + \delta d(f,g)$$

$$(Bg)\,(x) \leq [B(f + d(f,g))](x) \leq (Bf)\,(x) + \delta d(f,g)$$

(c) Combine these to show that $d(Bf, Bg) \leq \delta d(f, g)$.

$$(Bf)\,(x) - (Bg)\,(x) \leq \delta d(f,g)$$

$$(Bg)\,(x) - (Bf)\,(x) \leq \delta d(f,g)$$

$$|(Bf)(x) - (Bg)(x)| \leq \delta d(f,g)$$

$$\sup_{x} |(Bf)(x) - (Bg)(x)| \leq \delta d(f,g)$$

$$\therefore d(Bf, Bg) \leq \delta d(f,g)$$

Problem 4: Example of Blackwell's Conditions

Credit: David Laibson

We will now work out a simple example to illustrate these sufficient conditions. In particular, consider the Bellman operator in a consumption problem with stochastic asset returns, stochastic labor income, and a liquidity constraint:

$$(Bf)(x) = \sup_{c\in[0,x]} \left\{ u(c) + \delta \mathbb{E} f(\tilde{R}_{+1}(x-c) + \tilde{y}_{+1}) \right\} \quad \forall x$$

Notionally, $\tilde{R}$ and $\tilde{y}$ just underscore that these are random variables. The $_{+1}$ subscript underscores that these random variables are realized next period (in class, we used $'$ for this). The liquidity constraint is encoded in $c \in [0, x]$. (Why?)

(a) Interpret each term in the definition of this Bellman operator.

Each term in this operator has the following interpretation:

- $c \in [0, x]$: This represents the *liquidity constraint* on consumption. The agent can only consume up to their current wealth level, x, so c is restricted to the interval $[0, x]$.
- $u(c)$: This is the *utility function* of consumption, c.
- δ: This is the *discount factor* (where $0 < \delta < 1$), representing the agent's rate of time preference.
- $\mathbb{E} f\left(\tilde{R}_{+1}(x-c) + \tilde{y}_{+1}\right)$: This term represents the *expected continuation value* of future wealth.

(b) Explicitly write out the budget constraint that is used here.

The budget constraint in this setting reflects the agent's consumption and saving decision, given their income and asset returns. It can be expressed as follows:

$$x' = \tilde{R}_{+1}(x-c) + \tilde{y}_{+1}$$

where:

- x is the current wealth,
- c is the consumption choice in the current period,
- $\tilde{R}_{+1}$ is the stochastic return on assets, realized in the next period,
- $\tilde{y}_{+1}$ is the stochastic labor income in the next period,
- x' is the *wealth in the next period*, after accounting for asset returns and new income.

This budget constraint shows that next period's wealth x' depends on the remaining wealth after current consumption, multiplied by the asset return, and the additional labor income in the next period.

(c) Check the first of Blackwell's conditions: monotonicity.

Assume $f(x) \leq g(x)\ \forall x$. Suppose c_f^* is the optimal policy when the continuation value function is f.

$$\begin{aligned}
(Bf)(x) &= \sup_{c \in [0,x]} \left\{ u(c) + \delta E f(\tilde{R}_{+1}(x-c) + \tilde{y}_{+1}) \right\} \\
&= u(c_f^*) + \delta E f(\tilde{R}_{+1}(x - c_f^*) + \tilde{y}_{+1}) \\
&\leq u(c_f^*) + \delta E g(\tilde{R}_{+1}(x - c_f^*) + \tilde{y}_{+1}) \\
&\leq \sup_{c \in [0,x]} \left\{ u(c) + \delta E g(\tilde{R}_{+1}(x-c) + \tilde{y}_{+1}) \right\} \\
&= (Bg)(x)
\end{aligned}$$

(d) Check the second of Blackwell's conditions: discounting.

Adding a constant (Δ) to an optimization problem does not affect optimal choice, so

$$\begin{aligned}
[B(f+\Delta)](x) &= \sup_{c \in [0,x]} \left\{ u(c) + \delta E \left[f(\tilde{R}_{+1}(x-c) + \tilde{y}_{+1}) + \Delta \right] \right\} \\
&= \sup_{c \in [0,x]} \left\{ u(c) + \delta E f(\tilde{R}_{+1}(x-c) + \tilde{y}_{+1}) \right\} + \delta\Delta \\
&= (Bf)(x) + \delta\Delta
\end{aligned}$$

Problem 5: Neoclassical Growth Model with Log Utility

Credit: David Laibson

Recall the neoclassical growth model we discussed in class. Assuming log utility, full depreciation, and a decreasing-returns production function, preferences can be written as

$$V(k_0) = \max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \log(k_t^\alpha - k_{t+1})$$

where $0 < \alpha < 1$, subject to the constraint

$$k_{t+1} \in [0, k_t^\alpha] \equiv \Gamma(k_t).$$

Think of k_t^α as the resources you have available, and so the most you would be allowed to save is k_t^α. We represent this constraint by the *feasibility set* $\Gamma(k_t)$. (This is the more general notation you will find in Stokey-Lucas, for example.)

Consider the associated Bellman equation

$$V(k) = \max_{k' \in \Gamma(k)} \left\{ \ln(k^\alpha - k') + \beta V(k') \right\}.$$

(a) Try to solve the Bellman equation by *guessing* a solution. Specifically, start by guessing that the form of the solution is

$$V(k) = \psi + \phi \log k.$$

We will solve for the coefficients ψ and ϕ, and show that $V(k)$ solves the functional equation. Rewrite the functional equation substituting in $V(k) = \psi + \phi \log k$. Use the Envelope Theorem (ET) and the First Order Condition (FOC) to show

$$\phi = \frac{\alpha}{1 - \alpha\beta}.$$

Now use the FOC to show

$$k'(k) = \alpha\beta k^\alpha.$$

Finally, show that the functional equation is satisfied at all feasible values of k_0 if

$$\psi = \frac{1}{1-\beta}\left[\log(1 - \alpha\beta) + \frac{\alpha\beta}{1 - \alpha\beta} \log(\alpha\beta) \right].$$

You have now solved the functional equation by using the guess and verify method.

Substituting $v(k) = \psi + \phi \ln(k)$ into the Bellman equation and collecting the terms involving ψ on the LHS yields the following:

$$(1-\beta)\psi + \phi \ln(k) = \sup_{y \in [0, k^\alpha]} \left\{ \ln(k^\alpha - y) + \beta\phi \ln(y) \right\}.$$

Noting that the maximand is strictly concave on $[0, k^\alpha]$, there is a unique maximizer y^* that satisfies the first-order condition:

$$-\frac{1}{k^\alpha - y^*} + \frac{\beta\phi}{y^*} = 0 \quad \Rightarrow \quad y^* = \frac{\beta\phi}{1+\beta\phi} k^\alpha.$$

It follows from the envelope theorem that the derivative of the Bellman equation with respect to k is:

$$\frac{\phi}{k} = \frac{\alpha k^{\alpha-1}}{k^\alpha - y^*} \quad \Rightarrow \quad y^* = \frac{\phi - \alpha}{\phi} k^\alpha.$$

Combining the first-order condition and the envelope condition gives:

$$\frac{\beta\phi}{1+\beta\phi} = \frac{\phi - \alpha}{\phi} \quad \Rightarrow \quad \phi = \frac{\alpha}{1 - \alpha\beta}.$$

Thus, we obtain the optimal policy $y^* = \alpha\beta k^\alpha$. Substituting for ϕ and y into the Bellman equation yields:

$$(1-\beta)\psi + \frac{\alpha}{1-\alpha\beta}\ln(k) = \ln(k^\alpha - \alpha\beta k^\alpha) + \beta\frac{\alpha}{1-\alpha\beta}\ln(\alpha\beta k^\alpha)$$
$$= \frac{\alpha}{1-\alpha\beta}\ln(k) + \ln(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\ln(\alpha\beta)$$
$$\therefore \psi = \frac{1}{1-\beta}\left[\ln(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\ln(\alpha\beta)\right].$$

(b) We have derived the policy function

$$k'(k) = g(k) = \alpha\beta k^\alpha.$$

Derive the optimal sequence of state variables $\{k_t^*\}_{t=0}^\infty$ which would be generated by this policy function. Show that

$$V(k_0) = \sum_{t=0}^\infty \beta^t \log\left((k_t^*)^\alpha - k_{t+1}^*\right),$$

thereby confirming that this policy function is optimal.

The capital stock k_t^* evolves according to the recursive equation $k_t^* = \alpha\beta(k_{t-1}^*)^\alpha$ for each integer $t \geq 1$. The equation can be expressed in logs as:

$$\ln(k_t^*) = \ln(\alpha\beta) + \alpha\ln(k_{t-1}^*),$$

which is a first-order linear difference equation with some initial value $\ln(k_0)$. By backward induction, its (backward) solution is found to be:

$$\ln(k_t^*) = \ln(\alpha\beta) + \alpha\left[\ln(\alpha\beta) + \alpha\ln(k_{t-2}^*)\right]$$
$$= \ln(\alpha\beta) + \alpha\ln(\alpha\beta) + \alpha^2\left[\ln(\alpha\beta) + \alpha\ln(k_{t-3}^*)\right]$$
$$= \dots$$
$$= \ln(\alpha\beta)\sum_{s=0}^{t-1}\alpha^s + \alpha^t\ln(k_0)$$

Exponentiating both sides of the solution, we obtain:

$$k_t^* = (\alpha\beta)^{\frac{1-\alpha^t}{1-\alpha}} k_0^{\alpha^t}.$$

Thus, the present discounted value of the flow payoffs from the sequence k_t^* can be calculated as follows:

$$\sum_{t=0}^\infty \beta^t \ln[(k_t^*)^\alpha - k_{t+1}^*] = \sum_{t=0}^\infty \beta^t \ln[(k_t^*)^\alpha - \alpha\beta(k_t^*)^\alpha] = \sum_{t=0}^\infty \beta^t\left[\ln(1-\alpha\beta) + \alpha\ln(k_t^*)\right].$$

Using the earlier result for k_t^*, we get:

$$\sum_{t=0}^{\infty} \beta^t \left[\ln(1-\alpha\beta) + \alpha \left(\frac{1-\alpha^t}{1-\alpha} \ln(\alpha\beta) + \alpha^t \ln(k_0) \right) \right].$$

This simplifies to:

$$\begin{aligned}
&\frac{1}{1-\beta} \ln(1-\alpha\beta) + \frac{\alpha}{1-\alpha} \ln(\alpha\beta) \sum_{t=0}^{\infty} \left[\beta^t - (\alpha\beta)^t \right] + \alpha \ln(k_0) \sum_{t=0}^{\infty} (\alpha\beta)^t. \\
&= \frac{1}{1-\beta} \ln(1-\alpha\beta) + \frac{\alpha}{1-\alpha} \left(\frac{1}{1-\beta} - \frac{1}{1-\alpha\beta} \right) \ln(\alpha\beta) + \frac{\alpha}{1-\alpha\beta} \ln(k_0). \\
&= \frac{1}{1-\beta} \ln(1-\alpha\beta) + \frac{\alpha\beta}{(1-\beta)(1-\alpha\beta)} \ln(\alpha\beta) + \frac{\alpha}{1-\alpha\beta} \ln(k_0) \\
&= \psi + \phi \ln(k_0) \\
&= v(k_0)
\end{aligned}$$

This confirms that the policy function $g(k) = \alpha\beta k^\alpha$ is indeed optimal since following this policy attains the optimal value for any initial k_0.

(c) Consider the Bellman (functional) operator B defined by

$$(Bf)(k) = \sup_{k' \in \Gamma(k)} \left\{ \log(k^\alpha - k') + \beta f(k') \right\}.$$

Let $\hat{V}(k) = \frac{\alpha \log k}{1-\alpha\beta}$. Show that

$$(B^n \hat{V})(k) = \frac{1-\beta^n}{1-\beta} \left[\log(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta} \log(\alpha\beta) \right] + \frac{\alpha \log k}{1-\alpha\beta}.$$

To prove this you'll need to show that $k' = \alpha\beta k^\alpha$, and substitute this expression into the functional operator. Let,

$$\lim_{n\to\infty} (B^n \hat{V})(k) = V(k)$$

Confirm that $V(k)$ is the same solution to the the functional equation that you derived in part (a). You have now solved the functional equation by iterating the operator T on a starting guess.

The steady-state capital stock $\tilde{k}$ is the fixed point of the difference equation $k_t = \alpha\beta k_{t-1}^\alpha$. Hence, $\tilde{k}$ satisfies $\tilde{k} = \alpha\beta\tilde{k}^\alpha$ or, equivalently, $\alpha\beta\tilde{k}^{\alpha-1} = 1$. The first-order Taylor expansion of the policy function $k_{t+1} = \alpha\beta k_t^\alpha$ around $k_t = \tilde{k}$ yields:

$$k_{t+1} \approx \tilde{k} + \alpha^2\beta\tilde{k}^{\alpha-1}(k_t - \tilde{k}) \quad \Rightarrow \quad \frac{k_{t+1} - \tilde{k}}{k_t - \tilde{k}} \approx \alpha^2\beta\tilde{k}^{\alpha-1} = \alpha = e^{-[-\ln(\alpha)]}.$$

To see why α is the capital share (of output) in this model, note that the gross income/output derived from capital is k^α, and thus the marginal product of capital is $\alpha k^{\alpha-1}$. In a competitive market, the return rate r on a unit of capital is equal to this marginal product. Consequently, the share of output paid to capital is given by:

$$\frac{rk}{k^\alpha} = \frac{\alpha k^{\alpha-1}k}{k^\alpha} = \alpha.$$

A capital share α between 0.3 and 0.7 implies a convergence rate $-\ln(\alpha)$, at least as great as $-\ln(0.7) \approx 0.35$. One reason why the model fails to match the convergence rate of 0.05 in the data is that we have assumed that the depreciation rate of capital between periods is one. This assumption raises the convergence rate, because past investments in capital have a smaller effect on the current capital stock. Thus, an initial difference between the capital stocks of two economies declines at a more rapid rate. This intuition relies on an interpretation of the convergence rate as a measure of how quickly countries that started with different initial capital stocks will converge to the same steady-state capital stock. This interpretation is valid to the extent that the different economies can be modeled with the same production technologies and preferences.

Problem 6: A Model with Equity

Credit: David Laibson

Assume that a consumer with only equity wealth must choose period by period consumption in a discrete-time dynamic optimization problem. Specifically, consider the sequence problem

$$V(x_0) = \max_{\{c_t\}_{t=0}^\infty} \mathbb{E}_0 \sum_{t=0}^{\infty} e^{-\rho t} u(c_t)$$

subject to the constraints

$$x_{t+1} = e^{r+\sigma u_t - \frac{\sigma^2}{2}}(x_t - c_t)$$

where u_t is iid and $u_t \sim \mathcal{N}(0,1)$. There is a feasibility constraint $c_t \in [0, x_t]$. And we assume an endowment $x_0 > 0$. Here, x_t represents equity wealth at period t and c_t is consumption in period t. The consumer has discount rate ρ. The consumer can only invest in a risky asset with expected return $e^r = \mathbb{E}e^{r+\sigma u_t - \frac{\sigma^2}{2}}$. And we assume CRRA preferences with $u(c) = \frac{1}{1-\gamma}c^{1-\gamma}$, with $\gamma \in [0, \infty]$. We call this *constant* relative risk aversion because the relative risk aversion coefficient

$$-\frac{cu''(c)}{u'(c)} = \gamma$$

is constant. Notice that CRRA consumption preferences are *homothetic*, and this is what allows us to analytically solve this problem.

The associated Bellman equation is

$$V(x) = \max_{x' \in [0,x]} \left\{ u(x - x') + \mathbb{E} e^{-\rho} V\left(e^{r + \sigma u - \frac{\sigma^2}{2}} x' \right) \right\}.$$

(a) Explain all terms in this Bellman equation. Why is u not a state variable, i.e., why don't we have $V(x,u)$? Make sure that this equation makes sense to you.

The term $v(x)$ on the left-hand side of the Bellman equation represents the supremum of the present discounted value of flow payoffs that can be generated from a feasible consumption stream starting at a wealth level of x. On the right-hand side, the term $u(x - y)$ represents the current flow payoff from consuming the amount $x - y$, and the term

$$E\left[\exp(-\rho) v\left(\exp(r + \sigma u - \frac{\sigma^2}{2}) y\right)\right] = \exp(-\rho) E\left[v\left(\exp(r + \sigma u - \frac{\sigma^2}{2}) y\right)\right]$$

represents the present discounted value of the continuation payoff. The latter expression consists of the discount factor $\exp(-\rho)$ representing impatience and the expected value $E\left[v\left(\exp(r + \sigma u - \frac{\sigma^2}{2}) y\right)\right]$ of the optimization problem in the following period. Note that $\exp(r + \sigma u - \frac{\sigma^2}{2}) y$ is the wealth level in the following period given the realization of the return shock u. The current savings level y is chosen to maximize the value of the current flow payoff plus the discounted expected future value of the problem, subject to the constraint that savings cannot be negative or greater than the current wealth level x.

(b) Now guess that the value function takes the special form

$$V(x) = \phi \frac{x^{1-\gamma}}{1 - \gamma}.$$

Note the close similarity between this functional form and the functional form of the utility function. Assuming that the value function guess is correct, use the Envelope Theorem to derive the consumption function:

$$c = \phi^{-\frac{1}{\gamma}} x.$$

Now verify that the Bellman Equation is satisfied for a particular value of ϕ. Do not solve for ϕ (it's a very nasty expression). Instead, show that

$$\log(1 - \phi^{-\frac{1}{\gamma}}) = \frac{1}{\gamma}\Big[(1 - \gamma) r - \rho\Big] + \frac{1}{2}(\gamma - 1)\sigma^2.$$

Given that $v(x) = \frac{\phi x^{1-\gamma}}{1-\gamma}$, the envelope theorem implies that:

$$v'(x) = u'(x - y) \quad \Rightarrow \quad \phi x^{-\gamma} = (x - y)^{-\gamma} = c^{-\gamma}$$

so that the optimal consumption function is $c = \phi^{-\frac{1}{\gamma}} x$ and the corresponding saving function is $y = (1 - \phi^{-\frac{1}{\gamma}}) x$. Denoting $R = \exp(r + \sigma u - \frac{\sigma^2}{2})$, the first-order condition is:

$$u'(x - y) = \exp(-\rho) E\left[R v'(R y)\right] \quad \Rightarrow \quad (x - y)^{-\gamma} = \exp(-\rho) E\left[R^{1-\gamma} \phi y^{-\gamma}\right].$$

Substituting the consumption and saving functions into the last expression and rearranging yields:

$$\phi x^{-\gamma} = \exp(-\rho)\phi(1-\phi^{-\frac{1}{\gamma}})^{-\gamma} x^{-\gamma} E\left[R^{1-\gamma}\right] \quad \Rightarrow \quad (1-\phi^{-\frac{1}{\gamma}})^{\gamma} = \exp(-\rho) E\left[R^{1-\gamma}\right].$$

Noting that R is a lognormal random variable (i.e., $\ln(R)$ is normally distributed), the moment-generating function for the normal distribution implies that:

$$E\left[R^t\right] = E\left[\exp(t\ln R)\right] = \exp\left(tm + \frac{t^2}{2}s^2\right),$$

where m and s^2 are the mean and variance of $\ln(R)$. Since in our case, $m = r - \frac{\sigma^2}{2}$ and $s^2 = \sigma^2$, the preceding expression can be written as:

$$(1-\phi^{-\frac{1}{\gamma}})^{\gamma} = \exp\left[-\rho + (1-\gamma)(r - \frac{\sigma^2}{2}) + \frac{(1-\gamma)^2}{2}\sigma^2\right]$$

so that, taking the log of both sides and dividing by γ, yields:

$$\ln(1-\phi^{-\frac{1}{\gamma}}) = \frac{1}{\gamma}\left[(1-\gamma)r - \rho\right] + \frac{1}{2}(1-\gamma)\sigma^2.$$

(c) Now consider the log of the ratio of c_{t+1} and c_t. Show that

$$\mathbb{E}\log\left(\frac{c_{t+1}}{c_t}\right) = \frac{1}{\gamma}(r-\rho) + \frac{\gamma}{2}\sigma^2 - \sigma^2.$$

Using the consumption and saving functions derived above, we obtain that optimal consumption growth is given by:

$$\frac{c_{t+1}}{c_t} = \frac{\phi^{-\frac{1}{\gamma}} x_{t+1}}{\phi^{-\frac{1}{\gamma}} x_t} = R(1-\phi^{-\frac{1}{\gamma}}),$$

where the last equality follows from the feasibility constraint $x_{t+1} = Ry = R(1-\phi^{-\frac{1}{\gamma}})x$. Then, we have:

$$E\left[\ln\left(\frac{c_{t+1}}{c_t}\right)\right] = E\left[\ln(R)\right] + \ln(1-\phi^{-\frac{1}{\gamma}}) = \left(r - \frac{\sigma^2}{2}\right) + \left\{\frac{1}{\gamma}\left[(1-\gamma)r - \rho\right] + \frac{1}{2}(1-\gamma)\sigma^2\right\}.$$

Simplifying this expression:

$$E\left[\ln\left(\frac{c_{t+1}}{c_t}\right)\right] = \frac{1}{\gamma}(r-\rho) + \left(\frac{\gamma}{2} - 1\right)\sigma^2.$$

(d) Interpret the previous equation for the certainty case $\sigma = 0$. Note that $\log(\frac{c_{t+1}}{c_t}) = \log c_{t+1} - \log c_t$ is the growth rate of consumption. Explain why $\log c_{t+1} - \log c_t$ increases in r and decreases in ρ. Why does the coefficient of relative risk aversion γ appear in the denominator of the expression? Why does the coefficient of relative risk aversion regulate the consumer's willingness to substitute consumption between periods?

In the case where $\sigma = 0$, the Euler equation relating consumption across two periods can be expressed as $E\Delta \ln(c_{t+1}) = \frac{1}{\gamma}(r - \rho)$. A higher value of the interest rate r raises the price of current consumption relative to future consumption. In other words, current consumption becomes more costly because the opportunity cost represented by the expected return from investing in equity is higher. A higher value of the discount rate ρ means that a consumer is less patient. That is, the contribution of future utility flows to the discounted sum of utility flows becomes less important relative to current period utility. Hence, the growth rate $\Delta \ln(c_{t+1})$ of consumption is increasing in r and decreasing in ρ. The coefficient of relative risk aversion γ also regulates the consumer's willingness to substitute consumption between periods here because it measures the concavity of the flow utility function and thus the desire to smooth consumption across periods. Note that the elasticity of intertemporal substitution for this utility function is:

$$-\frac{d\ln\left(\frac{c_{t+1}}{c_t}\right)}{d\ln\left(\frac{u'(c_{t+1})}{u'(c_t)}\right)} = \frac{1}{\gamma}.$$

A higher value of γ corresponds to a more concave utility function and a lower willingness to substitute consumption across periods. Thus, the expected growth rate of consumption $E\Delta \ln(c_{t+1})$ is less sensitive to $r - \rho$.