202A: Dynamic Programming and Applications

Homework #2 Solutions

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Problem 1: Optimal Stopping

Consider the optimal stopping application from class: Each period $t = 0, 1, \ldots$ the consumer draws a job offer from a uniform distribution with support in the unit interval: $x \sim \text{unif}[0, 1]$. The consumer can either accept the offer and realize net present value x, or the consumer can wait another period and draw again. Once you accept an offer the game ends. Waiting to accept an offer is costly because the value of the remaining offers declines at rate $\rho = -\log(\beta)$ between periods. The Bellman equation for this problem is:

$$V(x) = \max \left\{ x,\ \beta \mathbb{E} V(x') \right\}$$

where x' is your next draw, which is a random variable.

(a) Explain the intuition behind this Bellman equation. Explain every term.

In each period, the consumer decides whether to accept the current offer of x or to continue searching for another offer. If the consumer accepts an offer of x, then she receives a payoff of x. If the consumer rejects the offer, then she faces an analogous problem next period, drawing an offer x' from the same distribution. Noting that x' is unknown in the current period and ρ is the rate at which the future is discounted, $\exp(-\rho)E[v(x')]$ is the continuation value from rejecting an offer. Thus, the value of drawing an offer of x is the greater of the payoff x from accepting the offer and the value $\exp(-\rho)E[v(x')]$ from continuing to search.

(b) Consider the associated functional operator:

$$(Bw)(x) = \max \left\{ x,\ \beta \mathbb{E} w(x') \right\}$$

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for all x. Using Blackwell's conditions, show that this Bellman operator is a contraction mapping.

Note that B is a mapping from the space of bounded functions into itself, since for any $w(x)$, the maximum value that $(Bw)(x)$ can take is 1. Thus, we can apply Blackwell's Theorem and show that the Bellman operator B is a contraction mapping, because it satisfies Blackwell's sufficient conditions. The monotonicity condition holds, because the following is true for functions f and g with $f \leq g$:

$$Bf(x) = \max\{x, \exp(-\rho)E[f(x')]\} \leq \max\{x, \exp(-\rho)E[g(x')]\} = Bg(x),$$

where the inequality results from $E[f(x')] \leq E[g(x')]$. Provided that $\rho > 0$, the discounting condition holds, because the following is true for a function f and a constant $\alpha \geq 0$:

$$\begin{aligned} B(f+\alpha)(x) &= \max\{x, \exp(-\rho)E[f(x')+\alpha]\} \\ &\leq \max\{x, \exp(-\rho)E[f(x')]\} + \exp(-\rho)\alpha \\ &= Bf(x) + \exp(-\rho)\alpha. \end{aligned}$$

(c) What does the contraction mapping property imply about $\lim_{n\to\infty} B^n w$, where w is *any* arbitrary function?

Because the Bellman operator B is a contraction mapping from a complete metric space into itself, the contraction mapping theorem implies that $\lim_{n\to\infty} B^n w = v$ for any arbitrary function w, where v is the unique fixed point of the Bellman operator and thus the solution to the Bellman equation.

(d) Suppose we make a (bad?) guess $w(x) = 1$ for all x. Analytically iterate on $B^n w$ and show that

$$\lim_{n\to\infty} (B^n w)(x) = V(x) = \begin{cases} x^* & \text{if } x \leq x^* \\ x & \text{if } x > x^* \end{cases}$$

where

$$x^* = e^{\rho}\left(1 - \left[1 - e^{-2\rho}\right]^{\frac{1}{2}}\right).$$

Let $w(x) = 1$ for all $x \in [0,1]$. It can be shown by induction that each iteration of the Bellman operator has the form:

$$B^n w(x) = \begin{cases} x_n, & x \leq x_n, \\ x, & x > x_n, \end{cases}$$

The first application of the Bellman operator to the function w yields:

$$Bw(x) = \max\{x, e^{-\rho}\} = \begin{cases} e^{-\rho}, & x \leq e^{-\rho}, \\ x, & x > e^{-\rho}, \end{cases}$$

confirming the claim for $n = 1$ with $x_1 = e^{-\rho}$. If the claim is true for some $n \geq 1$, then we obtain the following for $n + 1$:

$$\begin{aligned} B^{n+1}w(x) &= \max\{x, e^{-\rho}E[B^n w(x')]\} \\ &= \max\left\{x, e^{-\rho}\left(\int_0^{x_n} x_n\, dz + \int_{x_n}^1 z\, dz\right)\right\} \\ &= \max\left\{x, e^{-\rho}\left(\frac{1+x_n^2}{2}\right)\right\} = \begin{cases} x_{n+1}, & x \leq x_{n+1}, \\ x, & x > x_{n+1}, \end{cases} \\ &\text{where } \; x_{n+1} = e^{-\rho}\left(\frac{1+x_n^2}{2}\right) \end{aligned}$$

Thus, each iteration of the Bellman operator has the form claimed above. Because x_n is a sufficient statistic for $B^n w$, the convergence of $B^n w$ to the fixed point:

$$\lim_{n\to\infty} B^n w(x) = v(x) = \begin{cases} x^*, & x \leq x^* \\ x, & x > x^* \end{cases}$$

is equivalent to the convergence of x_n to the cutoff x^* defined by:

$$x^* = e^{-\rho}\frac{1+(x^*)^2}{2}.$$

Solving for x^* in the expression above yields:

$$(x^*)^2 - 2e^{\rho}x^* + 1 = 0 \quad \Rightarrow \quad x^* = e^{\rho} - \sqrt{e^{2\rho} - 1},$$

where the other root of the quadratic is greater than one and can be discarded.

Problem 2: Optimal Project Completion

Every period you draw a cost c distributed uniformly between 0 and 1 for completing a project. If you undertake the project, you pay c, and complete the project with probability $1 - p$. Each period in which the project remains uncompleted, you pay a late fee of l. The game continues until you complete the project.

(a) Write down the Bellman Equation assuming no discounting. Why is it ok to assume no discounting in this problem?

The Bellman equation for the problem is:

$$v(c) = \max\{-c + p[Ev(c') - l], Ev(c') - l\}.$$

This formulation of the Bellman equation is also valid:

$$v(c) = \min\{c + p[Ev(c') + l], Ev(c') + l\}.$$

In the utility maximization problems, discounting is usually needed to ensure a finite value function and policies that are interior. We need $l > 0$ (with or without discounting) to ensure that $c^* > 0$, and discounting or a large enough p (and small enough l) to ensure that $c^* < 1$. Note that in this problem discounting would imply an additional incentive for the agent to postpone paying the completion cost. But (for l small enough and p large enough) the agent has an incentive to postpone paying the completion cost for high enough cost draws even without discounting, since cost draws are iid so that it is always possible to get a smaller cost draw in the future.

(b) Derive the optimal threshold: $c^* = \sqrt{2l}$. Explain intuitively, why this threshold does not depend on the probability of failing to complete the project, p.

The optimal policy is defined by a cutoff c^*, such that the project is attempted iff $c \le c^*$. The value function for the problem is:

$$v(c) = \begin{cases} p\bar{v} - c, & \text{if } c \le c^*, \\ \bar{v}, & \text{if } c > c^*, \end{cases}$$

where $\bar{v} = Ev(c') - l$ is a constant with respect to c since c is i.i.d. Because the distribution of c is continuous, it's possible to get a draw of exactly $c = c^*$. At the cutoff, the agent is indifferent between the two choices, so:

$$v = pv - c^* \implies c^* = -(1-p)v.$$

Then, the solution $(\bar{v}, c^*)$ will solve the system given by (1), (2) and the definition of $\bar{v}$. First, let us solve for $\bar{v}$ in terms of c^* by evaluating the expectation of $v(c_{t+1})$ given by (1):

$$Ev(c_{t+1}) = \int_0^{c^*} (-c + p\bar{v})dc + \int_{c^*}^1 \bar{v}dc = -\frac{(c^*)^2}{2} + p\bar{v}c^* + (1-c^*)\bar{v}.$$

Substituting $Ev(c_{t+1}) = \bar{v} + l$ from the definition of $\bar{v}$ yields:

$$l = -(1-p)\bar{v}c^* - \frac{(c^*)^2}{2},$$

and substituting $\bar{v} = -c^*/(1-p)$ from (2) results in:

$$c^* = \sqrt{2l}.$$

The threshold c^* does not depend on the probability p of failing to complete the project for the following reason. Given any arbitrary threshold value $\tilde{c}$, the expected value of the problem is the sum of the total discounted expected project costs paid and the total discounted expected late fees paid. A higher threshold $\tilde{c}$ increases the total expected project costs while decreasing the total expected late fees. At the optimal threshold, the marginal effect of changing the threshold on each of these two components must cancel out. That is, if we're at the optimal threshold c^* and we consider a small perturbation $c^* + \epsilon$, the marginal reduction in the late fees component must equal the marginal increase in the project costs component. Then, c^* would only depend on $1-p$ if at the optimal c^*, the probability $1-p$ affected these two marginal effects differently. However, in the no discounting case, a decrease in $1-p$ increases the effect of c^* on both of these terms by the same proportion, effectively leaving the optimization problem unchanged.

Formally, one can show that, given a candidate threshold $\tilde{c}$, the present discounted value of expected completion costs and of expected late fees (from the perspective of an agent who hasn't yet observed the current period's draw of c) equals:

$$\begin{aligned} ECC(\tilde{c}) &= \frac{\tilde{c}}{2} + (1 - \tilde{c} + \tilde{c}p)\delta\frac{\tilde{c}}{2} + \dots \\ &= \frac{\tilde{c}^2}{2[1-\delta(1-(1-p)\tilde{c})]}, \end{aligned}$$

and

$$\begin{aligned} ELF(\tilde{c}) &= (1 - \tilde{c} + \tilde{c}p)\delta l + \dots \\ &= \frac{l\delta(1-(1-p)\tilde{c})}{1-\delta(1-(1-p)\tilde{c})} \end{aligned}$$

The value attained from policy $\tilde{c}$ would then be the negative of the sum of ECC and ELF. The optimal threshold satisfies:

$$\frac{dv(c^*)}{dc^*} = 0 \quad \Rightarrow \quad -\frac{\partial ECC}{\partial c^*} + \frac{\partial ELF}{\partial c^*} = 0. \qquad (*)$$

We can calculate the partials as:

$$\frac{\partial ECC}{\partial c^*} = \frac{2c^*(1-\delta) + \delta(1-p)c^*}{2[1-\delta(1-(1-p)c^*)]^2},$$

and

$$\frac{\partial ELF}{\partial c^*} = \frac{\delta l(1-p)}{[1-\delta(1-(1-p)c^*)]^2}.$$

p in general affects these marginal effects differently, but in the special case of $\delta = 1$, we have:

$$\frac{\partial ECC}{\partial c^*} = \frac{1}{2(1-p)}$$

and

$$\frac{\partial ELF}{\partial c^*} = \frac{l}{c^{*2}(1-p)}$$

so that p indeed affects the two marginal effects by the same proportion and thus does not affect the choice c^* determined by $(*)$. (Note that equation $(*)$ also yields $c^* = \sqrt{2l}$.) Intuitively, even though the sum of expected completion costs "starts today" and the sum of expected late fees "starts tomorrow," there is no asymmetry in the impact of p on the two infinite sums since all periods are weighted equally.

(c) How would these results change if we added discounting to the framework? Redo steps a and b, assuming that the agent discounts the future with discount factor $0 < \beta < 1$ and assuming that $p = 0$. Show that the optimal threshold is given by

$$c^* = \frac{1}{\beta}\left(\beta - 1 + \sqrt{(1-\beta)^2 + 2\beta^2 l}\right)$$

With a discount factor $\delta \in (0, 1)$ and $p = 0$, the Bellman equation becomes:

$$v(c) = \max\{-c, \delta[Ev(c') - l]\}.$$

The value function can be written as:

$$v(c) = \begin{cases} -c, & \text{if } c \leq c^*, \\ \delta\bar{v}, & \text{if } c > c^*. \end{cases}$$

where $\bar{v} = Ev(c') - l$. As above, we can obtain an indifference condition for the cutoff c^*:

$$-c^* = \delta[Ev(c') - l].$$

Solving for $Ev(c')$, we get:

$$Ev(c') = -\frac{(c^*)^2}{2} + (1 - c^*)\delta v.$$

For this threshold solution, we get:

$$Ev(c') = \int_0^{c^*} -c\, dc + \int_{c^*}^1 \delta\bar{v}\, dc = -\frac{(c^*)^2}{2} + (1 - c^*)\delta\bar{v}$$

Then, using the fact that $\bar{v} = Ev(c') - l$ and substituting in our indifference condition of $\delta\bar{v} = -c^*$ turns this into a quadratic in c^*:

$$-c^* = \delta[Ev(c') - l] = \delta\left(-\frac{(c^*)^2}{2} - (1 - c^*)c^* - l\right) = -\delta\left(c^* - \frac{(c^*)^2}{2} + l\right)$$

Solving for c^* in the expression above yields:

$$\delta(c^*)^2 + 2(1-\delta)c^* - 2\delta l = 0 \Rightarrow c^* = \frac{\delta - 1 + \sqrt{(1-\delta)^2 + 2\delta^2 l}}{\delta},$$

where the other root of the quadratic is negative and can be discarded.

(d) When $0 < \beta < 1$, is the optimal value of c^* still independent of the value of p? If not, how does c^* qualitatively vary with p? Provide an intuitive argument.

It is also possible to generalize the previous analysis to the case where $p \in (0,1)$. The Bellman equation for this problem is given by:

$$v(c) = \max\{-c + p\delta[Ev(c_{t+1}) - l], \delta[Ev(c_{t+1}) - l]\};$$

so that, the cutoff c^* is defined by the equation:

$$-c^* + p\delta\bar{v} = \delta\bar{v} = \delta[Ev(c_{t+1}) - l] \implies \frac{-c^*}{1-p} = \delta\bar{v} = \delta[Ev(c_{t+1}) - l]$$

This gives us:

$$Ev(c') = \int_0^{c^*} (-c + p\bar{v})dc + \int_{c^*}^1 \delta\bar{v}dc = -\frac{(c^*)^2}{2} + p\delta\bar{v}c^* + (1-c^*)\delta\bar{v}$$

Then, noting again that $\bar{v} = Ev(c_{t+1}) - l$ and using the fact that $\frac{-c^*}{1-p} = \delta\bar{v}$ from the indifference condition gives us:

$$\begin{aligned}\frac{-c^*}{1-p} &= \delta\left[-\frac{(c^*)^2}{2} + \delta\bar{v} - (1-p)\delta\bar{v}c^* - l\right] \\ &= \delta\left[\frac{-(c^*)^2}{2} - \frac{c^*}{1-p} + (c^*)^2 - l\right]\end{aligned}$$

which is a quadratic equation with one sensible solution:

$$\delta(c^*)^2 + 2\frac{1-\delta}{1-p}c^* - 2\delta l = 0 \implies c^* = \frac{1}{\delta}\left(\sqrt{\left(\frac{1-\delta}{1-p}\right)^2 + 2\delta^2 l} - \frac{1-\delta}{1-p}\right)$$

Differentiating the quadratic equation totally with respect to p, we get:

$$\frac{\partial c^*}{\partial p} = \frac{-\frac{1-\delta}{(1-p)^2}c^*}{\delta c^* + \frac{1-\delta}{1-p}} < 0$$

so that c^* is decreasing in p.

In the presence of discounting, an increase in p has a greater impact on the present discounted value of expected completion costs than the present discounted value of late fees. This is because the agent pays the late fee for the current period regardless of whether or not he attempts to complete the project today, so that he trades off expected completion costs starting today and expected late fees starting tomorrow. See the formalization of this point in part (b). Therefore, because the sum of expected completion costs depends positively on the level of the threshold and the sum of expected late fees depends negatively on the level of the threshold, an increase in the failure rate p lowers the threshold c^* below which the project is attempted (this follows from ($*$)).

(e) Take the perspective of an agent who has not yet observed the current period's draw of c. Prove that the expected delay until completion is given by:

$$\frac{1}{c^*(1-p)} - 1$$

Conditional on the project remaining uncompleted at the start of a period, the probability of completing the project in that period is $c^*(1-p)$. The time T for the project to be completed is a geometric random variable with the probability mass function:

$$\Pr(T = t) = [1 - c^*(1-p)]^t c^*(1-p),$$

and expectation:

$$\begin{aligned} E(T) &= c^*(1-p) \sum_{t=0}^{\infty} [1 - c^*(1-p)]^t \cdot t \\ &= c^*(1-p) \frac{1 - c^*(1-p)}{[c^*(1-p)]^2} \\ &= \frac{1 - c^*(1-p)}{c^*(1-p)}, \end{aligned}$$

where the second line follows from the fact that, for $\beta < 1$,

$$\sum_{t=0}^{\infty} \beta^t = \frac{1}{1-\beta}$$

and differentiating with respect to β gives:

$$\sum_{t=0}^{\infty} \beta^t t = \frac{\beta}{(1-\beta)^2}.$$

Problem 3: Consumption-Savings with Deterministic Income

Consider an economy populated by a continuum of infinitely lived households. There is no uncertainty in this economy for now. Households' preferences are given by

$$\max \int_0^{\infty} e^{-\rho t} u(c_t) dt.$$

That is, households discount future consumption c_t at a rate ρ. Oftentimes, we will use constant relative risk aversion (CRRA) preferences, given by

$$u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}.$$

A special form of these preferences are log preferences,

$$u(c_t) = \log(c_t).$$

The household's flow budget constraint in this economy is given by

$$\frac{d}{dt}(P_t a_t) = i_t(P_t a_t) - P_t c_t + P_t y_t,$$

where P_t is the nominal price level, c_t is real consumption expenditures, a_t is the real wealth of the household and $\{y_t\}$ is an **exogenous** stream of income whose future path the household knows at any time point with certainty (because there is no uncertainty or risk for now).

(a) Derive the budget constraint for real wealth, i.e., $\frac{d}{dt}a_t = \dot{a}_t$. Define the real interest rate as $r_t = i_t - \pi_t$, where $\pi_t \equiv \frac{\dot{P}_t}{P_t}$ is price inflation.

We can write the budget constraint directly in real terms. We have:

$$\dot{P}_t a_t + P_t \dot{a}_t = i_t P_t a_t - P_t c_t + P_t y_t$$

$$P_t \dot{a}_t = (i_t P_t - \dot{P}_t) a_t - P_t c_t + P_t y_t,$$

and finally, defining inflation as:

$$\pi_t = \frac{\dot{P}_t}{P_t} \tag{1}$$

and the real interest rate as:

$$r_t = i_t - \pi_t \tag{2}$$

we have:

$$\dot{a}_t = r_t a_t - c_t + y_t. \tag{3}$$

(b) Derive the lifetime budget constraint.

We can write the lifetime budget constraint here as:

$$\int_0^\infty e^{-R_t} c_t dt = a_0 + \int_0^\infty e^{-R_t} y_t dt,$$

where

$$R_t = \int_0^t r(s) ds.$$

This follows from time-integrating the flow budget constraint.

(c) In class, we have so far always worked with the flow budget constraint as our constraint. And then we used either calculus of variations or optimal control theory. Alterantively, we can use the lifetime budget constraint as our constraint in this setting. (Why? When would you not be able to work with a lifetime budget constraint?) Set up the optimization problem with the lifetime budget constraint (i.e., write down the Lagrangian and introduce a multiplier) and take the first-order conditions. Solve for a consumption Euler equation.

The Lagrangian would be:

$$L(a_0) = \max \int_0^\infty e^{-\rho t} u(c_t) dt + \lambda \left[a_0 + \int_0^\infty e^{-R_t} y_t dt - \int_0^\infty e^{-R_t} c_t dt \right].$$

Why is the Lagrange multiplier λ not indexed by time t? Because this problem now only has one constraint (a lifetime constraint) rather than a sequence of constraints, one for each time period.

So now we can derive the Euler equation directly from this lifetime budget constraint. We have:

$$e^{-\rho t} u'(c_t) = \lambda e^{-R_t}$$

$$-\rho t + \log[u'(c_t)] = \log \lambda - R_t.$$

Taking derivatives:

$$-\rho + \frac{u''(c_t)\dot{c}_t}{u'(c_t)} = -r_t.$$

And so, by rearranging, we now have the Euler equation.

$$\frac{\dot{c}_t}{c_t} = \frac{u'(c_t)}{u''(c_t)c_t} \left(\rho - r_t \right).$$

Note: We could have alternatively directly worked with the budget constraint for *nominal* wealth. This would give us:

$$-\rho t + \log[u'(c_t)] = \log(\lambda) - I_t + \log(P_t)$$

$$-\rho + \frac{u''(c_t)\dot{c}_t}{u'(c_t)} = -i_t + \frac{\dot{P}_t}{P_t}.$$

(d) Consider the two functional forms given earlier for utility, $u(c_t)$. Plug them into the Euler equation and solve for the term $\frac{u'(c_t)}{u''(c_t)c_t}$.

We consider two functional forms of constant relative risk aversion (CRRA) preferences, defined as follows:

(General CRRA Preferences)

$$u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}.$$

In this case, the relative risk aversion is given by:

$$\frac{u'(c_t)}{u''(c_t)c_t} = \frac{c_t^{-\gamma}}{-\gamma c_t^{-\gamma-1} c_t} = -\frac{1}{\gamma}.$$

(Logarithmic Preferences) When $\gamma = 1$, the utility function takes the logarithmic form:

$$u(c_t) = \log(c_t).$$

For logarithmic preferences, the relative risk aversion is calculated as:

$$\frac{u'(c_t)}{u''(c_t)c_t} = \frac{1/c_t}{(-1)(1/c_t^2)c_t} = -1.$$

We will now derive the simple Euler equation using two different approaches. The first approach will be using optimal control theory. In Problem 2, we will then use dynamic programming and confirm that the two approaches are equivalent.

(e) Write down the optimal control problem. Identify the state, control variables and multipliers.

Let's rewrite the household's optimization problem:

$$\max_{\{c_t\}_{t>0}} \int_0^\infty e^{-\rho t} u(c_t) dt$$

subject to

$$\dot{a}_t = r a_t + y_t - c_t.$$

We will distinguish between four different types of objects.

1. State variables: $\{a_t\}$ (heuristically: variables for which we have a law of motion in the constraint set)
2. Control variables: $\{c_t\}$ (heuristically: variables the agent chooses
3. Exogenous variables: $\{y_t\}$
4. Costate / multiplier variables: $\{\mu_t\}$

(f) Write down the (current-value) Hamiltonian.

It is given by

$$\mathcal{H}_t(a_t, c_t, y_t, \mu_t) = u(c_t) + \mu_t\Big(ra_t + y_t - c_t\Big). \tag{4}$$

(g) Find the FOCs. Rearrange and again find the consumption Euler equation. Confirm that it's the same equation we derived above.

The solution to the optimization problem is then characterized by three sets of equations:
(Optimality) Differentiate the Hamiltonian with respect to the choice variable, and set it to 0.

$$0 = \partial_{c_t}\mathcal{H} = u'(c_t) - \mu_t. \tag{5}$$

(Multiplier) Differentiate the Hamiltonian with respect to the state variables, and set that equal to:

$$\rho\mu_t - \dot{\mu}_t = \partial_{a_t}\mathcal{H} = r\mu_t. \tag{6}$$

(State equations) Finally, there are the original laws of motion that characterize the economy. Here, we simply have the law of motion for wealth.

$$\dot{a}_t = ra_t + y_t - c_t. \tag{7}$$

This now gives us a system of 3 equations, (5) through (7), in the 3 unknowns $\{c_t, a_t, \mu_t\}$. For convenience, let's again write down this system:

$$\begin{aligned} \mu_t &= u'(c_t) \\ (\rho - r)\mu_t &= \dot{\mu}_t \\ \dot{a}_t &= ra_t + y_t - c_t. \end{aligned}$$

All we have to do now is to combine the first (optimality) and second (multiplier) equations. This yields:

$$(\rho - r)u'(c_t) = u''(c_t)\dot{c}_t,$$

where I used

$$\dot{\mu}_t = \frac{d}{dt}\mu_t = \frac{d}{dt}u'(c_t) = \frac{du'(c_t)}{dc_t}\frac{dc_t}{dt} = u''(c_t)\dot{c}_t.$$

The complete solution to this model, giving us the allocation of consumption and assets over time, is thus given by:

$$\dot{c}_t = \frac{u'(c_t)}{u''(c_t)}(\rho - r)$$

$$\dot{a}_t = ra_t + y_t - c_t,$$

for a given exogenous stream of income $\{y_t\}$.

Problem 4: Consumption-Savings using Dynamic Programming

We concluded Problem 1 by deriving the household's consumption Euler equation using optimal control theory. We will now do the same using dynamic programming.

Consider again the preferences of our household, given by

$$V_0 = \max_{\{c_t\}_{t>0}} \int_0^\infty e^{-\rho t} u(c_t) dt,$$

as well as the flow budget constraint

$$\dot{a}_t = r_t a_t + y_t - c_t,$$

where the household takes the (deterministic) paths of real interest rates $\{r_t\}$ and income $\{y_t\}$ as given.

We will now derive a recursive representation of the household's value function, and use the resulting Hamilton-Jacobi-Bellman (HJB) equation to derive the consumption Euler equation.

(a) We will derive a recursive representation for the household value function $V_t(a) = V(t, a)$. Why does the value function take this form in this context? I.e., why does V depend explicitly on calendar time, and why are we working with a representation of the problem where a is the only state variable?

The value function takes the form $V_t(a) = V(t, a)$ because the income path $\{y_t\}$ is given and may vary over time, making it an exogenous and potentially non-stationary factor. This time variation in income implies that the value function depends on calendar time explicitly, as future income changes directly impact the household's optimal consumption and savings decisions. Since a (assets) is the only state variable that evolves over time based on consumption and income, we represent the problem with a as the sole state variable.

(b) In class, we derived the HJB heuristically, starting with a discrete time Bellman equation and taking the continuous time limit. We will now derive the HJB equation from the sequence problem. Recall in Lecture 1 we derived the Bellman equation from the sequence problem in discrete time. Please follow the same general proof strategy and derive the HJB in continuous time. You should arrive at the following HJB equation:

$$\rho V_t(a) = \partial_t V_t(a) + \max_c \left\{ u(c) + \Big[r_t a + y_t - c \Big] \partial_a V_t(a) \right\}$$

where ∂_x denotes the partial derivative with respect to x.

We start by breaking down the value function over an infinitesimally small time interval $[t, t + \Delta t]$.

$$\begin{aligned} V(a,t) &= \max_{\{c_s\}_{s>t}} \left\{ \int_t^{t+\Delta t} e^{-\rho(s-t)} u(c_s) ds + \int_{t+\Delta t}^{\infty} e^{-\rho(s-t)} u(c_s) ds \right\} \\ &= \max_{\{c_s\}_{s>t}} \left\{ u(c_t)\Delta t + e^{-\rho \Delta t} \int_{t+\Delta t}^{\infty} e^{-\rho(s-(t-\Delta t))} u(c_s) ds \right\} \end{aligned}$$

The value function $V(a,t)$ can be expressed over this interval as:

$$V(a,t) = \max_c \left\{ u(c)\Delta t + e^{-\rho \Delta t} V(a_{t+\Delta t}, t + \Delta t) \right\}.$$

To approximate $V(a_{t+\Delta t}, t + \Delta t)$, we expand it around (a,t) using a Taylor series:

$$V(a_{t+\Delta t}, t + \Delta t) \approx V(a,t) + \frac{\partial V(a,t)}{\partial t}\Delta t + \frac{\partial V(a,t)}{\partial a}\Delta a,$$

where $\Delta a = (r_t a + y_t - c)\Delta t$, based on the budget constraint. Substituting this into the expression, we get:

$$V(a,t) = \max_c \left\{ u(c)\Delta t + e^{-\rho \Delta t} \left(V(a,t) + \frac{\partial V(a,t)}{\partial t}\Delta t + \frac{\partial V(a,t)}{\partial a}(r_t a + y_t - c)\Delta t \right) \right\}.$$

Expanding $e^{-\rho \Delta t} \approx 1 - \rho \Delta t$ for small Δt, we have:

$$V(a,t) = \max_c \left\{ u(c)\Delta t + (1 - \rho \Delta t) \left(V(a,t) + \frac{\partial V(a,t)}{\partial t}\Delta t + \frac{\partial V(a,t)}{\partial a}(r_t a + y_t - c)\Delta t \right) \right\}.$$

Dividing by Δt and taking the limit as $\Delta t \to 0$:

$$0 = \rho V(a,t) - \frac{\partial V(a,t)}{\partial t} - \max_c \left\{ u(c) + (r_t a + y_t - c)\frac{\partial V(a,t)}{\partial a} \right\}.$$

Rearranging, we obtain the Hamilton-Jacobi-Bellman (HJB) equation:

$$\rho V(a,t) = \frac{\partial V(a,t)}{\partial t} + \max_c \left\{ u(c) + (r_t a + y_t - c)\frac{\partial V(a,t)}{\partial a} \right\}.$$

(c) Why is there a $\partial_t V_t(a)$ term?

The term $\partial_t V_t(a)$ appears because the value function $V_t(a)$ depends explicitly on calendar time due to the non-stationary path of income $\{y_t\}$. Since income may change over time, the household's optimal decisions are time-dependent, necessitating the inclusion of $\partial_t V_t(a)$ to capture the effect of time on the value function.

(d) Write down the FOC for consumption and interpret every term. Define and discuss the consumption policy function.

The first-order condition (FOC) with respect to consumption c is:

$$u'(c) = \partial_a V_t(a).$$

This condition implies that the marginal utility of consumption must equal the marginal value of assets, reflecting the trade-off between consuming today and saving for future utility. The consumption policy function $c_t = c(a,t)$ maps the current asset level and time to the optimal consumption choice, balancing current consumption benefits against future value.

(e) Plug the consumption policy function back into the HJB. Discuss why this is now a non-linear partial differential equation.

Substituting $c(a,t)$ back into the HJB equation:

$$\rho V_t(a) = \partial_t V_t(a) + u(c(a,t)) + (r_t a + y_t - c(a,t))\,\partial_a V_t(a).$$

This equation is a non-linear partial differential equation (PDE) because $u(c(a,t))$ introduces non-linearity through the utility function $u(\cdot)$, which is generally non-linear in c.

(f) Take the envelope condition by differentiating the HJB with respect to a.

Differentiating the HJB with respect to a gives:

$$\begin{aligned}
\rho\partial_a V_t(a) &= \partial_a(\partial_t V_t(a)) + u'(c(a,t))\frac{\partial c(a,t)}{\partial a} + \left(r_t - \frac{\partial c(a,t)}{\partial a}\right)\partial_a V_t(a) + (r_t a_t + y_t - c(a,t))\,(\partial_a)^2 V_t(a) \\
&= \partial_a(\partial_t V_t(a)) + r_t\partial_a V_t(a) + (r_t a_t + y_t - c(a,t))\,(\partial_a)^2 V_t(a) \\
&\therefore (\rho - r_t)\partial_a V_t(a) = \partial_a(\partial_t V_t(a)) + (r_t a_t + y_t - c(a,t))\,(\partial_a)^2 V_t(a)
\end{aligned}$$

(g) Use the chain rule to characterize $\frac{d}{dt}V(t,a_t)$, taking into account explicitly that a_t is also a function of time.

Using the chain rule:

$$\frac{d}{dt}V(t,a_t) = \partial_t V(t,a_t) + \partial_a V(t,a_t) \cdot \dot{a}_t.$$

Since $\dot{a}_t = r_t a_t + y_t - c_t$, we have:

$$\frac{d}{dt}V(t,a_t) = \partial_t V(t,a_t) + \partial_a V(t,a_t) \cdot (r_t a_t + y_t - c_t).$$

(h) You will now arrive at the consumption Euler equation by combining 3 equations: the characterization of $\frac{d}{dt}V_t(a_t)$ from the previous part, the FOC, and the envelope condition.

Using the envelope condition and the expression for $\frac{d}{dt}V(t,a_t)$:

$$\begin{aligned}(\rho - r_t)\partial_a V_t(a) &= \partial_a \left(\partial_t V_t(a)\right) + \left(r_t a_t + y_t - c(a,t)\right)(\partial_a)^2 V_t(a) \\ &= \partial_a \left(\frac{d}{dt}V(t,a_t)\right).\end{aligned}$$

Using the FOC:

$$\begin{aligned}(\rho - r_t)u'(c) &= \frac{d}{dt}u'(c) \\ &= u''(c)\frac{dc}{dt}.\end{aligned}$$

Assuming CRRA utility, $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$, we find that:

$$-\frac{c\,u''(c)}{u'(c)} = \gamma.$$

Thus, the consumption Euler equation is:

$$\frac{\dot{c}}{c} = \frac{r_t - \rho}{\gamma}.$$

Problem 5: Oil Extraction

Credit: Pablo Kurlat (https://sites.google.com/view/pkurlat/teaching)

Time is continuous and indexed by $t \in [0,\infty)$. At time $t = 0$, there is a finite amount of oil x_0. Denote by c_t the rate of oil consumption at date t. Oil is non-renewable, so the remaining amount of oil at date t is

$$x_t = x_0 - \int_0^t c_s ds.$$

A social planner (the government) wants to set the rate of oil consumption to maximize utility of the representative household given by

$$\int_0^\infty e^{-\rho t} u(c_t) dt,$$

with $u(c) = \log(c)$.

(a) Explain the expression for the remaining amount of oil: $x_t = x_0 - \int_0^t c_s ds$

The LOM of the remaining oil is: $\dot{x}_t = -c_t$, so this is the solution: oil remaining = initial- all past consumption.

(b) Set up the (present-value) Hamiltonian for this problem. List all state variables, control variables, and multipliers.

Take derivative with respect to t to the equation of remaining oil, we could derive the law of motion for change in oil stock:

$$\dot{x}(t) = -c(t)$$

The present-value Hamiltonian for this problem is given by:

$$\mathcal{H}(c(t), x(t), \lambda(t)) = e^{-\rho t} \log(c(t)) + \lambda(t)(-c(t))$$

State variable $x(t)$, control $c(t)$ and co-state/multiplier $\lambda(t)$

(c) Write down the first-order necessary conditions. Solve for the optimal policy c_t

Take FOCs of the Hamiltonian, we have:

$$\frac{\partial \mathcal{H}}{\partial c(t)} = 0 \Rightarrow e^{-\rho t} \frac{1}{c(t)} = \lambda(t)$$

$$\frac{\partial \mathcal{H}}{\partial x(t)} = -\dot{\lambda}(t) \Rightarrow 0 = -\dot{\lambda}(t)$$

The second FOC implies that $\lambda(t)$ does not vary over time, which would require $c(t) = c(0) \cdot e^{-\rho t}$ for the first FOC to hold. The transversality condition is

$$\lim_{t \to \infty} x(t)\lambda(t) = 0$$

so:

$$\int_0^\infty c(s) ds = x_0$$

$$c(0) \int_0^\infty e^{-\rho s} ds = x_0$$

$$c(0) = \rho x_0$$

Therefore:

$$c(t) = \rho x_0 e^{-\rho t}$$

(d) Write down the HJB equation for this problem

The HJB equation for this problem is:

$$\rho V(x) = \max_{c} \log(c) + V'(x)(-c)$$

(e) Guess and verify that the value function is $V(x) = a + b\log(x)$. Solve for a and b

Take FOC of the HJB equation, we could get:

$$\frac{1}{c} = V'(x) \Rightarrow c = \frac{1}{V'(x)}$$

Substitute c into the HJB equation:

$$\rho V(x) = -\log\left(V'(x)\right) - 1$$

Guess $V(x) = b\log(x) + a$ and substitute into the HJB equation:

$$\rho(b\log(x) + a) = \log\left(\frac{x}{b}\right) - 1$$

$$\rho b\log(x) + a\rho = \log(x) - \log(b) - 1$$

Match the coefficients and we have:

$$\rho b = 1 \Rightarrow b = \frac{1}{\rho}$$

$$a\rho = -1 - \log(b) \Rightarrow a = -\frac{1 - \log(\rho)}{\rho}$$