Dynamic Programming and Applications

Discrete Time Dynamics and Optimization

Lecture 2

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Last lecture, we introduced the Bellman equation:

$$V(k) = \max_{k'} \left\{ u\Big(f(k) - k'\Big) + \beta V(k') \right\}$$

- The value today is = flow payoff + continuation value
- This assumes there is no uncertainty: We can make perfect forecasts about the future
  Not how the world works!
- Suppose production also depends on productivity $y_t = f(k_t, A_t)$ and A_{t+1} is uncertain

In an uncertain world, we care about the *expected* continuation value

$$V(k, A) = \max_{k'} \left\{ u\Big(f(k, A) - k'\Big) + \beta \mathbb{E} V(k', A') \right\}$$

- Now the only question is: How do we compute the expectation $\mathbb{E}$?
- We have to study stochastic processes (and stochastic calculus) to answer this

Outline

Part 1: Difference equations

1. Stochastic processes
2. Markov chains
3. Difference equations
4. Stochastic difference equations

Part 2: Stochastic dynamic programming

1. Stochastic dynamic programming
2. History notation
3. The stochastic neoclassical growth model

Part 3: Optimal stopping

Part 1: Difference Equations

1. Stochastic processes

- Let X_t be a random variable that is time t adapted
- Discrete time: We index time discretely $t = 0, 1, 2, \ldots, T \leq \infty$
- Stochastic process in discrete time: a sequence of random variables indexed by t, $\{X_t\}_{t=0}^{T}$
- Continuous time: We index time continuously $t \in [0, T]$ with $T \leq \infty$
- Stochastic process in continuous time: a sequence of random variables indexed by t, $\{X_t\}_{t \geq 0}$

2. Markov chains

- A stochastic process $\{X_t\}$ has the *Markov property* if for all $k \geq 1$ and all t:

$$\mathbb{P}(X_{t+1} = x \mid X_t, X_{t-1}, \ldots, X_{t-k}) = \mathbb{P}(X_{t+1} = x \mid X_t)$$

- *State space* of the Markov process = set of events or states that it visits
- A Markov chain is a Markov process (stochastic process with Markov property) that visits a finite number of states (*discrete state space*)
- Simplest example: Individual i is randomly hit by earnings (employment) shocks and switches between $X_t \in \{X^L, X^H\}$

- Markov chains have a *transition matrix* P that describes the probability of transitioning from state i to state j

- Simplest example with state space $\{X^L, X^H\}$

$$P = \begin{pmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{pmatrix}$$

- This says: P of staying in employment state = 0.8, P of switching = 0.2

- P_{ij} is the probability of switching from state i to state j (one period)

- P^2 characterizes transitions over two periods: $(P^2)_{ij}$ is prob of going from i to j in two periods

- The rows of the transition matrix have to sum to 1 (definition of probability measure)

3. Difference equations

- We start with deterministic (non-random) dynamics and then conclude with stochastic (random) dynamics

- The *first-order linear difference equation* is defined by

$$x_{t+1} = bx_t + cz_t \tag{1}$$

  where $\{z_t\}$ is an exogenously given, bounded sequence

- For now, all objects are (real) scalars (easy to extend to vectors and matrices)

- Suppose we have an *initial condition* (i.e., given initial value) x_0

- When $c = 0$, (1) is a *time-homogeneous* difference equation

- When cz_t is constant for all t, (1) is an *autonomous* difference equation

Autonomous equations

- Consider the autonomous equation with $z_t = 1$
- A particular solution is the constant solution with $x_t = \frac{c}{1-b}$ when $b \neq 1$
- Such a point is called a *stationary point* or *steady state*
- General solution of the autonomous equation (for some constant x):

$$x_t = (x_0 - x)b^t + x \tag{2}$$

- Important question is long-run behavior (stability / convergence)
- When $|b| < 1$, (2) converges asymptotically to steady state x for any initial value x_0 (steady state x is globally stable)
- If $|b| > 1$, (2) explodes and is not stable (except when $x_0 = x$)

Examples in macro

Capital accumulation:

$$K_{t+1} = (1-\delta)K_t + I_t$$

- δ is depreciation and I_t is investment
- This is a *forward equation* and requires an initial condition K_0
- If $I_t = 0$ and $0 < \delta < 1$, $K_t \to 0$
- If $I_t = c$ constant, then K_t converges to $\frac{c}{\delta}$: $K_{t+1} = (1-\delta)\frac{c}{\delta} + c = \frac{c}{\delta}$

Wealth dynamics:

$$a_{t+1} = R_t a_t + y_t - c_t$$

- R_t is the gross real interest rate, y_t is income, c_t is consumption
- This is a *forward equation* and requires an initial condition a_0
- We will study this as a *controlled* process because c_t will be chosen optimally
- Work out the following: $R_t = R$ and $y_t = y$ constant, and

$$c_t = \left(1 - \frac{1}{R}\right)\left(a_t + \sum_{s=t}^{\infty} R^{-(s-t)} y\right)$$

  What are the dynamics of a_t?

Consumption Euler equation:

$$\frac{1}{C_t} = \beta R_t \frac{1}{C_{t+1}}$$

- $\frac{1}{C_t} = u'(C_t)$ is marginal utility with log preferences
- This is a *backward equation* and requires a terminal condition or transversality condition, i.e., c_T must converge to something
- Suppose there exists time T s.t. for all $t \geq T$, $C_t = C$
- Then solve *backwards* from: $\frac{1}{C_{T-1}} = \beta R_{T-1} \frac{1}{C_T}$ or expressed as *time-homogeneous first-order linear difference equation*

$$C_{T-1} = \frac{1}{\beta R_{T-1}} C_T$$

- Difference between *forward* and *backward* equations is critical! This is closely related to the idea of *boundary conditions* (much more to come)

4. Stochastic difference equations

- Consider the process $\{X_t\}$ with

$$X_{t+1} = AX_t + Cw_{t+1} \tag{3}$$

where w_{t+1} is an iid. process with $w_{t+1} \sim \mathcal{N}(0,1)$

- Equation (3) is a *first-order, linear stochastic difference equation*
- Let $\mathbb{E}_t$ the *conditional expectation* operator (conditional on time t information)
- For example:

$$\begin{aligned}\mathbb{E}_t(X_{t+1}) &= \mathbb{E}(X_{t+1} \mid X_t) = \mathbb{E}(AX_t + Cw_{t+1} \mid X_t) \\ &= AX_t + C\mathbb{E}(w_{t+1} \mid X_t) = AX_t + C\mathbb{E}(w_{t+1}) = AX_t\end{aligned}$$

- Rational expectations: agents' beliefs about stochastic processes are consistent with the true distribution of the process

- Key equation: wealth dynamics with income fluctuations:

$$a_{t+1} = R_t a_t + y_t - c_t,$$

  where y_t is a stochastic process

- Consumption Euler equation with uncertainty (e.g., stochastic income):

$$u'(C_t) = \beta R \mathbb{E}_t \Big[u'(C_{t+1}) \Big]$$

Part 2: Stochastic Dynamic Programming

1. Stochastic dynamic programming

- Follow Ljungqvist-Sargent notation, Chapter 3.2
- Under uncertainty, household problem takes the form

$$\max_{\{c_t\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to $k_{t+1} = g(k_t, c_t, \epsilon_{t+1})$ (*first-order stochastic difference equation*)

- Notice: $\{c_t\}$ now denotes a stochastic process, no longer simple sequence!
- $\{\epsilon_t\}_{t=0}^{\infty}$ is sequence of iid random variables (*stochastic process*)
- Initial condition x_0 given

- Usually best to start with sequence problem, then derive recursive representation

- To derive recursive representation, your first question must be:
  - Recursive representation means we go from thinking about sequences (stochastic processes) to thinking about functions
  - But functions *of what*? I.e., what is the domain of the functions we're interested in?
  - Answer: functions of the *state variables*

- What are state variables?
  - In the (deterministic) neoclassical growth model: just k
  - Generally: state variables = set of information you need today to compute the continuation value for tomorrow
  - That's why they're called "states"

- Dynamic programming: look for recursive representation with state variable k
- Q: Why is k the state variable here, not (k, ϵ)? (Think about structure of $g(\cdot)$.)
- The problem is to look for a *policy function* $c(k)$ that solves

$$V(k) = \max_c \left\{ u(c) + \beta \mathbb{E}\left[V\Big(g(k,c,\epsilon)\Big) \,\Big|\, k \right] \right\}, \quad \text{where } \mathbb{E}[V(\cdot) \mid k] = \int V(\cdot) dF(\epsilon)$$

- $V(k)$ is (lifetime) value that agent obtains from solving this problem starting from k
- FOC that characterizes the consumption policy function $c(k)$ is

$$0 = u'(c(k)) + \beta \mathbb{E}\left\{ \partial_k V\Big(g(k,c(k),\epsilon)\Big) \cdot \partial_c g(k,c(k),\epsilon) \,\Big|\, k \right\} = 0$$

2. History notation

- A very popular approach to deal with uncertainty in macro is to use history notation (Ljungqvist-Sarget, e.g., chapters 8, 12)
- Time is discrete and indexed by $t = 0, 1, \ldots$
- At every t, there is a realization of a stochastic event $s_t \in \mathcal{S}$
- We denote the **history** of such events up to t by $s^t = \{s_0, s_1, \ldots, s_t\}$
- The unconditional probability of history s^t is given by $\pi_t(s^t \mid s_0)$
- If Markov, $\pi_t(s^t \,|\, s_0) = \pi(s_t \mid s_{t-1})\pi(s_{t-1} \mid s_{t-2}) \ldots \pi(s_0)$

- Crucial to understand notation:
  - $\{c_t\}_{t\geq 0}$ is the stochastic process
  - c_t is the random variable
  - $c_t(s^t)$ is the realization of the random variable at date t in history s^t

- The **lifetime value** of representative household is then defined as

$$V(s_0) = \sum_{t=0}^{T} \beta^t \sum_{s^t} \pi_t\left(s^t \mid s_0\right) u\left(c_t\left(s^t\right), \ell_t\left(s^t\right)\right)$$

- Here we also allow household to choose labor supply ℓ_t

- *Generalizations*: heterogeneous beliefs, general preferences (Epstein-Zin), recursive formulation, multiple commodities, intergenerational considerations

3. Stochastic Growth Model

- Discrete time: $t \in \{0, 1, \ldots, T\}$, where $T \leq \infty$
- At t, event $s_t \in \mathcal{S}$ is realized; history $s^t = (s_0, \ldots, s_t)$ has probability $\pi_t(s^t)$
- Representative household has preferences over consumption $c_t(s^t)$ and labor $\ell_t(s^t)$

$$\sum_{t=0}^{\infty} \beta^t \sum_{s^t} \pi_t\left(s^t\right) u\left(c_t(s^t), \ell_t(s^t)\right)$$

- Inada conditions $\lim_{c \to 0} u_c(c, \ell) = \lim_{\ell \to 0} u_\ell(c, \ell) = \infty$
- At $t = 0$, household endowed with k_0

- Technology, capital accumulation, and budget / resource constraint:

$$c_t(s^t) + i_t(s^t) = A_t(s^t) F(k_t(s^{t-1}), \ell_t(s^t))$$
$$k_{t+1}(s^t) = (1-\delta) k_t(s^{t-1}) + i_t(s^t)$$

- $F(\cdot)$ is twice continuously differentiable and constant returns to scale

- Source of uncertainty is stochastic process for TFP $A_t(s^t)$

- Standard regularity conditions on $F(\cdot)$ (see LS)

Lagrangian approach to sequence problem

$$L = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t(s^t) \Big\{ u(c_t(s^t), l_t(s^t))$$
$$+ \lambda_t(s^t) \Big[A_t(s^t) F(k_t(s^{t-1}), \ell_t(s^t)) - c_t(s^t) + (1-\delta) k_t(s^{t-1}) - k_{t+1}(s^t) \Big] \Big\}$$

- FOCs for $c_t(s^t)$, $\ell_t(s^t)$ and $k_{t+1}(s^t)$ are given by

$$u_c(s^t) = \lambda_t(s^t)$$
$$u_\ell(s^t) = u_c(s^t) A_t(s^t) F_\ell(s^t)$$
$$u_c(s^t)\pi_t(s^t) = \beta \sum_{s^{t+1}|s^t} u_c(s^{t+1}) \pi_{t+1}(s^{t+1}) \Big[A_{t+1}(s^{t+1}) F_k(s^{t+1}) + (1-\delta) \Big]$$

- Summation over $(s^{t+1} \mid s^t)$ is like conditional expectation
  summing over histories that branch out from s^t

Dynamic programming approach

- Assume time-homogeneous Markov process:

$$\mathbb{E}_t(A_{t+1}) = \mathbb{E}\left[A(s^{t+1}) \mid s^t\right] = \mathbb{E}\left[A(s_{t+1}) \mid s_t\right] = \sum_{s'} \pi(s' \mid s_t) A(s')$$

- Drop t subscripts: s is current state, s' denotes next period's draw
- Denote by X_t *endogenous state* (assume for now there is such a representation)
- Intuitively: s is the exogenous state and X is the endogenous state

Bellman equation becomes:

$$V(X, s) = \max_{c,\ell} \left\{ u(c, \ell) + \beta \sum_{s'} \pi(s' \mid s) V(X', s') \right\} \quad \text{where } X' = g(X, c, \ell, s, s')$$

Part 3: Optimal Stopping

Application: optimal stopping problem

Problem: Every period t, an agent draws an offer x from a uniform distributon over the unit interval $[0, 1]$. The agent can accept the offer, in which case her payoff is x, and the game ends, or the agent can reject the offer and draw again a period later. Draws are independent. Rejections are costly because the agent discounts the future at β. The game continues until the agent receives an offer she accepts.

Many applications (problems in life) look like this:

- buying a house
- searching for a partner
- closing a production plant
- exercising an option
- adopting a new technology

What is recursive / dynamic programming representation of optimal stopping problem?

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Agent's dynamic optimization problem given recursively by Bellman equation

$$V(x) = \max\left\{x,\, \beta\mathbb{E}V(x')\right\}$$

where the expectation (operator) $\mathbb{E}$ is taken over the next draw x'

Our problem is to find the value function $V(x)$ that solves the Bellman equation. We'll also want to find the associated policy rule.

Definition: *A policy is a function that maps every point in state space* $[0,1]$ *to an action*
There are 2 actions: ACCEPT and REJECT

Definition: *An optimal policy achieves payoff* $V(x)$ *for all feasible* $x \in [0,1]$

Let's try to understand the shape of $V(x)$ intuitively:

- For large values $\hat{x}$ where you ACCEPT, what's the value $V(\hat{x})$?

- For small values $\tilde{x}$ where you REJECT and instead choose the continuation value, $\beta \mathbb{E} V(x') > \tilde{x}$, does the continuation value depend on $\tilde{x}$? Why not?

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Shape of $V(x)$ must therefore be:

$$V(x) = \begin{cases} x & \text{if} \quad x \geq x^* \\ x^* & \text{if} \quad x < x^* \end{cases}$$

- Solution to the problem: there is be threshold $x^* \in [0, 1]$ s.t. agent accepts for $x \geq x^*$
- Also called **free boundary problem** (have to find endogenous boundary x^*)

Lemma: *In the optimal stopping problem, a policy is a best response to a continuation value function $\widehat{v}(x)$ if and only if the policy is a threshold rule with cutoff*

$$x^* \equiv \beta \mathbb{E}[\widehat{v}(x')]$$

Proof: Show by contradiction that optimization must imply

$$\begin{array}{lll} \text{ACCEPT} & \text{if} & x > \beta \mathbb{E}[\widehat{v}(x')] \equiv x^* \\ \text{REJECT} & \text{if} & x < \beta \mathbb{E}[\widehat{v}(x')] \equiv x^* \end{array}$$

If $x = \beta \mathbb{E}[\widehat{v}(x')]$, then ACCEPT and REJECT generate the same payoff. ■

- Why no jump in $V(x)$ at x^*? (lim from RHS must be x^*, from LHS by contradiction)
- Continuation value must be $V(x')$ because problem tmr is repeat of today

- We just concluded: at $x = x^*$, indifferent between ACCEPT and REJECT

- This is enough information to solve the problem!

$$\begin{aligned} V(x^*) &= x^* \\ &= \beta \mathbb{E} V(x') \\ &= \beta \int_0^{x^*} x^* f(x) dx + \beta \int_{x^*}^1 x f(x) dx \\ &= \beta x^* [x]_0^{x^*} + \beta \frac{1}{2} [x^2]_{x^*}^1 \\ &= \beta (x^*)^2 + \beta \frac{1}{2} [1 - (x^*)^2] \end{aligned}$$

Solution:

$$x^* = \frac{1}{\beta}(1 - \sqrt{1 - \beta^2})$$

Always sanity-check comparative statics: What happens as $\beta \to 0$ and $\beta \to 1$?

Why is this threshold rule a *solution to the Bellman Equation*?
If you REJECT, your continuation payoff is

$$x^* = \beta \mathbb{E} V(x') = \beta \int_0^{x^*} x^* f(x)\,dx + \beta \int_{x^*}^1 x\,f(x)\,dx.$$

So it's optimal to REJECT if $x \leq x^*$ and it's optimal to ACCEPT if $x \geq x^*$. Hence, for all values of x

$$V(x) = \max\{x, x^*\} = \max\{x, \beta\mathbb{E}[V(x')]\}$$