

# Graph Theory - Part II

Foundation Course on Discrete Mathematics for GATE

Find the odd one in the group

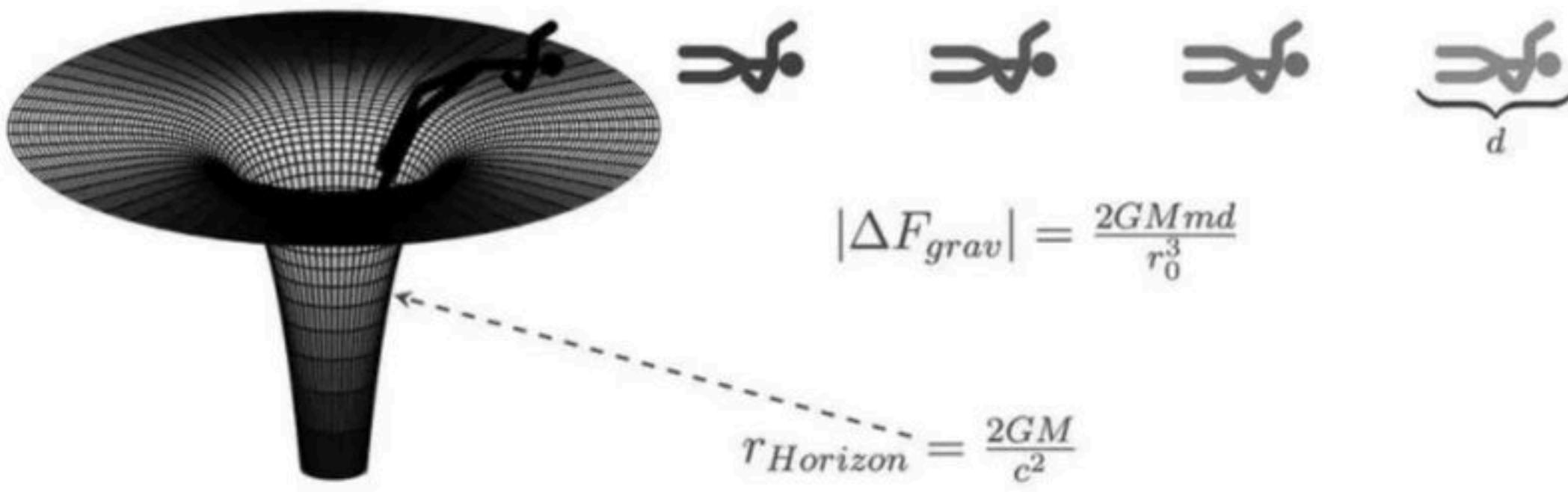


## **Group Theory**

- Group theory is very important mathematical tool which is used in a number of areas in research and application. Using group theory, we can estimate the strength of a set with respect to an operator.

- This idea will further help us in research field to identify the correct mathematical system to work in a particular research area. E.g. can we use natural numbers in complex problem area like soft computing or studying black holes.

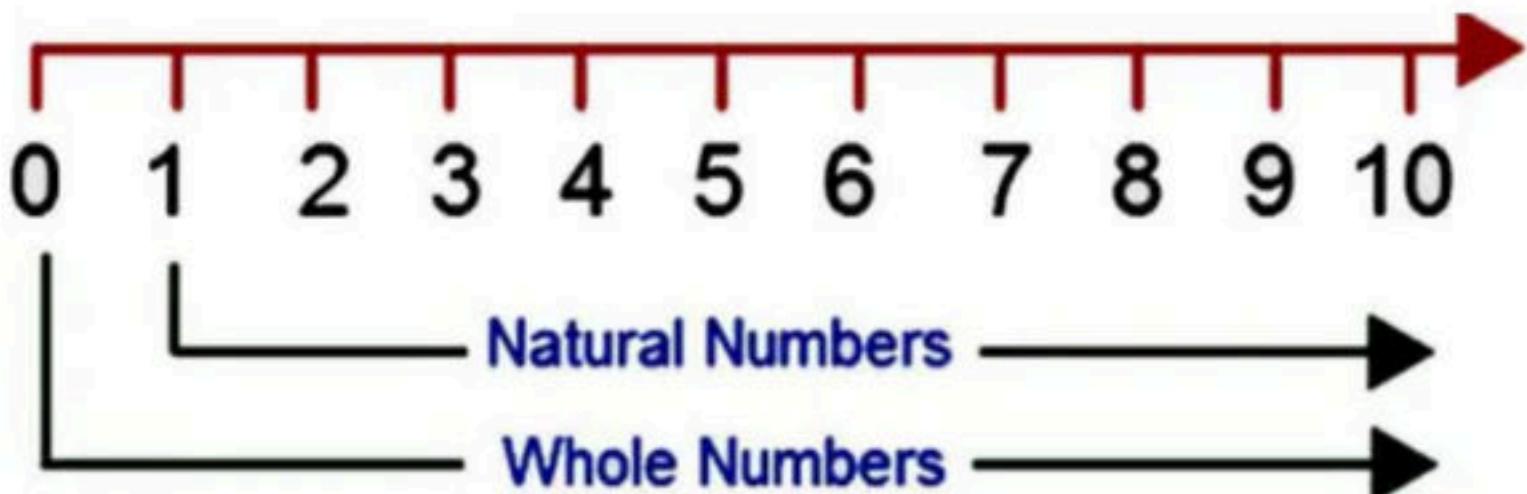
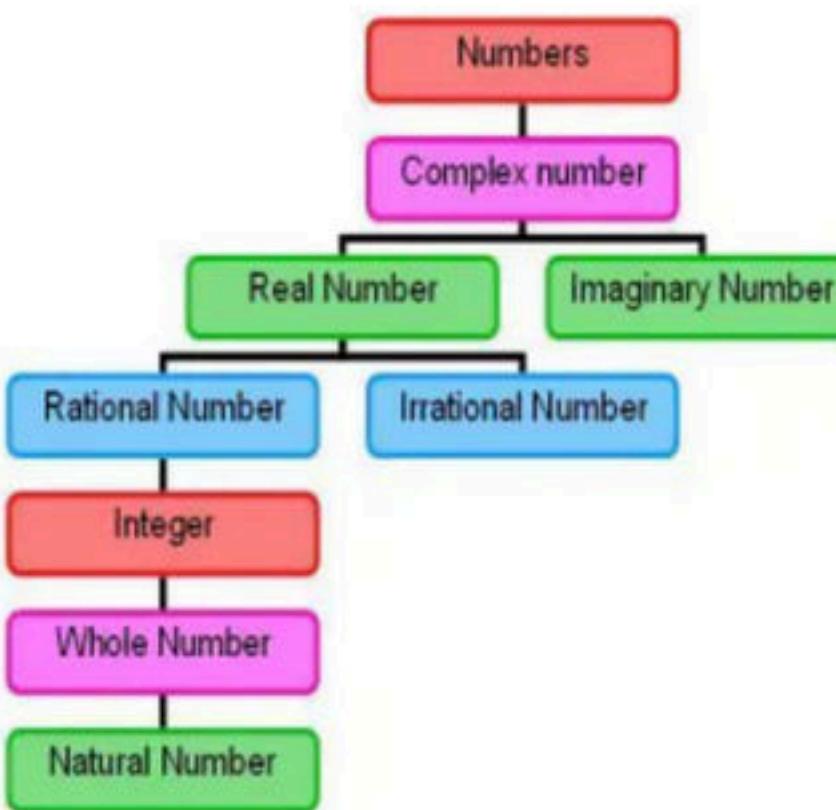
$$ds^2 = - \left(1 - \frac{2GM}{c^2 r}\right) dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2 d\Omega^2$$



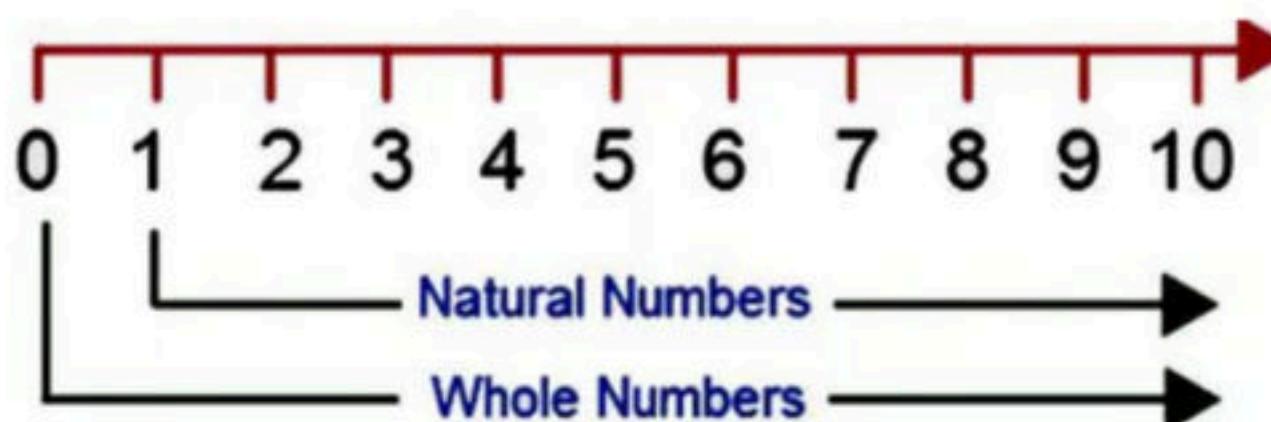
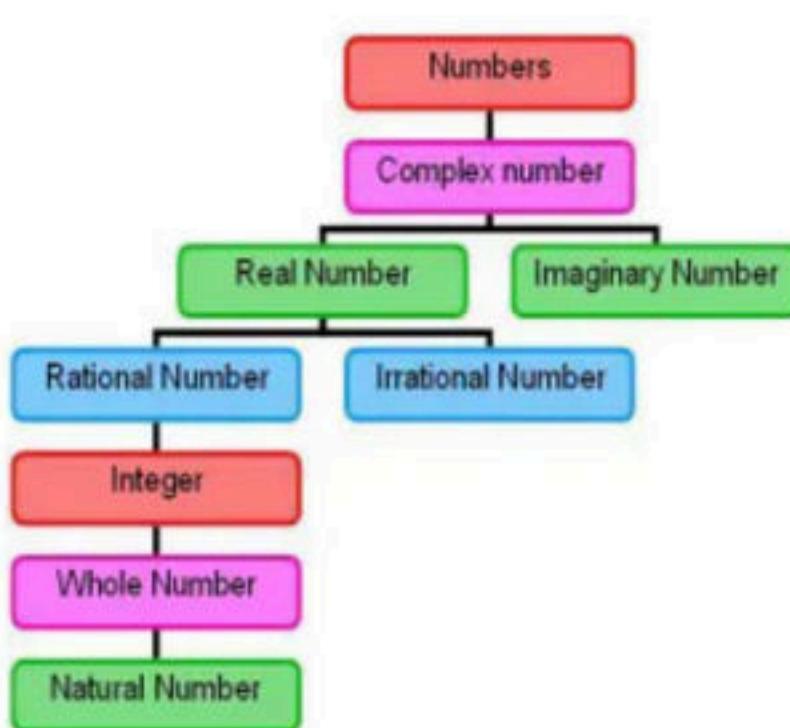
- Now we will directly study some of the basic set related properties and will define some structure of set and operator based on the properties and will check those properties on basics number systems like natural numbers, integers, real numbers etc.



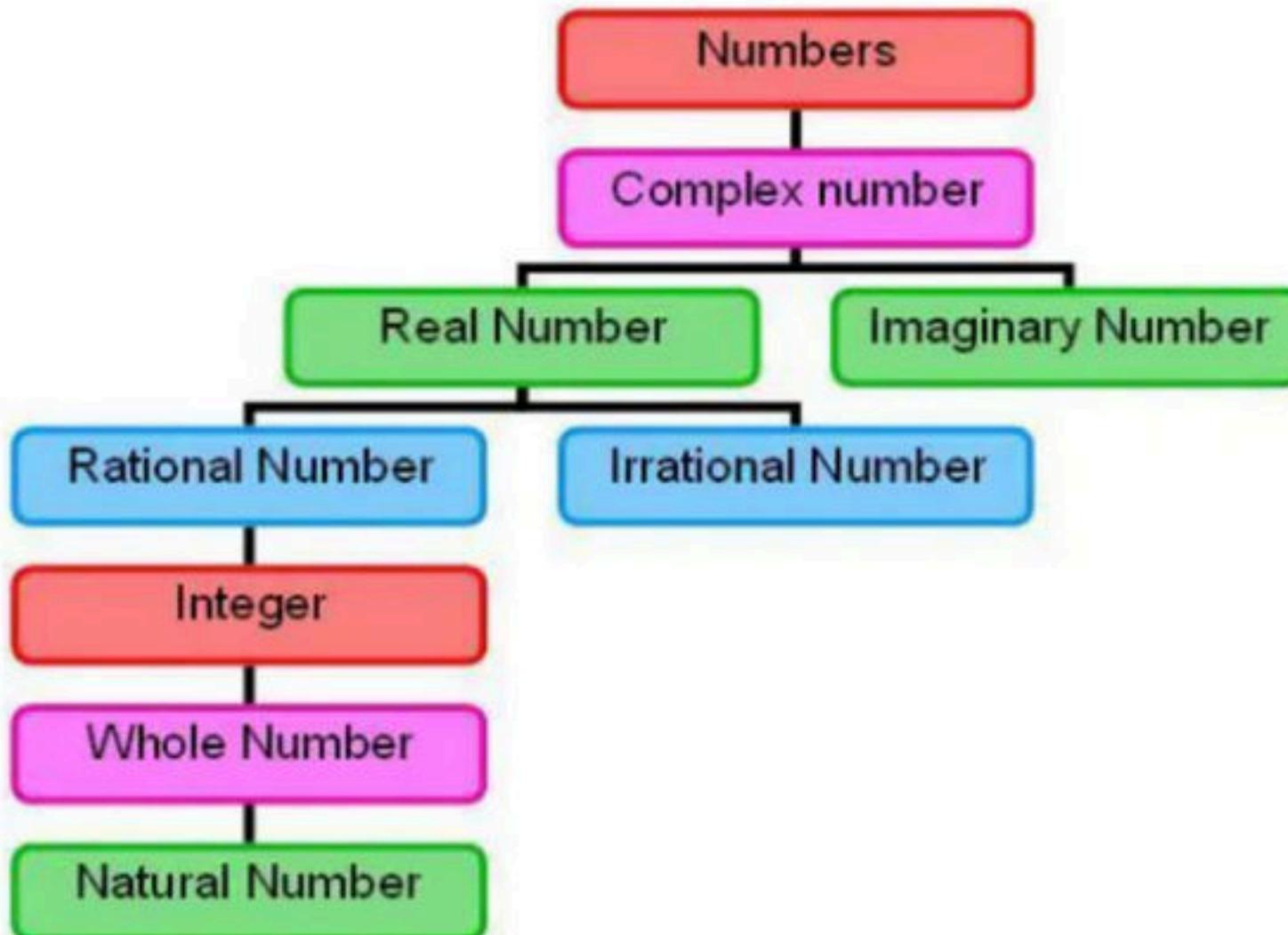
- **Set of all-Natural number(N)** - A natural number is a number that occurs commonly and obviously in nature. The set of natural numbers, can be defined as  $N = \{1, 2, 3, 4, \dots, \infty\}$



- **Set of all Integer(z)** - An integer is a number that can be written without a fractional component.
- **Set of all Whole number(w)** - A whole number is a science expanded natural number. Set of natural number and zero



- **Set of all Real number(R)** - A real number is a value that represents a quantity along a continuous line, containing all of the rational numbers and all of the irrational numbers.



**Break**

**1. Closure property:** - Consider a non-empty set A and a binary operation \* on A. A is said to be closed with respect to \*, if  $\forall a, b \in A$ , then  $a * b \in A$ .

**2. Algebraic Structure:** - A non-empty set A is said to be an algebraic structure with respect to a binary operation \*, if A satisfy closure property with respect to \*.

		Algebraic Structure
1	(N, +)	
2	(N, -)	
3	(N, /)	
4	(N, x)	
5	(Z, +)	
6	(Z, -)	
7	(Z, /)	
8	(Z, x)	
9	(R, +)	
10	(R, -)	
11	(R, /)	
12	(R, x)	
13	(M, +)	
14	(M, x)	
15	(E, +)	
16	(E, x)	
17	(O, +)	
18	(O, x)	
19	(R-O, x)	
20	(R-O, /)	
21	(Non-Singular Matrix, x)	

**1. Associative property:** - Consider a non-empty set A and a binary operation \* on A. A is said to be associative with respect to \*, if  $\forall a, b, c \in A$ , then  $(a*b)*c = a*(b*c)$

**2. Semi-Group:** - A non-empty set A is said to be a Semi-group with respect to a binary operation \*, if A satisfy closure, Associative property with respect to \*.

		Algebraic Structure	Semi Group
1	(N, +)	Y	
2	(N, -)	N	
3	(N, /)	N	
4	(N, x)	Y	
5	(Z, +)	Y	
6	(Z, -)	Y	
7	(Z, /)	N	
8	(Z, x)	Y	
9	(R, +)	Y	
10	(R, -)	Y	
11	(R, /)	N	
12	(R, x)	Y	
13	(M, +)	Y	
14	(M, x)	Y	
15	(E, +)	Y	
16	(E, x)	Y	
17	(O, +)	N	
18	(O, x)	Y	
19	(R-0, x)	Y	
20	(R-0, /)	Y	
21	(Non-Singular Matrix, x)	Y	

**1. Identity property:** - Consider a non-empty set A and a binary operation \* on A. A is said to satisfy identity property with respect to \*, if  $\forall a \in A$ , there must be unique  $e \in A$ , such that  $a * e = e * a = a$

**2.** There is exactly one Identity element in the set and will be same for all element in the set.

**3. Monoid:** - A non-empty set A is said to be a Monoid with respect to a binary operation \*, if A satisfy closure, Associative, identity property with respect to \*.

		Algebraic Structure	Semi Group	Monoid
1	(N, +)	Y	Y	
2	(N, -)	N	N	
3	(N, /)	N	N	
4	(N, x)	Y	Y	
5	(Z, +)	Y	Y	
6	(Z, -)	Y	N	
7	(Z, /)	N	N	
8	(Z, x)	Y	Y	
9	(R, +)	Y	Y	
10	(R, -)	Y	N	
11	(R, /)	N	N	
12	(R, x)	Y	Y	
13	(M, +)	Y	Y	
14	(M, x)	Y	Y	
15	(E, +)	Y	Y	
16	(E, x)	Y	Y	
17	(O, +)	N	N	
18	(O, x)	Y	Y	
19	(R-O, x)	Y	Y	
20	(R-O, /)	Y	N	
21	(Non-Singular Matrix, x)	Y	Y	

1. **Inverse property**: - Consider a non-empty set A and a binary operation \* on A. A is said to satisfy inverse property with respect to \*, if  $\forall a \in A$ , there must be unique element  $a^{-1} \in A$ , such that  $a * a^{-1} = a^{-1} * a = e$

2. Every element has exactly one unique inverse which is also present in the same set.

3. If a is the inverse of b, then b will be inverse of a.

4. No two elements can have the same inverse

5. Identity element is its own inverse.

6. **Group**: - A non-empty set A is said to be a group with respect to a binary operation \*, if A satisfies closure, Associative, identity, inverse property with respect to \*.

		AS	Semi Group	Monoid	Group
1	(N, +)	Y	Y	N	
2	(N, -)	N	N	N	
3	(N, /)	N	N	N	
4	(N, x)	Y	Y	Y	
5	(Z, +)	Y	Y	Y	
6	(Z, -)	Y	N	N	
7	(Z, /)	N	N	N	
8	(Z, x)	Y	Y	Y	
9	(R, +)	Y	Y	Y	
10	(R, -)	Y	N	N	
11	(R, /)	N	N	N	
12	(R, x)	Y	Y	Y	
13	(M, +)	Y	Y	Y	
14	(M, x)	Y	Y	Y	
15	(E, +)	Y	Y	Y	
16	(E, x)	Y	Y	N	
17	(O, +)	N	N	N	
18	(O, x)	Y	Y	Y	
19	(R-O, x)	Y	Y	Y	
20	(R-O, /)	Y	N	N	
21	(Non-Singular Matrix, x)	Y	Y	Y	

1. If the total number of elements in a group is even then there exists at least one element in the group who is the inverse of itself.
2. Some time it is also possible that every element is inverse of itself in a group.
3. In a group  $(a * b)^{-1} = b^{-1} * a^{-1}$  for  $\forall a, b \in A$
4. Cancelation law holds good
  1.  $a * b = a * c \rightarrow b = c$
  2.  $a * c = b * c \rightarrow a = b$

**1. Commutative property:** - Consider a non-empty set A and a binary operation \* on A. A is said to satisfy commutative property with respect to \*, if  $\forall a, b \in A$ , such that  $a * b = b * a$

**2. Abelian Group:** - A non-empty set A is said to be a group with respect to a binary operation \*, if A satisfy closure, Associative, identity, inverse, commutative property with respect to \*.

		AS	SG	Monoid	Group	Abelian Group
1	(N, +)	Y	Y	N	N	
2	(N, -)	N	N	N	N	
3	(N, /)	N	N	N	N	
4	(N, x)	Y	Y	Y	N	
5	(Z, +)	Y	Y	Y	Y	
6	(Z, -)	Y	N	N	N	
7	(Z, /)	N	N	N	N	
8	(Z, x)	Y	Y	Y	N	
9	(R, +)	Y	Y	Y	Y	
10	(R, -)	Y	N	N	N	
11	(R, /)	N	N	N	N	
12	(R, x)	Y	Y	Y	N	
13	(M, +)	Y	Y	Y	Y	
14	(M, x)	Y	Y	Y	N	
15	(E, +)	Y	Y	Y	Y	
16	(E, x)	Y	Y	N	N	
17	(O, +)	N	N	N	N	
18	(O, x)	Y	Y	Y	N	
19	(R-0, x)	Y	Y	Y	Y	
20	(R-0, /)	Y	N	N	N	
21	(Non-Singular Matrix, x)	Y	Y	Y	Y	

**Break**

**Q** let  $A = \{1, 3, 5, \dots, \infty\}$  and  $B = \{2, 4, 6, \dots, \infty\}$ , what is the highest structure achieved by anyone of them?

**1)**  $(A, +)$

**2)**  $(A, *)$

**3)**  $(B, +)$

**4)**  $(B, *)$

a) Algebraic Structure

b) Semi Group

c) Monoid

d) Group

**Q** Consider a set of natural numbers N, with respect to \*, such that  
 $a * b = a^b$  which of the following is true?

- a)** semi group but not monoid      **b)** A monoid but not a group
- c)** A group      **d)** not a semi group

**Q** let  $\{p, q, r, s\}$  be the set. A binary operation \* is defined on the set and is given by the following table:

*	p	q	r	s
p	p	r	s	p
q	p	q	r	s
r	p	q	p	r
s	p	q	q	q

Which of the following is true about the binary operation?

- a) it is commutative but not associative
- b) it is associative but not commutative
- c) it is both associative and commutative
- d) it is neither associative nor commutative

**Q** Consider a set of integers Z, with respect to \*, such that  
 $a * b = \max(a, b)$  which of the following is true?

- a) Algebraic structure**
- b) semi-group**
- c) Monoid**
- d) group**

**Q** Consider a set of integers Z, with respect to \*, such that  
 $a * b = \min(a, b)$  which of the following is true?

- a) Algebraic structure**
- b) semi-group**
- c) Monoid**
- d) group**

**Q** which of the following is not a group?

a)  $\{ \dots, -6, -4, -2, 0, 2, 4, 6, \dots \}, +$

b)  $\{ \dots, -3k, -2k, -k, 0, k, 2k, 3k, \dots \}, + [k \in \mathbb{Z}]$

c)  $\{2^n, n \in \mathbb{Z}\}, *$

d) set of complex number, \*

**Q** Consider the set of all integers( $Z$ ) with the operation defined as  $m * n = m + n + 2$ ,  $m, n \in Z$

if  $(Z, *)$  forms a group, then determine the identity element

a) 0

b) -1

c) -2

d) 2

**Q** Consider a set of positive rational number with respect to an operation  $*$ , such that  $a * b = (a \cdot b)/3$ , it is known that the it is an abelian group, which of the following is not true?

- a)** identity element  $e = 3$
- b)** inverse of  $a = 9/a$
- c)** inverse of  $2/3 = 6$
- d)** inverse of  $3 = 3$

**Break**

**Q** The binary operator  $\neq$  is defined by the following truth table (GATE-2015) (1 Marks)

Which one of the following is true about the binary operator  $\neq$ ?

(A) Both commutative and associative

(B) Commutative but not associative

(C) Not commutative but associative

(D) Neither commutative nor associative

p	q	$P \neq q$
0	0	0
0	1	1
1	0	1
1	1	0

**Q** Which of the following properties a Group G must hold, in order to be an Abelian group? **(Net Dec 2015)**

- (a)** The distributive property
  - (b)** The commutative property
  - (c)** The symmetric property
- a)** (a) and (b)      **b)** (b)  
**c)** (a) and (b)      **d)** (a), (b) and (c)

**Q** A binary operation  $\alpha$  on a set of integers is defined as  $x * y = x^2 + y^2$ . Which one of the following statements is TRUE about  $*$ ? **(GATE-2013)**

**(1 Marks)**

- (A)** Commutative but not associative
- (B)** Both commutative and associative
- (C)** Associative but not commutative
- (D)** Neither commutative nor associative

**Q** Which one of the following is NOT necessarily a property of a Group? (GATE-2009) (2 Marks)

**(A)** Commutativity

**(B)** Associativity

**(C)** Existence of inverse for every element

**(D)** Existence of identity

**Q** Consider the set H of all  $3 \times 3$  matrices of the type

$$\begin{matrix} a & f & e \\ 0 & b & d \\ 0 & 0 & c \end{matrix}$$

where a, b, c, d, e and f are real numbers and abc ≠ 0.

Under the matrix multiplication operation, the set H is **(GATE-2005) (2 Marks)**

- (A) a group
- (B) a monoid but not a group
- (C) a semigroup but not a monoid
- (D) neither a group nor a semigroup

**Q** Consider the set  $\Sigma^*$  of all strings over the alphabet  $\Sigma = \{0, 1\}$ .  $\Sigma^*$  with the concatenation operator for strings **(GATE-2003) (1 Marks)**

- (A)** does not form a group
- (B)** forms a non-commutative group
- (C)** does not have a right identity element
- (D)** forms a group if the empty string is removed from  $\Sigma^*$

**Q** Which of the following is true? (GATE-2002) (2 Marks)

- (A)** The set of all rational negative numbers forms a group under multiplication.
- (B)** The set of all non-singular matrices forms a group under multiplication.
- (C)** The set of all matrices forms a group under multiplication.
- (D)** Both (2) and (3) are true.

**Q Which of the following statements is FALSE? (GATE-1996) (1 Marks)**

- a) The set of rational numbers is an abelian group under addition
- b) The set of integers in an abelian group under addition
- c) The set of rational numbers form an abelian group under multiplication
- d) The set of real numbers excluding zero is an abelian group under multiplication

**Q** Let A be the set of all non-singular matrices over real number and let \* be the matrix multiplication operation. Then **(GATE-1994) (2 Marks)**

- a) A is closed under \* but  $\langle A, * \rangle$  is not a semigroup.
- b)  $\langle A, * \rangle$  is a semigroup but not a monoid.
- c)  $\langle A, * \rangle$  is a monoid but not a group.
- d)  $\langle A, * \rangle$  is a group but not an abelian group.

**Break**

- **Finite Group**: - A group with finite number of elements is called a finite group.
- **Order of group**: - Order of a group is denoted by  $O(G)$  = no of elements in  $G$ 
  - If there is only one element in the Group, it must be an identity element.

**Q** Check out which of the following is a finite group?

**1-  $\{0\}$ , +**

+	0
0	

**2-  $\{0\}$ , \***

*	0
0	

**3-  $\{1\}$ , +**

+	1
1	

**4-  $\{1\}$ , \***

*	1
1	

**5-  $\{0,1\}$ , +**

+	0	1
0		
1		

**6-  $\{0,1\}$ , \***

*	0	1
0		
1		

**7-  $\{-1, 0, 1\}$ , +**

+	-1	0	1
-1			
0			
+1			

**8-  $\{-1, 0, 1\}$ , \***

*	-1	0	1
-1			
0			
1			

**Q** Check out which of the following is a finite group?

**9-**  $\{-1, 1\}$ , +

**10-**  $\{-1, 1\}$ , \*

**11-**  $\{-2, -1, 0, 1, 2\}$ , +

+	-1	1
-1		
1		

*	-1	1
-1		
1		

+	-2	-1	0	1	2
-2					
-1					
0					
1					
2					

**Q** Check out which of the following is a finite group?

**12-**  $\{-2, -1, 0, 1, 2\}$ , \*

*	-2	-1	0	1	2
-2					
-1					
0					
1					
2					

**13-**  $\{1, \omega, \omega^2\}$ , \*

*	1	$\omega$	$\omega^2$
1			
$\omega$			
$\omega^2$			

**14-**  $\{-1, 1, i, -i\}$ , \*

*	-1	1	i	-i
-1				
1				
i				
-i				

1. **Conclusion:** - it is very difficult to design finite group as with number greater than 2 closure property fails with simple addition and multiplication operation.
2. So we will try to develop new modified addition and multiplication operators with which closure and other properties can be satisfied.

**Break**

- **Addition modulo**: - addition modulo is a binary operator denoted by  $+_m$  such that
- $a +_m b = a + b$  if  $(a + b < m)$
- $a +_m b = a + b - m$  if  $(a + b \geq m)$

$\{0,1,2,3\}$ ,  $+_4$

$+_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

- **Multiplication modulo**: - Multiplication modulo is a binary operator denoted by  $*_m$  such that
- $a *_m b = a * b$  if  $(a * b < m)$
- $a *_m b = (a * b) \% m$  if  $(a * b \geq m)$

$\{1,2,3,4\}, *_5$

$*_5$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

**Q** Check out which of the following is a group?

**1-  $\{0,1,2,3\}$ ,  $+_4$**

$+_4$	0	1	2	3
0				
1				
2				
3				

**2-  $\{0,1,2,3\}$ ,  $*_4$**

$*_4$	0	1	2	3
0				
1				
2				
3				

**3-  $\{1,2,3\}$ ,  $+_4$**

$+_4$	1	2	3
1			
2			
3			

**4-  $\{1,2,3\}$ ,  $*_4$**

$*_4$	1	2	3
1			
2			
3			

5-  $\{0,1,2,3,4\}$ ,  $+_5$

$+_5$	0	1	2	3	4
0					
1					
2					
3					
4					

6-  $\{0,1,2,3,4\}$ ,  $*_5$

$*_5$	0	1	2	3	4
0					
1					
2					
3					
4					

7-  $\{1,2,3,4\}$ ,  $+_5$

$+_5$	1	2	3	4
1				
2				
3				
4				

8-  $\{1,2,3,4\}$ ,  $*_5$

$*_5$	1	2	3	4
1				
2				
3				
4				

**9-  $\{0,1,2,3,4,5,6\}$ ,  $+$ <sub>7</sub>**

$+$ <sub>7</sub>	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

**10-  $\{0,1,2,3,4,5,6\}$ ,  $*$ <sub>7</sub>**

$*$ <sub>7</sub>	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	1	3	5	6
3	0	3	6	2	4	1	5
4	0	4	1	5	3	6	2
5	0	5	6	4	2	3	1
6	0	6	5	3	1	4	2

**11-  $\{1,2,3,4,5,6\}$ ,  $+$ <sub>7</sub>**

$+$ <sub>7</sub>	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	3	4	5	6	0
3	3	4	5	6	0	1
4	4	5	6	0	1	2
5	5	6	0	1	2	3
6	6	0	1	2	3	4

**12-  $\{1,2,3,4,5,6\}$ ,  $*$ <sub>7</sub>**

$*$ <sub>7</sub>	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	1	3	5	6
3	3	1	6	4	2	5
4	4	3	2	5	6	0
5	5	6	0	1	4	3
6	6	5	4	3	0	2

**13-  $\{1,3,5,7\}$ ,  $*_8$**

$*_8$	1	3	5	7
1				
3				
5				
7				

**14-  $\{1,2,4,7,8,11,13,14\}$ ,  $*_{15}$**

$*_{15}$	1	2	4	7	8	11	13	14
1								
2								
4								
7								
8								
11								
13								
14								

**15-**  $\{1, 2, 3, 4, \dots, p-1\}, *_p$

**16-**  $\{0, 1, 2, 3, 4, \dots, p-1\}, *_p$

**17-**  $\{1, 2, 3, 4, \dots, p-1\}, +_p$

**18-**  $\{0, 1, 2, 3, 4, \dots, p-1\}, +_p$

**Break**

**Q**  $\{0,1,2,3,4,5\}$ ,  $+_6$  is a group which of the following is not true?

- a)  $1^{-1} = 5$       b)  $2^{-1} = 4$       c)  $3^{-1} = 6$       d)  $0^{-1} = 0$

$+_6$	0	1	2	3	4	5
0						
1						
2						
3						
4						
5						

**Q**  $\{1,2,3,4,5,6\}$ ,  $*_7$  is a group which of the following is not true?

- a)  $1^{-1} = 1$       b)  $2^{-1} = 4$       c)  $3^{-1} = 5$       d)  $6^{-1} = 6$

$*_7$	1	2	3	4	5	6
1						
2						
3						
4						
5						
6						

**Q**  $\{1,3,5,7\}$ ,  $*_8$  is a group which of the following is not true?

a)  $1^{-1} = 1$

b)  $3^{-1} = 3$

c)  $5^{-1} = 5$

d)  $7^{-1} = 7$

$*_8$	1	3	5	7
1				
3				
5				
7				

**Q**  $\{1, 2, 4, 7, 8, 11, 13, 14\}$ ,  $*_{15}$  is a group which of the following is not true?

a)  $2^{-1} = 8$

b)  $4^{-1} = 4$

c)  $7^{-1} = 13$

d)  $11^{-1} = 14$

$*_{15}$	1	2	4	7	8	11	13	14
1								
2								
4								
7								
8								
11								
13								
14								

**Break**

**Q** The set  $\{1, 2, 3, 5, 7, 8, 9\}$  under multiplication modulo 10 is not a group. Given below are four plausible reasons. Which one of them is false? **(GATE-2006) (1 Marks)**

- (A) It is not closed
- (B) 2 does not have an inverse
- (C) 3 does not have an inverse
- (D) 8 does not have an inverse

$*$ 10	1	2	3	5	7	8	9
1							
2							
3							
5							
7							
8							
9							

**Q** The set  $\{1, 2, 4, 7, 8, 11, 13, 14\}$  is a group under multiplication modulo 15. The inverses of 4 and 7 are respectively (GATE-2005) (2 Marks)

- (A) 3 and 13      (B) 2 and 11      (C) 4 and 13      (D) 8 and 14

$*_{15}$	1	2	4	7	8	11	13	14
1								
2								
4								
7								
8								
11								
13								
14								

**Q** The following is the incomplete operation table a 4-element group.  
**(GATE-2004) (2 Marks)**

The last row of the table is

**(A) c a e b**

**(B) c b a e**

**(C) c b e a**

**(D) c e a b**

*	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b				
c				

**Q** Consider the binary operation  $\oplus$  over set  $Z_n = \{0, 1, 2, \dots, n-1\}$

$$a \oplus b = a + b \quad \text{if } (a + b < n)$$

$$a \oplus b = a + b - n \quad \text{if } (a + b \geq n)$$

- a) it is closed
- b) it does not form a group
- c) it forms a group but not an abelian group
- d) it is an abelian group

**Break**

## **Sub Group**

1. The subset of a group may or may not be a group.
2. When the subset of a group is also a group then it is called sub group.
3. The identity element of a group and its sub group is always same.
4. Union of two subgroup may or may not be a subgroup.
5. Intersection of two subgroup is always a subgroup.
6. Lagrange's theorem: - the order of a group is always exactly divisible by the order of a sub group.

**Q** consider a group  $G = \{1,3,5,7\}$ ,  $*_8$  which of the following sub set of this set does not form is sub group?

**a)**  $\{0,1\}$

**b)**  $\{1,3\}$

**c)**  $\{1,5\}$

**d)**  $\{1,7\}$

**e)**  $\{1,3,7\}$

$*_8$	0	1
0		
1		

$*_8$	1	3
1		
3		

$*_8$	1	5
1		
5		

$*_8$	1	7
1		
7		

$*_8$	1	3	7
1			
3			
7			

**Q** Let  $G$  be a group with 15 elements. Let  $L$  be a subgroup of  $G$ . It is known that  $L \neq G$  and that the size of  $L$  is at least 4. The size of  $L$  is \_\_\_\_\_.

- (A) 3      (B) 5      (C) 7      (D) 9**

**Q** Let  $G$  be a finite group on 84 elements. The size of a largest possible proper subgroup of  $G$  is \_\_\_\_\_. **(GATE-2014) (1 Marks)**

**Q** let  $(A, *)$  be a group of prime order, how many proper-subgroups are possible for A?

- a) 0**
- b) 1**
- c)  $P-1$**
- d)  $P$**

**Break**

**Order of an element:** -  $(A, *)$  be a group, then  $\forall a \in A$ , order of  $a$  is denoted by  $O(a)$ .

1.  $O(a) = n$  (smallest positive integer), such that  $a^n = e$
2. Order of identity element is always one.
3. Order of an element and its inverse is always same.
4. Order of an element in an infinite group does not exist or infinite except identity.

**Q** consider a group  $\{0,1,2,3\}$ ,  $+_4$  and find the order of each element?

$+_4$	0	1	2	3
0				
1				
2				
3				

**Q** consider a set on cube root of unity  $\{1, \omega, \omega^2\}$ , \* and find the order of each element?

*	1	$\omega$	$\omega^2$
1			
$\omega$			
$\omega^2$			

**Q** consider a set on forth root of unity  $\{-1, 1, i, -i\}$ \* and find the order of each element?

$$i = i \rightarrow \text{O}(i) = 1$$

$$(-i)^1 = -i$$

$$(-i)^2 = -1$$

$$(-i)^3 = i$$

$$(-i)^4 = 1$$

$$\text{O}(-i) = 4$$

*	-1	1	i	-i
-1				
1				
i				
-i				

$$(-1)^1 = -1$$

$$(-1)^2 = 1 \rightarrow \text{O}(-1) = 2$$

$$(i)^1 = i$$

$$(i)^2 = -1$$

$$(i)^3 = -i$$

$$(i)^4 = 1$$

$$\text{O}(i) = 4$$

**Break**

**Generating element or Generator:** - A element 'a' is said to be a generating element, if every element of A is an integral power of a, i.e. every element of A can be represented using power of a.

$$A = \{a^1, a^2, a^3, a^4, a^5, \dots\}$$

**Cyclic group:** - A group  $(A, *)$  is said to be a cyclic group if it contains at least one generator.

- ~~1. In a cyclic group if an element is a generator than its inverse will also be a generator.~~
- ~~2. The order of a cyclic group is always the order of the generating element of G.~~

Q consider a group  $\{1, 2, 4, 7, 8, 11, 13, 14\}$ ,  $*_{15}$  and find the order of each element?

$*_{15}$	1	2	4	7	8	11	13	14
1								
2								
4								
7								
8								
11								
13								
14								

$$\begin{array}{l} \text{Order of } 1 = 1 \\ \text{Order of } 2 = 8 \\ \text{Order of } 4 = 4 \\ \text{Order of } 7 = 8 \\ \text{Order of } 8 = 8 \\ \text{Order of } 11 = 1 \\ \text{Order of } 13 = 1 \\ \text{Order of } 14 = 1 \end{array}$$

$$u^8 = 16/15 = 1$$

$$O(u) = 8$$



$$u^8 = 1$$

$$O(1) = 1$$

**Q** For the composition table of a cyclic group shown below

$$\begin{aligned}d' &= d \\d' &= d * d = b \\d' &= b * d = c \\d' &= c * d = a\end{aligned}$$

*	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	b	a
d	d	c	a	b

$$\begin{aligned}c' &= c \\c^2 &= c * c = b \\c^3 &= b * c = d \\c^4 &= d * c = \underline{a}\end{aligned}$$

Which one of the following choices is correct?

**(GATE-2009) (2 Marks)**

~~(A) a, b are generators~~ - 18

~~(B) b, c are generators~~ - 9

~~(C) c, d are generators~~ - 71

~~(D) d, a are generators~~ - 3

**Break**

## Number of generators

Lagrange's theorem: - let  $A$  be a cyclic group of order  $n$ , number of Generator in  $A$  is denoted by  $\phi(n) = \frac{n(p_1-1)(p_2-1)(p_3-1)\dots\dots(p_k-1)}{(p_1p_2p_3\dots\dots p_k)}$

$$\phi(8) = \frac{8(2-1)^1}{2} = 4$$

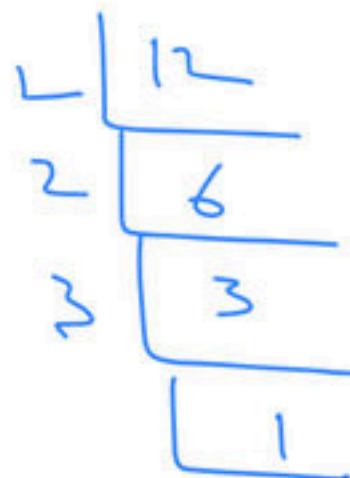
Q let G be a cyclic group,  $O(G) = 8$ , number of generators in G =?

$$\begin{array}{r} 2 \mid 8 \\ 2 \mid 4 \\ 2 \mid 2 \\ \hline 1 \end{array}$$

Q let G be a cyclic group,  $O(G) = 12$ , number of generators in G =?

$$\phi(12) = \frac{n(\kappa_1 - 1)(\kappa_2 - 1)}{\kappa_1 \cdot \kappa_2}$$

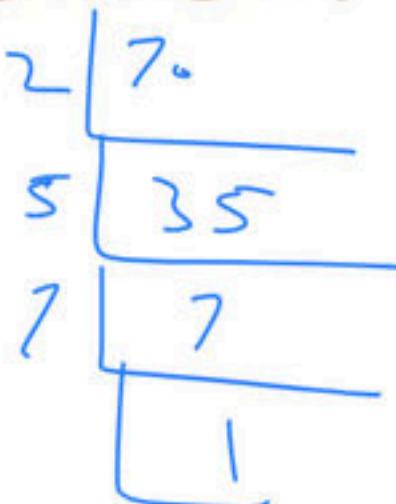
$$= \cancel{2} \times (\cancel{2-1}) \cancel{(3-1)} = \textcircled{6}$$



Q let G be a cyclic group,  $O(G) = 70$ , number of generators in G =?

$$= \cancel{70} \times (\cancel{2}) (\cancel{5}) (\cancel{7})$$

~~2 × 5 × 7~~

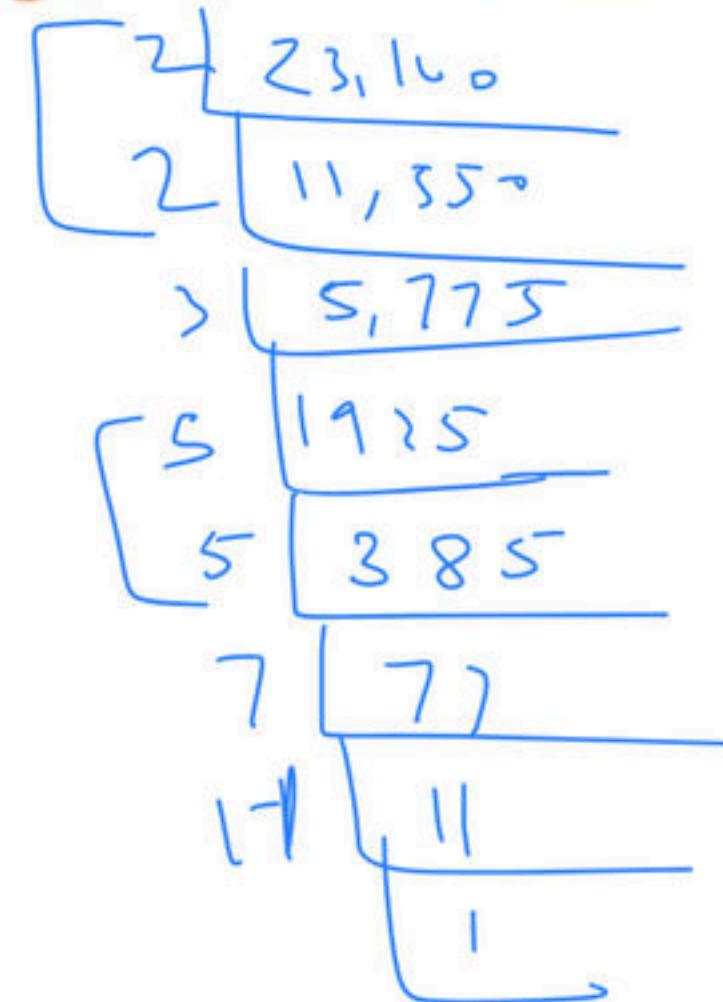


$$= \underline{\underline{2}} \underline{\underline{7}}$$

Q let G be a cyclic group,  $O(G) = 23,100$ , number of generators in G =?

$$= \frac{23,100 \times (2-1)(3-1)(5-1)(7-1)(11-1)}{2 \times 3 \times 5 \times 7 \times 11}$$

$$= \underline{\underline{9800}}$$



**Break**

**Q** Let  $S$  = set of all integers. A binary operation  $*$  is defined by

$$a * b = a + b + 3$$

consider the following statements

**S<sub>1</sub>:**  $(S, *)$  is a group

**S<sub>2</sub>:** -3 is identity element of  $(S, *)$

**S<sub>3</sub>:** the inverse of -6 is 0

which of the following are true

a) Only  $S_1$  and  $S_2$  —  $\cancel{43}$

b) Only  $S_2$  and  $S_3$  —  $\cancel{16}$

c) Only  $S_1$  and  $S_3$  —  $\cancel{7}$

d) Only  $S_1, S_2$  and  $S_3$  —  $\cancel{40}$

$$a * e = a \quad \text{--- } \textcircled{1}$$

$$a * e = a + e + 3 \quad \text{--- } \textcircled{2}$$

$$a = a + e + 3$$

$$e = -3$$

$$-6 * 0 = -6 + 0 + 3$$

$$= -3$$

$$a^{-1} = -3 - 3 - a$$

$$\boxed{a^{-1} = -a - 6}$$

$$a * a^{-1} = e \quad \text{--- } \textcircled{1}$$

$$a * a^{-1} = a + a^{-1} + 3 \quad \text{--- } \textcircled{2}$$

$$e = a + a^{-1} + 3$$

$$a^{-1} = e - 3 - a$$

Q  $(D_{12}, *)$  where  $a^*b = \text{g.c.d of } (a, b)$   $\forall a, b \in D_{12}$  then  $(D_{12}, *)$  is

a) a semigroup but not monoid - 14  
~~b) a monoid but not a group~~  
 c) a group - 19  
 d) not a semi group - 8

Gcd(a, b)	1	2	3	4	6	12
1	1	1	1	1	1	1
2	1	2	1	2	2	2
3	1	1	3	1	3	3
4	1	2	1	4	2	4
6	1	2	3	2	6	6
12	1	2	3	4	6	12

$$D_{12} = \{1, 2, 3, 4, 6, 12\}$$

**Q** In a group  $(G, *)$  if  $a * a = a$ , then proof that  $a = e$ , where  $e$  is identity element of  $a$ ?

$$\cancel{a * a = a} \longrightarrow \textcircled{1}$$

$$a * e = a \longrightarrow \textcircled{2}$$

$$\cancel{a * a =} \quad \cancel{a * e}$$

$$a = e$$

**Q** In a group if  $\underline{x}' = \underline{x}$  for  $\forall \underline{x} \in G$  in  $G$ , then  $G$  is an abelian group?

$$(a * b)^{-1} = b^{-1} * a^{-1} \quad ?$$

$\downarrow \qquad \downarrow \qquad \downarrow$

$a * b = b * a$

$\swarrow \qquad \searrow$

$$a \in G$$
$$\downarrow \in G$$

$$\underline{a * b} \in G \quad ?$$

$$(a * b)^{-1} = (a * b)$$

$$a = a^{-1}$$

$$b = b^{-1}$$

Q In a group  $(G, *)$ , if  $(a * b)^2 = a^2 * b^2$ , then prove that G is an abelian group?

$$(a * b)^2 = a^2 * b^2$$

$$(a * b) * (a * b) = (a * a) * (b * b)$$

$$\cancel{a} \cancel{*} b * a \cancel{*} \cancel{a} = \cancel{a} \cancel{*} a * b \cancel{*} \cancel{a}$$

$$b * a = a * b$$

$$a * b = b * a$$

Hence Proved

**Break**

**Q** Let  $G$  be an arbitrary group. Consider the following relations on  $G$ :

- $R_1: \forall a, b \in G, aR_1b \text{ if and only if } \exists g \in G \text{ such that } a = g^{-1}bg$
- $R_2: \forall a, b \in G, aR_2b \text{ if and only if } a = b^{-1}$

Which of the above is/are equivalence relation/relations? **(GATE-2019) (2 Marks)**

- (A)**  $R_1$  and  $R_2$       **(B)**  $R_1$  only      **(C)**  $R_2$  only      **(D)** Neither  $R_1$  nor  $R_2$

**Q** There are two elements  $x, y$  in a group  $(G, *)$  such that every element in the group can be written as a product of some number of  $x$ 's and  $y$ 's in some order. It is known that  $x * x = y * y = x * y * x * y = y * x * y * x = e$  where  $e$  is the identity element. The maximum number of elements in such a group is y. (GATE-2014) (2 Marks)

Ans: 3

b) 2 ← 55

( ) 9 ← 19

a) none of

6 ~

$$\{x, y, \frac{x * y}{y * x}, e\}$$

$$x * x = e$$

$$y * y = e$$

$$\cancel{(x * y)} * \cancel{(x * y)} = e$$

$$x \cancel{(x * y)} * (x * y) = \cancel{(y * x)} * (y * x)$$

$$\cancel{(y * x)} * \cancel{(y * x)} = e$$

$$\underline{y * x} = \underline{x * y}$$

$$\boxed{(x * y) * (y * x) = (x * y) * \cancel{(x * y)}}$$

$$\cancel{x * x * y} * \cancel{y} = \cancel{x * y} * \cancel{x * y}$$

$$\cancel{x * y} = \cancel{y * x}$$

**Q** Consider the set  $S = \{1, \omega, \omega^2\}$ , where  $\omega$  and  $\omega^2$  are cube roots of unity. If \* denotes the multiplication operation, the structure  $(S, *)$  forms **(GATE-2010) (1 Marks)**

- (A)** A group      **(B)** A ring      **(C)** An integral domain      **(D)** A field

A **ring** is a set  $R$  equipped with two binary operations<sup>[a]</sup>  $+$  (addition) and  $\cdot$  (multiplication) satisfying the following three sets of axioms, called the **ring axioms**<sup>[b]</sup>:

1.  $R$  is an **abelian group** under addition, meaning that:

- $(a + b) + c = a + (b + c)$  for all  $a, b, c$  in  $R$  (that is,  $+$  is associative).
- $a + b = b + a$  for all  $a, b$  in  $R$  (that is,  $+$  is commutative).
- There is an element  $0$  in  $R$  such that  $a + 0 = a$  for all  $a$  in  $R$  (that is,  $0$  is the additive identity).
- For each  $a$  in  $R$  there exists  $-a$  in  $R$  such that  $a + (-a) = 0$  (that is,  $-a$  is the additive inverse of  $a$ ).

2.  $R$  is a **monoid** under multiplication, meaning that:

- $(a \cdot b) \cdot c = a \cdot (b \cdot c)$  for all  $a, b, c$  in  $R$  (that is,  $\cdot$  is associative).
- There is an element  $1$  in  $R$  such that  $a \cdot 1 = a$  and  $1 \cdot a = a$  for all  $a$  in  $R$  (that is,  $1$  is the multiplicative identity).<sup>[b]</sup>

3. Multiplication is **distributive** with respect to addition, meaning that:

- $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$  for all  $a, b, c$  in  $R$  (left distributivity).
- $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$  for all  $a, b, c$  in  $R$  (right distributivity).

An [integral domain](#) is a nonzero commutative ring in which the product of any two nonzero elements is nonzero. Equivalently:

- An integral domain is a nonzero commutative ring with no nonzero zero divisors.
- An integral domain is a commutative ring in which the zero ideal  $\{0\}$  is a prime ideal.
- An integral domain is a nonzero commutative ring for which every non-zero element is cancellable under multiplication.
- An integral domain is a ring for which the set of nonzero elements is a commutative monoid under multiplication (because a monoid must be closed under multiplication).
- An integral domain is a nonzero commutative ring in which for every nonzero element  $r$ , the function that maps each element  $x$  of the ring to the product  $xr$  is injective. Elements  $r$  with this property are called *regular*, so it is equivalent to require that every nonzero element of the ring be regular.
- An integral domain is a ring that is isomorphic to a subring of a field. (Given an integral domain, one can embed it in its field of fractions.)

Informally, a field is a set, along with two operations defined on that set: an addition operation written as  $a + b$ , and a multiplication operation written as  $a \cdot b$ , both of which behave similarly as they behave for rational numbers and real numbers, including the existence of an additive inverse  $-a$  for all elements  $a$ , and of a multiplicative inverse  $b^{-1}$  for every nonzero element  $b$ . This allows one to also consider the so-called inverse operations of subtraction,  $a - b$ , and division,  $a / b$ , by defining:

$$a - b = a + (-b),$$

$$a / b = a \cdot b^{-1}.$$

## Classic definition [edit]

Formally, a field is a set  $F$  together with two binary operations on  $F$  called addition and multiplication.<sup>[1]</sup> A binary operation on  $F$  is a mapping  $F \times F \rightarrow F$ , that is, a correspondence that associates with each ordered pair of elements of  $F$  a uniquely determined element of  $F$ .<sup>[2][3]</sup> The result of the addition of  $a$  and  $b$  is called the sum of  $a$  and  $b$ , and is denoted  $a + b$ . Similarly, the result of the multiplication of  $a$  and  $b$  is called the product of  $a$  and  $b$ , and is denoted  $ab$  or  $a \cdot b$ . These operations are required to satisfy the following properties, referred to as field axioms (in these axioms,  $a$ ,  $b$ , and  $c$  are arbitrary elements of the field  $F$ ):

- **Associativity of addition and multiplication:**  $a + (b + c) = (a + b) + c$ , and  $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ .
- **Commutativity of addition and multiplication:**  $a + b = b + a$ , and  $a \cdot b = b \cdot a$ .
- **Additive and multiplicative identity:** there exist two different elements  $0$  and  $1$  in  $F$  such that  $a + 0 = a$  and  $a \cdot 1 = a$ .
- **Additive inverses:** for every  $a$  in  $F$ , there exists an element in  $F$ , denoted  $-a$ , called the additive inverse of  $a$ , such that  $a + (-a) = 0$ .
- **Multiplicative inverses:** for every  $a \neq 0$  in  $F$ , there exists an element in  $F$ , denoted by  $a^{-1}$  or  $1/a$ , called the multiplicative inverse of  $a$ , such that  $a \cdot a^{-1} = 1$ .
- **Distributivity of multiplication over addition:**  $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ .

This may be summarized by saying: a field has two operations, called addition and multiplication; it is an abelian group under addition with  $0$  as the additive identity; the nonzero elements are an abelian group under multiplication with  $1$  as the multiplicative identity; and multiplication distributes over addition.

Even more summarized: a field is a commutative ring where  $0 \neq 1$  and all nonzero elements are invertible.

## Alternative definition [edit]

Fields can also be defined in different, but equivalent ways. One can alternatively define a field by four binary operations (addition, subtraction, multiplication, and division) and their required properties. Division by zero is, by definition, excluded.<sup>[4]</sup> In order to avoid existential quantifiers, fields can be defined by two binary operations (addition and multiplication), two unary operations (yielding the additive and multiplicative inverses respectively), and two nullary operations (the constants  $0$  and  $1$ ). These operations are then subject to the conditions above. Avoiding existential quantifiers is important in constructive mathematics and computing.<sup>[5]</sup> One may equivalently define a field by the same two binary operations, one unary operation (the multiplicative inverse), and two constants  $1$  and  $-1$ , since  $0 = 1 + (-1)$  and  $-a = (-1)a$ .<sup>[nb 1]</sup>

**Q** Which one of the following is false? (GATE-1996) (2 Marks)

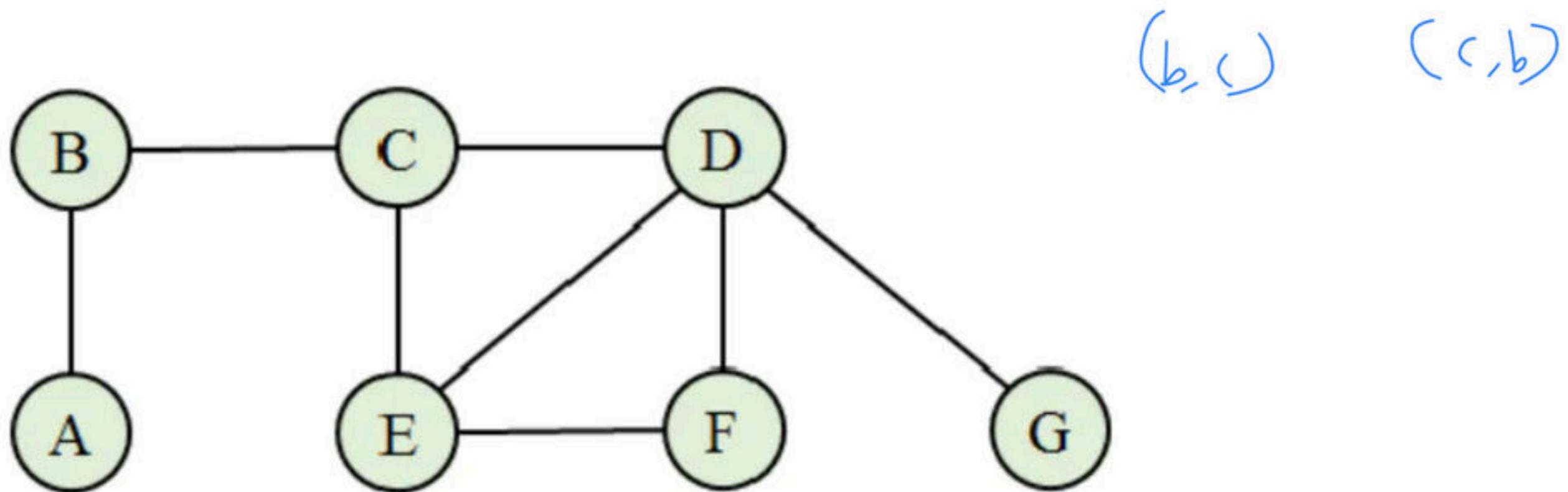
- (A) The set of all bijective functions on a finite set forms a group under function composition.
- (B) The set  $\{1, 2, \dots, p-1\}$  forms a group under multiplication mod  $p$  where  $p$  is a prime number
- (C) The set of all strings over a finite alphabet  $\Sigma$  forms a group under concatenation
- (D) A subset  $S \neq \emptyset$  of  $G$  is a subgroup of the group if and only if for any pair of element  $a, b \in S$ ,  $a * b^{-1} \in S$

**Q** Some group  $(G, o)$  is known to be abelian. Then, which one of the following is true for  $G$ ? **(GATE-1994) (2 Marks)**

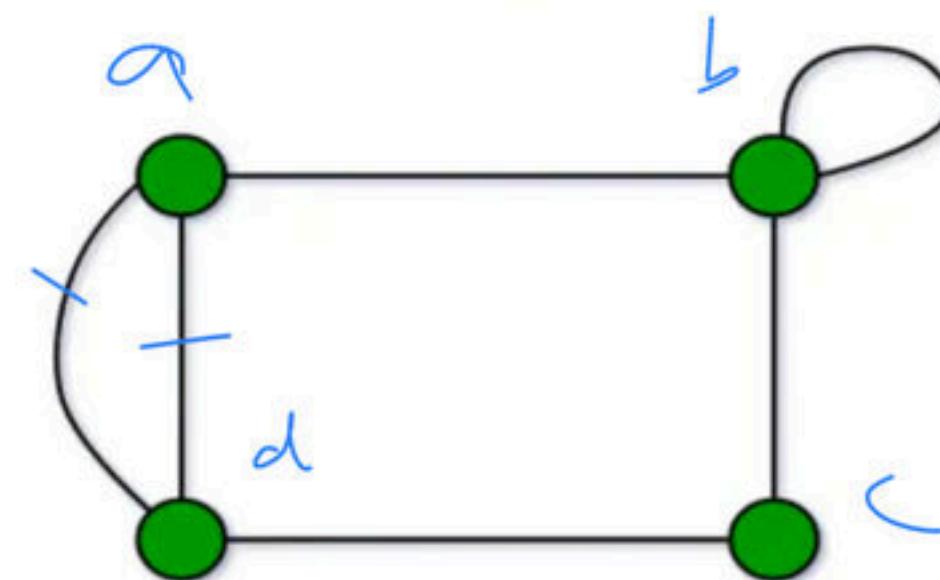
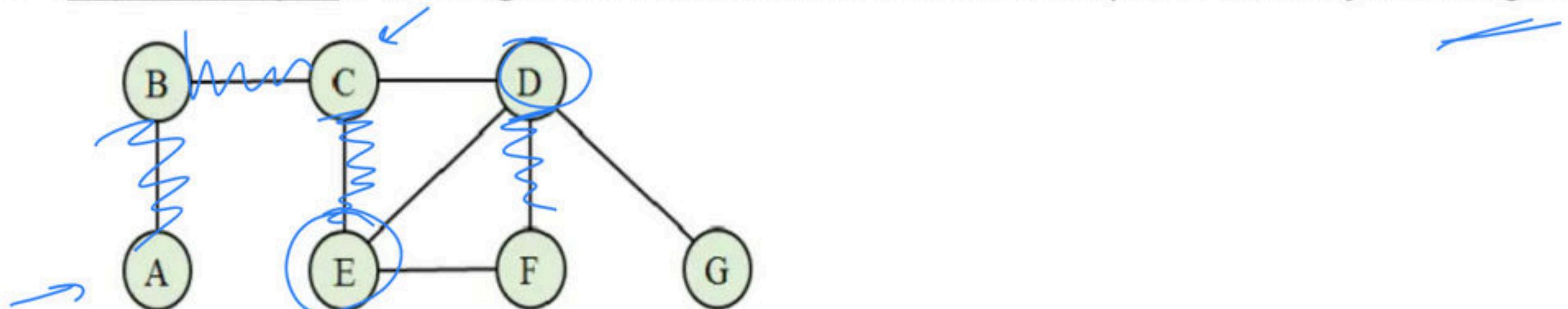
- A)  $g = g^{-1}$  for every  $g \in G$
- B)  $(goh)^2 = g^2oh^2$  for every  $g, h \in G$
- C)  $g = g^2$  for every  $g \in G$
- D)  $G$  is of finite order

## Graph Theory

1. A graph  $G(V, E)$  consists of a set of objects  $V = \{V_1, V_2, V_3, \dots, V_N\}$  called vertices and another set  $E = \{E_1, E_2, E_3, \dots, E_n\}$  whose elements are called edges.
2. Each edge  $e_k$  is identified with an unordered pair  $(v_i, v_j)$  of vertices.
3. The vertices  $v_i, v_j$  associated with edge  $e_k$  are called the end vertices of  $e_k$ .

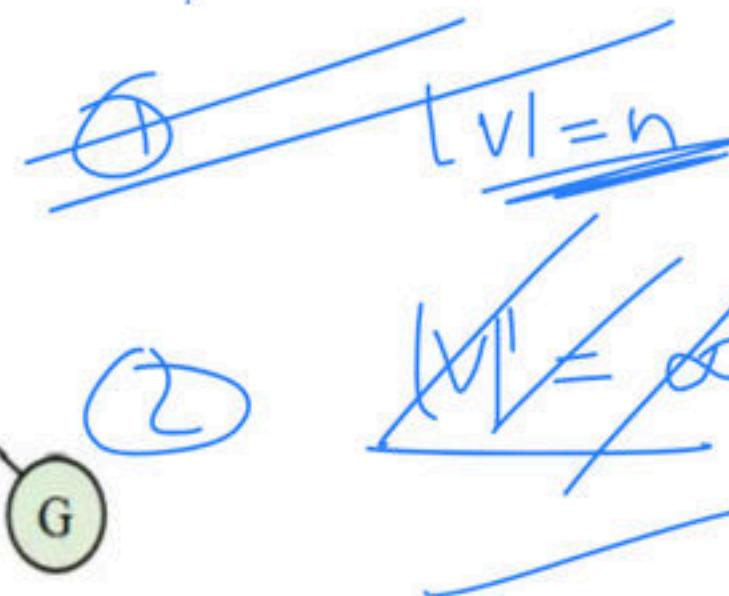
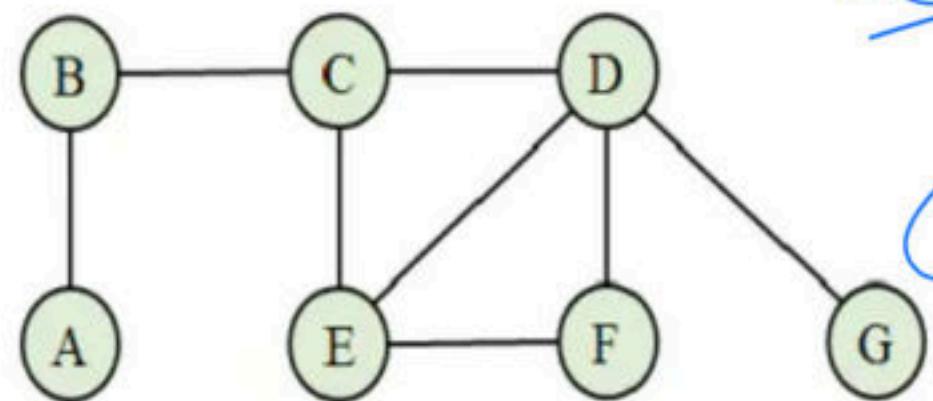


1. **Self-Loop**: Edge having the same vertex ( $v_i, v_i$ ) as both its end vertices is called self-loop.
2. **Parallel Edge**: When more than one edge associated with a given pair of vertices such edges are referred as parallel edges.
3. **Adjacent Vertices**: If two vertices are joined by the same edges, they are called adjacent vertices.
4. **Adjacent Edges**: If two edges are incident on some vertex, they are called adjacent edges.

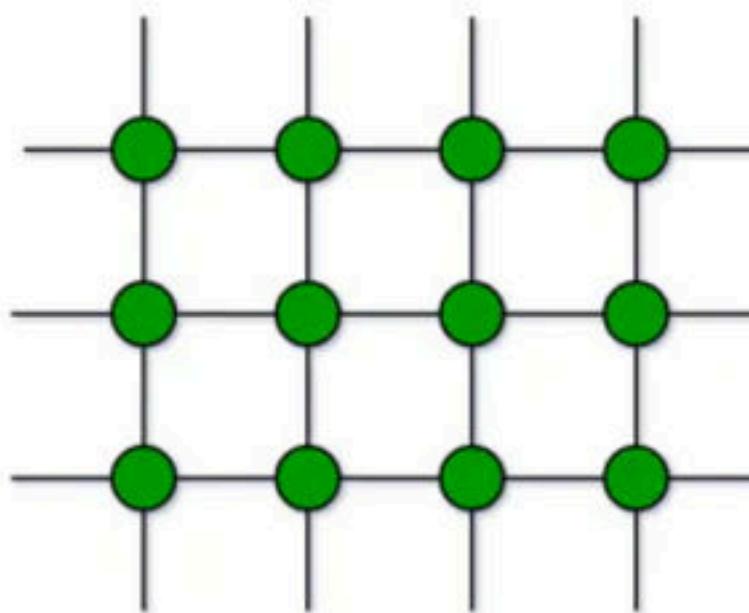
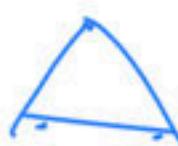


	<b>Self-Loop</b>	<b>Parallel Edge</b>
<b>Simple Graph</b>	No	No
<b>Multi/Pseudo Graph</b>	Yes	Yes

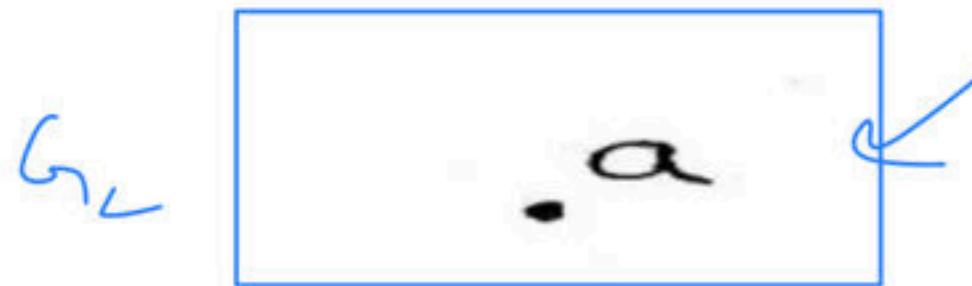
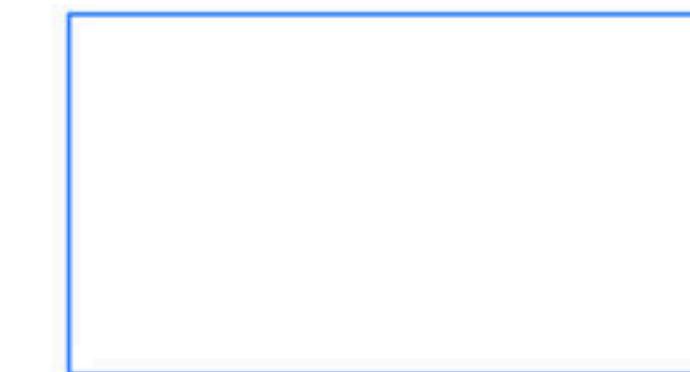
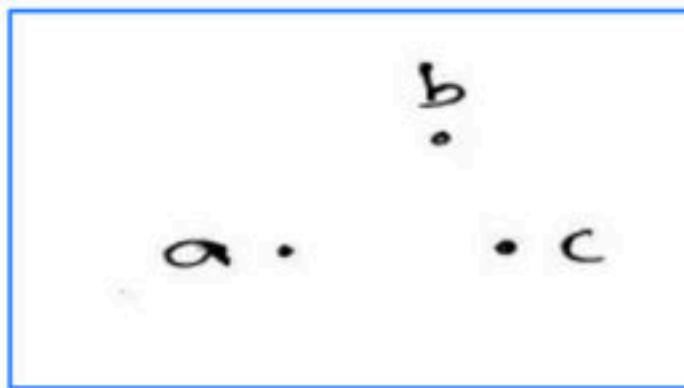
1. **Finite graph:** - A graph with finite number of vertices as well as the finite number of edges is called a finite graph.
2. For simple graph we can say if the number of vertices are finite then number of edges will also be finite.



$$\rightarrow |E| = \cancel{n} \quad |E| = n$$

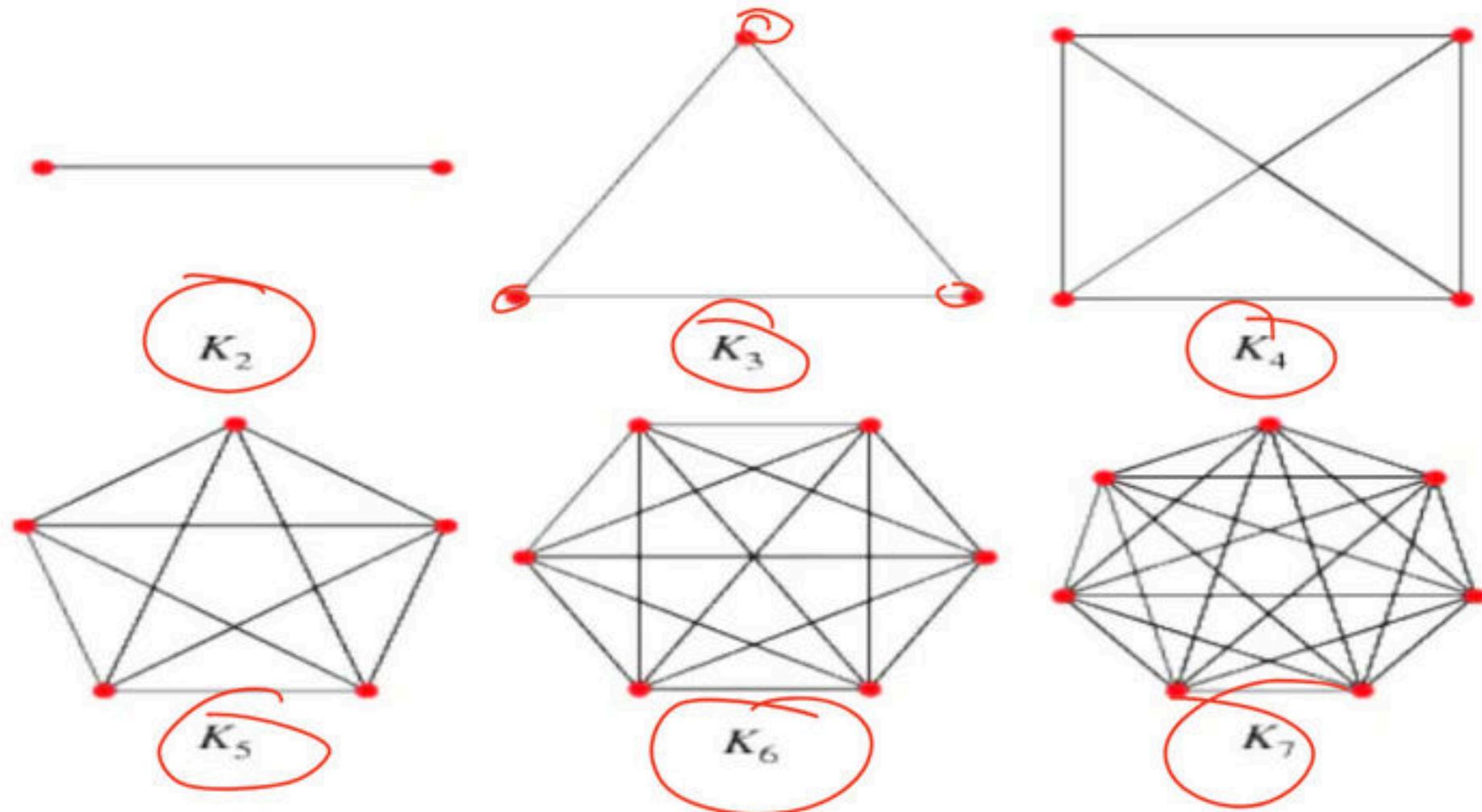


1. **Null Graph:** A graph is said to be null if edge set is empty  $E = \{\}$ , that is a graph with only vertices but no edges.
2. **Trivial Graph:** A graph with only one vertex without an edge is called trivial graph. It is the smallest possible.

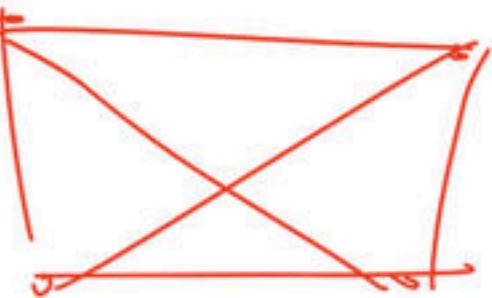


$$A = \{ \quad \}$$

- **Complete or Full Graph**: In a simple graph there exist an edge between each and every pair of vertices i.e. every vertex are adjacent to each other, then the graph is said to be a complete graph, denoted by  $K_n$ .



$$|V| = n \quad |E| = \frac{n(n-1)}{2}$$



1. A simple graph with maximum number of edges are called Complete Graph.
2. Number of edges in a simple graph is  $n(n-1)/2$

**Q1** The number of edges in a complete graph with N vertices is equal to: **(NET-DEC-2006) (NET-DEC-2007)**

- a)  $N(N - 1)$
- b)  $N(N-1)]/2$
- c)  $N^2$
- d)  $2N-1$

**Q2** The number of edges in a complete graph of n vertices is **(NET-DEC-2009)**

- (A)  $n$
- (B)  $n(n - 1)/2$
- (C)  $n(n + 1)/2$
- (D)  $n^2/2$

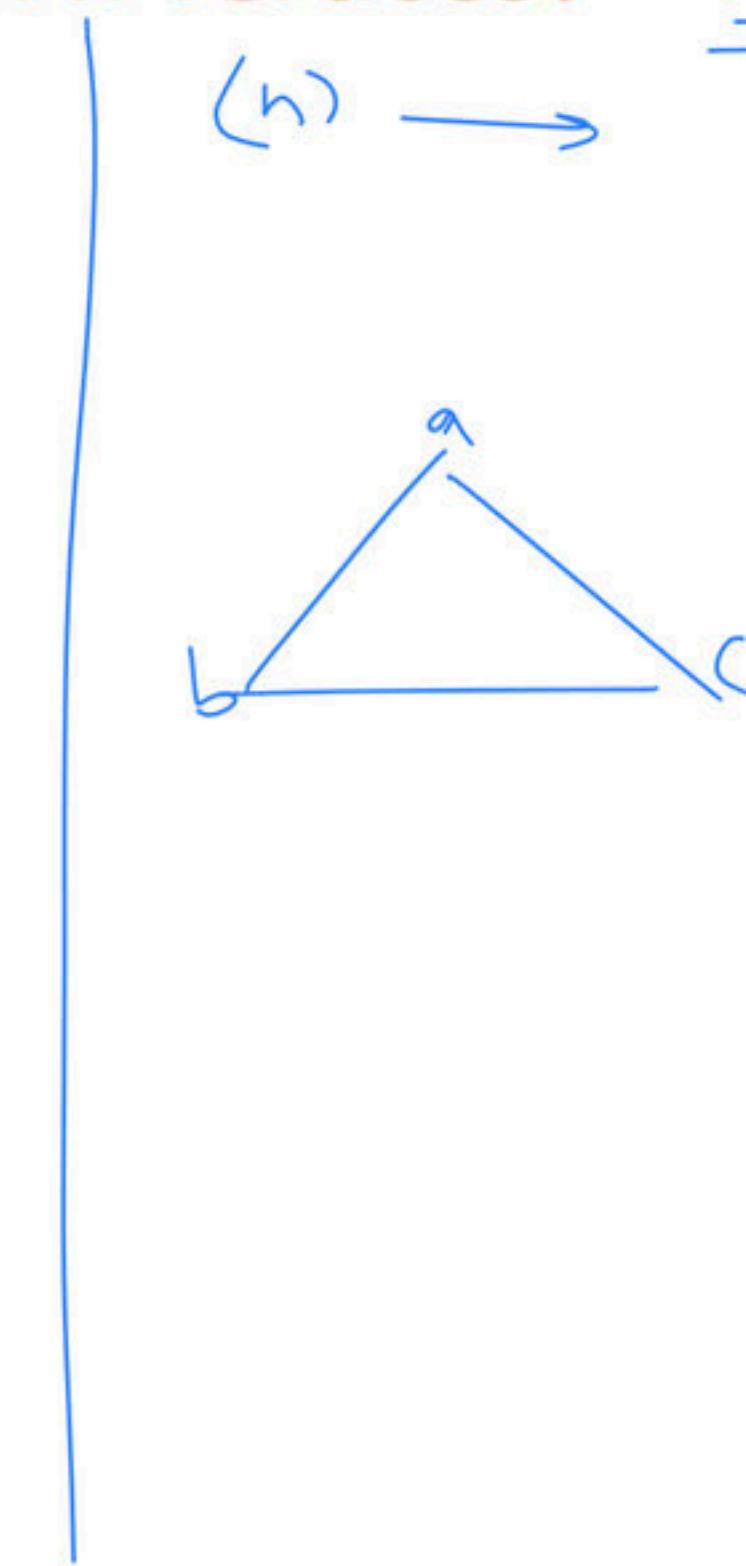
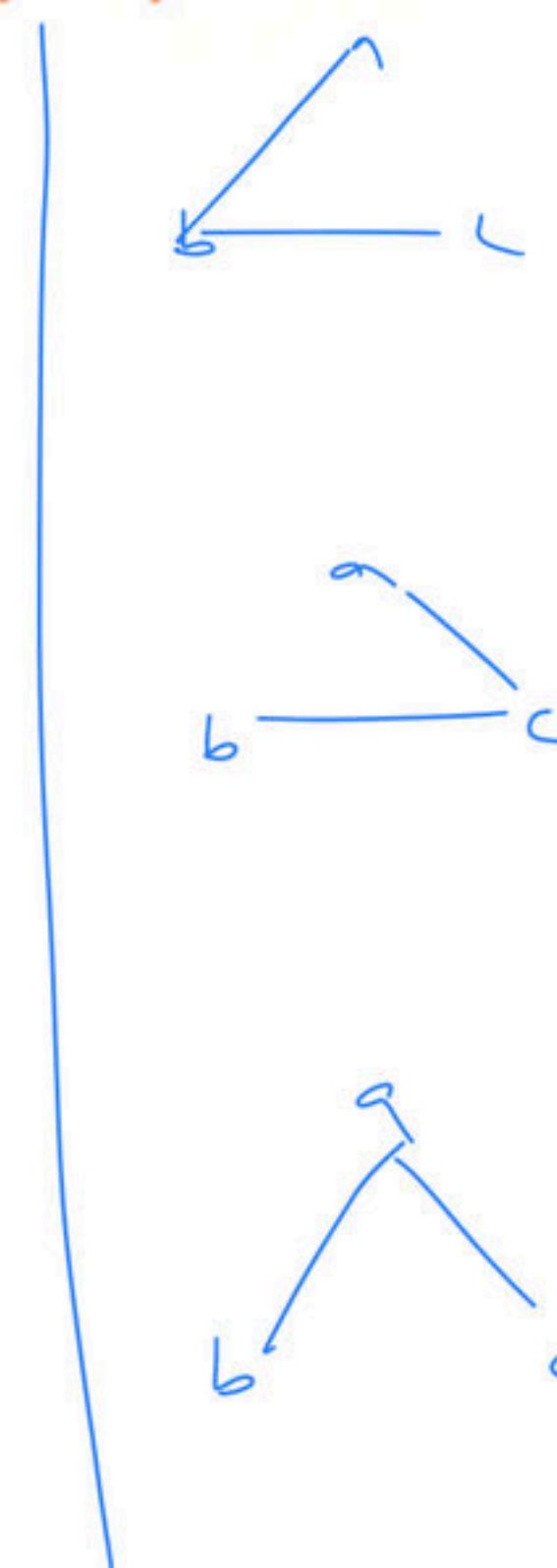
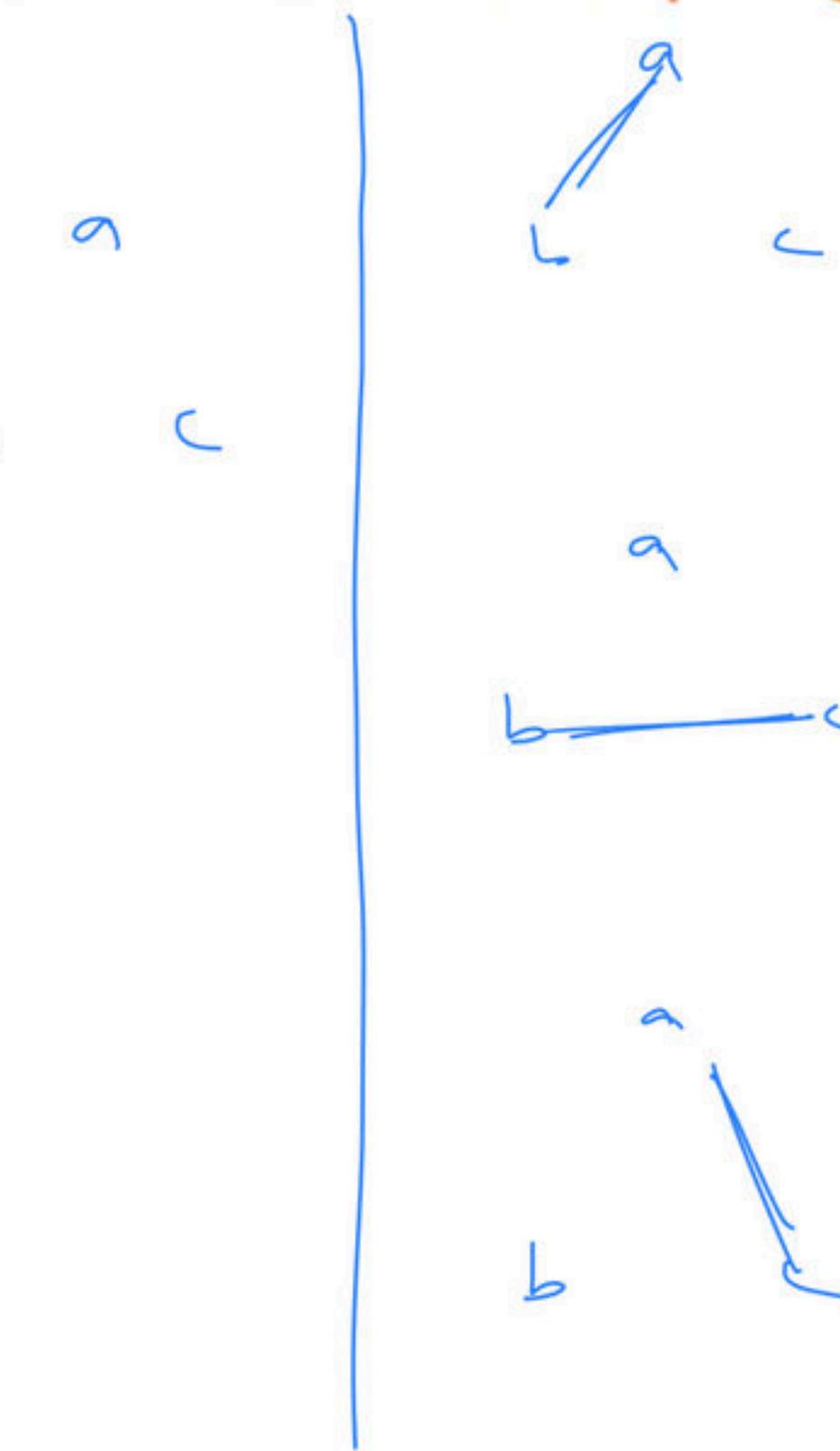
**Q3** The complete graph with four vertices has k edges where k is: **(NET-JUNE-2009)**

- a) 3
- b) 4
- c) 5
- d) 6

~~Q Maximum number of edges in a n node undirected graph without self-loops is (GATE-2002) (1 Marks) (NET-DEC-2011)~~

- (A)  $n^2$       (B)  $n(n - 1)/2$       (C)  $n - 1$       (D)  $(n + 1)(n)/2$

Q Number of simple graph possible with n vertices?  $|n|=3$



$$|V|=3 \rightarrow \frac{3(3-1)}{2} = 6 = 2^3 = 8$$

$$|V|=n \rightarrow \frac{n(n-1)}{2} \rightarrow \boxed{2^{\frac{n(n-1)}{2}}} \quad K$$

$$|V|=4 \rightarrow \frac{4(4-1)}{2} = 6 = 2^3 = 64$$

**Q** Number of simple graph possible with n vertices and e edges?

**Q** The number of distinct simple graphs with up to three nodes is (GATE-1994) (1 Marks)

- a) 15
- b) 11
- c) 8
- d) 9

**Q** How many undirected graphs (not necessarily connected) can be constructed out of a given set  $V = \{v_1, v_2, \dots, v_n\}$  of  $n$  vertices? **(GATE-2001) (2 Marks)**

- (A)**  $n(n-1)/2$       **(B)**  $2^n$       **(C)**  $n!$       **(D)**  $2^{n(n-1)/2}$

**Q** Let G be a complete undirected graph on 6 vertices. If vertices of G are labeled, then the number of distinct cycles of length 4 in G is equal to **(GATE-2012) (2 Marks)**

- (A) 15**      **(B) 30**      **(C) 90**      **(D) 360**

**Q** Consider an undirected graph G where self-loops are not allowed. The vertex set of G is  $\{(i, j) : 1 \leq i \leq 12, 1 \leq j \leq 12\}$ . There is an edge between  $(a, b)$  and  $(c, d)$  if  $|a - c| \leq 1$  and  $|b - d| \leq 1$ . **(GATE-2014) (2 Marks) (NET-AUG-2016)**  
The number of edges in this graph is \_\_\_\_\_.

**Q** Consider an undirected random graph of eight vertices. The probability that there is an edge between a pair of vertices is  $1/2$ . What is the expected number of unordered cycles of length three? **(GATE-2013) (1 Marks)**

**(A)**  $1/8$

**(B)** 1

**(C)** 7

**(D)** 8

**Q** How many graphs on  $n$  labeled vertices exist which have at least  $(n^2 - 3n)/2$  edges? **(GATE-2004) (2 Marks)**

a)  $\binom{(n^2 - n)/2}{(n^2 - 3n)/2} C_{((n^2 - n)/2)}$

b)  $\sum_{K=0}^{((n^2 - 3n)/2)} \binom{((n^2 - n)/2)}{K} C_K$

c)  $\binom{(n^2 - n)/2}{n} C_n$

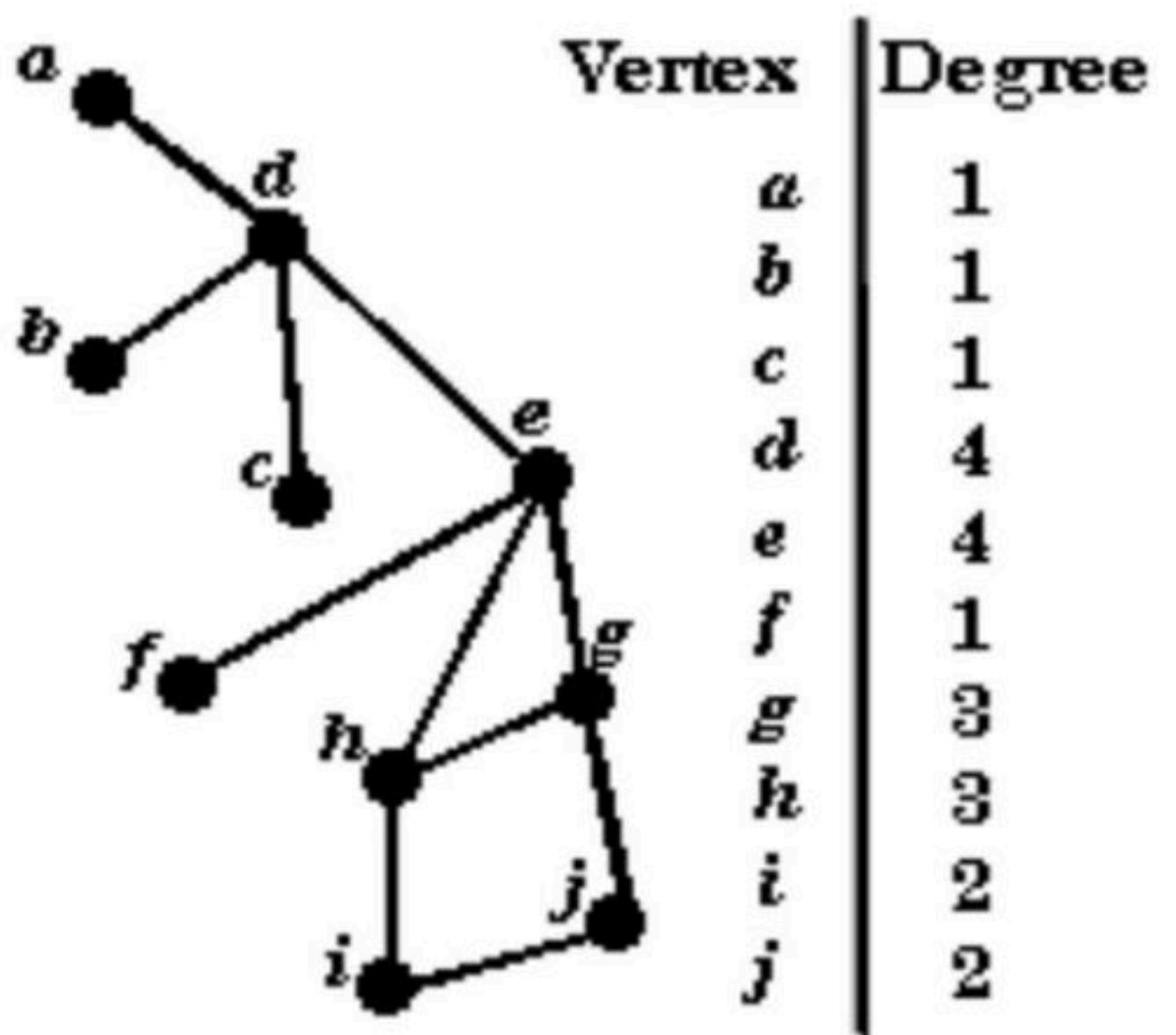
d)  $\sum_{K=0}^n \binom{((n^2 - n)/2)}{K} C_K$

**Q** The  $2^n$  vertices of a graph G corresponds to all subsets of a set of size n, for  $n \geq 6$ . Two vertices of G are adjacent if and only if the corresponding sets intersect in exactly two elements. The number of vertices of degree zero in G is **(GATE-2006) (2 Marks)**

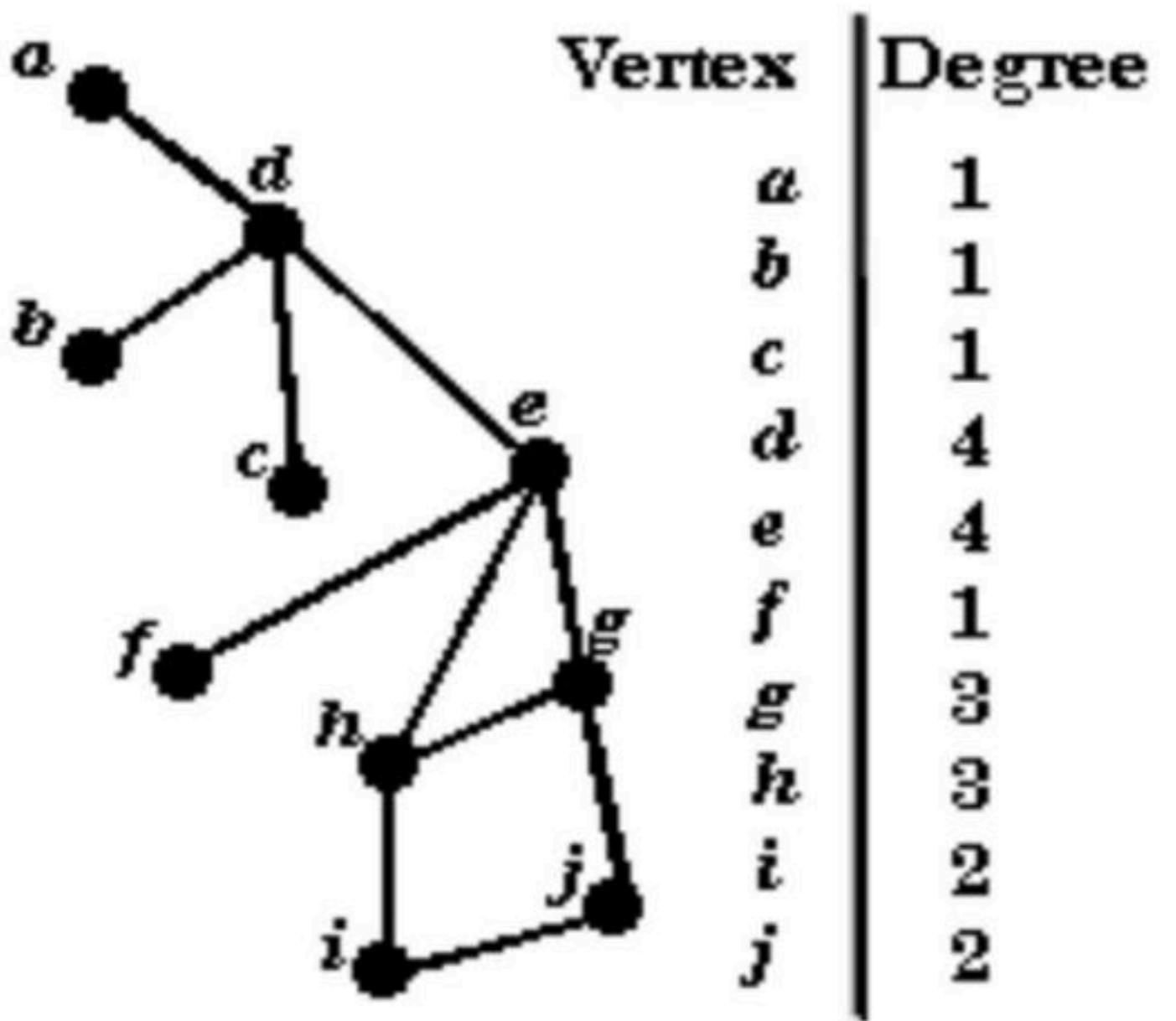
- (A) 1
- (B) n
- (C)  $n+1$
- (D)  $2n$

## Degree

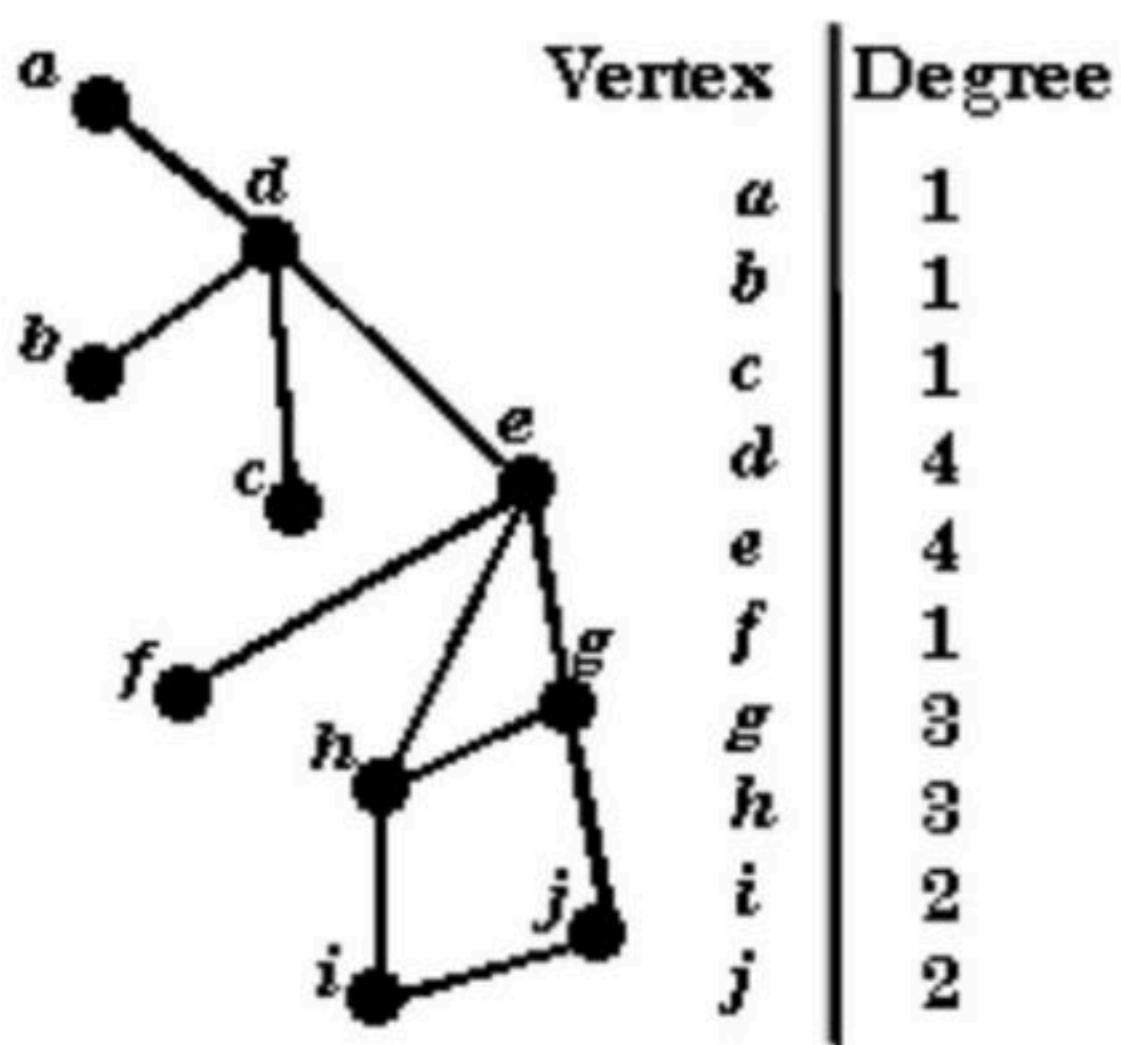
- **Degree of a Vertex:** The degree of a vertex in an undirected graph is the number of edges associated with it, denoted by  $\deg(v_i)$ .



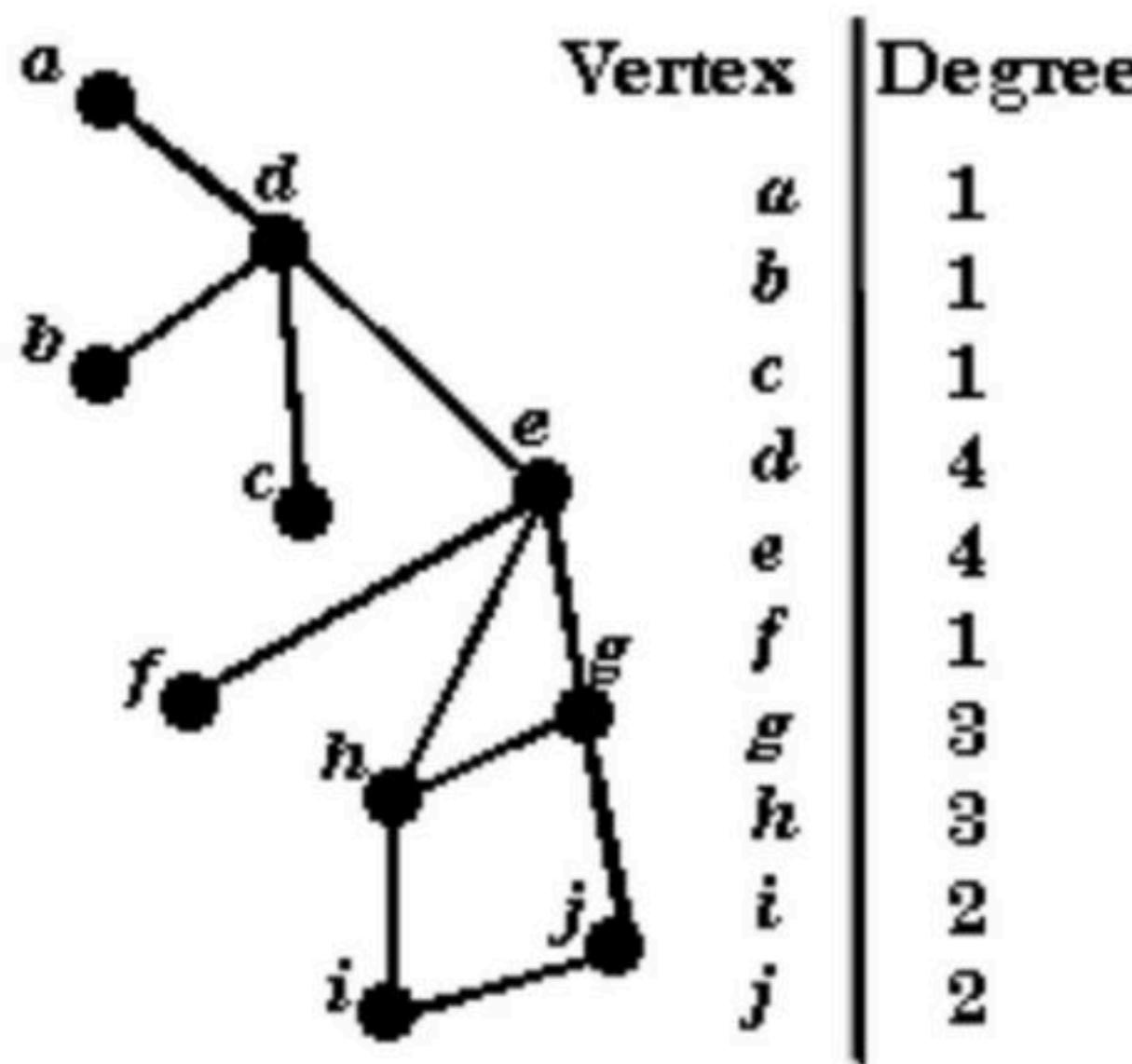
- **Isolated vertex**: A vertex with degree zero is called isolated vertex.
- **Pendant vertex**: A vertex with degree one is called pendant vertex.



- **Hand-shaking theorem:** - Since each edge contribute two degree in the graph, the sum of the degree of all vertices in G is twice the number of edges in g.
- $\sum_{i=1}^n d(v_i) = 2 |E|$



- The number of vertices of odd degree in a graph is always even.
  - $\sum_{i=1}^n d(v_i) = \sum_{\text{even}} d(v_i) + \sum_{\text{odd}} d(v_i)$



**Q** Which of the following statements is/are TRUE for undirected graphs? (GATE-2013) (1 Marks)

**P:** Number of odd degree vertices is even.

**Q:** Sum of degrees of all vertices is even.

a) P only

b) Q only

c) Both P and Q

d) Neither P nor Q

**Q** Which one of the following is TRUE for any simple connected undirected graph with more than 2 vertices? **(GATE-2009) (1 Marks)**

**(A)** No two vertices have the same degree.

**(B)** At least two vertices have the same degree.

**(C)** At least three vertices have the same degree.

**(D)** All vertices have the same degree.

**Q** A simple graph  $G$  contains 21 edges, 3 vertices of degree 4 and all the remaining vertices are of degree 2. Then number of vertices  $|v|$  is?

**Q** A simple non-directed graph G has 24 edges and degree of each vertex is 4, then find the  $|v|$ ?

**Q** Consider a simple graph with 35 edges such that 4 vertex of degree 5, 5 vertex of degree 4, 4 vertex of degree 3, find the number of vertex with degree 2?

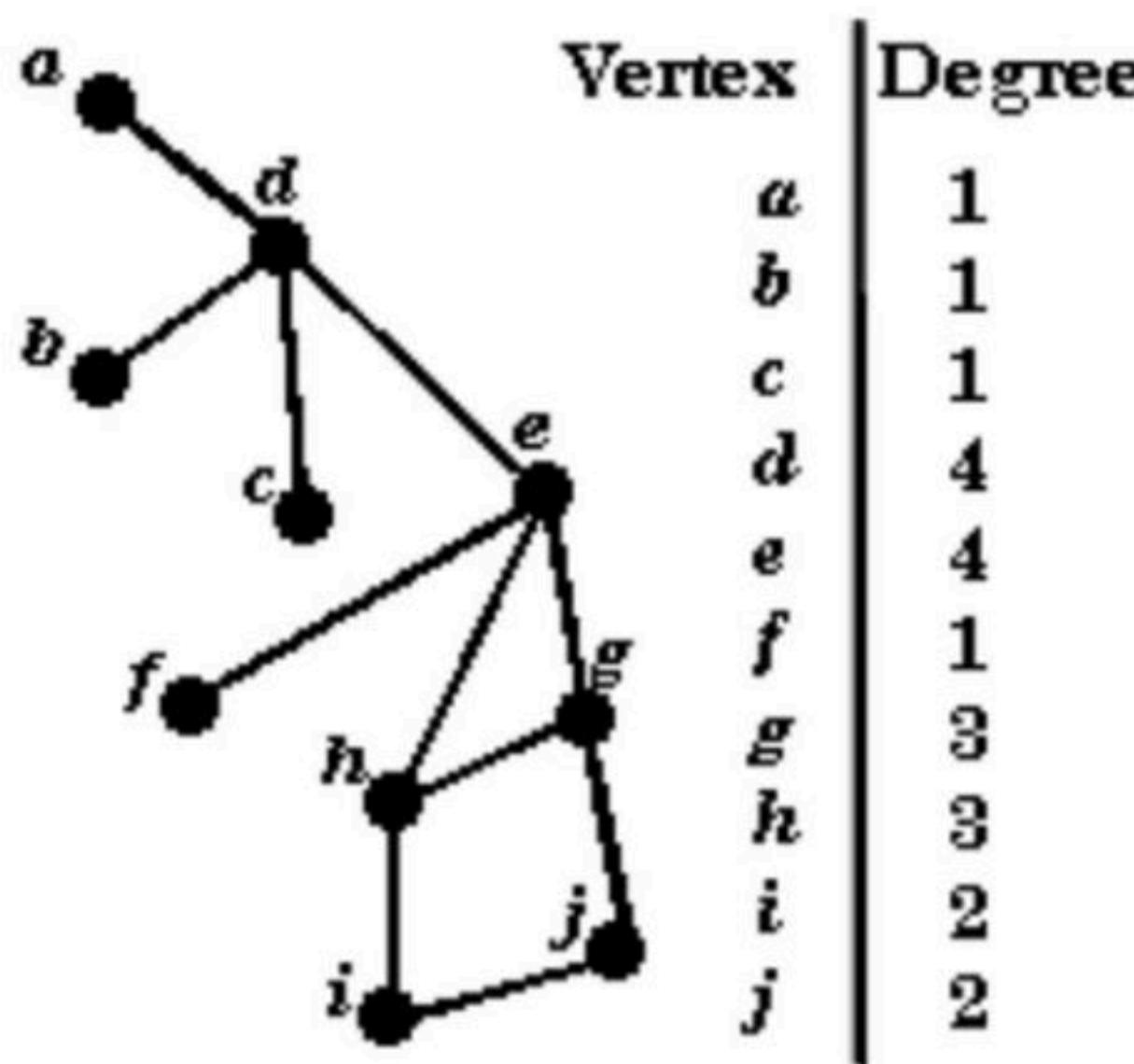
**Q** What is the number of vertices in an undirected connected graph with 27 edges, 6 vertices of degree 2, 3 vertices of degree 4 and remaining of degree 3? **(GATE-2004) (2 Marks)**

- (A) 10
- (B) 11
- (C) 18
- (D) 19

Q simple non-directed graph G has 24 edges and degree of each vertex is K, the which of the following is possible no of vertices?

- a) 20
- b) 15
- c) 10
- d) 8

- $\delta(G)$  is the minimum possible degree of any vertex in a graph
- $\Delta(G)$  is the maximum possible degree of any vertex in a graph.



$$\delta(G) * |V(G)| \leq 2|E| \leq \Delta(G) * |V(G)|$$

**Q** G is undirected graph with n vertices and 25 edges such that each vertex has degree at least 3. Then the maximum possible value of n is

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**(GATE-2017) (2 Marks)**

**Q** Maximum no of vertex in a simple graph with 35 edges and degree of each vertex is at least 3 is \_\_\_\_\_?

**Q** Minimum number of vertices possible in a simple graph if 41 edges and degree of each vertex is at most 5?

**Q** Which of the following degree sequence represent a simple non-directed graph?

1) {2, 3, 3, 4, 4, 5}

2) {2, 3, 4, 4, 5}

3) {3, 3, 3, 1}

4) {1, 3, 3, 4, 5, 6, 6}

5) {2, 3, 3, 3, 3}

6) {6, 6, 6, 6, 4, 3, 3, 0}

7) {6, 5, 5, 4, 3, 3, 2, 2, 2}

**Q** An ordered n-tuple  $(d_1, d_2, \dots, d_n)$  with  $d_1 \geq d_2 \geq \dots \geq d_n$  is called graphic if there exists a simple undirected graph with n vertices having degrees  $d_1, d_2, \dots, d_n$  respectively. Which of the following 6-tuples is NOT graphic? **(GATE-2014) (2 Marks)**

- (A) (1, 1, 1, 1, 1, 1)**      **(B) (2, 2, 2, 2, 2, 2)**
  
- (C) (3, 3, 3, 1, 0, 0)**      **(D) (3, 2, 1, 1, 1, 0)**

**Q** The degree sequence of a simple graph is the sequence of the degrees of the nodes in the graph in decreasing order. Which of the following sequences cannot be the degree sequence of any graph? **(GATE-2010) (2 Marks)**

I. 7, 6, 5, 4, 4, 3, 2, 1

II. 6, 6, 6, 6, 3, 3, 2, 2

III. 7, 6, 6, 4, 4, 3, 2, 2

IV. 8, 7, 7, 6, 4, 2, 1, 1

**(A)** I and II

**(B)** III and IV

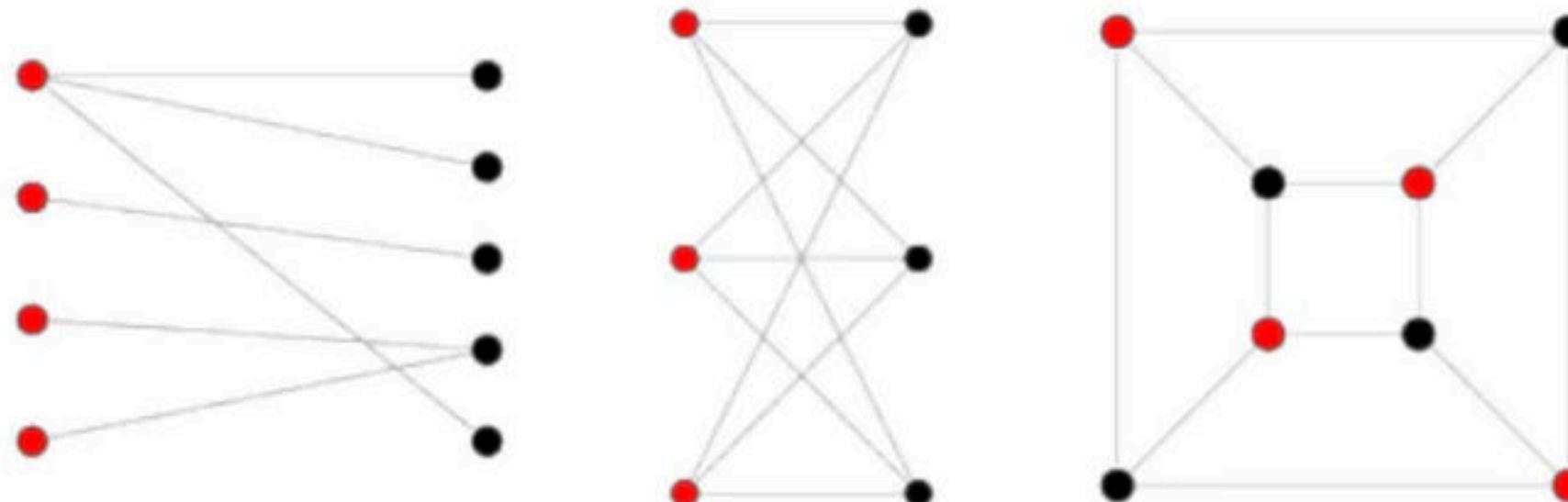
**(C)** IV only

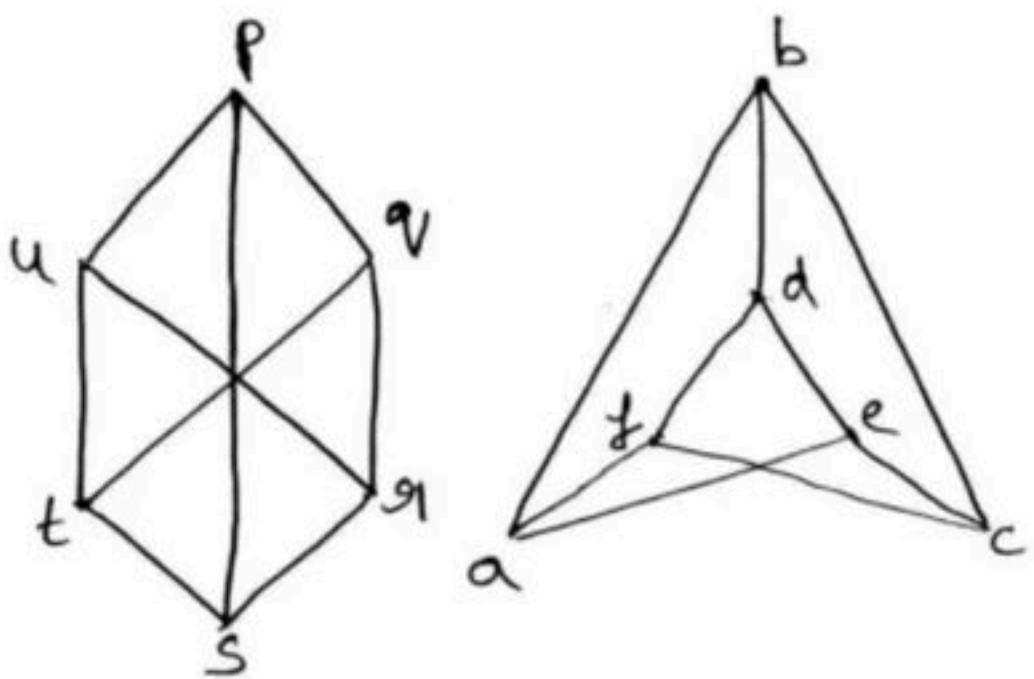
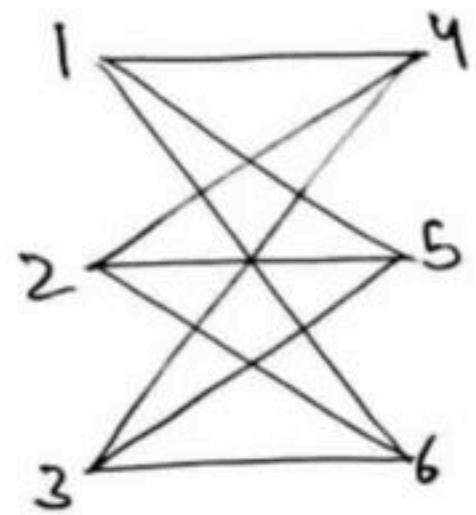
**(D)** II and IV

1. The **Havel–Hakimi algorithm** is an algorithm in graph theory solving the graph realization problem. That is, it answers the following question: Given a finite list of nonnegative integers, is there a simple graph such that its degree sequence is exactly this list.
2. Here, the "degree sequence" is a list of numbers that for each vertex of the graph states how many neighbors it has. For a positive answer the list of integers is called *graphic*.
3. The algorithm constructs a special solution if one exists or proves that one cannot find a positive answer. This construction is based on a recursive algorithm. The algorithm was published by Havel (1955), and later by Hakimi (1962).

## Some Popular Graph

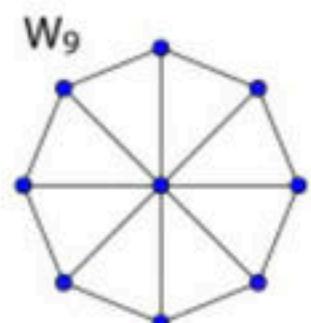
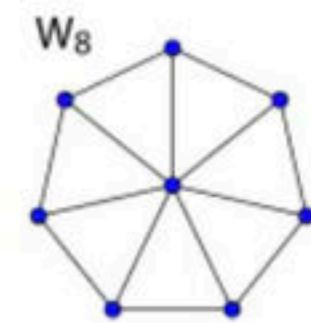
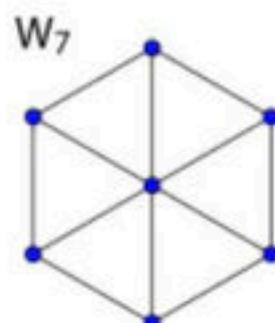
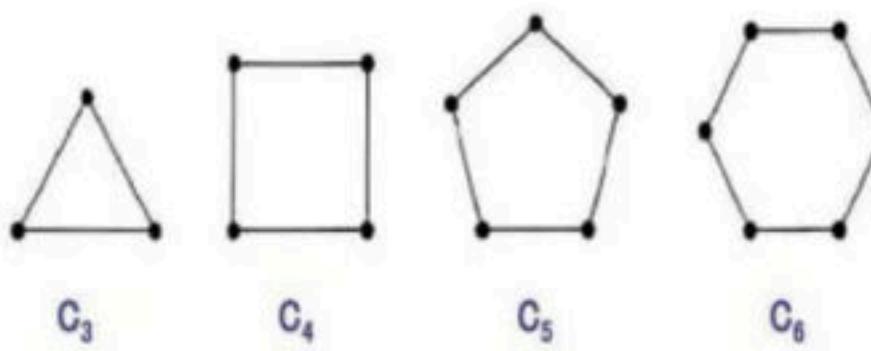
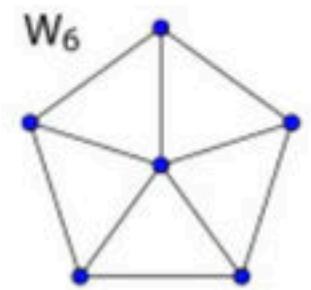
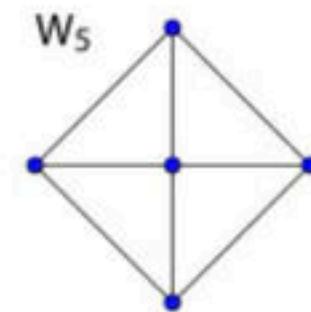
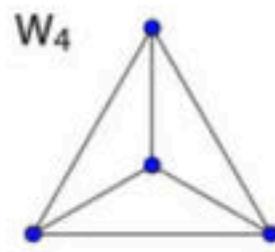
1. **Bi-partite graph:** - A graph  $G(V, E)$  is called bi-partite graph if it's vertex set  $V(G)$  can be partitioned into two non-empty disjoint subset  $V_1(G)$  and  $V_2(G)$  in such a way that each edge  $e \in E(G)$  has it's one end point in  $V_1(g)$  and other end point in  $V_2(g)$ . The partition  $V = V_1 \cup V_2$  is called bipartition of  $G$ .
2. **Complete Bi-partite graph:** - A Bi-partite graph  $G(V, E)$  is called Complete bi-partite graph if every vertex in the first partition is connected to every vertex in the second partition, denoted by  $K_{m,n}$ .





**Q** The maximum number of edges in a bipartite graph on 12 vertices is \_\_\_\_\_ . **(GATE-2014) (1 Marks)**

1. **Cycle Graph:** - A cycle graph or circular graph is a graph that consists of a single cycle, or in other words, some number of vertices (at least 3) connected in a closed chain. The cycle graph with  $n$  vertices is called  $C_n$ . The number of vertices in  $C_n$  equals the number of edges, and every vertex has degree 2; that is, every vertex has exactly two edges incident with it.
2. **Wheel graph:** - A wheel graph is a graph formed by connecting a single universal vertex to all vertices of a cycle. Some authors write  $W_n$  to denote a wheel graph with  $n$  vertices ( $n \geq 4$ ); other authors instead use  $W_n$  to denote a wheel graph with  $n+1$  vertices ( $n \geq 3$ ), which is formed by connecting a single vertex to all vertices of a cycle of length  $n$ .



**Q** the graph  $K_{3,4}$  has: (NET-DEC-2008)

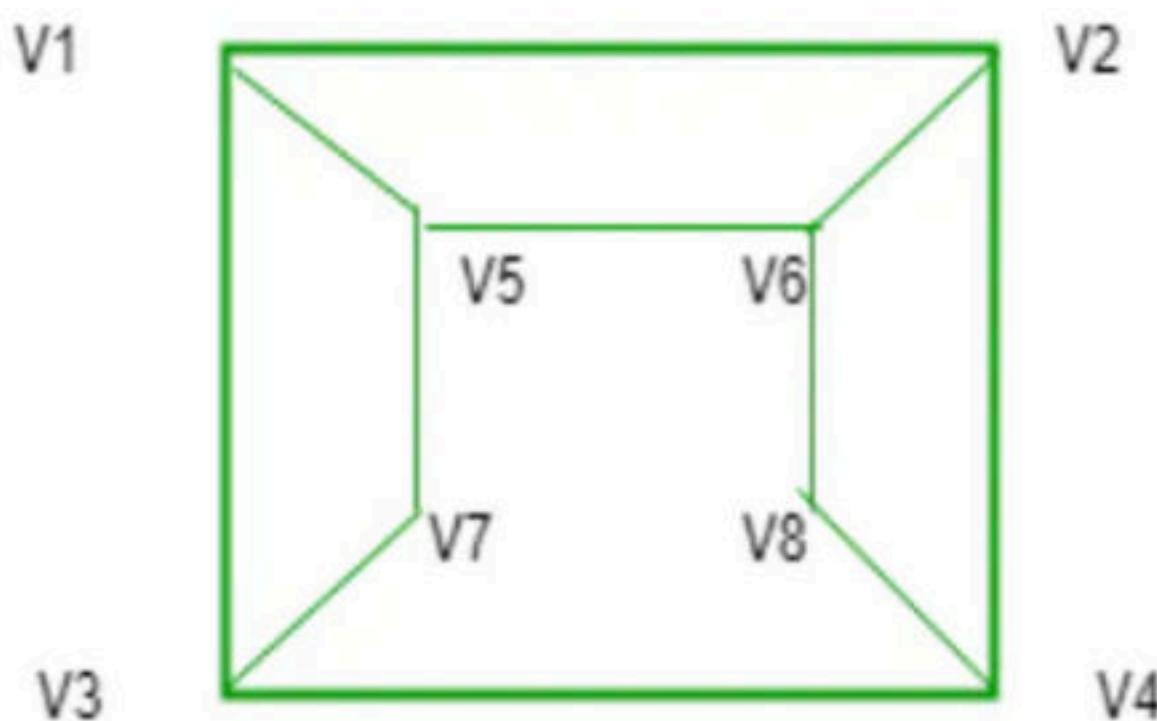
- a) 3 edges
- b) 4 edges
- c) 7 edges
- d) 12 edges

**Q** Consider the graph given below: (NET-DEC-2015)

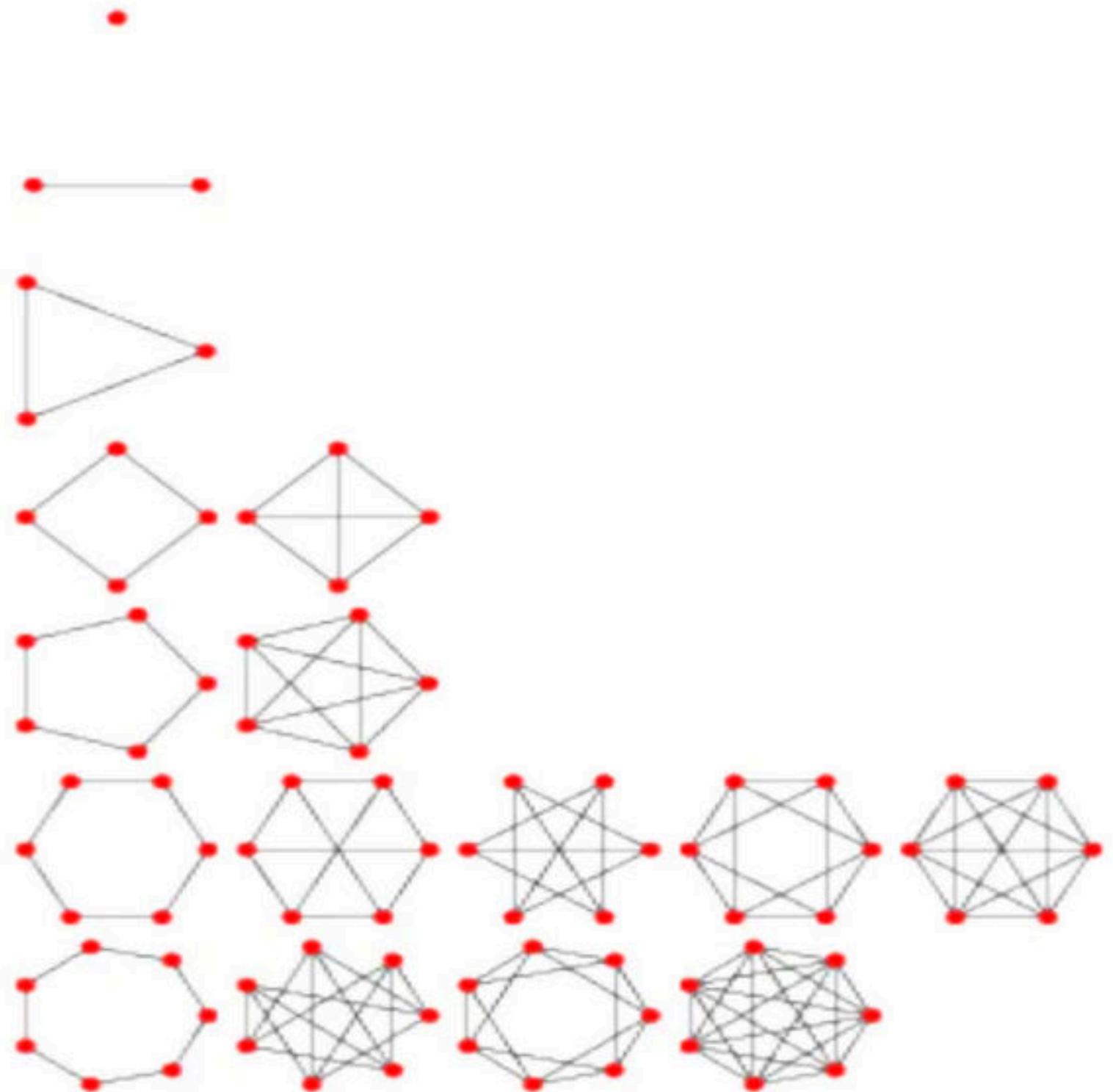
The two distinct sets of vertices, which make the graph bipartite are:

- a)  $(v_1, v_4, v_6); (v_2, v_3, v_5, v_7, v_8)$
- c)  $(v_1, v_4, v_6, v_7); (v_2, v_3, v_5, v_8)$

- b)  $(v_1, v_7, v_8); (v_2, v_3, v_5, v_6)$
- d)  $(v_1, v_4, v_6, v_7, v_8); (v_2, v_3, v_5)$

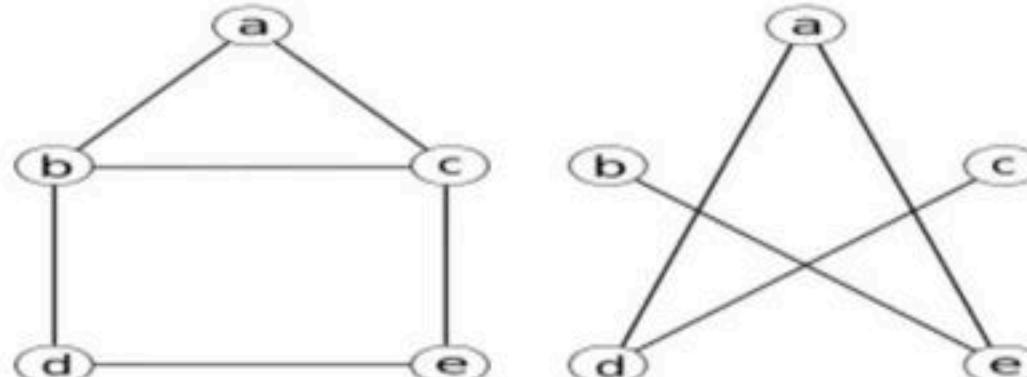
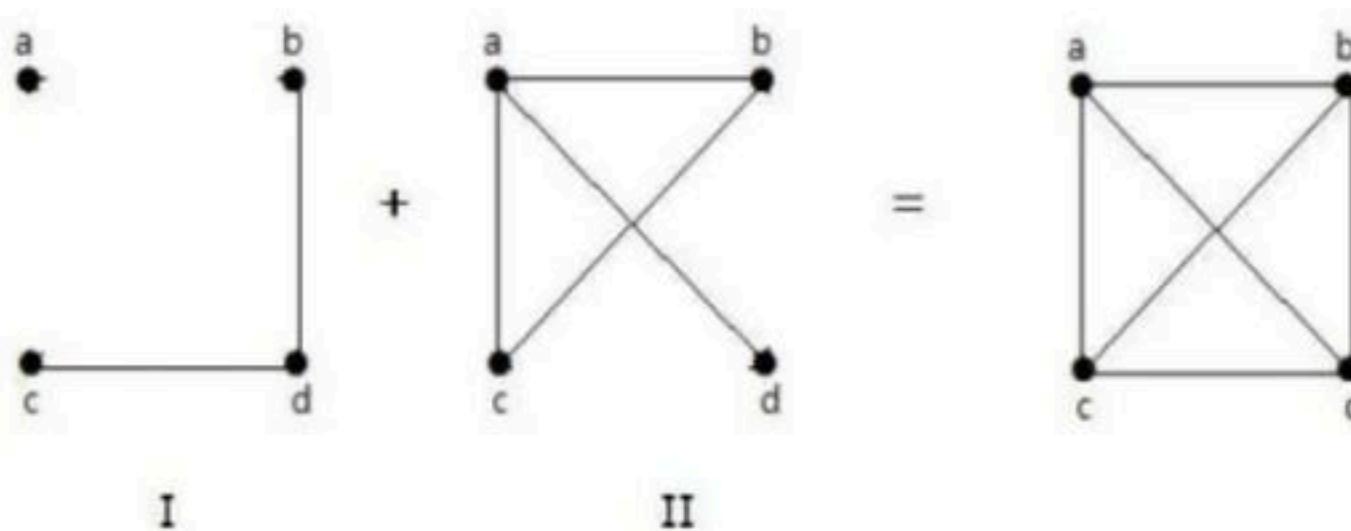


- **Regular graph:** - A graph in which all the vertices are of equal degree is called a regular graph.  
E.g. 2-regular graph, 3-regular graph.



## Complement of a Graph

1. The complement of a simple graph  $G(V, E)$  is a graph  $G^c(V, E^c)$  on the same vertices set as of  $G$ , such that there will be an edge between two vertices  $u, v$  in  $G^c$  if and only if there is no edge between  $u, v$  in  $G$ . i.e. two vertices of  $G^c$  are adjacent iff they are not adjacent in  $G$ .
2.  $V(G) = V(G^c)$
3.  $E(G^c) = \{(u, v) \mid (u, v) \notin E(G)\}$
4.  $E(G^c) = E(K_n) - E(G)$



## Properties

1.  $G \cup G^c = K_n$
2.  $G \cap G^c = \text{null graph}$
3.  $|E(G)| + |E(G^c)| = E(K_n) = n(n-1)/2$

**Q** A simple graph  $G$  has 30 edges and  $G^c$  has 36 edges, the number of vertices in  $G$  will be?

**Q** A simple graph  $G$  has 56 edges and  $G^c$  has 80 edges, the number of vertices in  $G$  will be?

**Q** a simple graph G has  $|v|=8$  and  $|E|=12$ , find number of edges in  $|E(G^c)|$ ?