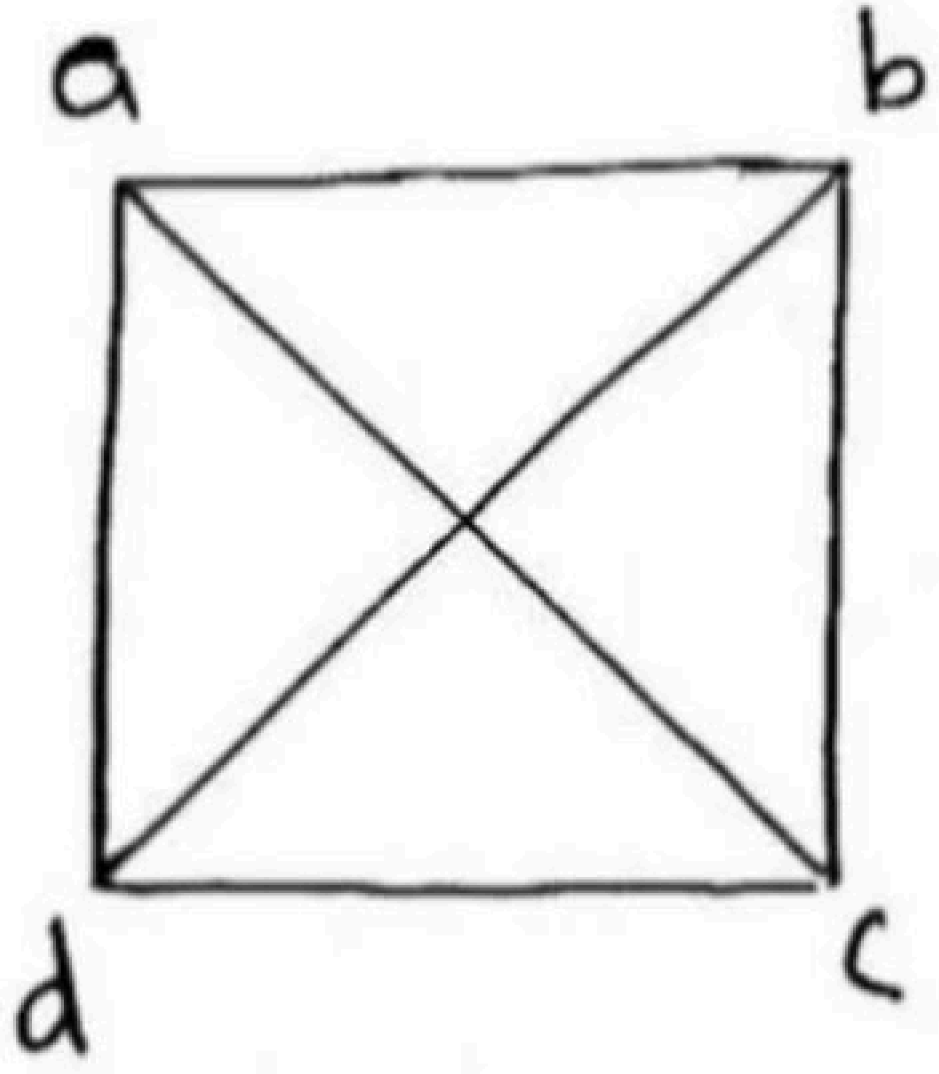
Graph Theory - Part IV

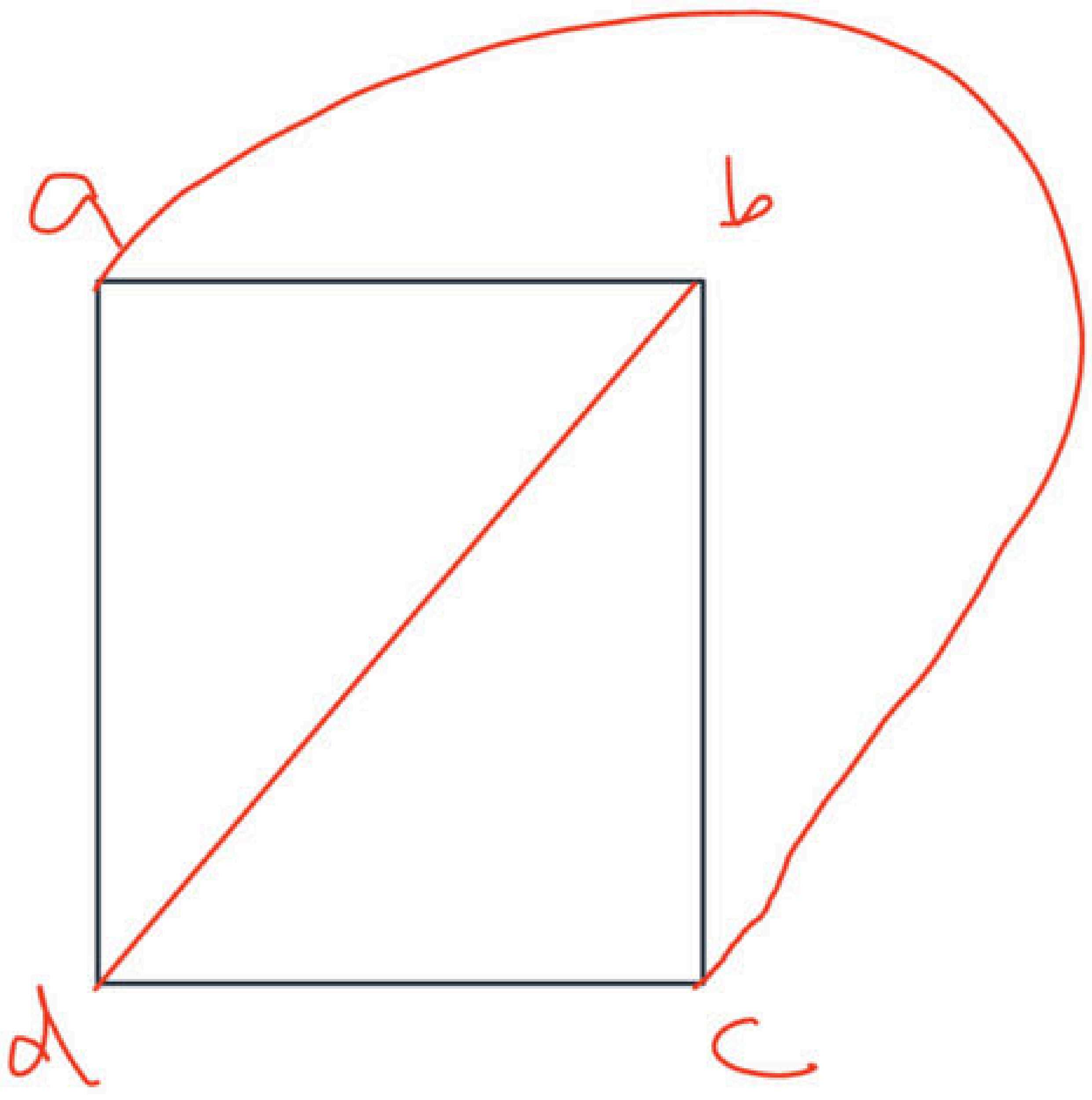
Foundation Course on Discrete Mathematics for GATE

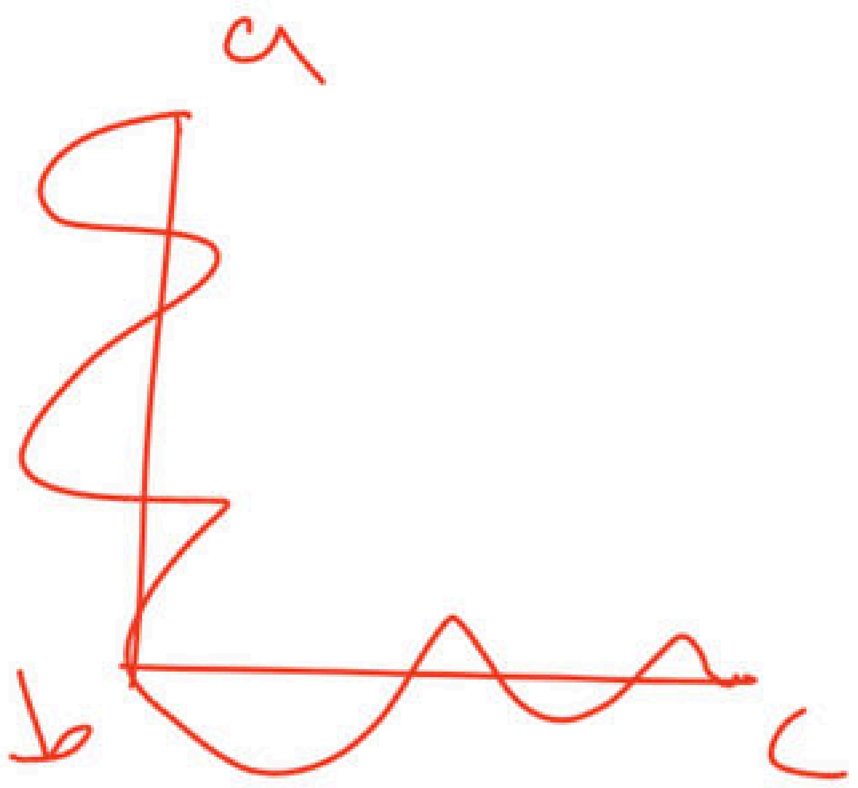
Sanchit Jain • Lesson 22 • Mar 2, 2022

Planer Graph

Planer Graph: - A graph is called a planer graph if it can be drawn on a plan in such a way that no edges cross each other, otherwise it is called non-planer. Application: civil engineering, circuit designing



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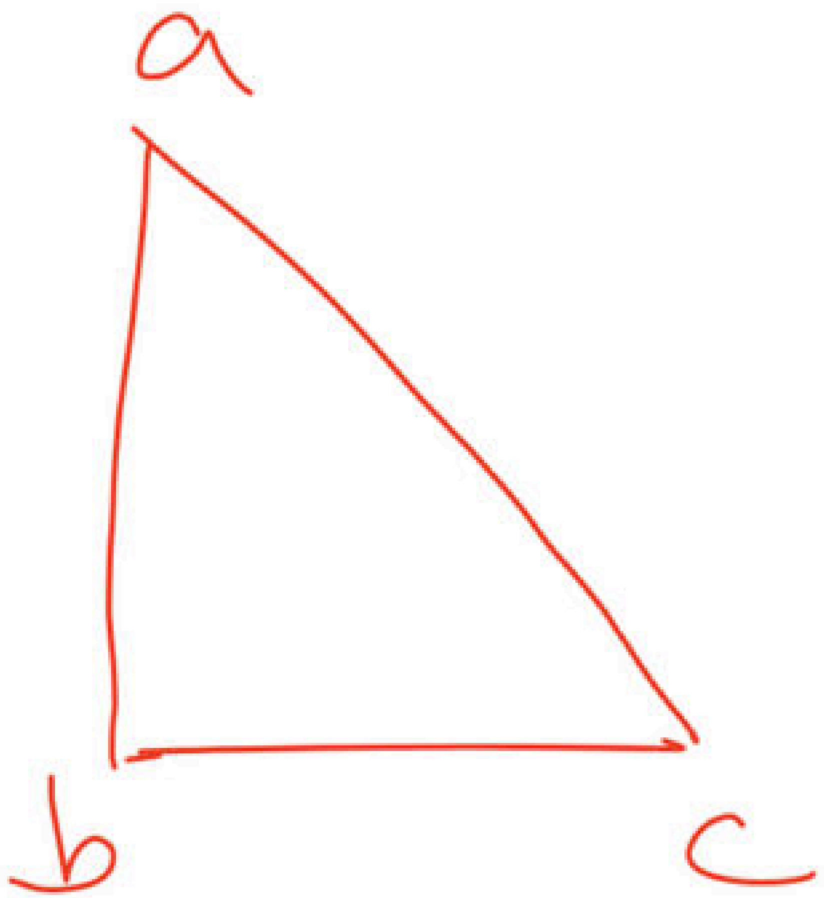
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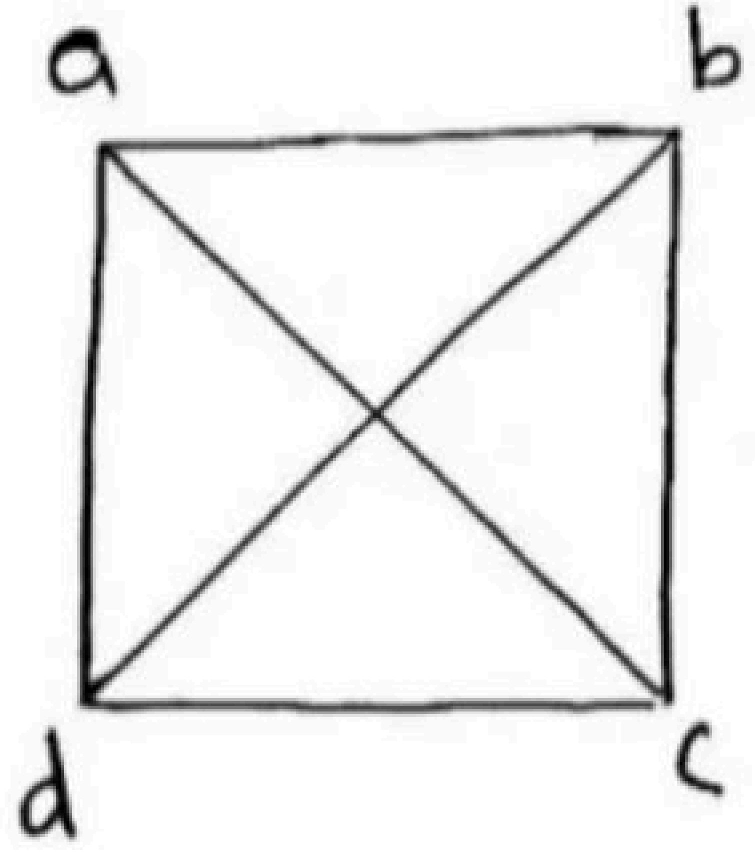
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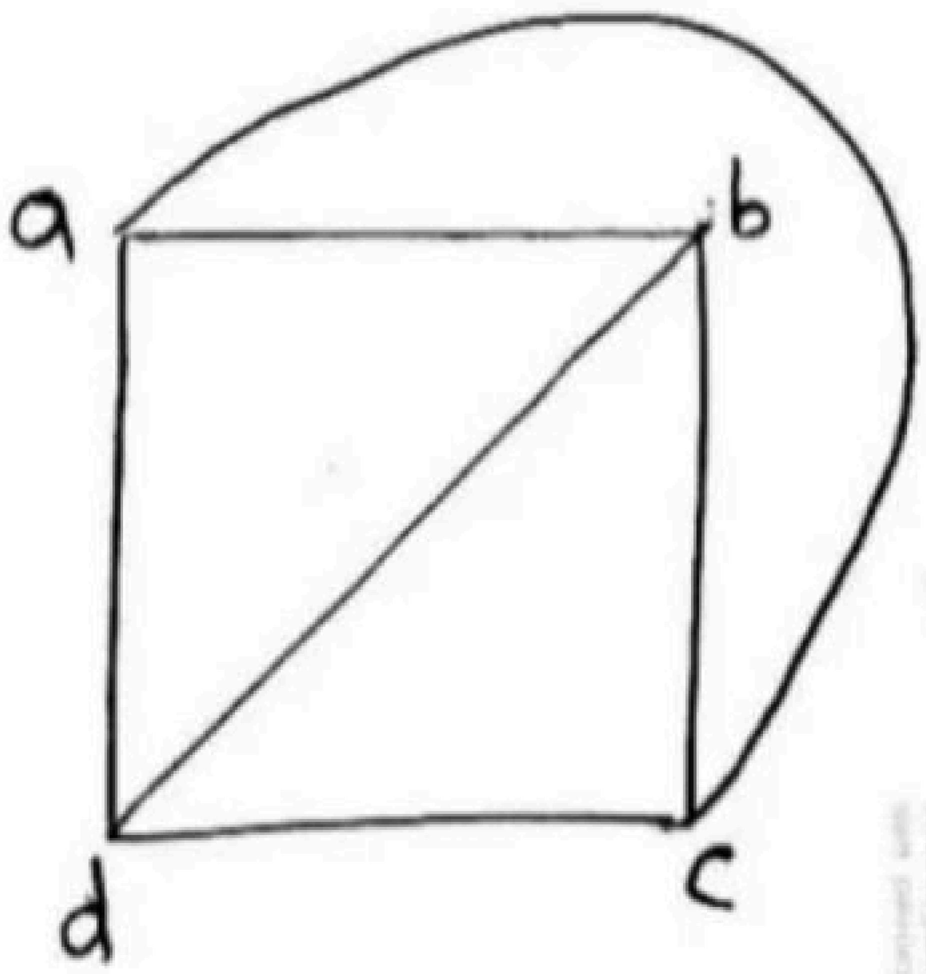
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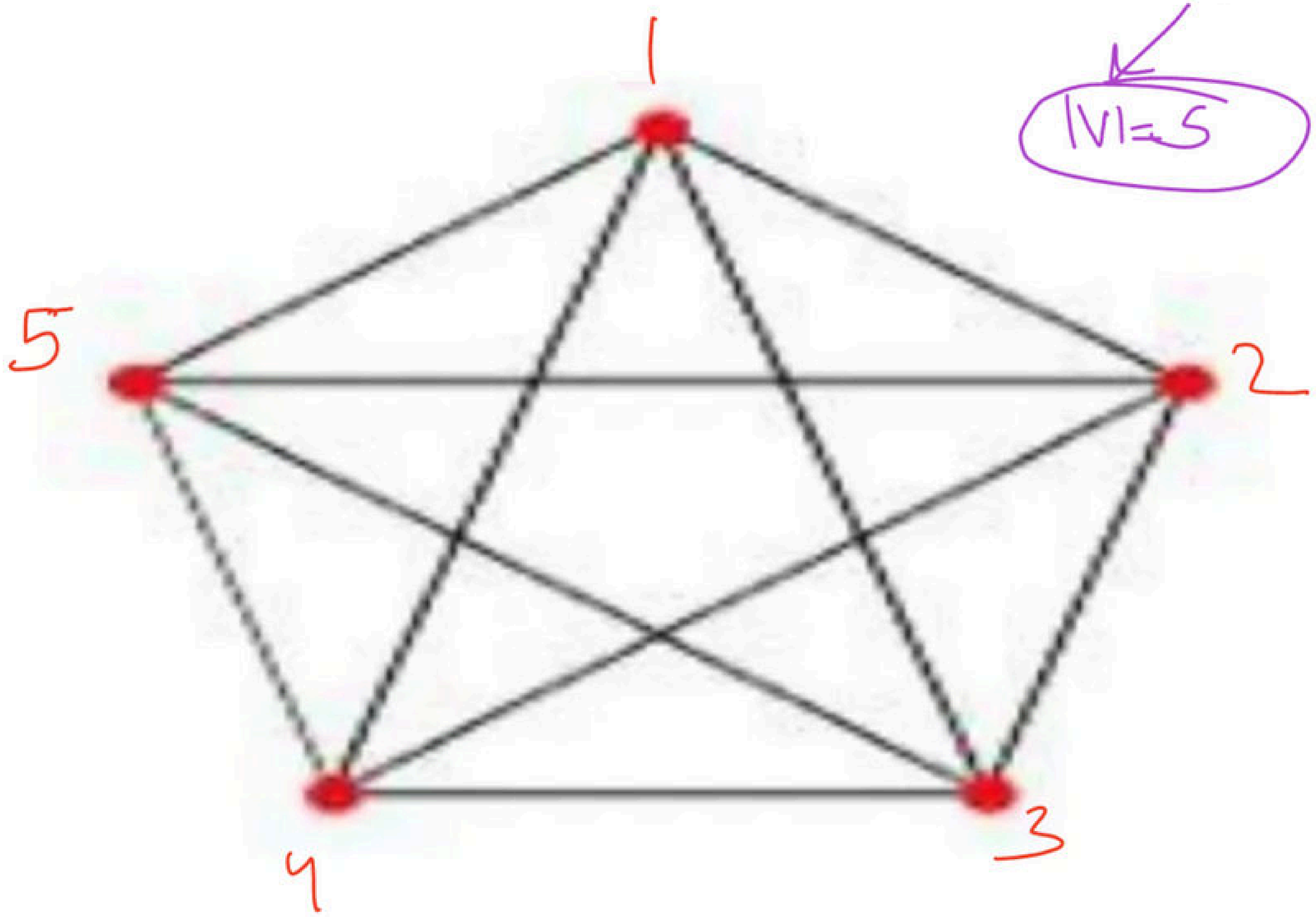
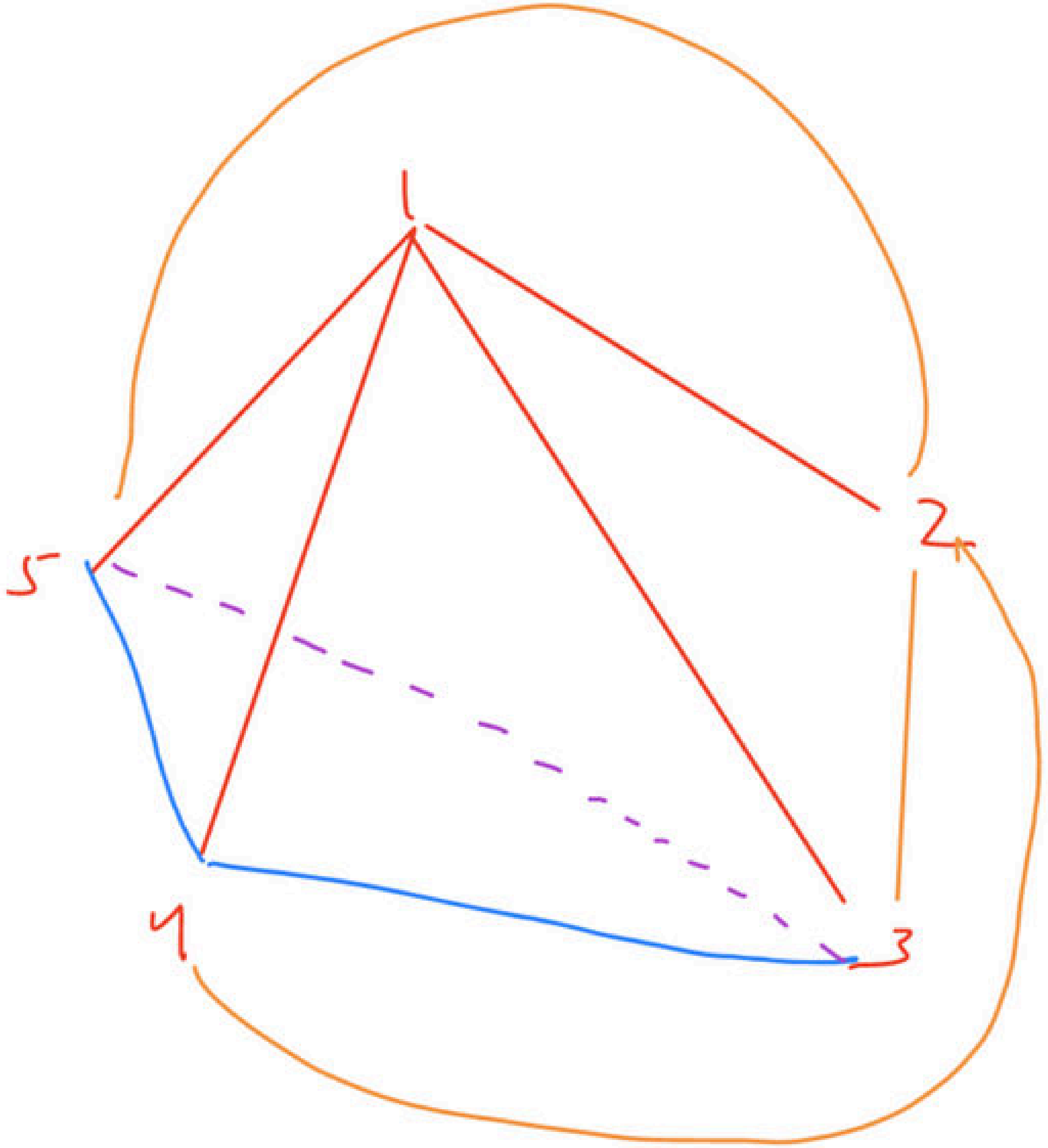
K_5

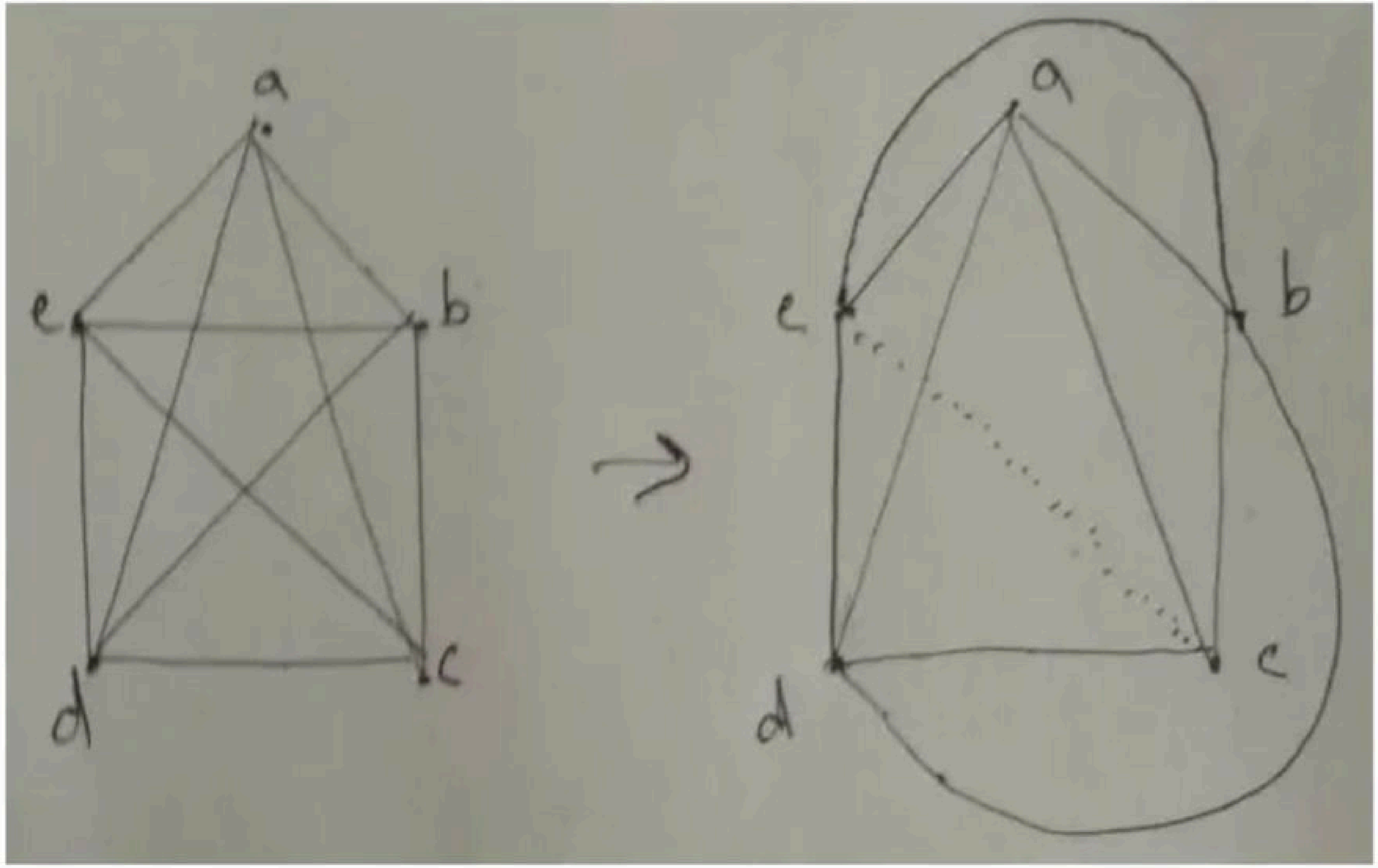
$|V|=5$





K_5

K_2, | K_3 | K_4 | $K_5 \rightarrow |V|=5$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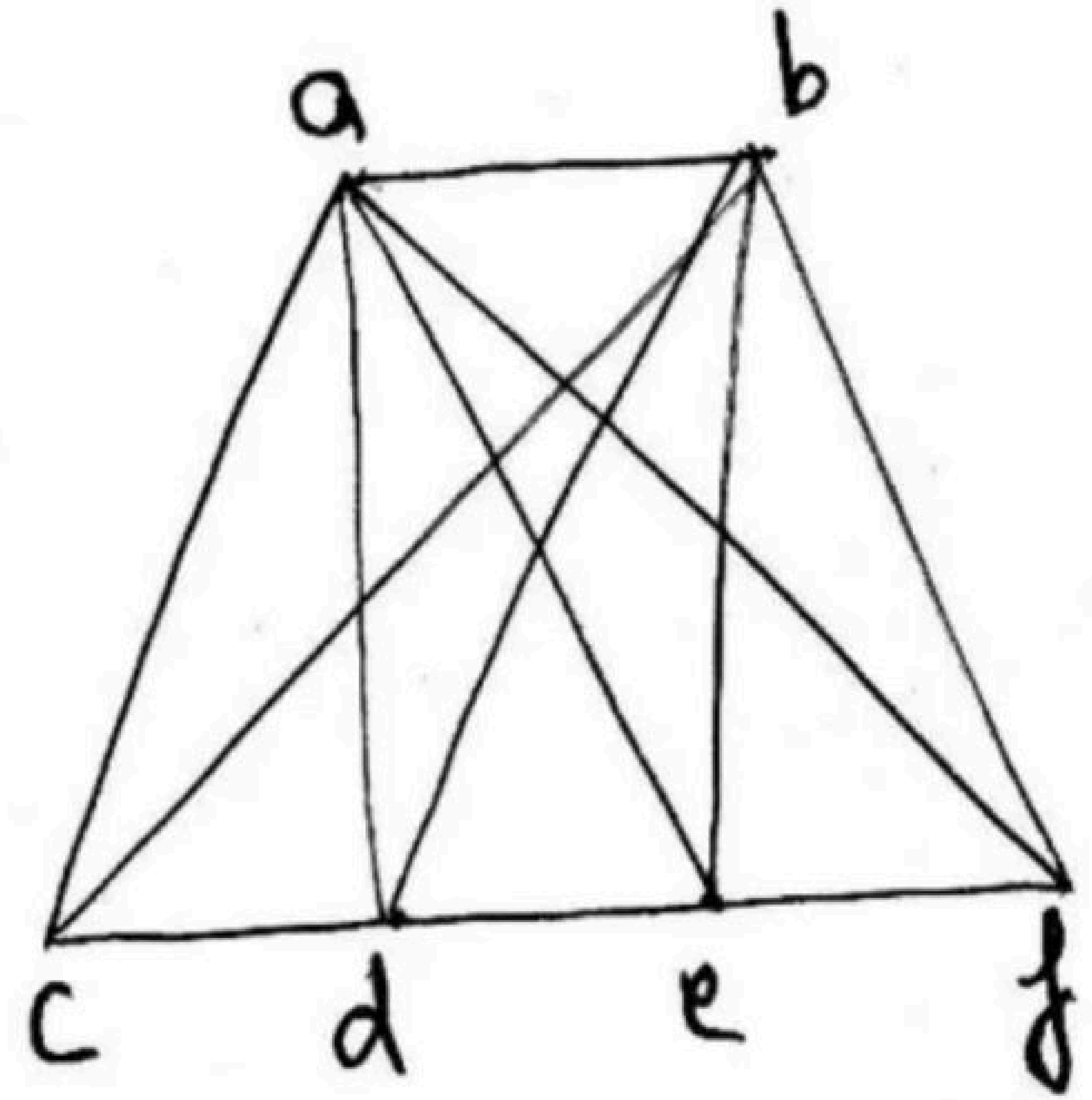
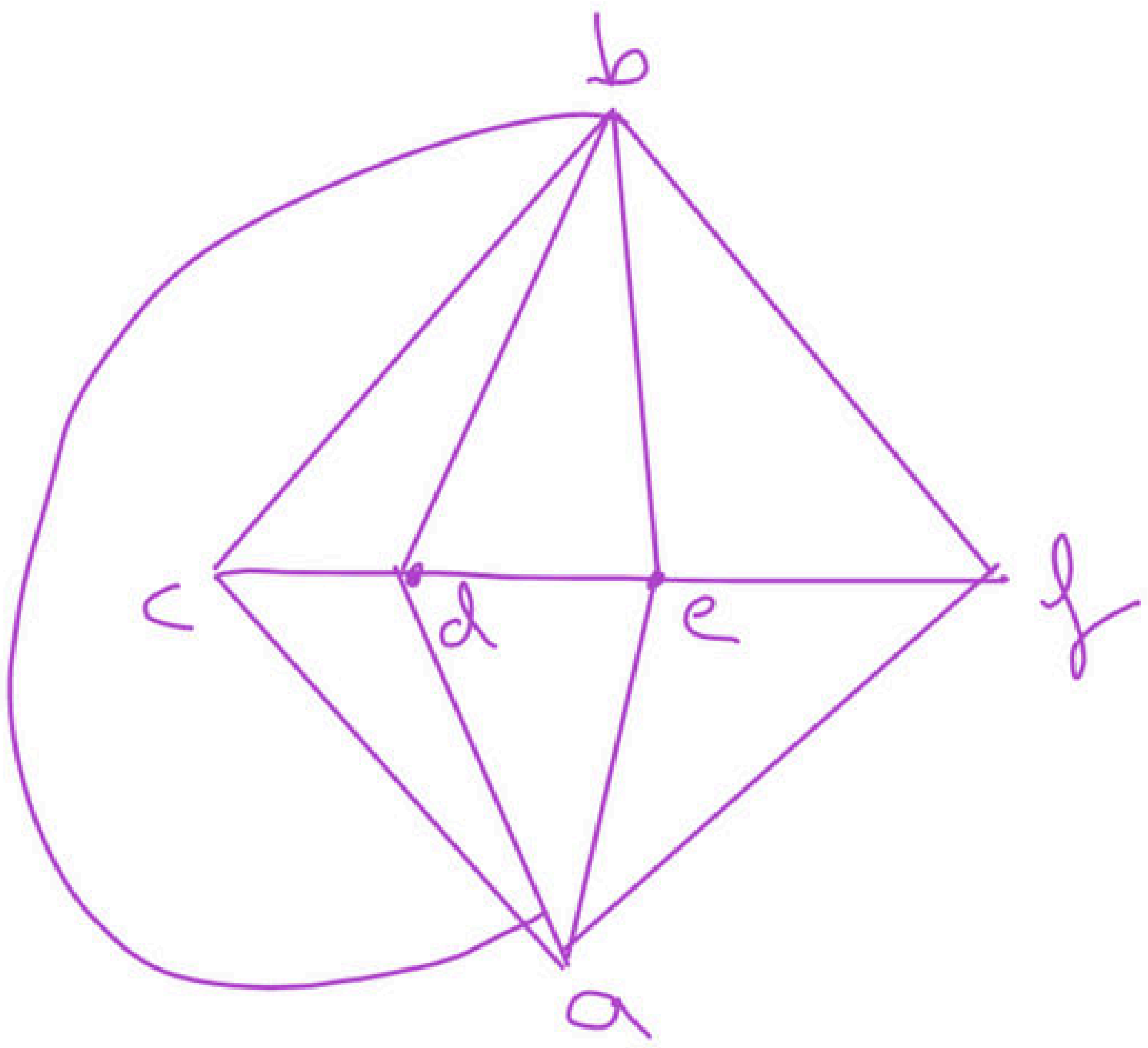
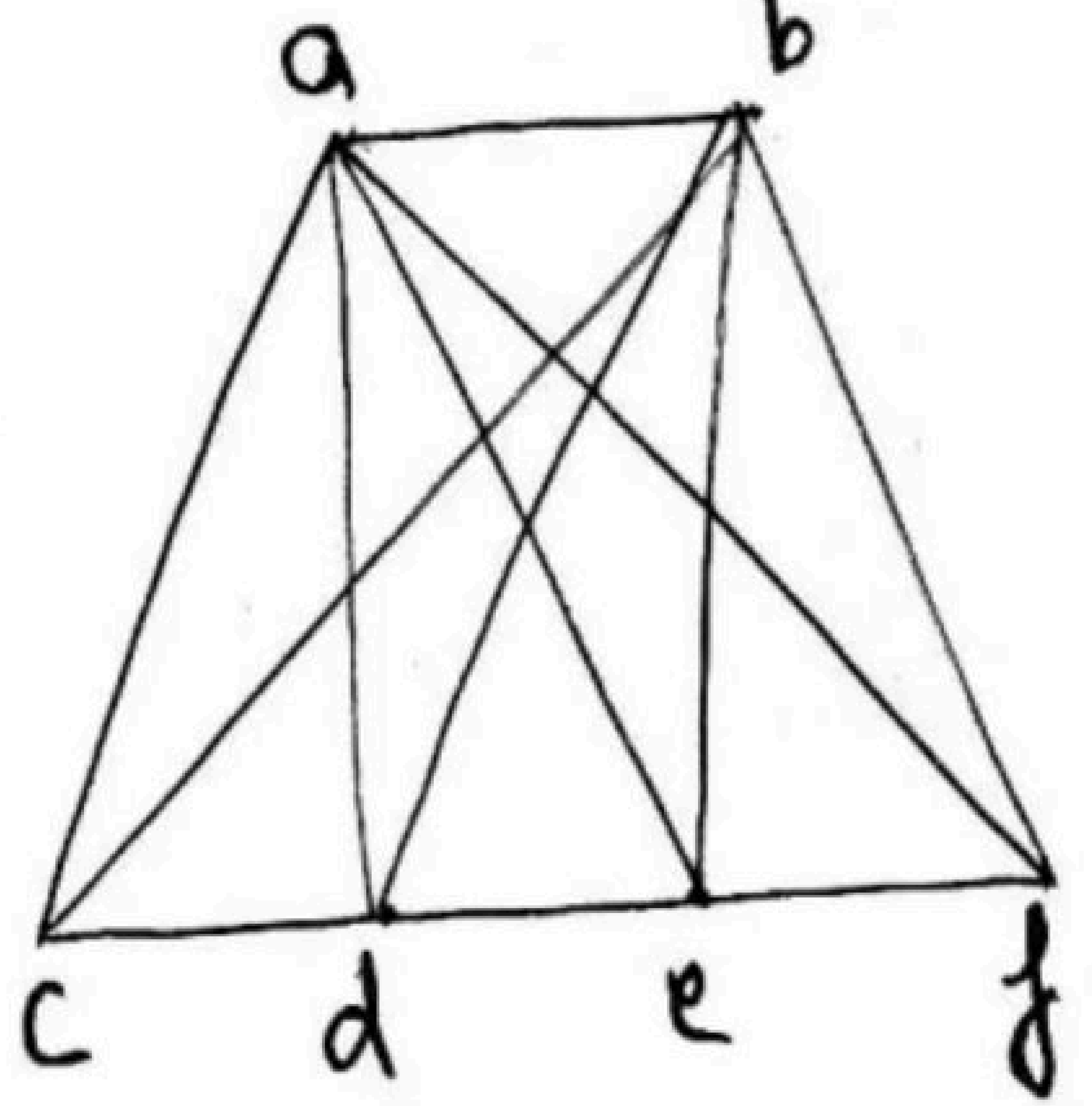


T = Planar

F = Non plr

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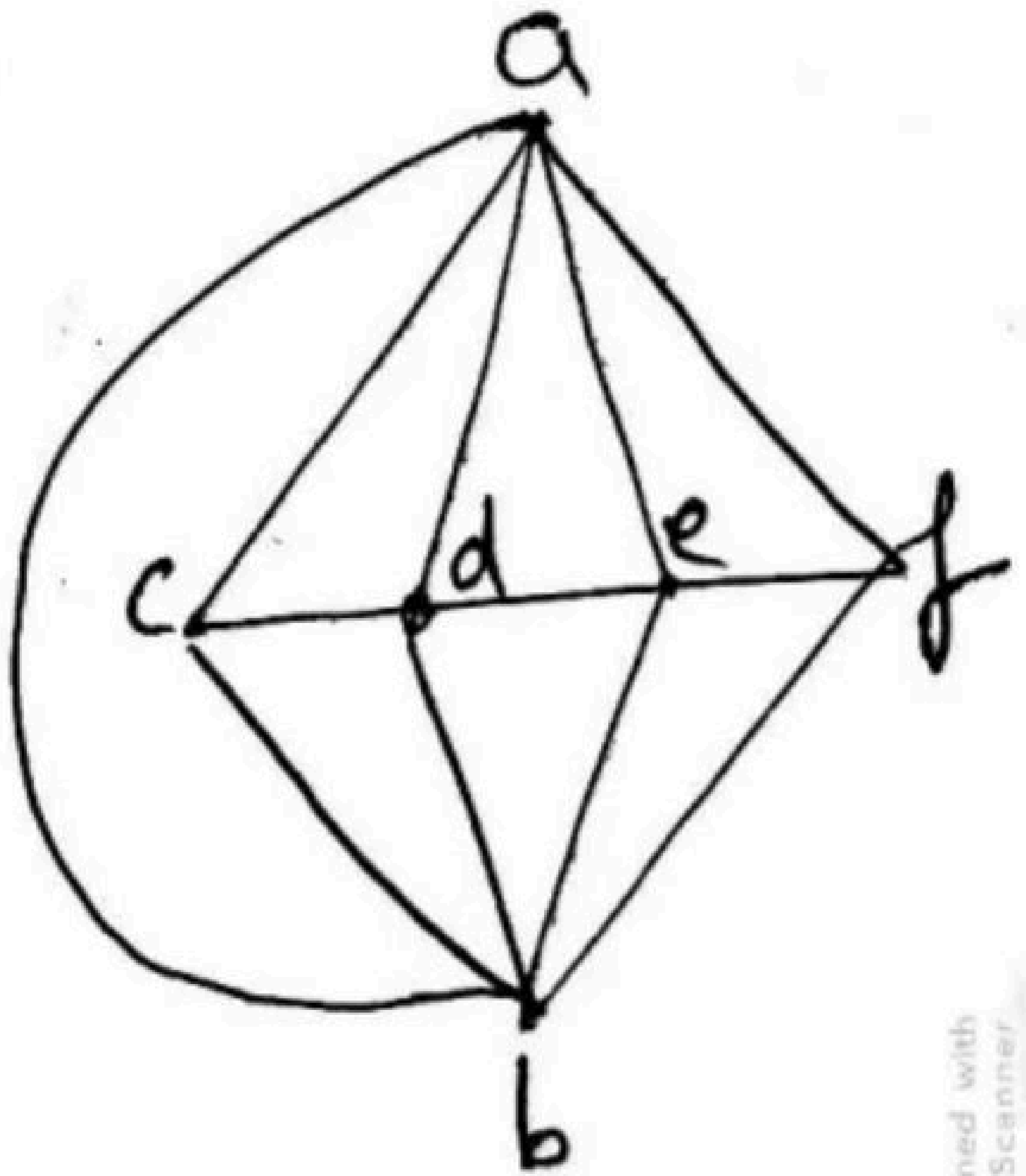
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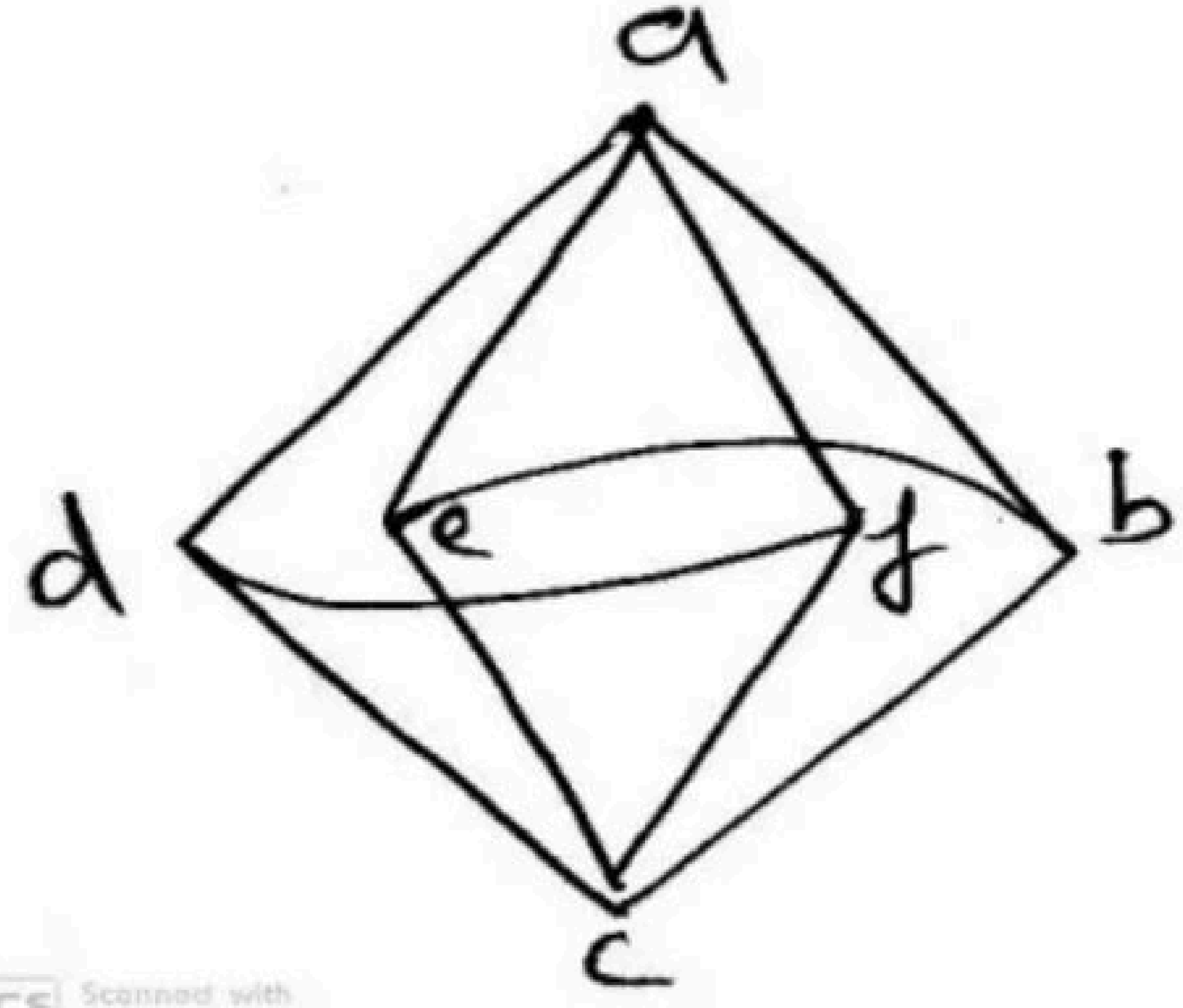
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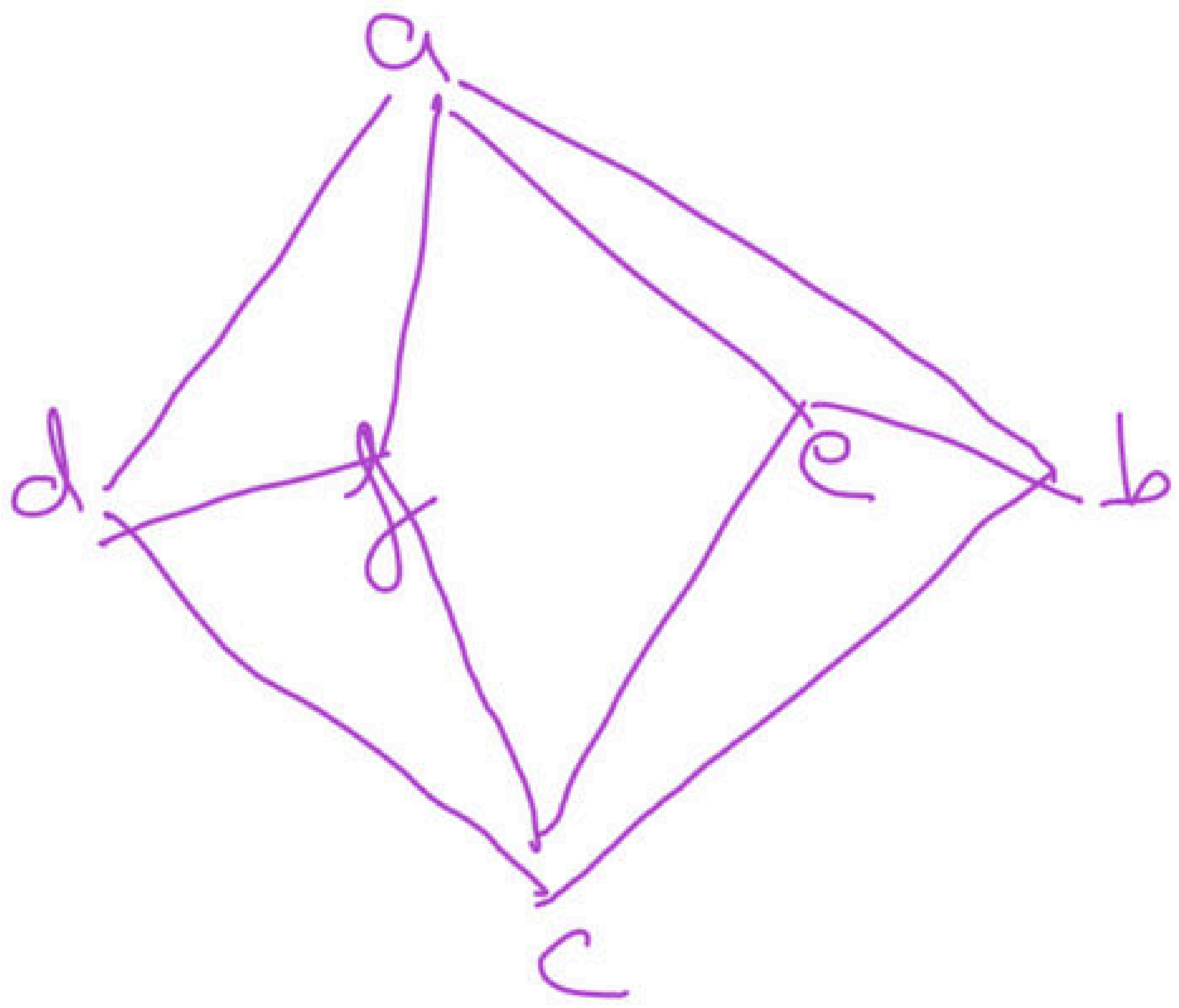
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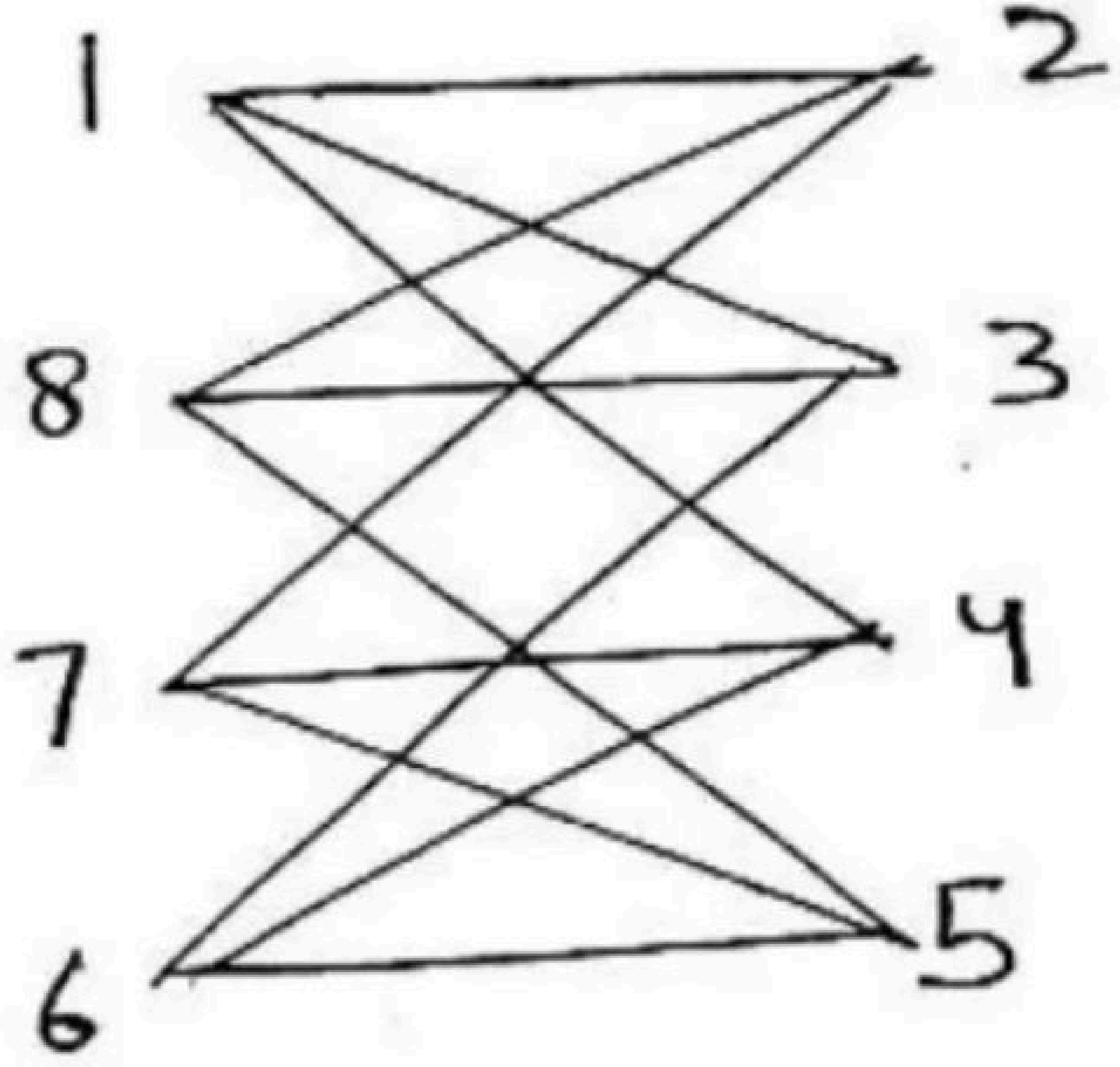
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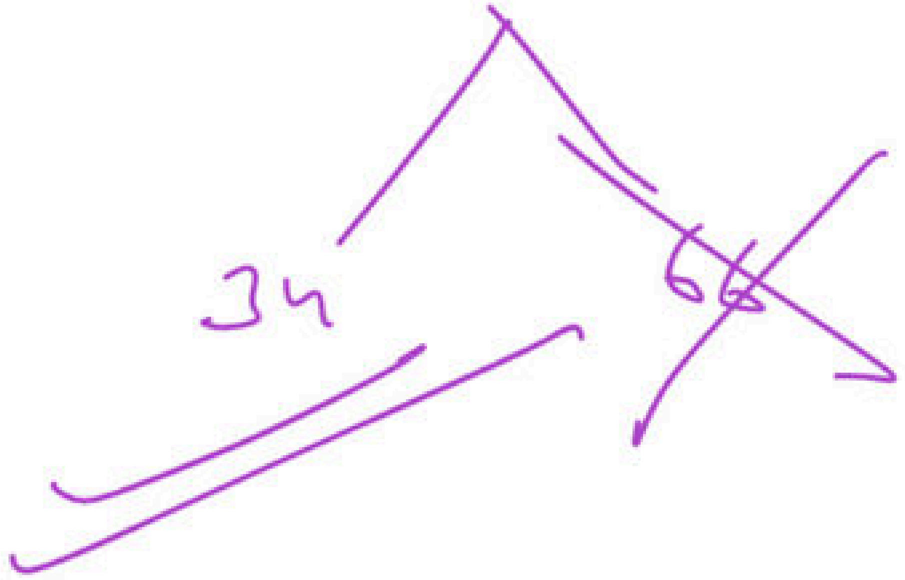
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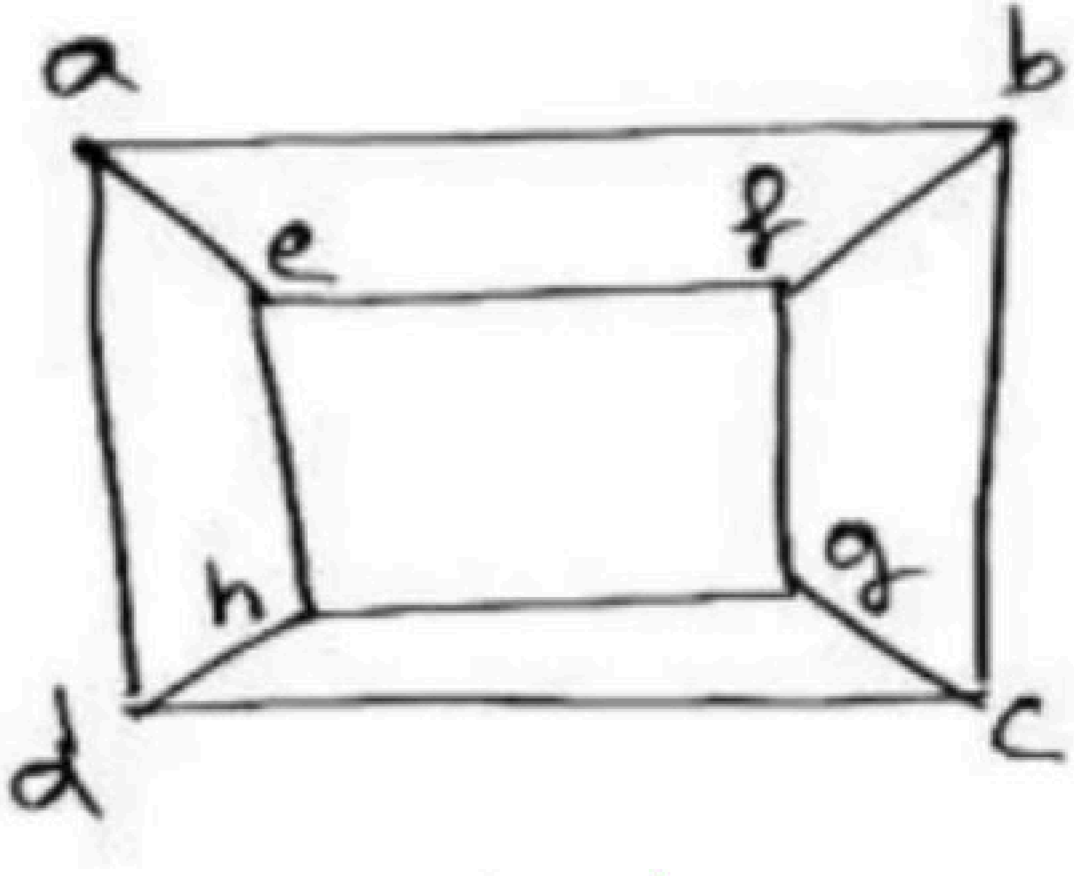

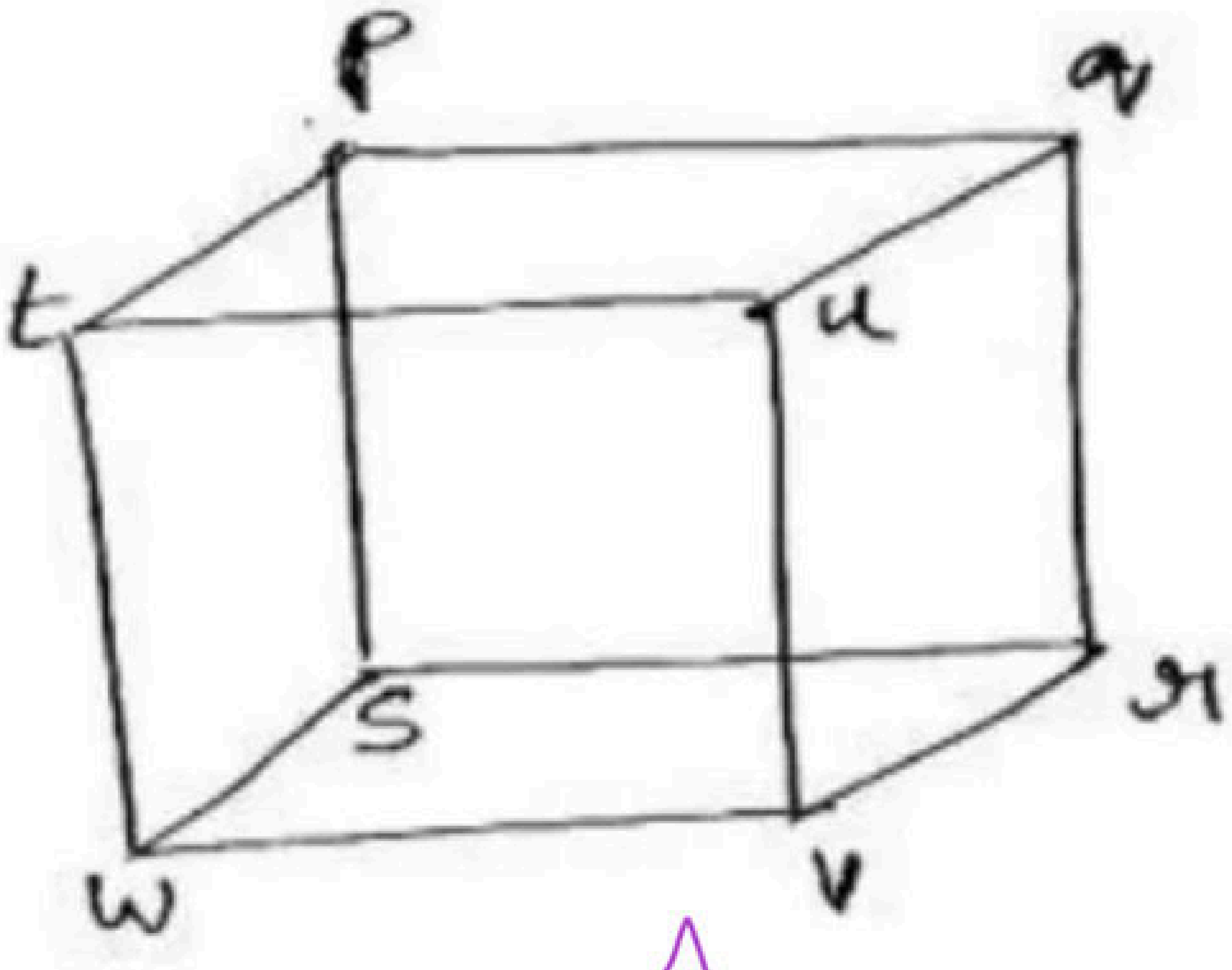

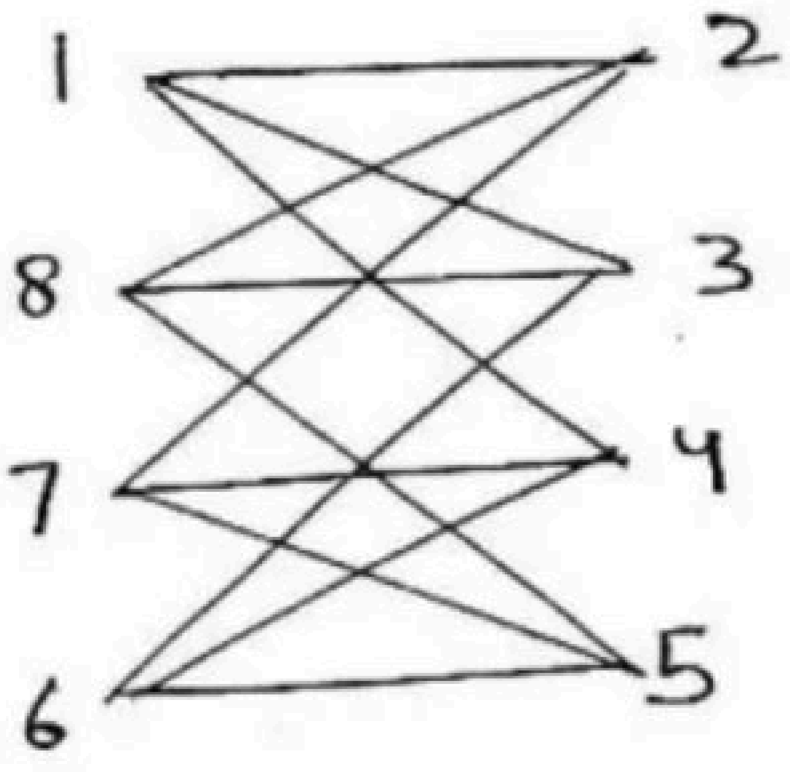

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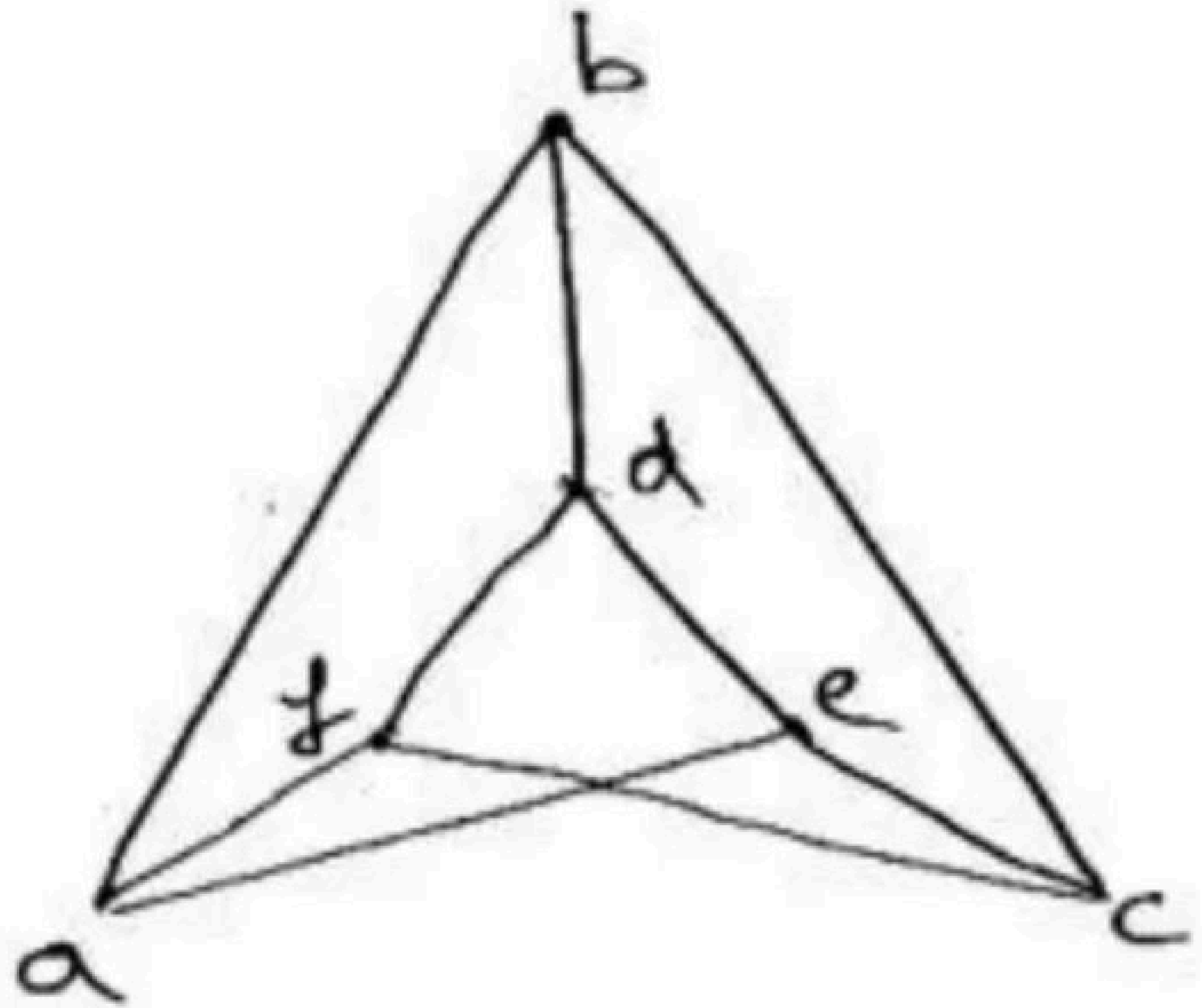
G_1 G_2 G_3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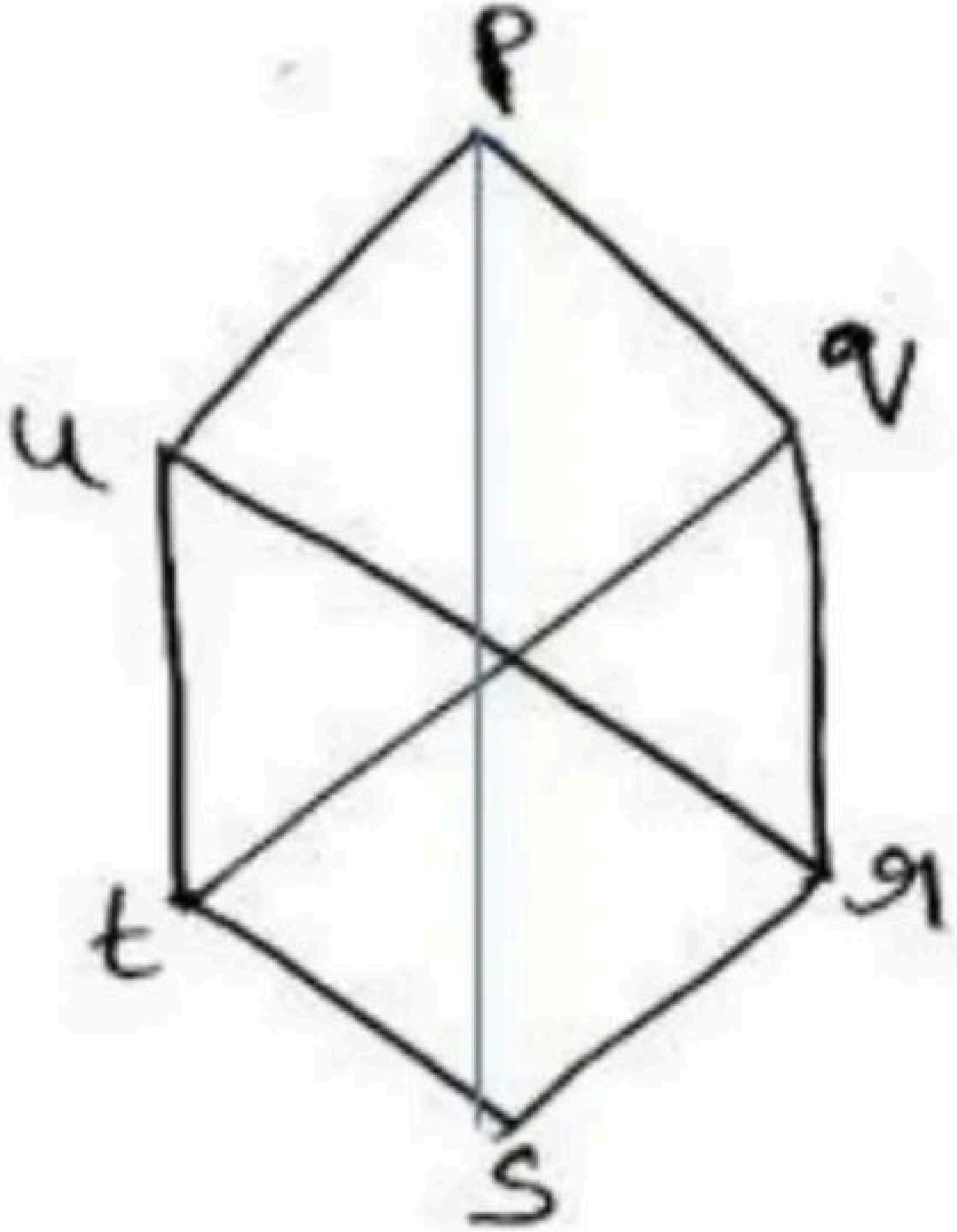
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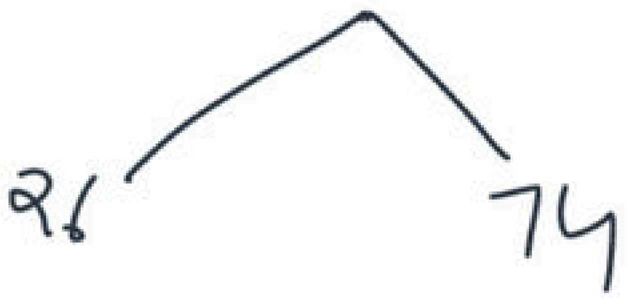
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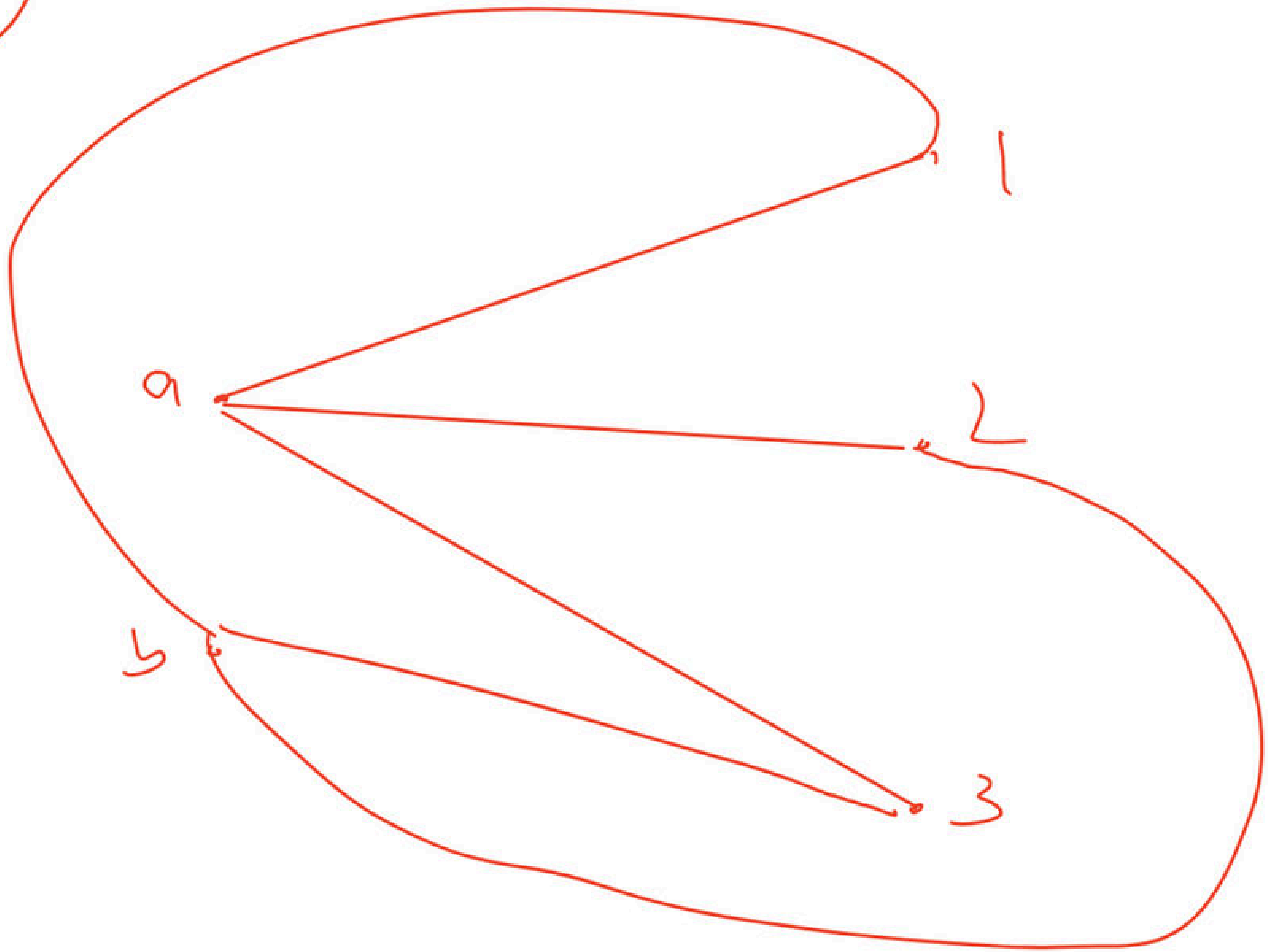
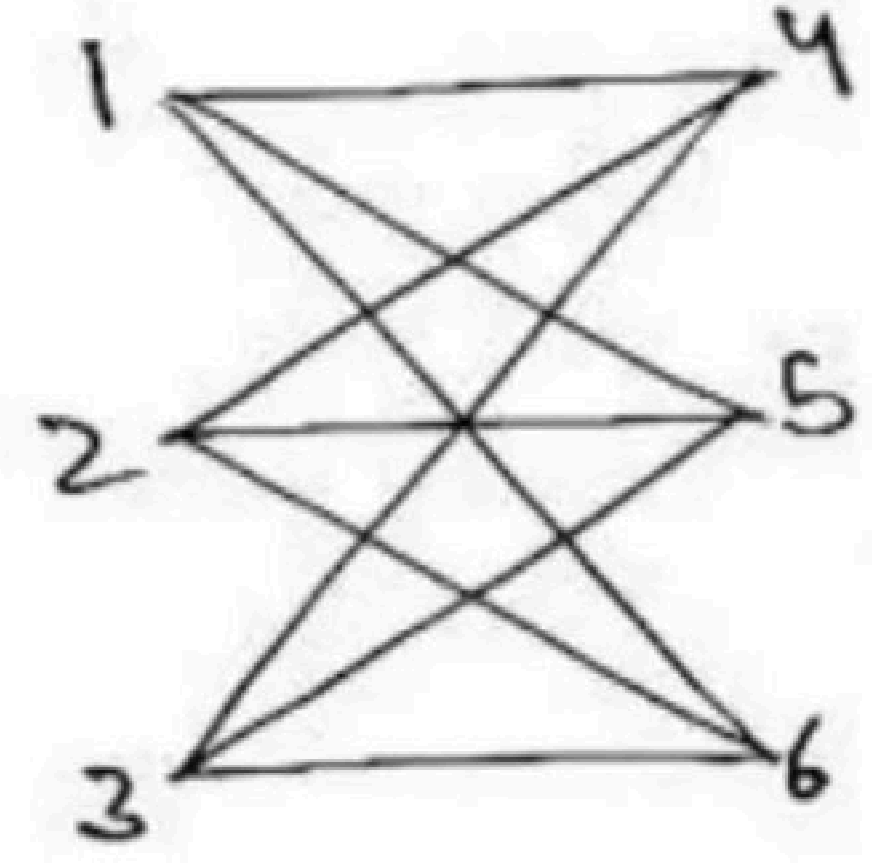
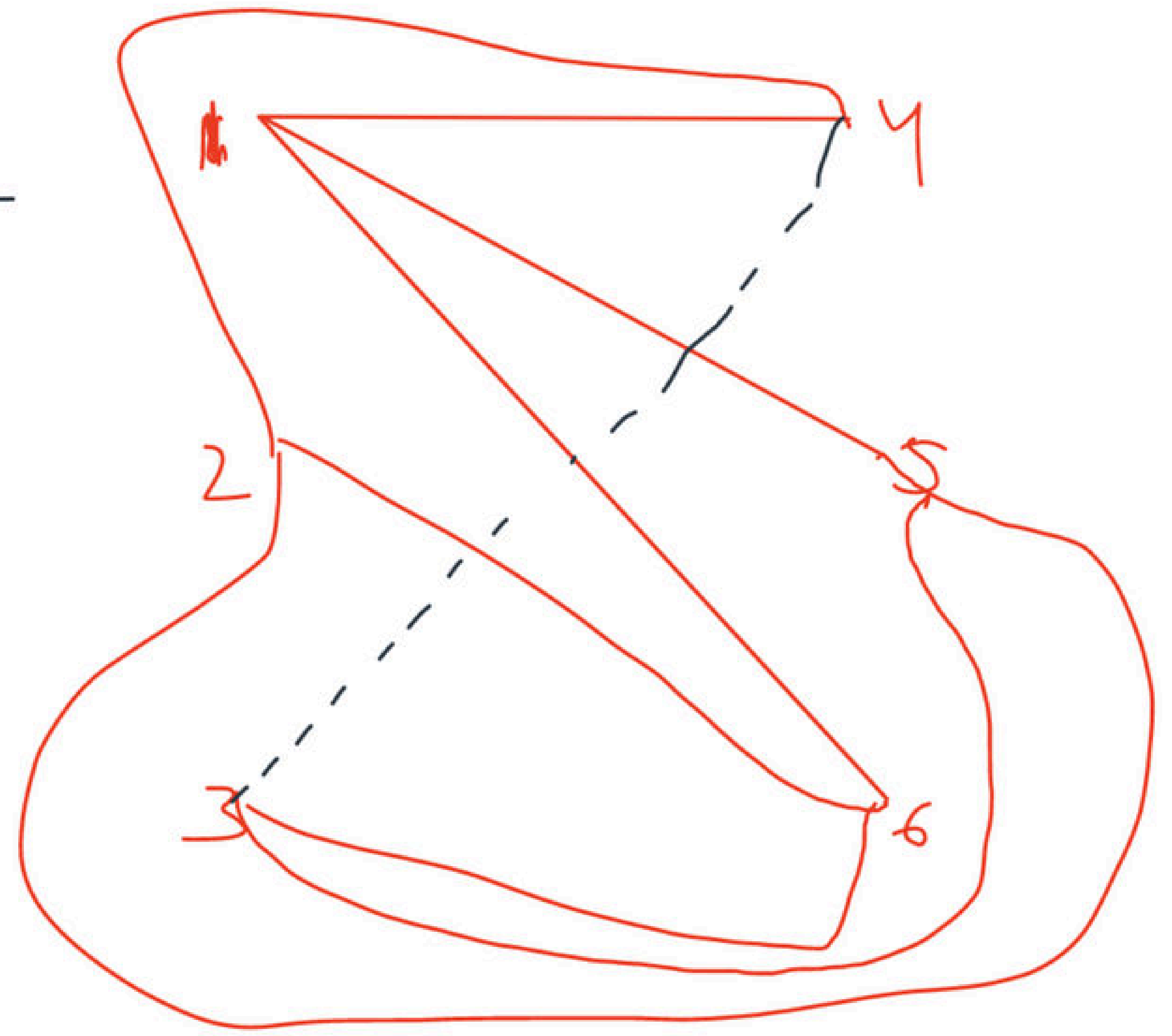
$K_{2,3}$

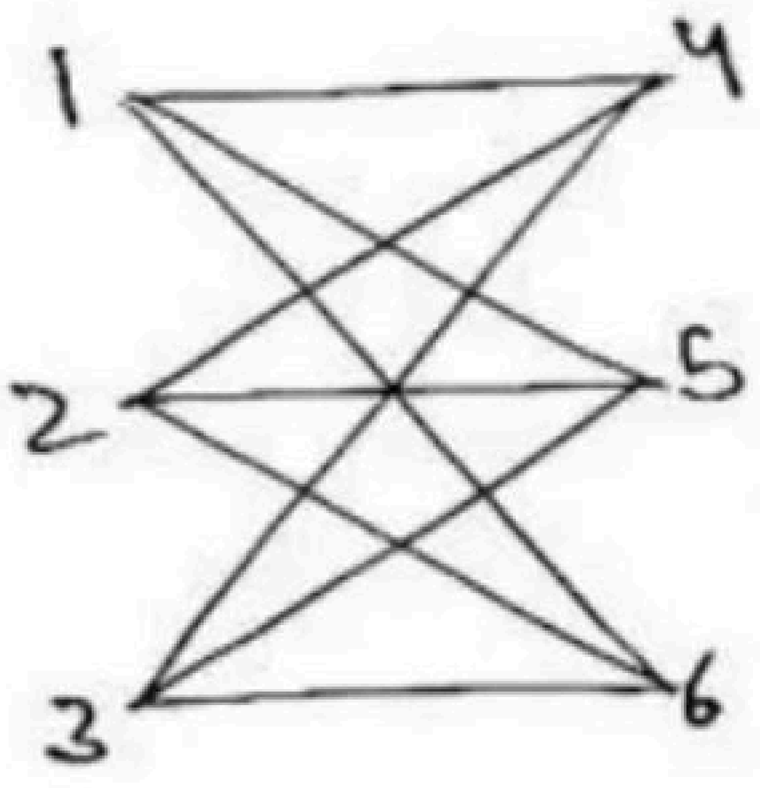


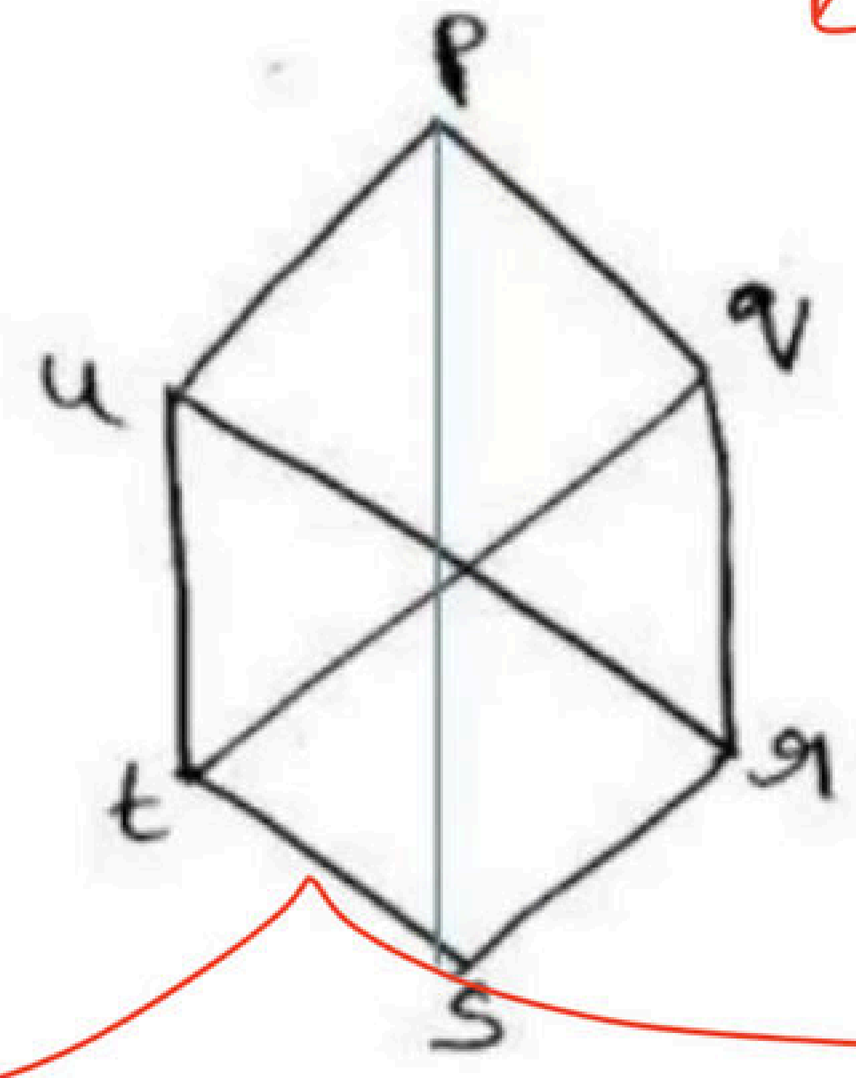
$K_5 \longrightarrow |V| = 5 \quad |E| = 10$

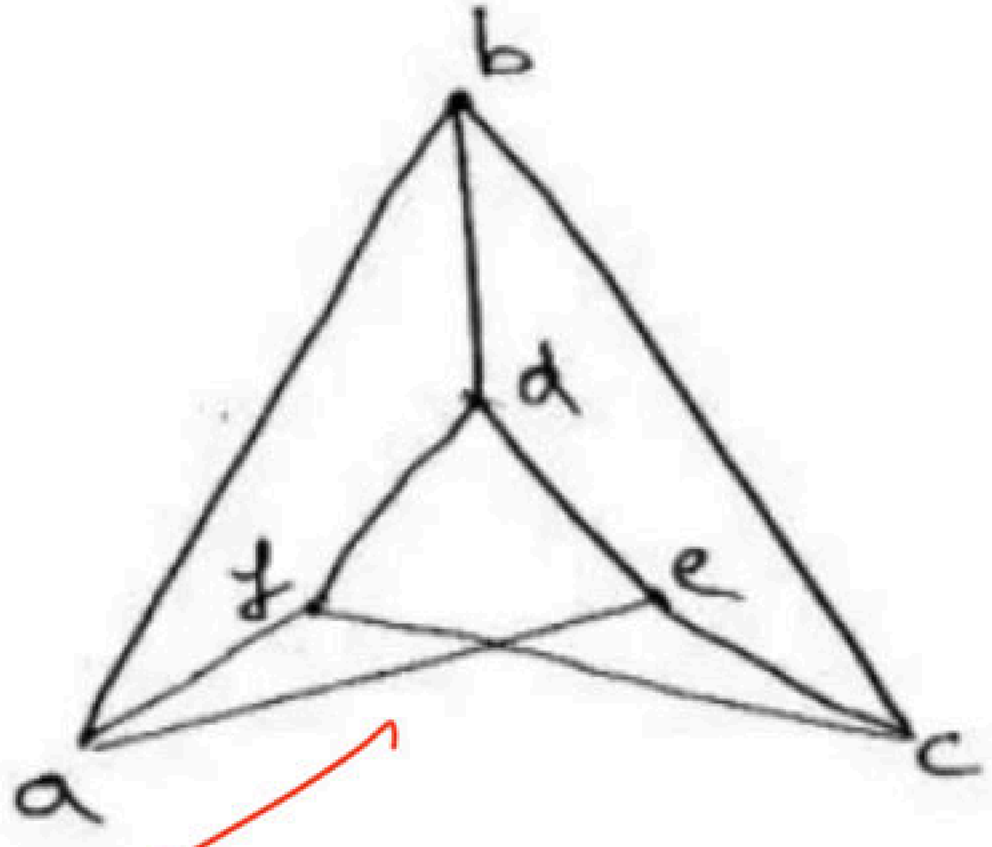




$K_{3,3} \quad |V| = 6 \quad |E| = 9$



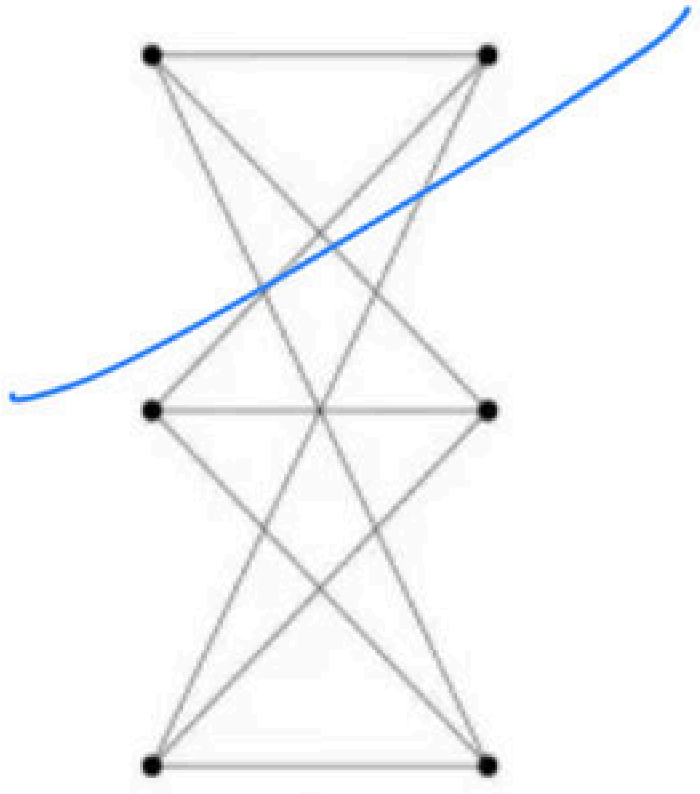




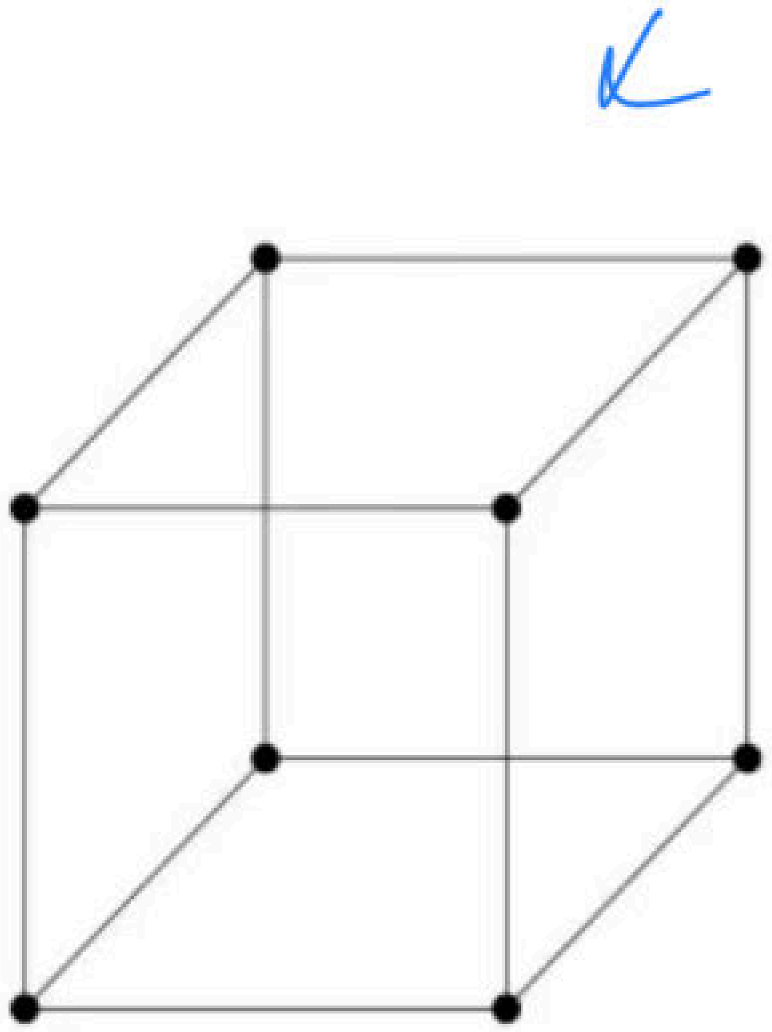
K 3x3

Q Which of the following graphs is/are planner? **(GATE-19) (2 Marks)**

$K_{3,3}$

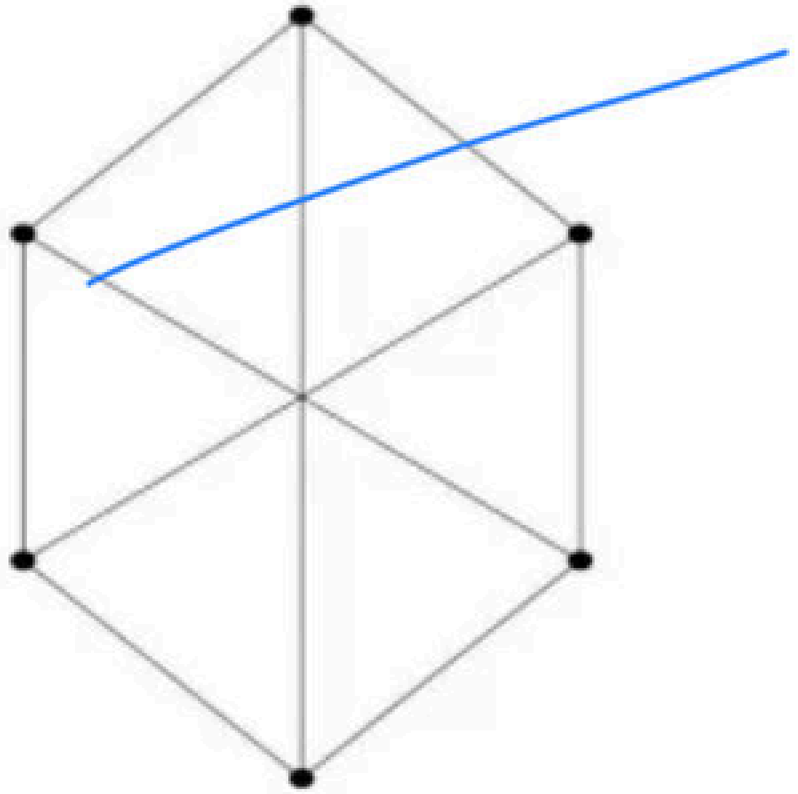


G_1



G_2

$K_{8,3}$



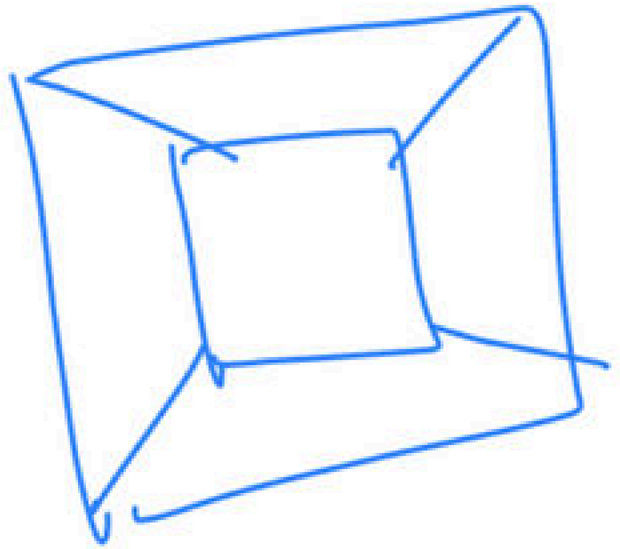
G_3

a) G1 only — 5

b) G1 and G2 — 8

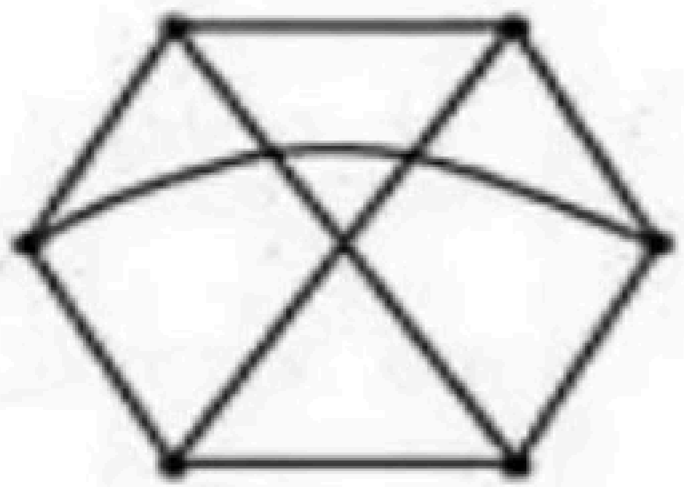
c) G2 only — 7 8

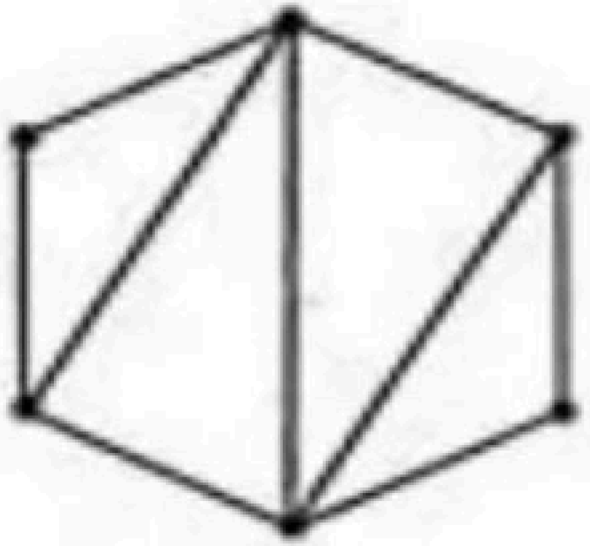
d) G2 and G3 — 9

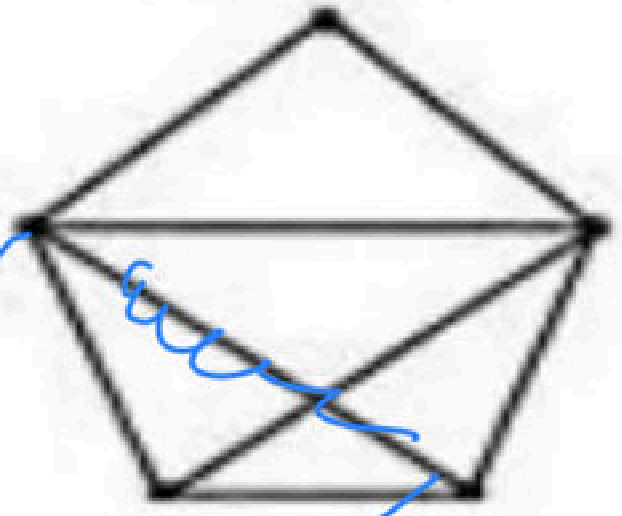


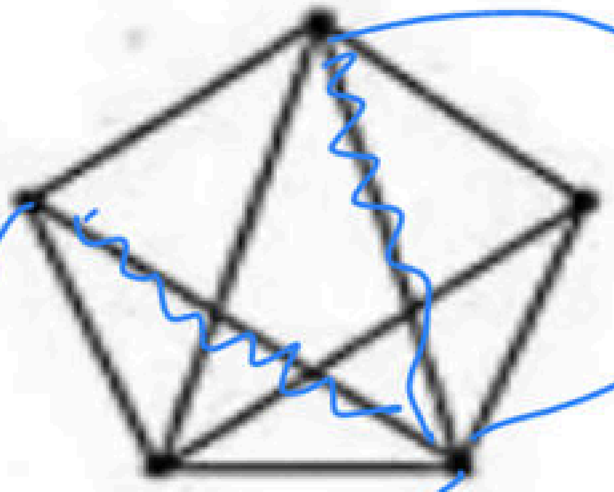
Q Which one of the following graphs is NOT planar? **(GATE-2005) (2 Marks)**

$K_{3,3}$

G1: 

G2: 

G3: 

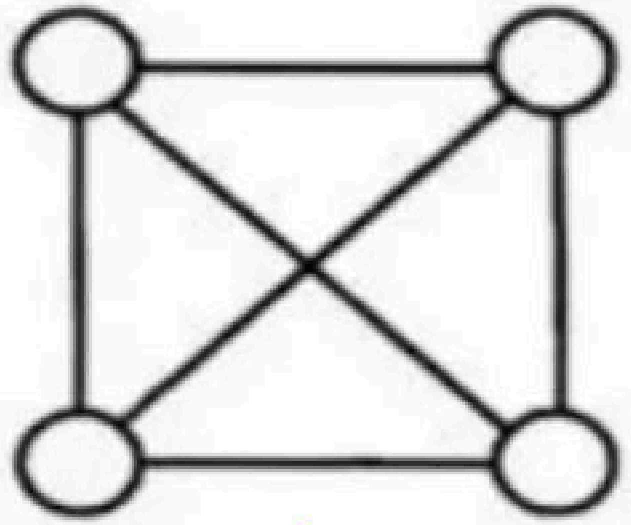
G4: 

(A) G1 **(B)** G2 **(C)** G3 **(D)** G4

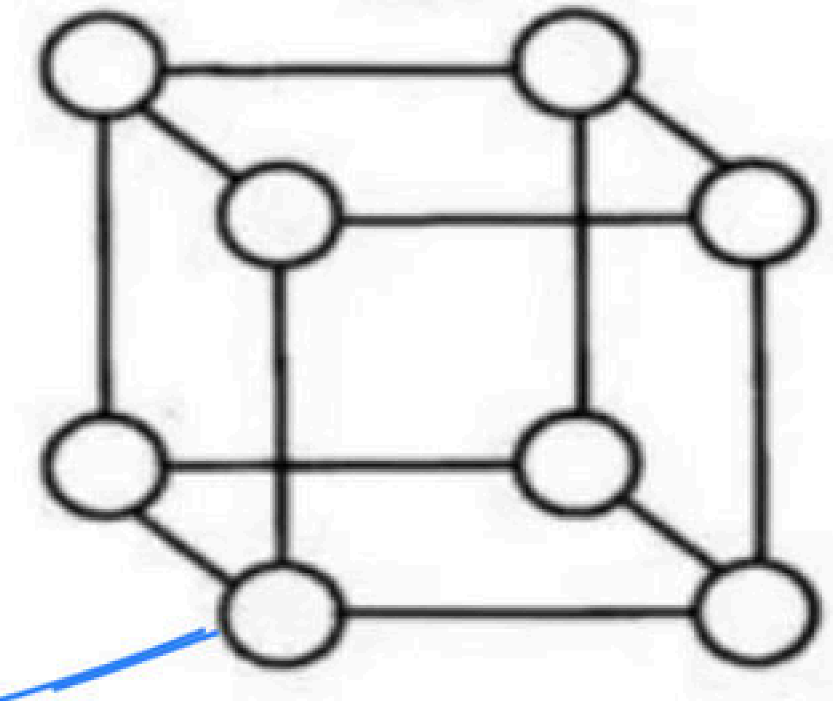
61 5 6 29

Q (GATE-2010) (2 Marks)

K4



Q3



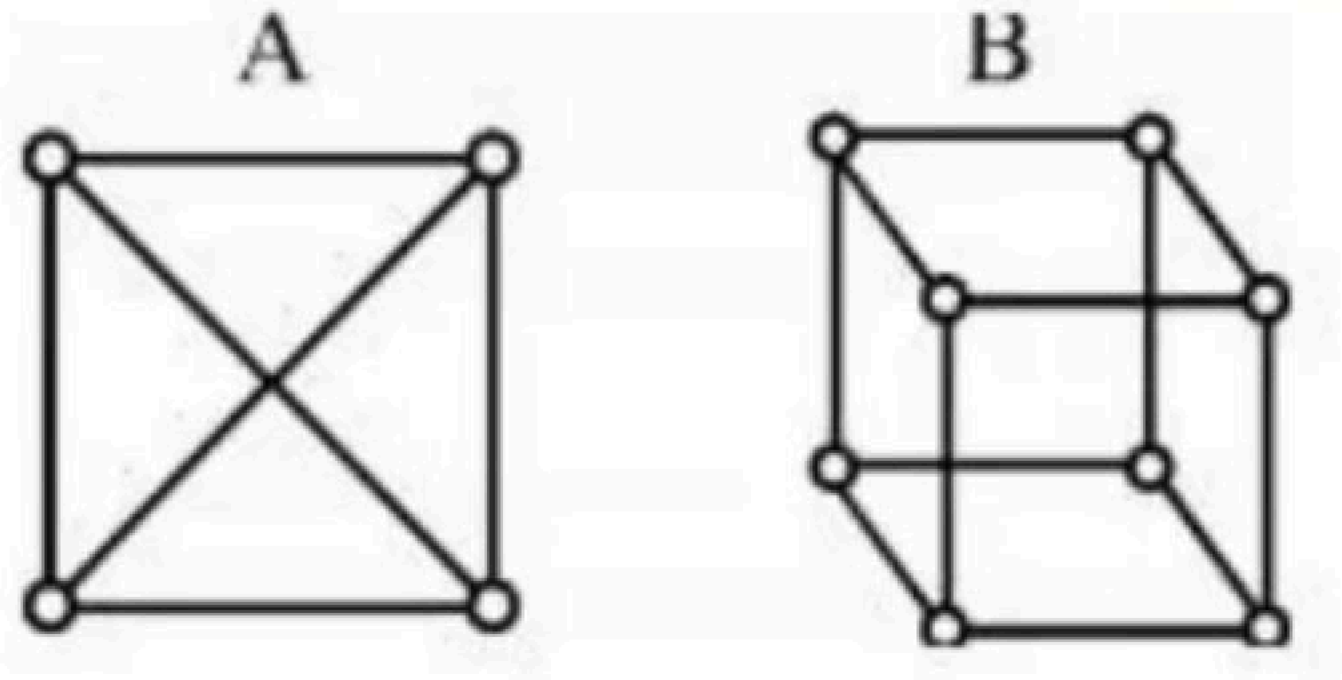
(A) K4 is planar while Q3 is not — 2

(B) ~~Both K4 and Q3 are planar~~ — 8 6

(C) Q3 is planar while K4 is not — 8

(D) Neither K4 nor Q3 are planar — 4

Q Two graphs A and B are shown below: Which one of the following statements is true? **(NET-DEC-2015)**



(a) Both A and B are planar

b) Neither A nor B is planar

(c) A is planar and B is not

d) B is planar and A is not

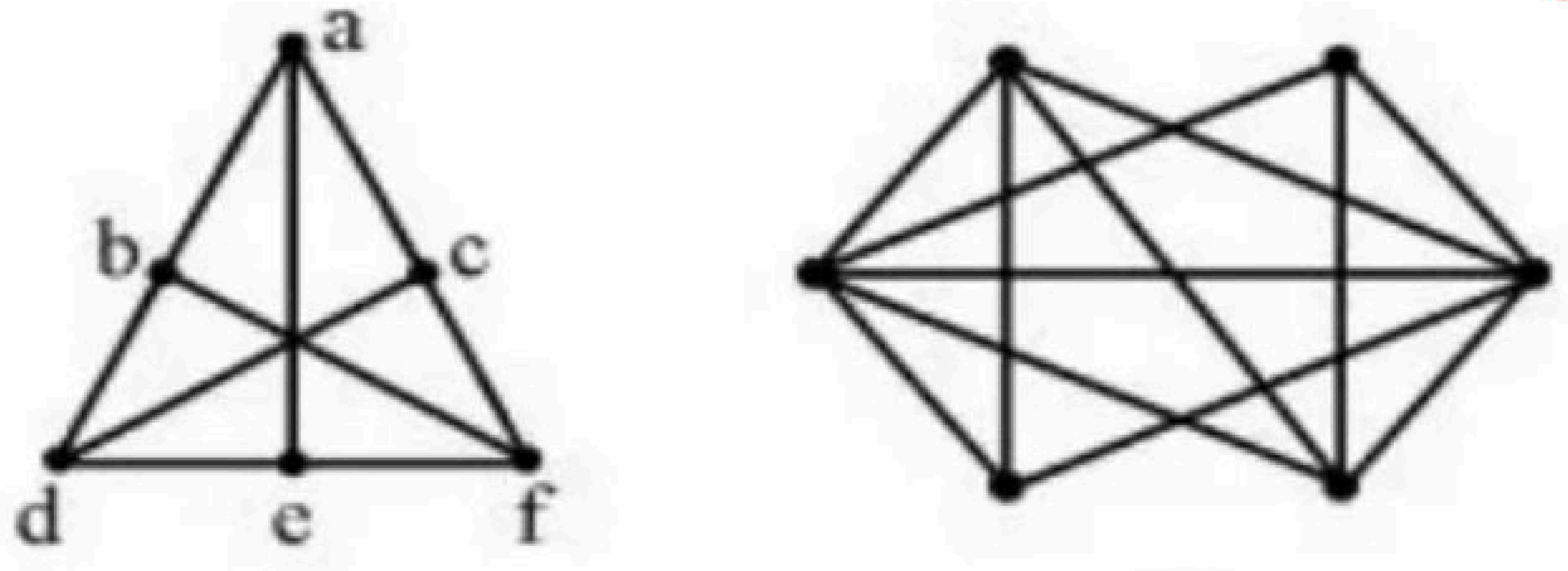
Q G1 and G2 are two graphs as shown: **(NET-JUNE-2012)**

a) ~~Both G1 and G2 are planar graphs~~ – 10

b) Both G1 and G2 are not planar graphs – 38

c) ~~G1 is planar and G2 is not planar~~ – 12

d) G1 is not planar and G2 is planar – 39

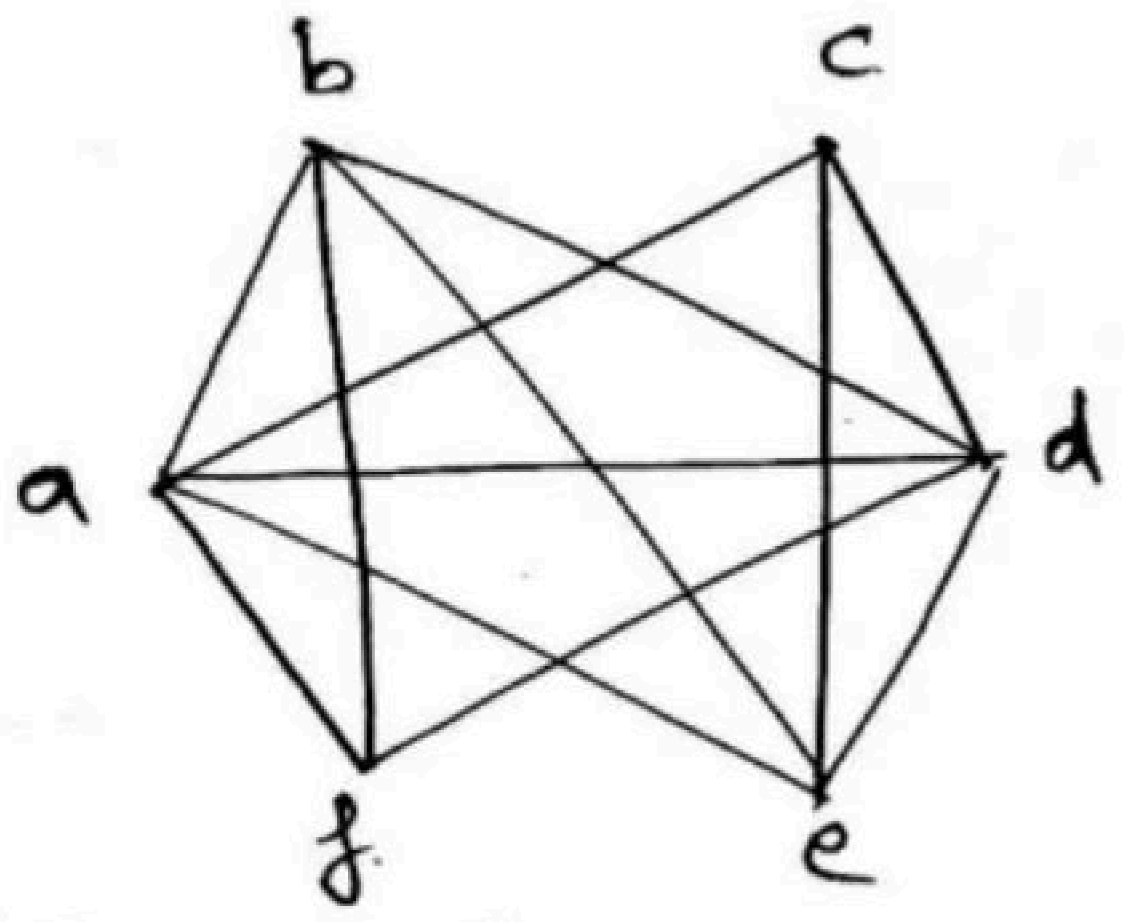


G1 G2

$K_{3,3}$

9 , 15, 12 , 11 ,

13, 12, 10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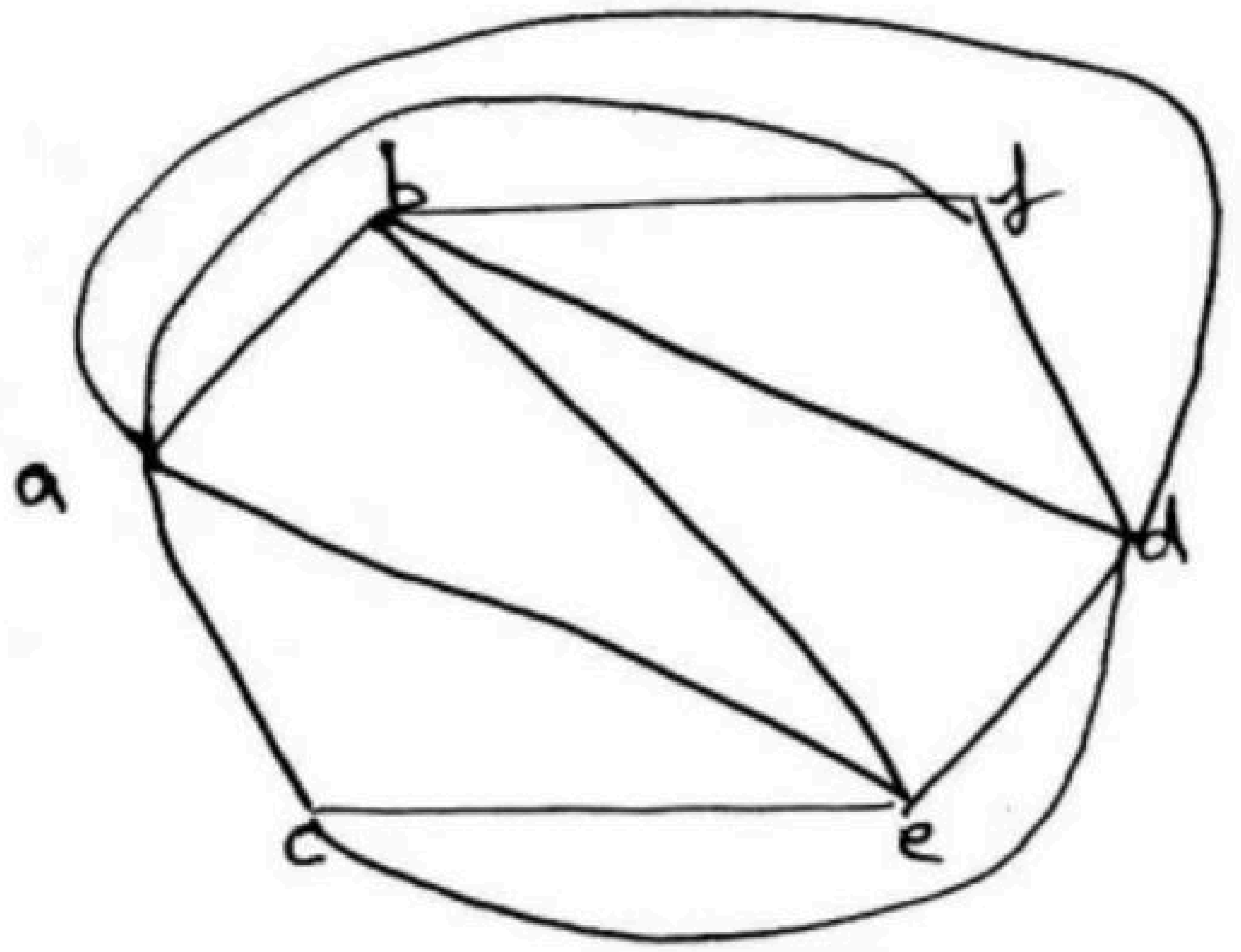
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Simplest Non-Planer Graphs

1. **Kuratowski's case I:** - K_5
2. **Kuratowski's case II:** - $K_{3,3}$
3. Both are simplest non-planer graph
4. Both are regular graph
5. If we delete either an edge or a vertex from any of the graph, they will become planer

Q Let G be the non-planar graph with the minimum possible number of edges. Then G has **(GATE-1992) (1 Marks) (GATE-2007) (1 Marks)**

(A) 9 edges and 5 vertices 20

K_{33} K_5

(B) ~~9 edges and 6 vertices~~ 45

(C) 10 edges and 5 vertices 31

(D) 10 edges and 6 vertices 4

$|V| \rightarrow K_5$

$|E| \rightarrow K_{33}$

Q A graph is planar if and only if, **(GATE-1990) (2 Marks)**

a) It does not contain subgraphs homeomorphic to k5 and k3,3.

b) It does not contain subgraphs isomorphic to k5 or k3,3.

c) It does not contain a subgraph isomorphic to k5 or k3,3

d) It does not contain a subgraph homeomorphic to k5 or k3,3.

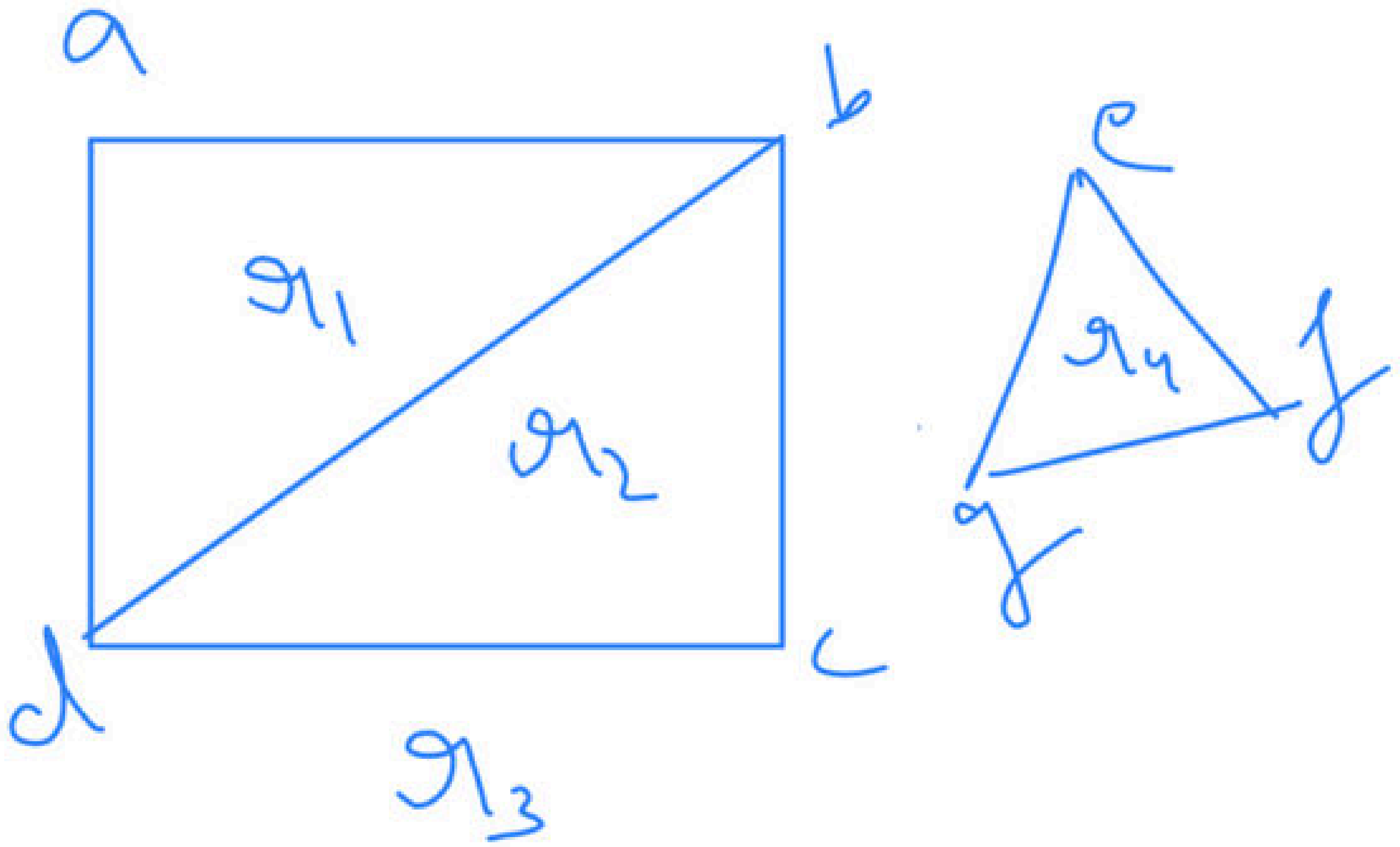
Q A graph is non-planar if and only if it contains a subgraph homomorphic to **(NET-DEC-2013)**

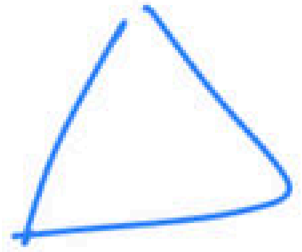
(A) $K_{3,2}$ or K_5 **(B)** $K_{3,3}$ and K_6 **(C)** $K_{3,3}$ or K_5 **(D)** $K_{2,3}$ and K_5

- How to find weather a graph is planer or non-planer
    - A finite graph is planar if and only if it does not contain a subgraph that is a subdivision(homorphism) of the complete graph K_5 or the complete bipartite graph. In practice, it is difficult to use Kuratowski's criterion to quickly decide whether a given graph is planar.
    - A finite graph is planar if and only if it does not have K_5 or $K_{3,3}$ as a minor. A minor of a graph results from taking a subgraph and repeatedly contracting an edge into a vertex, with each neighbor of the original end-vertices becoming a neighbor of the new vertex. (Wanger's theorem)

Further Analysis

1. A planer graph divides the plane into number or regions (faces, planer embedding), which are further divided into bounded(internal) and unbounded region(external).
2. **Euler's formula** states that if a finite, connected, planar graph with *v* is the number of vertices, *e* is the number of edges and *r* is the number of faces (regions bounded by edges, including the outer, infinitely large region), then
3. $\boxed{r = e - v + 2}$
4. Euler's formula can be proved by mathematical induction
5. Euler's formula (Disconnected graph): $V - e + r - k = 1$





$\cancel{7} - \cancel{8} + 4 - 2 = 1$

$-1 + 4 - 2$

$+1 = +1$

$r = e - v + 2$

$= \cancel{5} - \cancel{4} + \cancel{2}$

$= 3$

$r_1 \quad r_2 \qquad r_3$

Q suppose that a connected planar graph has six vertices, each of degree four, into how many regions is the plane divided by a planner representation of this graph? **(NET-JULY-2019)**

a) 6 **b)** 8 **c)** 12 **d)** 20

$|V| = 6$

$\deg(v_i) = 4$

$6 \times 4 = 24 = 2|E|$

$12 = E$

31.5%

$= e - v + 2$

$= 12 - 6 + 2$

$6 + 2 = 8$

Q Let G be a simple connected planar graph with 13 vertices and 19 edges. Then, the number of faces in the planar embedding of the graph is **(GATE-2005) (2 Marks)**

(A) 6 — 10

(B) 8 — 83

(C) 9 — 7

(D) 13 — 0

$v = 13$

$e = 19$

$= e - v + 2$

$= \cancel{19} - \cancel{13} + 2$

$= 6 + 2$

$= 8$

Q Let G be a simple undirected planar graph on 10 vertices with 15 edges. If G is a connected graph, then the number of bounded faces in any embedding of G on the plane is equal to **(GATE-2012) (1 Marks)**

(A) 3 (B) 4 (C) 5 (D) ~~6~~

1 2 7 67

24

$v = 10$

$e = 15$

$r = e - v + 2$

$= \cancel{15} - \cancel{10} + 2$

$5 + 2$

$= 7$

6 1

Other formula derived from Euler's formula

1. Connected planar graphs with more than one edge obey the inequality $2e>=3r$, because each face has at least three face-edge incidences and each edge contribute exactly two incidences.
2. Degree of the region is number of edges covering the region. Sum of degree of regions = $2|E|$
3. Using $\boxed{r = e - v + 2}$ and $3r<=2e$, eliminating r we get, $e \le 3v - 6$
4. Using $r = e - v + 2$ and $3r<=2e$, Eliminating e we get, $r \le 2v - 4$

$$r = e - v + 2 \quad \text{—— (1)}$$

$$2e \geq 3r \quad \text{—— (2)}$$

$$\boxed{e \geq \frac{3}{2} r}$$

$$\boxed{\frac{2e}{3} \geq r}$$

$$r \geq \frac{3}{2} r - V + 2$$

$$r - \frac{3}{2} r \geq -V + 2$$

$$\frac{-r}{2} \geq -V + 2$$

$$-r \geq -2v + 4$$

$$\boxed{2v - 4 \geq r}$$

$$\frac{2e}{3} \geq e - V + 2$$

$$\frac{-e}{3} \geq -V + 2$$

$$-e \geq -3V + 6$$

$$\boxed{3v - 6 \geq e}$$

Q maximum number of edges in a planar graph with n vertices **(GATE-1992) (1 Marks)**

Q A graph $G = (V, E)$ satisfies $|E| \leq 3|V| - 6$. The min-degree of G is defined as $\min_{v \in V} \{\text{degree}(v)\}$

Therefore, min-degree of G cannot be **(GATE-2003) (2 Marks)**

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6

Q Let δ denote the minimum degree of a vertex in a graph. For all planar graphs on n vertices with $\delta \geq 3$, which one of the following is TRUE? **(GATE-2014) (2 Marks)**

(A) In any planar embedding, the number of faces is at least $n/2 + 2$

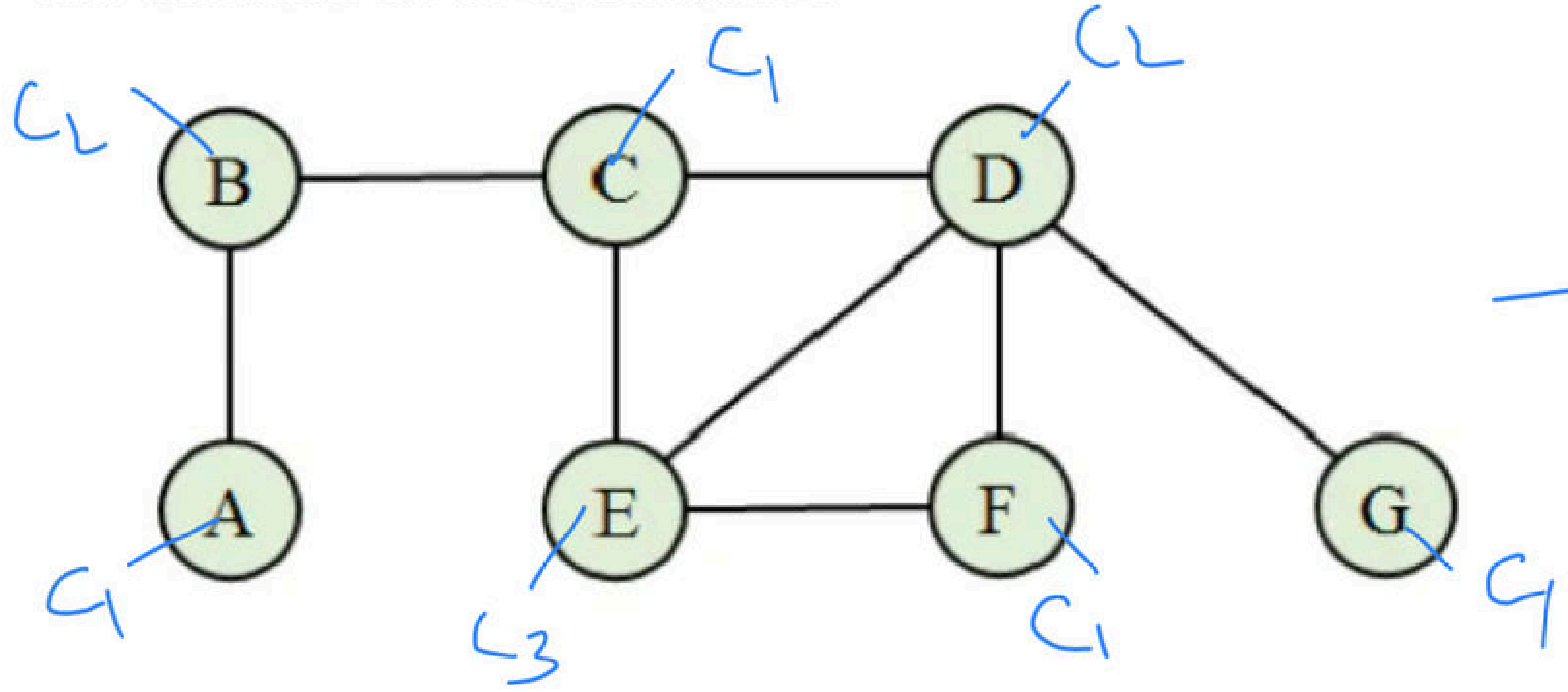
(B) In any planar embedding, the number of faces is less than $n/2 + 2$

(C) There is a planar embedding in which the number of faces is less than $n/2 + 2$

(D) There is a planar embedding in which the number of faces is at most $n/(\delta + 1)$

Graph Coloring

1. Graph coloring can be of two types vertex coloring and edge coloring.
2. Associating a color with each vertex of the graph is called vertex coloring.
3. **Proper Vertex coloring:** - Associating all the vertex of a graph with colors such that no two adjacent vertices have the same color is called proper vertex coloring.
4. **Chromatic number of the graph:** - Minimum number of colors required to do a proper vertex coloring is called the chromatic number of the graph, denoted by $\chi(G)$. the graph is called K-chromatic or K-colorable.

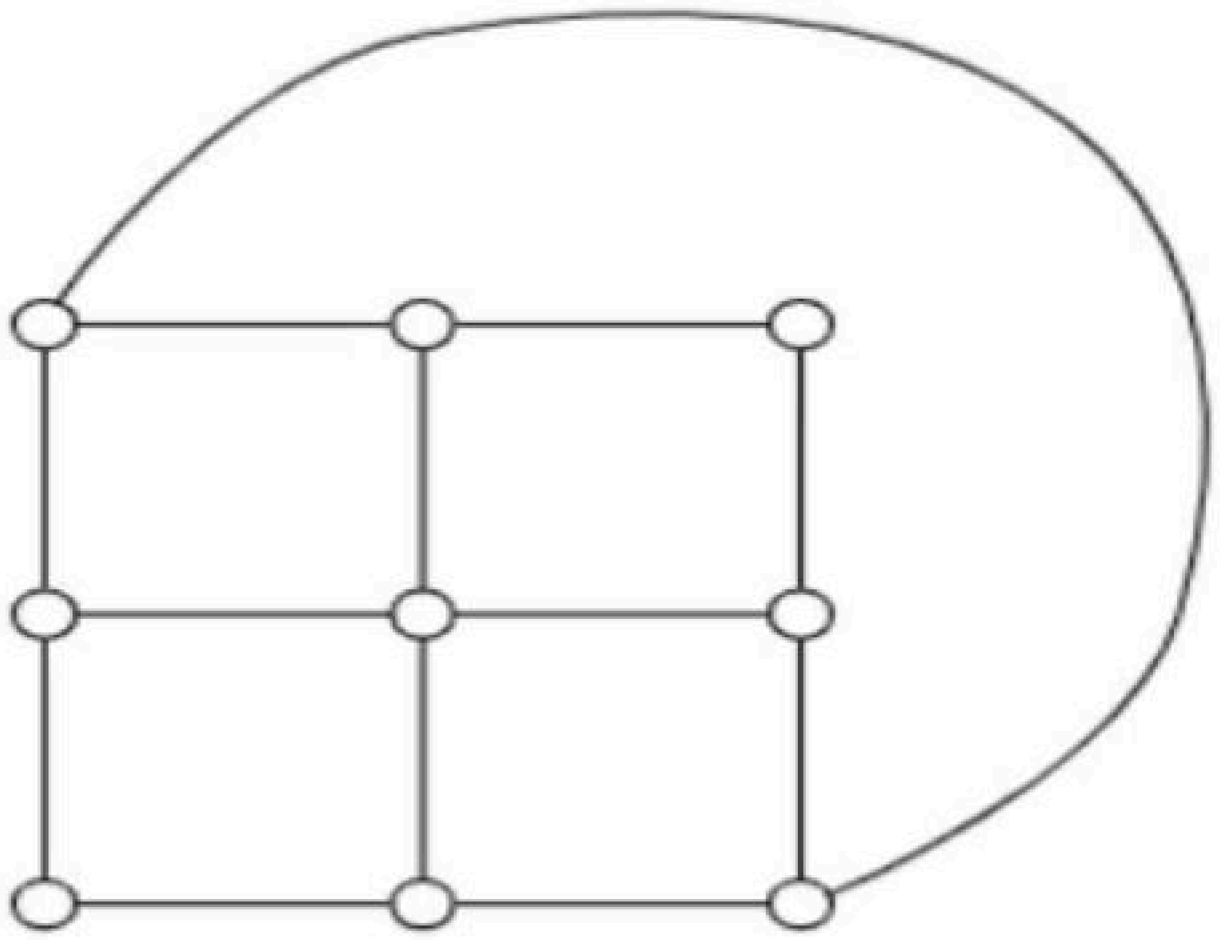


$\chi(G) = 3$

- Cost of finding chromatic number is an NPC problem and there exists no polynomial algorithm to do that. There exists some greedy approach which try to solve it in P time, but they do not guarantee optimal solution.

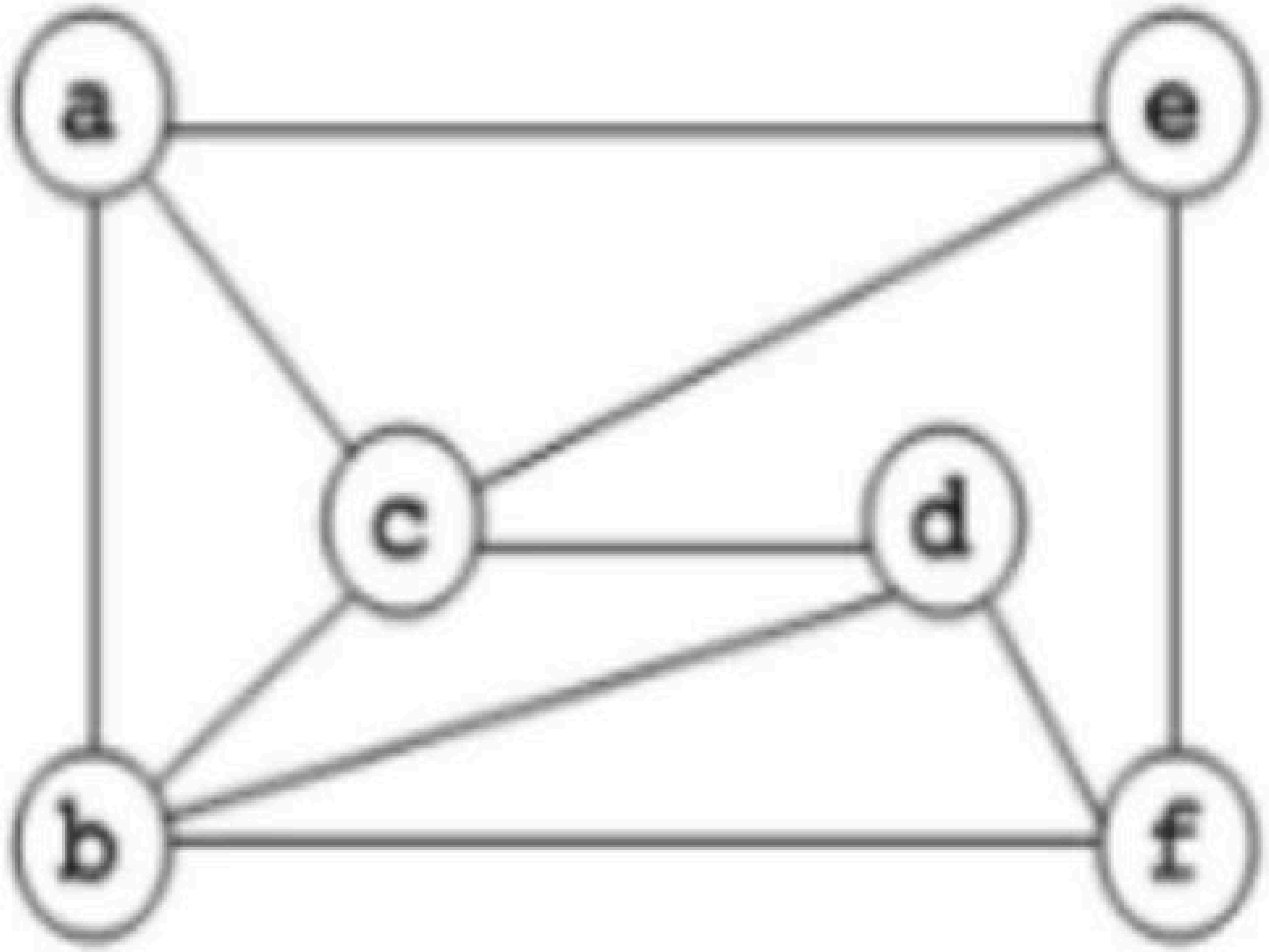
1. Trivial graph is 1-chromatic
2. A graph with 1 or more edge is at least 2-chromatic
3. A complete graph K_n is n-chromatic
4. Tree is always 2-chromatic
5. Bi-partite graph is 2-chromatic
6. C_n is 2-chromatic if n is even, C_n is 3-chromatic if n is odd
7. 5-color theorem-any planer graph is at most 5-chromatic
8. 4-colour theorem/hypothesis- any planer graph is 4-chromatic
9. If $\Delta(G)$ is the maximum degree of any vertex in a graph then, $\chi(G) <= 1 + \Delta(G)$

Q What is the chromatic number of the following graph? **(GATE-2008) (1 Marks)**

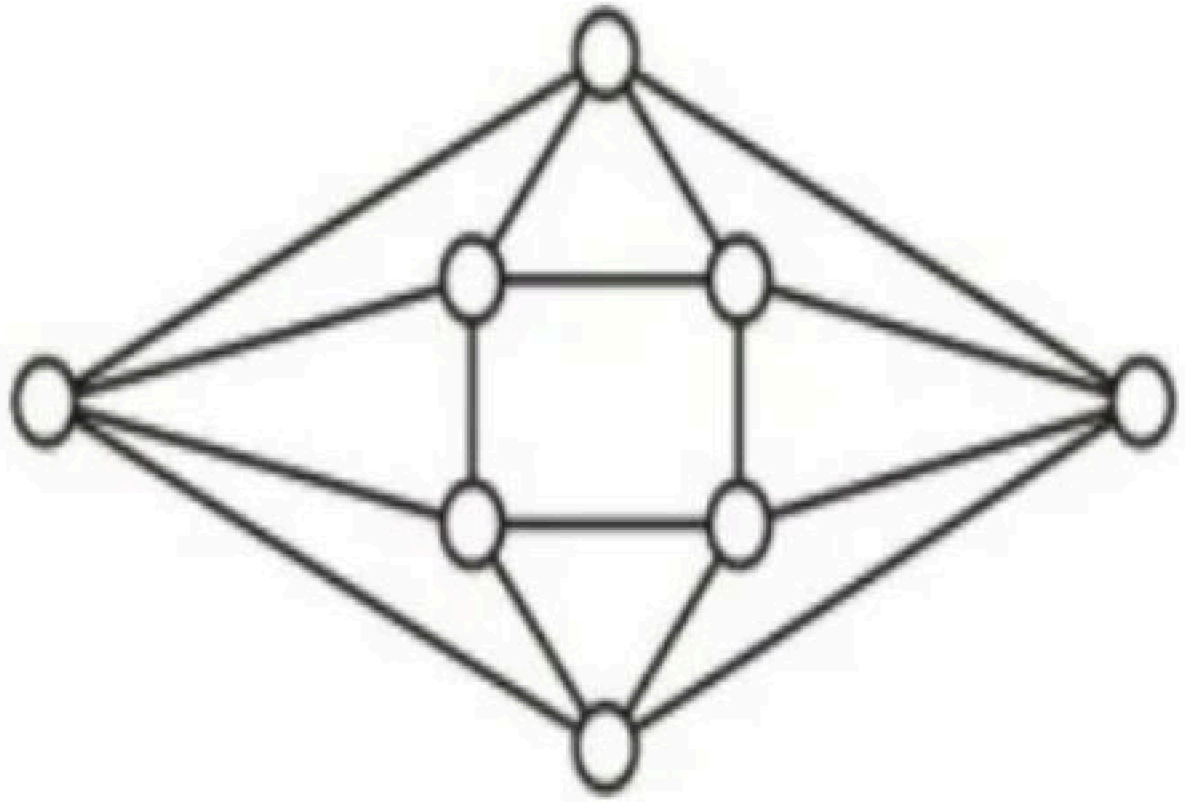


(A) 2 **(B)** 3 **(C)** 4 **(D)** 5

Q The chromatic number of the following graph is _______ **(GATE-2018) (2 Marks)**



Q The minimum number of colors required to color the following graph, such that no two adjacent vertices are assigned the same color, is **(GATE-2004) (2 Marks)**



(A) 2 **(B)** 3 **(C)** 4 **(D)** 5

Q The minimum number of colors that is sufficient to vertex color any planar graph is ________________**(GATE-2016) (1 Marks)**

Q What is the chromatic number of an n-vertex simple connected graph which does not contain any odd length cycle? Assume n >= 2. **(GATE-2009) (1 Marks)**

(A) 2 **(B)** 3 **(C)** n-1 **(D)** n

Q The minimum number of colors required to color the vertices of a cycle with η nodes in such a way that no two adjacent nodes have the same color is **(GATE-2002) (1 Marks)**

(A) 2 **(B)** 3 **(C)** 4 **(D)** $n - 2\lfloor n/2 \rfloor + 2$

Q The number of colors required to properly color the vertices of every planer graph is **(NET-JUNE-2012)**

a) 2 **b)** 3 **c)** 4 **d)** 5

Q In k-coloring of an undirected graph G = (V, E) is a function c: V → {0, 1, ..., K-1} such that c(u) !=c (v) for every edge (u,v) ∈ E. Which of the following is not correct? **(NET-DEC-2018)**

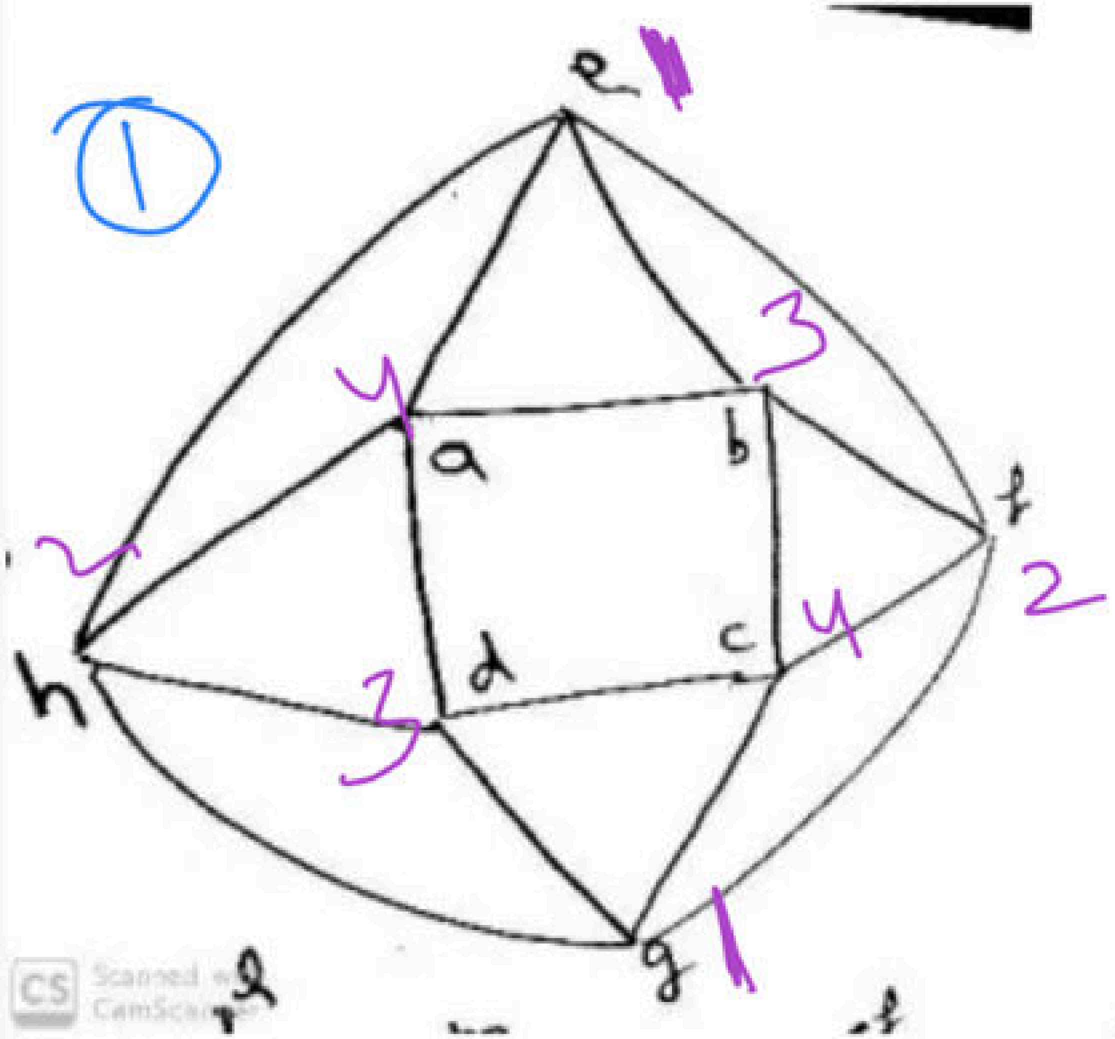
a) G is bipartite

b) G is 2-colorable

c) G has cycles of odd length

d) G has no cycles of odd length

1



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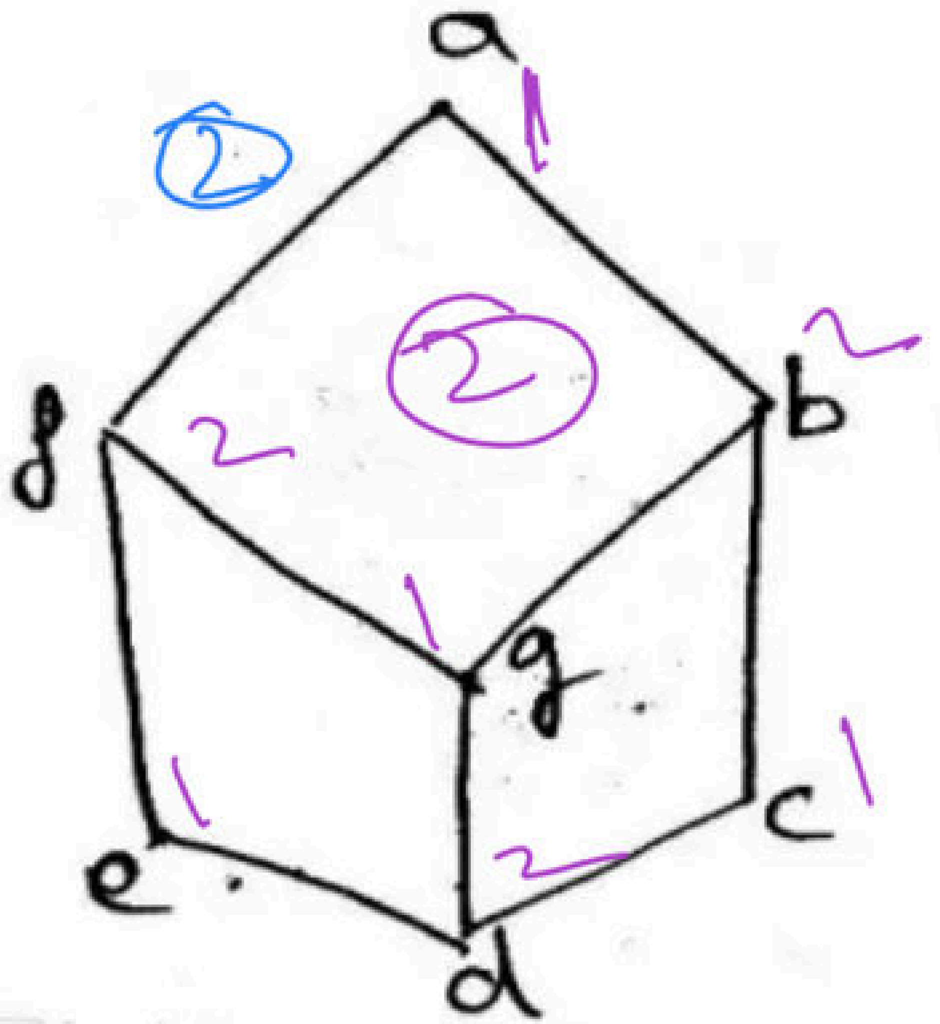

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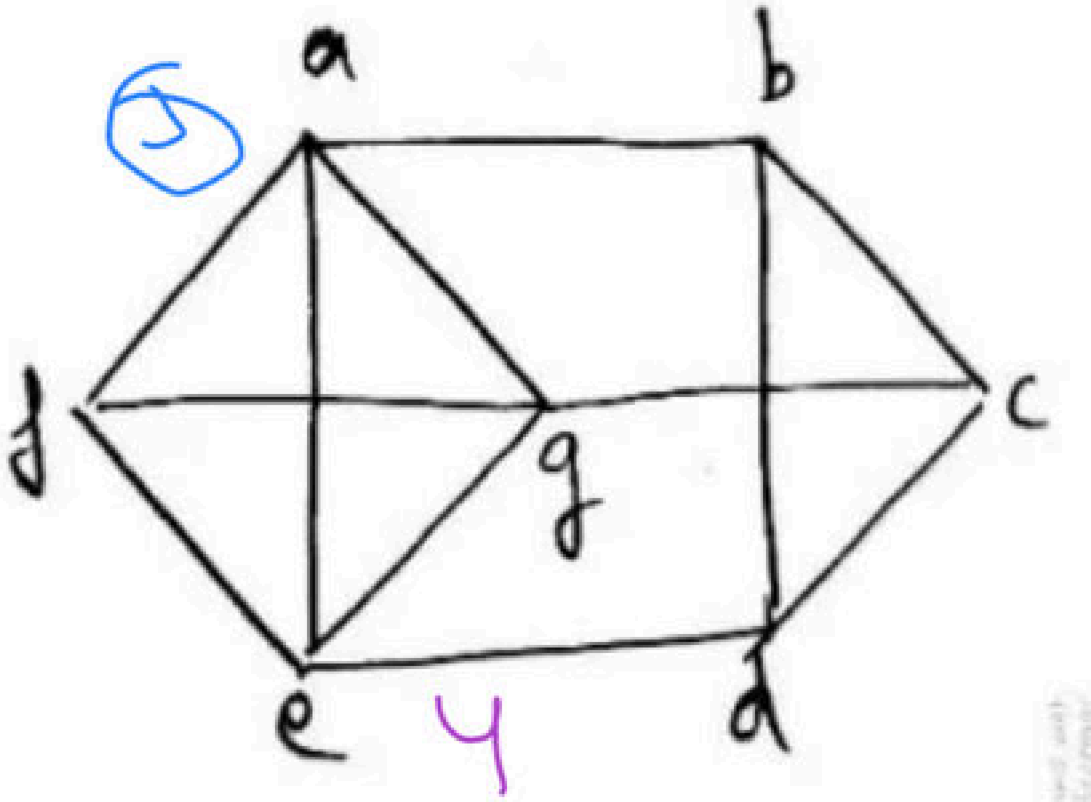

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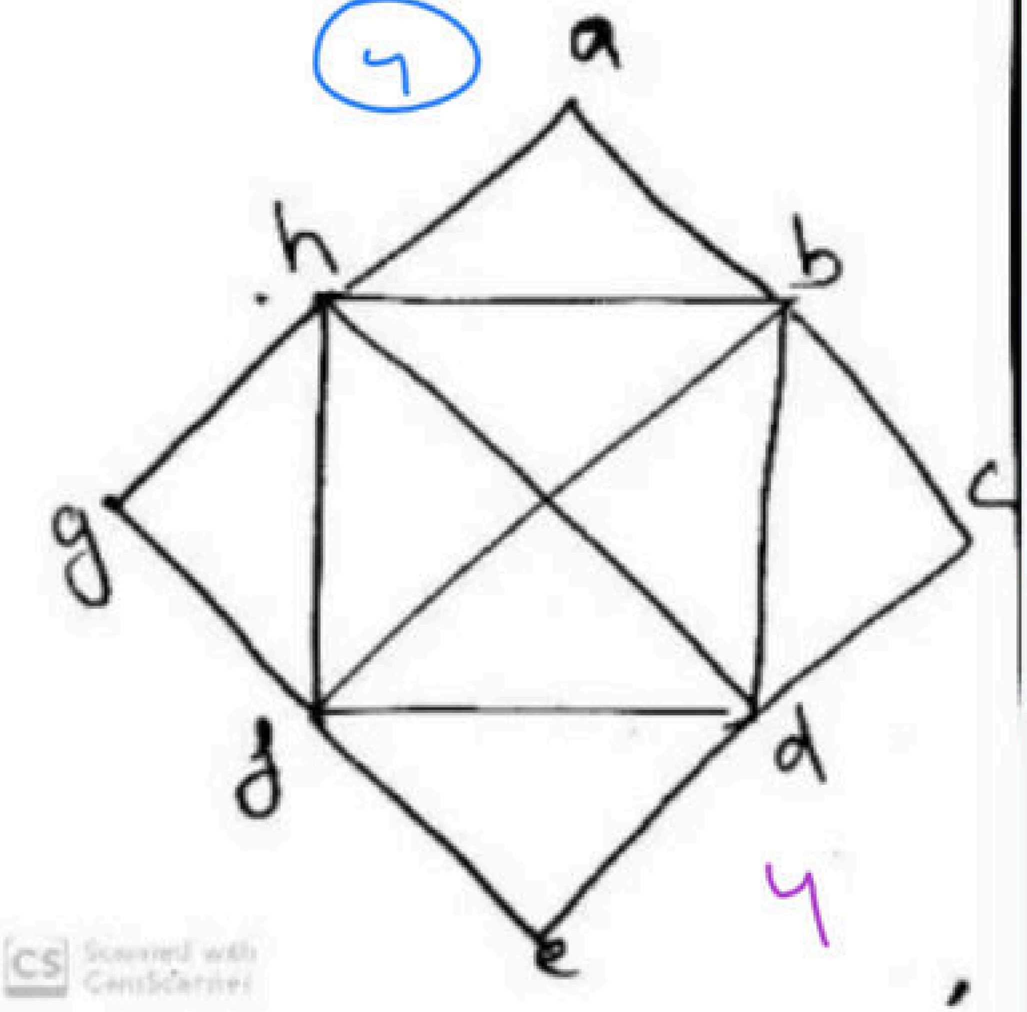

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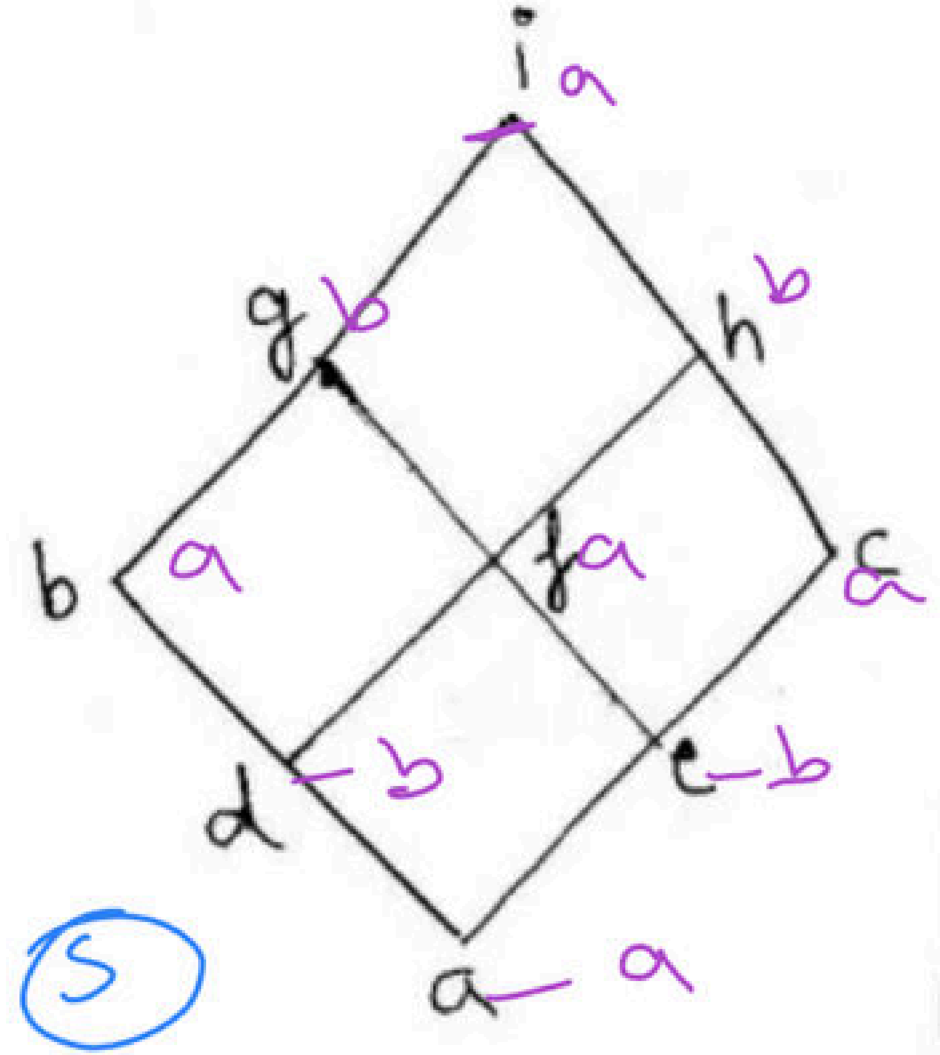

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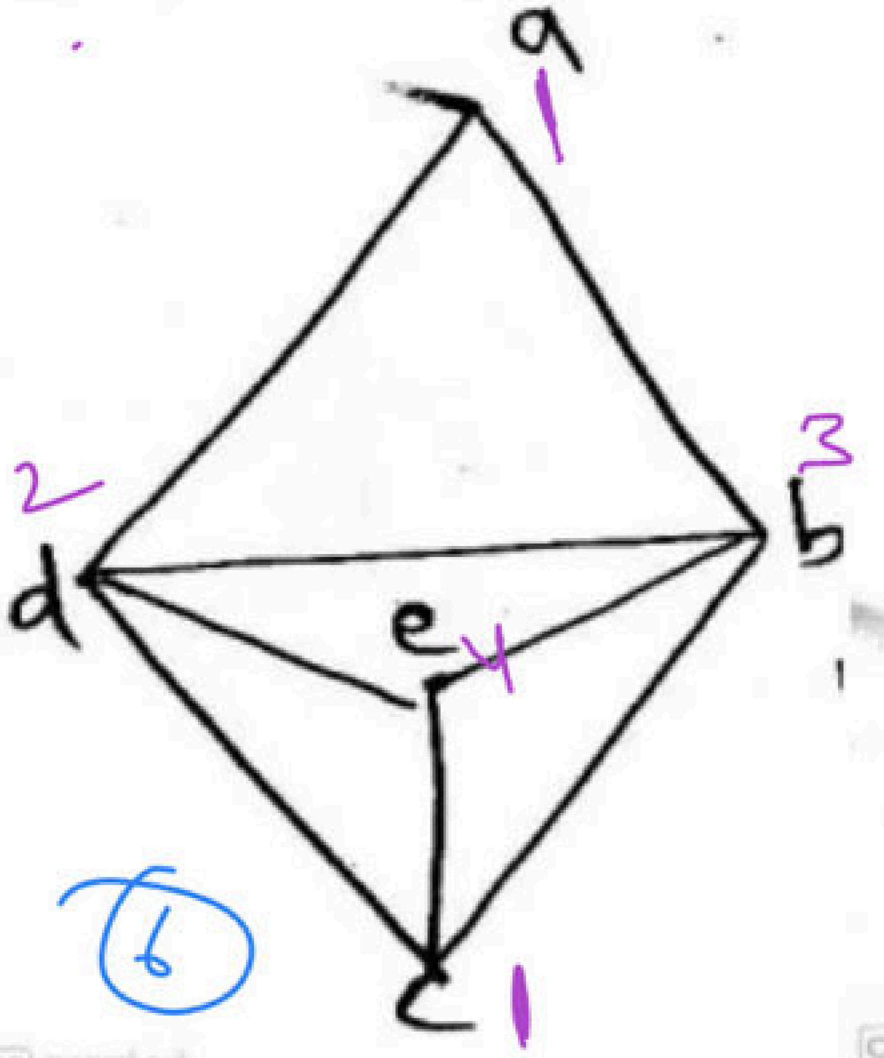

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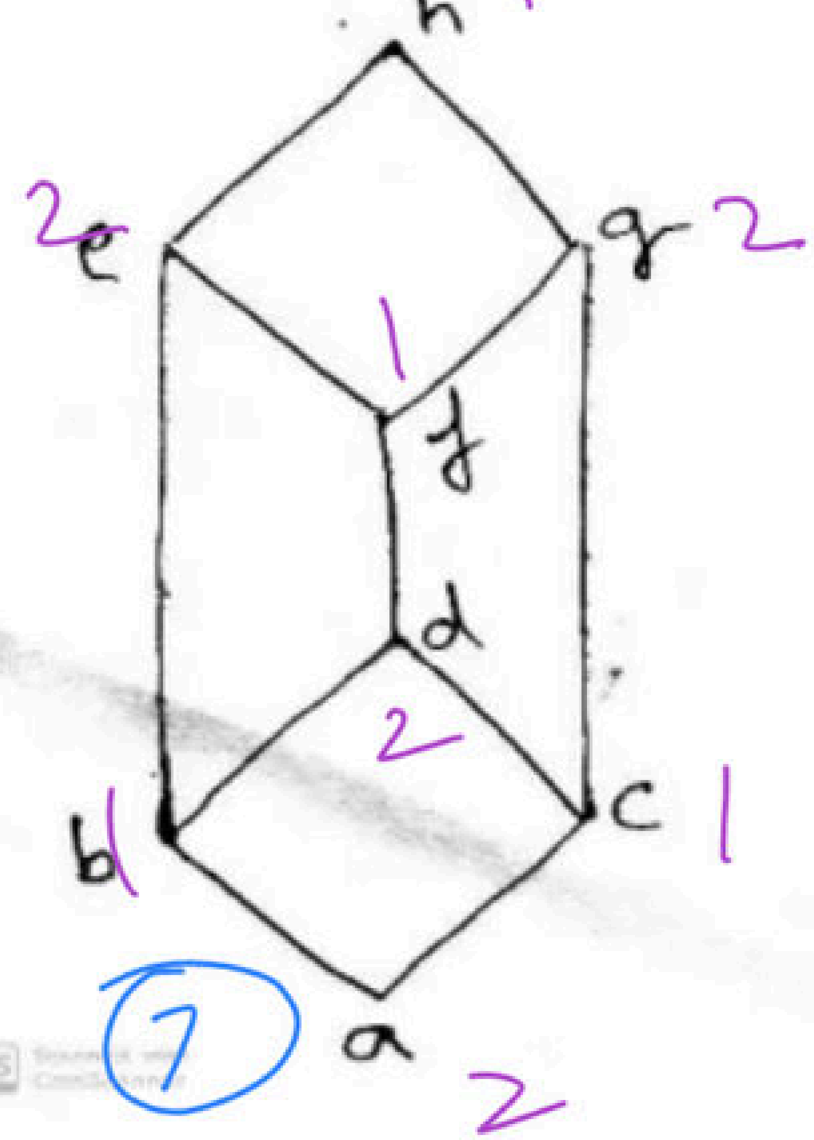
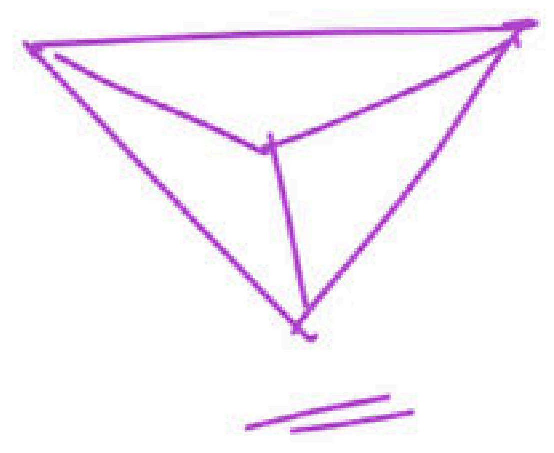
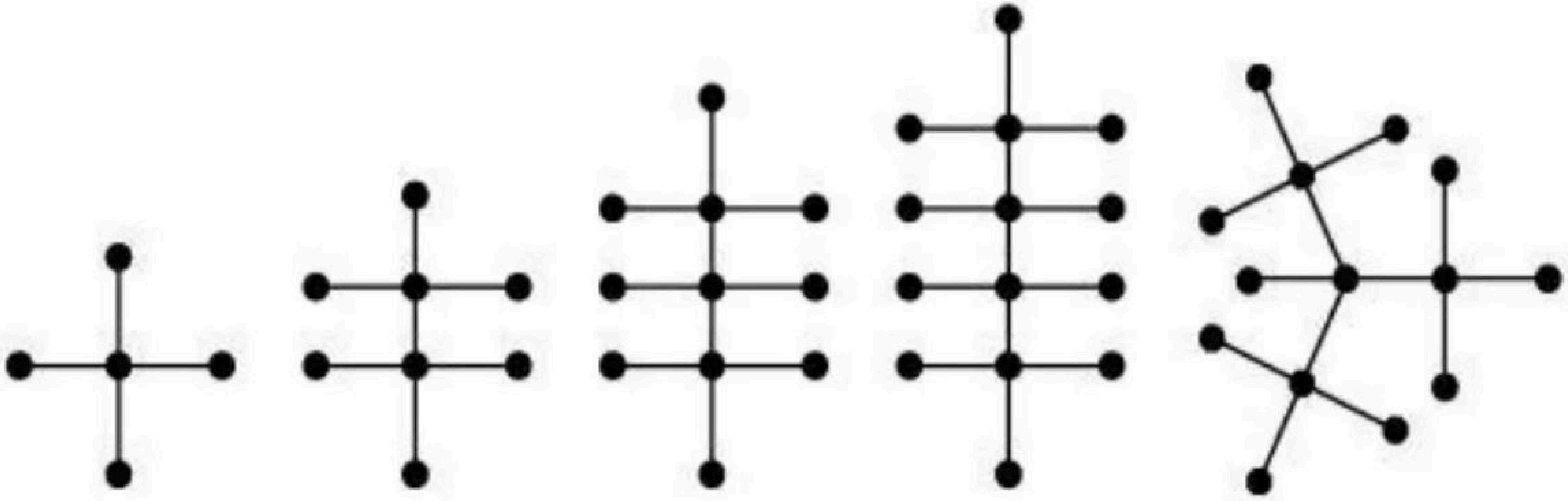




K_4

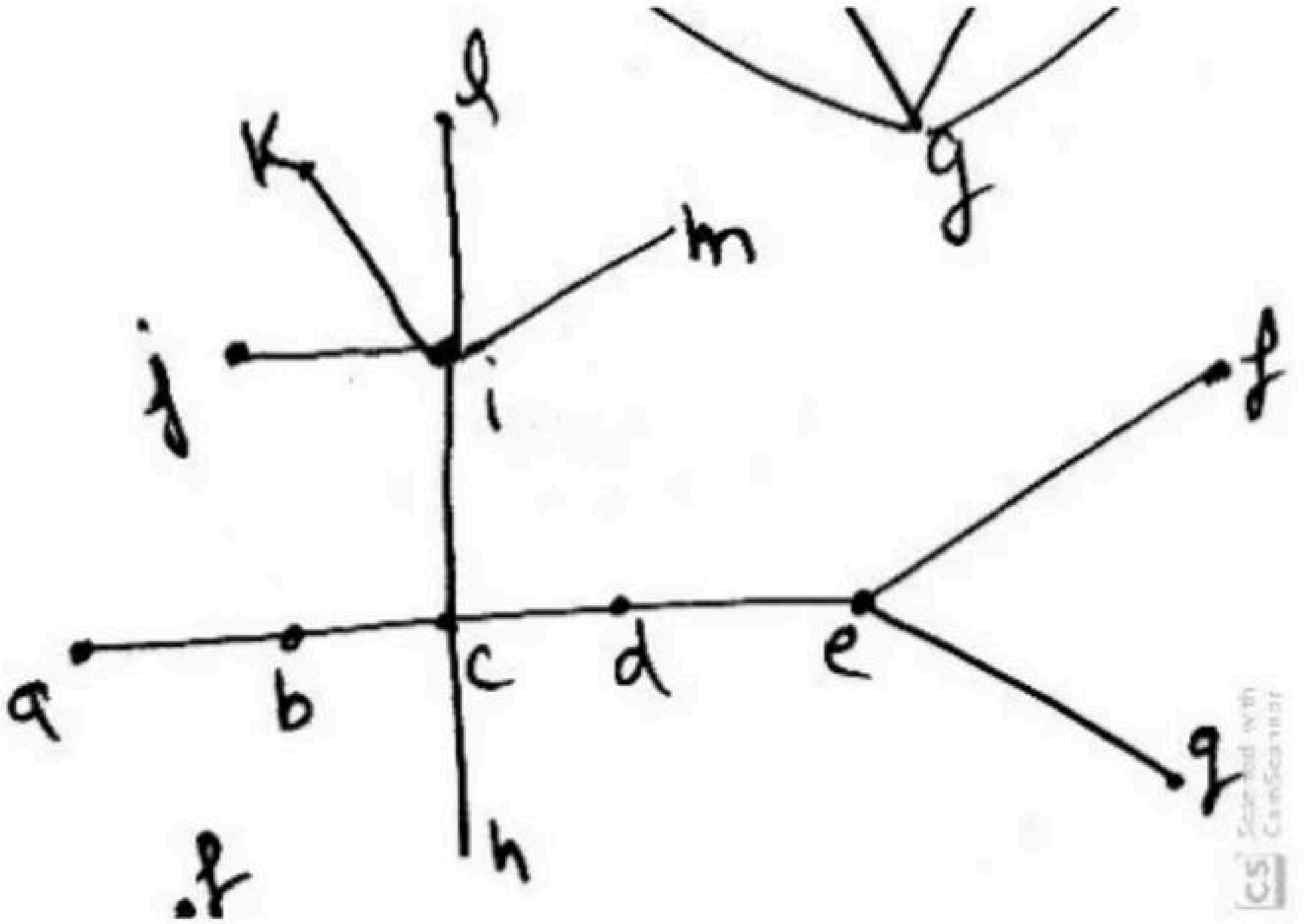
Tree

- A tree is a connected graph without any circuit.

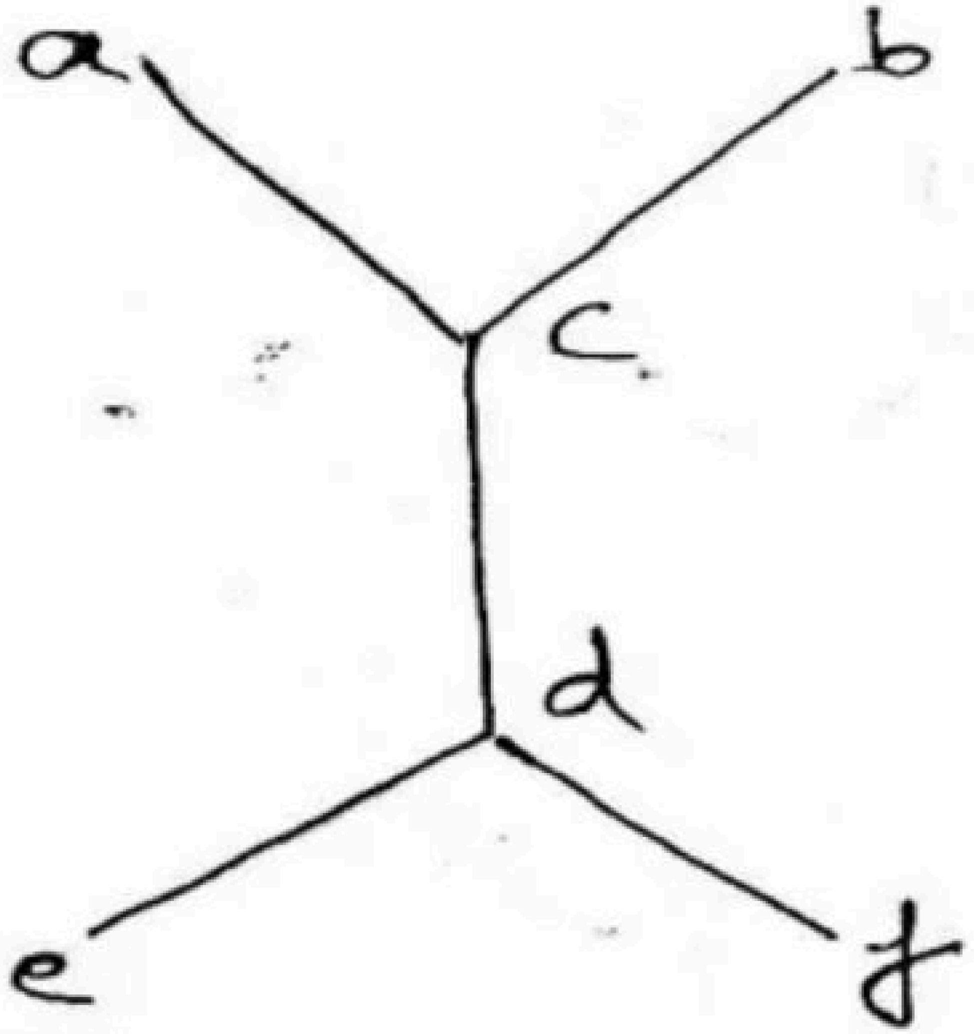


1. There is one and only one path between every pair of vertices in a tree
2. If in a graph G, there is one and only one path between every pair of vertices then G is a tree
3. A tree with n vertices has n-1 edges
4. Any connected graph with n vertices and n-1 edges in a tree
5. A graph is a tree if and only if it is minimally connected
6. A graph G with n vertices and n-1 edges and no circuit is connected

Eccentricity: - Eccentricity of a vertex is denoted by E(v) of a vertex v in a graph G, it is the distance from V to the vertex farthest from V in G. E(v) = max d(v, v_i) $v_i \in$ G

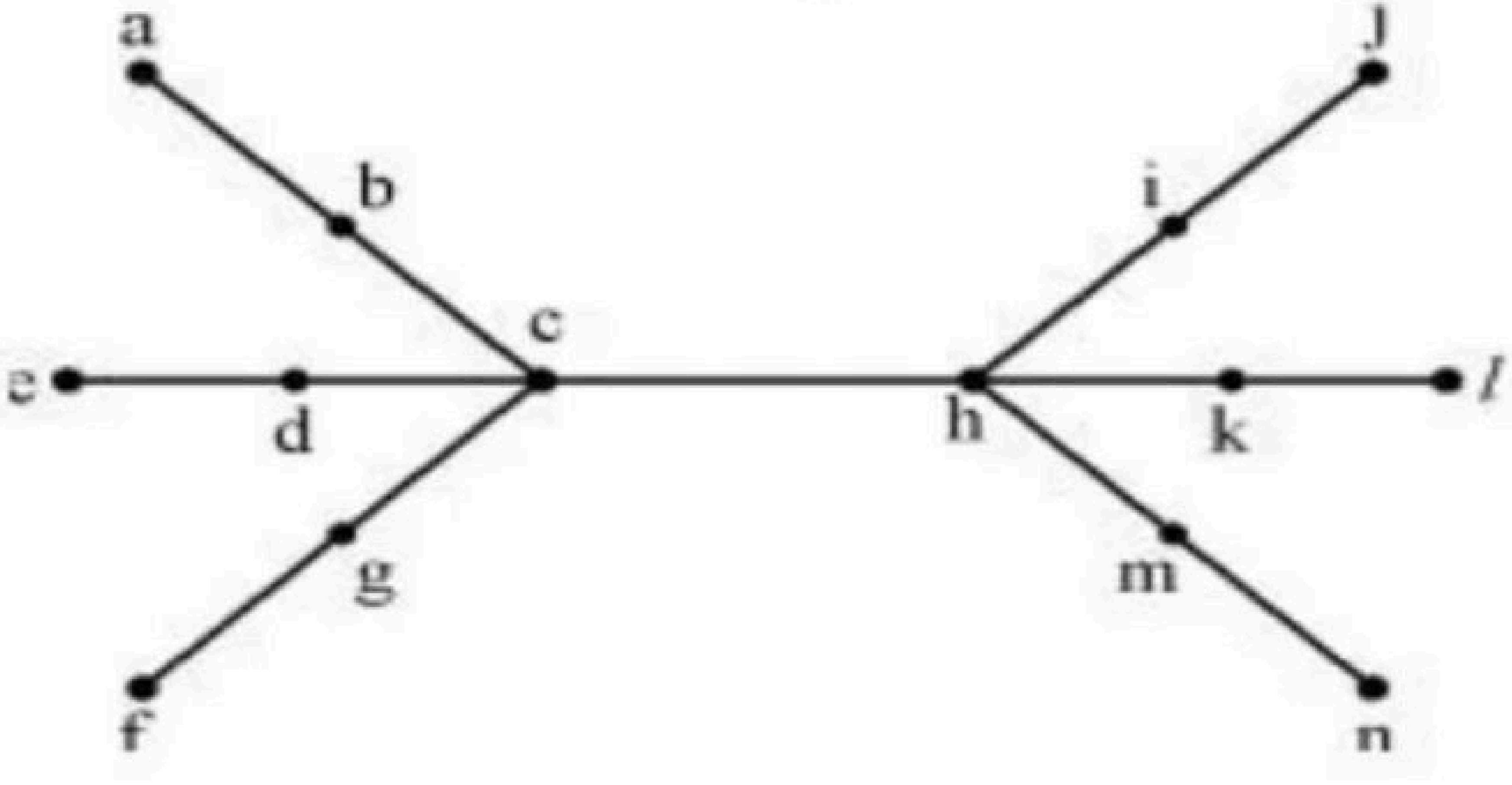


- A vertex with minimum eccentricity in a tree T is called center of T.
- Minimum eccentricity of any vertex in a tree T is called radius of tree. (eccentricity of center)
- Maximum eccentricity of any vertex in a tree T is called diameter of tree. (length of the longest path)
- Every tree has either one or two centers.



Q Consider the tree given below: **(NET-DEC-2012)**
Using the property of eccentricity of a vertex, find every vertex that is the center of the given tree:

a) d & h
b) c & k
c) g, b, c, h, i, m
d) c & h



Q Let T be a tree with 10 vertices. The sum of the degrees of all the vertices in T is _____. **(GATE-2017) (1 Marks)**

Q A certain tree has two vertices of degree 4, one vertex of degree 3 and one vertex of degree 2. If the other vertices have degree 1, how many vertices are there in the graph? **(NET-DEC-2014)**

a) 5 **b)** n–3 **c)** 20 **d)** 11

Q T is a graph with n vertices. T is connected and has exactly n-1 edges, then: **(NET-DEC-2005)**

a) T is a tree

b) T contains no cycles

c) Every pairs of vertices in T is connected by exactly one path

d) All of these

Q How many edges are there in a forest of t-trees containing a total of n vertices? **(NET-DEC-2013)**

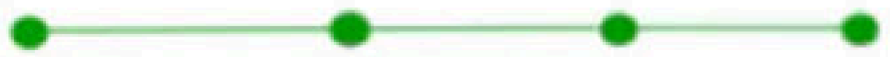
(A) $n + t$ **(B)** $n - t$ **(C)** $n * t$ **(D)** n^t

Q A tree with n vertices is called graceful, if its vertices can be labeled with integers 1, 2,....n such that the absolute value of the difference of the labels of adjacent vertices are all different. Which of the following trees are graceful? **(NET-DEC-2015)**

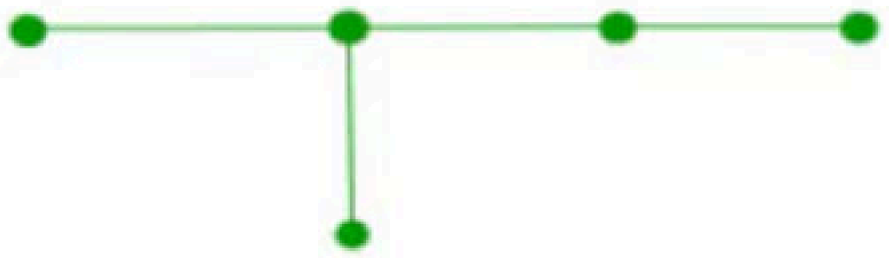
a) (a) and (b) **b)** (b) and (c)

c) (a) and (c) **d)** (a), (b) and (c)

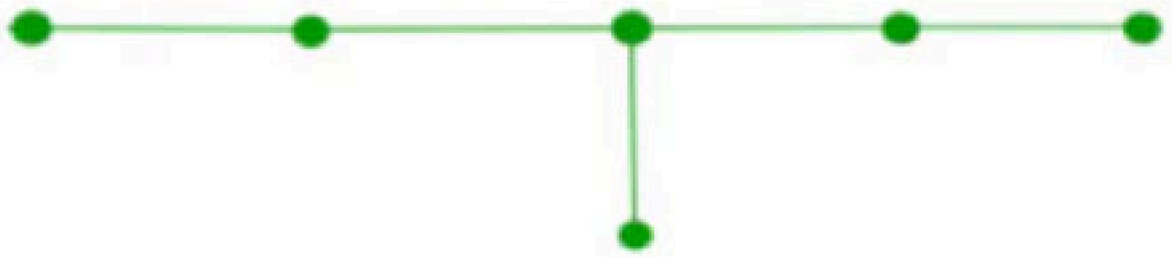
(a)



(b)



(c)



Q What is the maximum number of edges in an acyclic undirected graph with n vertices? **(GATE-2004) (1 Marks)**

(A) n-1 **(B)** n **(C)** n + 1 **(D)** 2n-1

Q The minimum number of edges in a connected graph with 'n' vertices is equal to **(NET-DEC-2010)**

(A) n (n – 1) **(B)** n $(n-1)^2$ **(C)** n^2 **(D)** n – 1

Q which of the following statement is false? **(NET-JUNE-2006)**

a) Every tree is a bipartite graph

b) A tree contains a cycle

c) A tree with n nodes contains (n-1) edges

d) A tree is connected graph

Q Which of the following does not define a tree? **(NET-JUNE-2008)**

a) a tree is a connected acyclic graph.

b) A tree is a connected graph with n-1 edges where ‘n’ is the number of vertices in the graph.

c) A tree is an acyclic graph with n-1 edges where ‘n’ is the number of vertices in the graph.

d) A tree is a graph with no cycles.

Q which two of the following are equivalent for an undirected graph G? **(NET-JUNE-2009)**

i) G is a tree

ii) There is at least one path between any two distinct vertices of G

iii) G contains no cycles and has (n-1) edges

iv) G has n edges

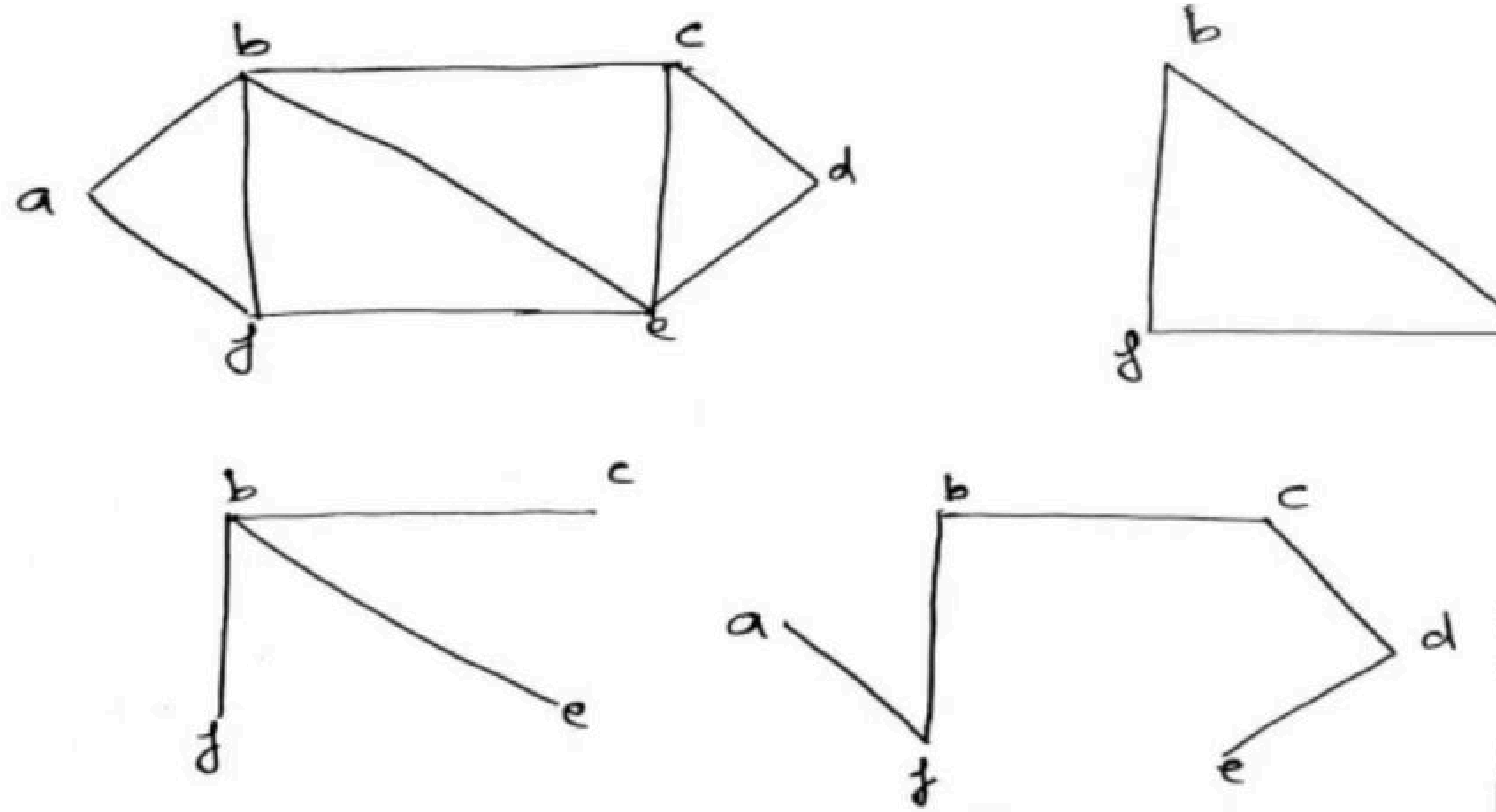
a) (i) and (ii) **b)** (i) and (iii) **c)** (i) and (iv) **d)** (ii) and (iii)

Q Consider the undirected graph G defined as follows. The vertices of G are bit strings of length n. We have an edge between vertex u and vertex v if and only if u and v differ in exactly one-bit position (in other words, v can be obtained from u by flipping a single bit). The ratio of the chromatic number of G to the diameter of G is **(GATE-2006) (2 Marks)**

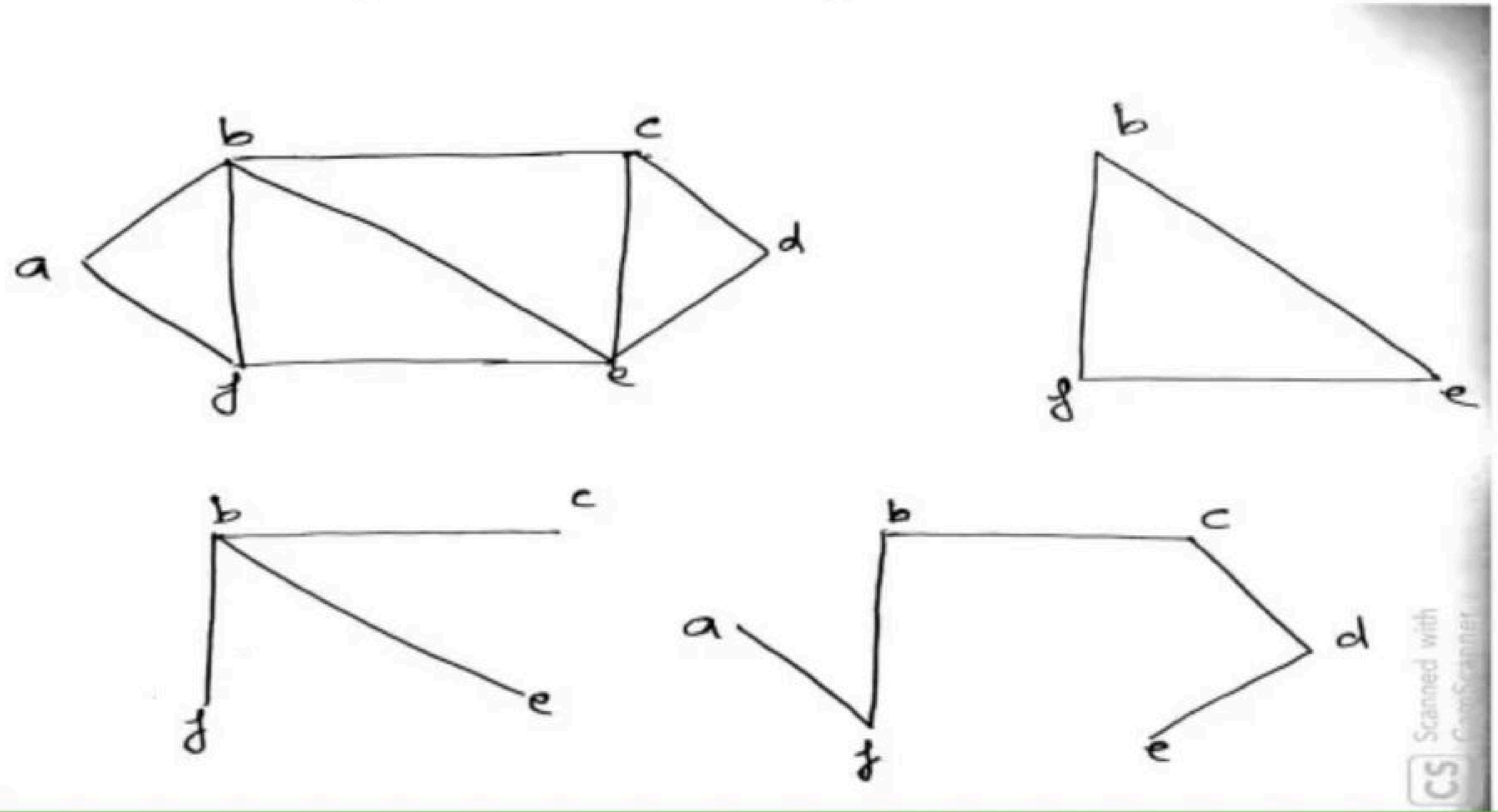
(A) $1/(2^{n-1})$ **(B)** $1/n$ **(C)** $2/n$ **(D)** $3/n$

Spanning tree

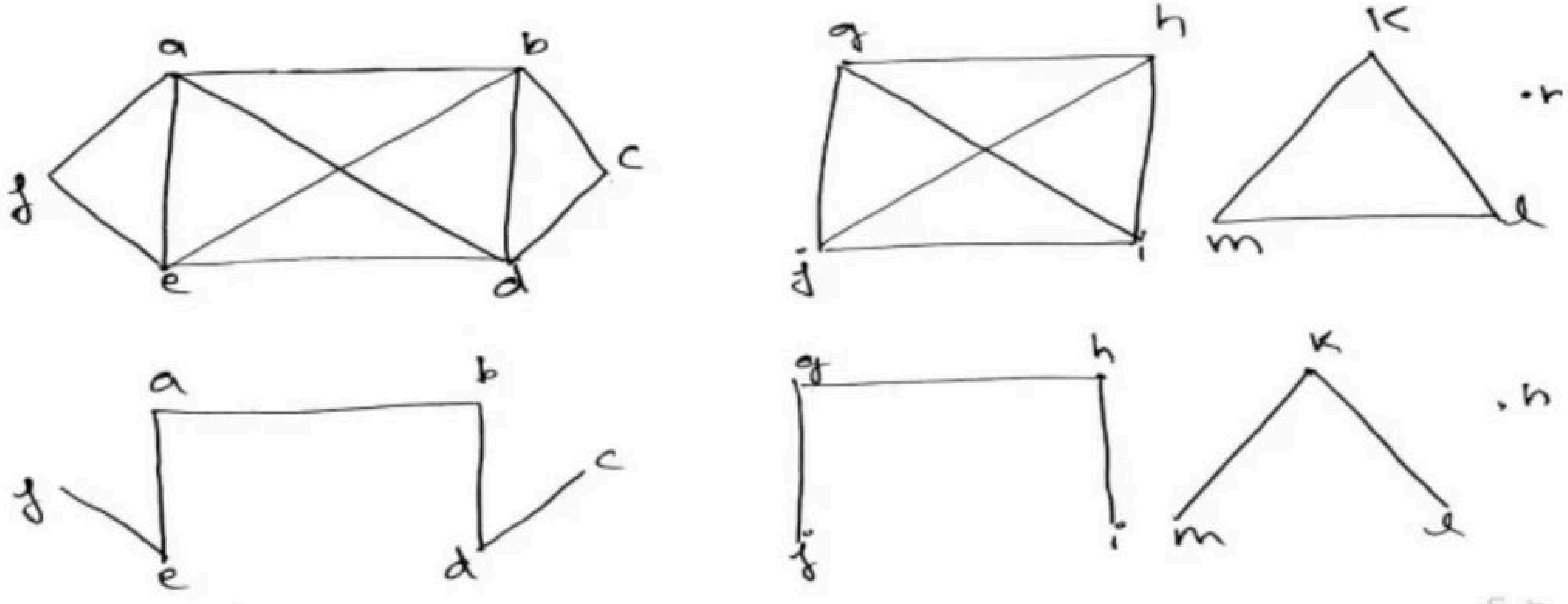
1. A tree T is said to be spanning tree of a connected graph G, if T is a subgraph of G and T contains all vertices of G.
2. An edge in a spanning tree T is called a branch of T
3. An edge that is not in the given spanning tree T is called a chord.
4. Branch and Chord are defined with respect to a given spanning tree.



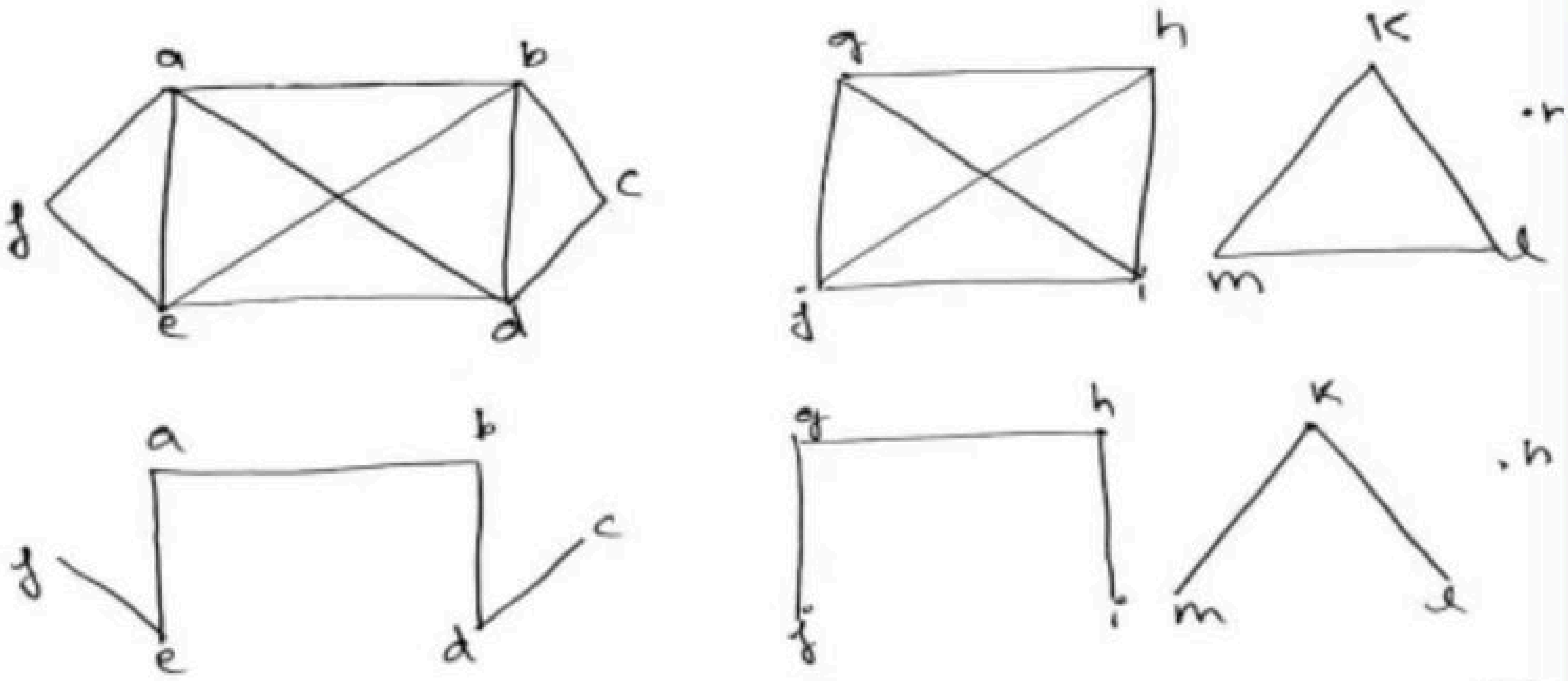
1. With respect to any of its spanning tree, a connected graph of n vertices and e edges has n-1 branches and e-n+1 chord
2. A connected graph G is a tree if and only if adding an edge between any two vertices in g creates exactly one cycle.
3. Rank(r) = n-1
4. Nullity(μ) = e – n + 1
5. Rank + nullity = number of edges in G



Spanning Forest: - if a graph is not connected, then there is no possibility of finding a spanning tree, but we can find a spanning forest. If a graph is not connected then we can find connected components, finding a spanning tree in each component we can find spanning forest.
A disconnected graph with K components has a spanning forest consisting of K spanning tree.



1. Rank(r) = n-k
2. Nullity(μ) = e – n + k
3. Rank + nullity = number of edges in G



Fundamental circuit: - With respect to a spanning tree T in a connected graph G, adding any one chord to T will create exactly one circuit such a circuit formed by adding a chord to a spanning tree is called fundamental circuit.

Q for a complete graph with N vertices, the total number of spanning tree is given by: **(NET-DEC-2006)**

a) 2^{N-1} **b)** N^{N-1} **c)** N^{N-2} **d)** 2^{N+1}

Q How many edges must be removed to produce the spanning forest of a graph with N vertices, M edges and C connected components? **(NET-JUNE-2013)**

(A) M+N–C **(B)** M–N–C **(C)** M–N+C **(D)** M+N+C

Q Which of the following connected simple graph has exactly one spanning tree? **(NET-JUNE-2013)**

(A) Complete graph

(B) Hamiltonian graph

(C) Euler graph

(D) None of the above

Q The number of different spanning trees in complete graph, K4 and bipartite graph, K2,2 have _______ and ________ respectively. **(NET-JULY-2016)**

a) 14, 14 **b)** 16, 14 **c)** 16, 4 **d)** 14, 4

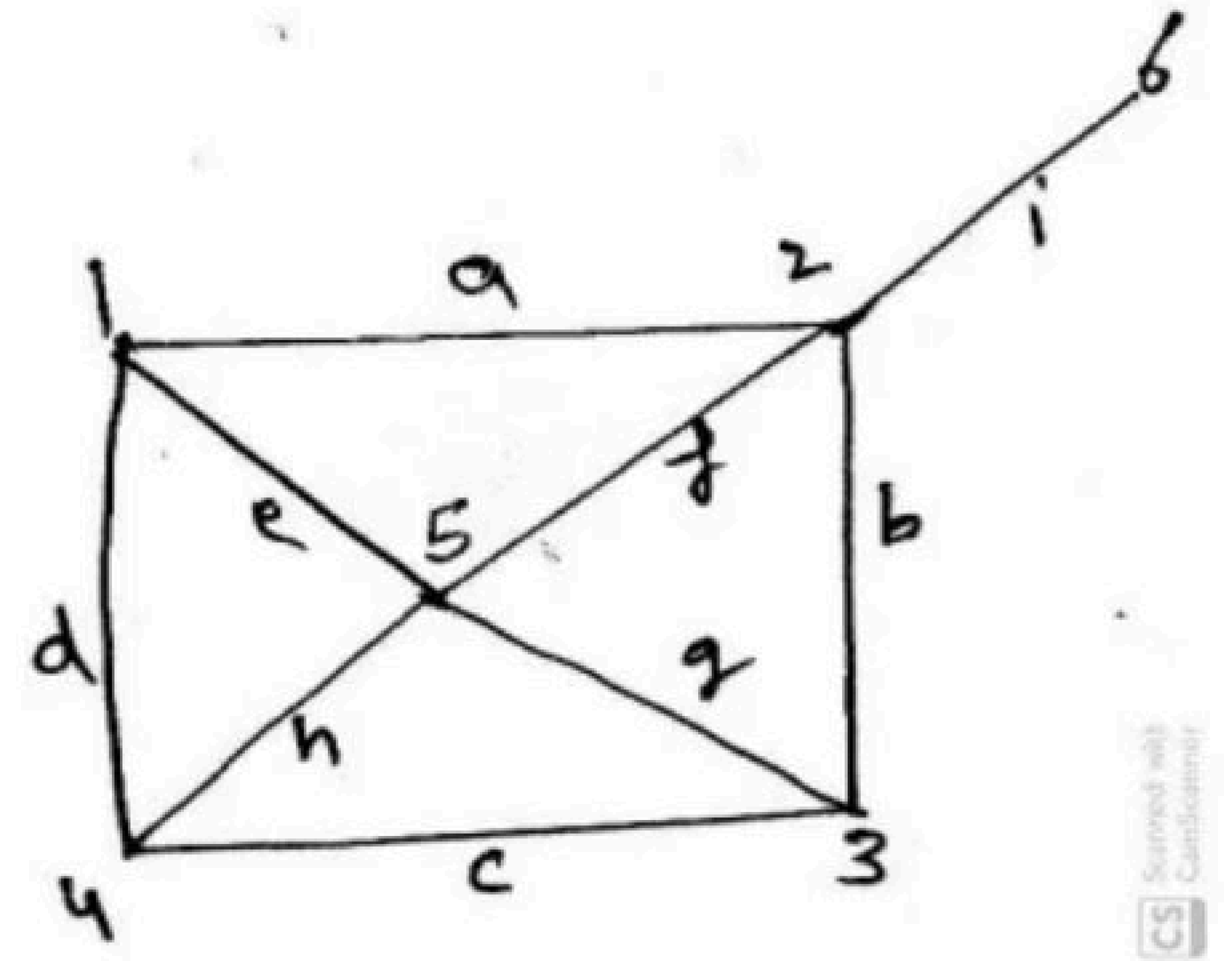
Q if G is a forest with n vertices and k connected components, how many edges does G have? **(GATE-2014) (2 Marks)**

(A) floor(n/k) **(B)** ceil(n/k) **(C)** n-k **(D)** n-k+1

Cut-Set (edge and vertex connectivity)

Cut-Set (Edges)

Cut Set: - In a connected graph G, a cut set is a set of edges whose removal from g leaves G disconnected, provided removal of no proper subset of these edges disconnects G.



Cut Set	Validity	Reason
{a, f, g}		
{a, e, h, c}		
{a, i}		
{e, h, f, g}		
{d, h, c, g}		
{d, e, f}		

1. Every Cut Set in a connected graph G must contain at least one branch of every spanning tree of G.
2. Every circuit has an even number of edges in common with any Cut-Set.

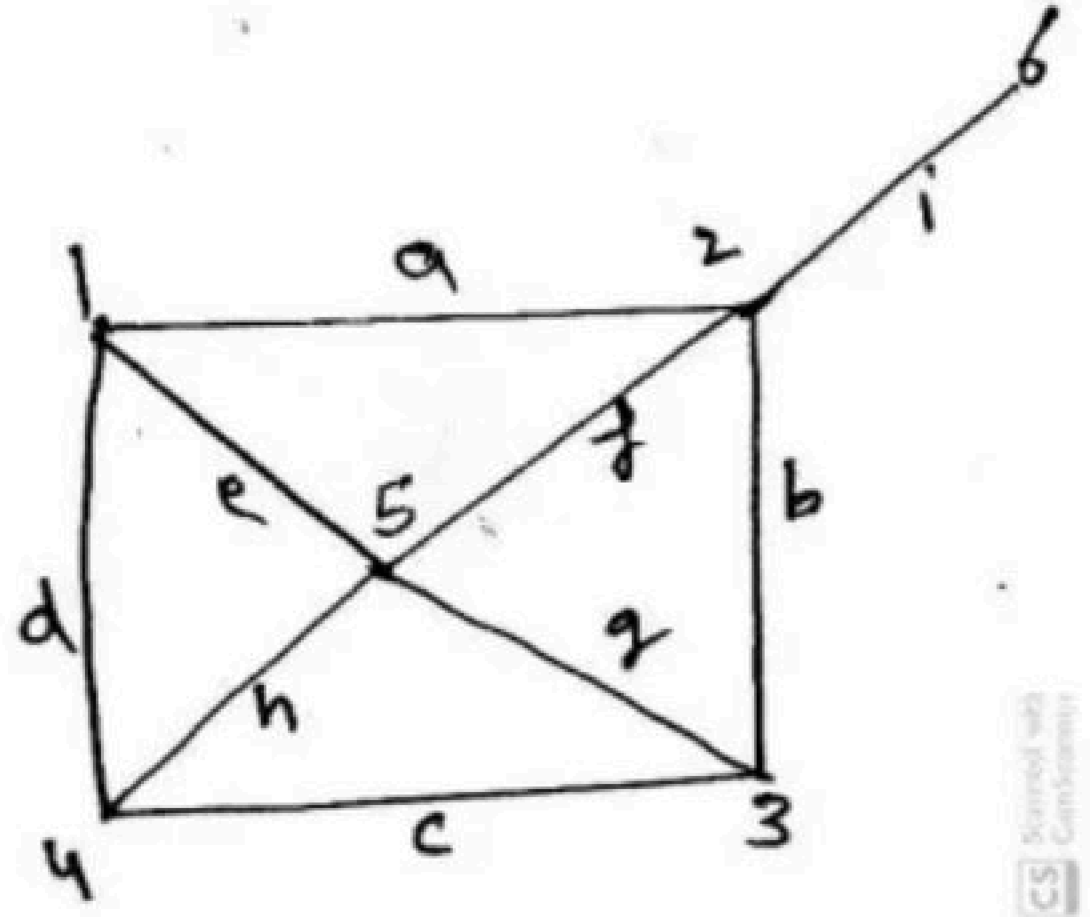
<u>Connectivity:</u> - each cut-set of a connected graph G consist of a certain number of edges. The number of edges in the smallest cut-set is defined as the edges connectivity of G. It is denoted by $\lambda(G)$.

- if the edge connectivity from a graph is one, then that edge how's removal disconnect the graph is called a bridge.

Cut-Set (Vertex)

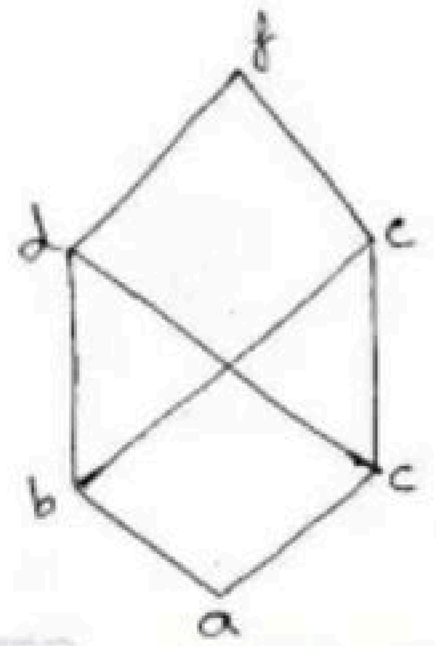
Cut Set: - In a connected graph G, a cut set is a set of vertices whose removal from g leaves G disconnected, provided removal of no proper subset of these vertices disconnects G.

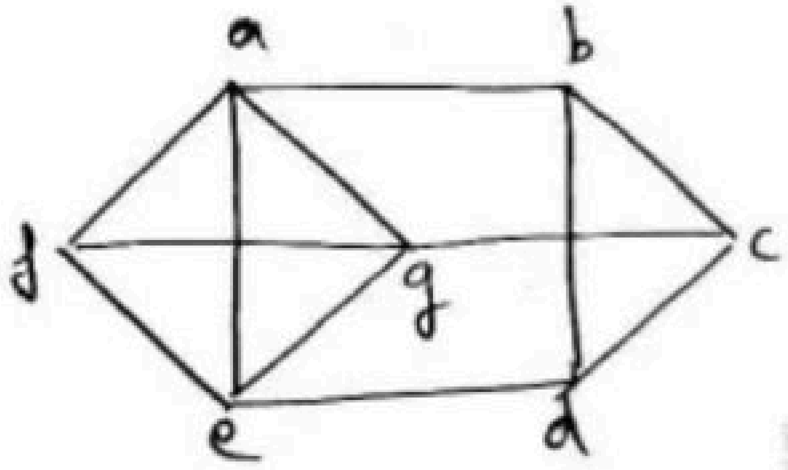
Cut Set	Validity	Reason
{5, 3}		
{6}		
{5, 2}		
{2}		
{1, 5, 3}		

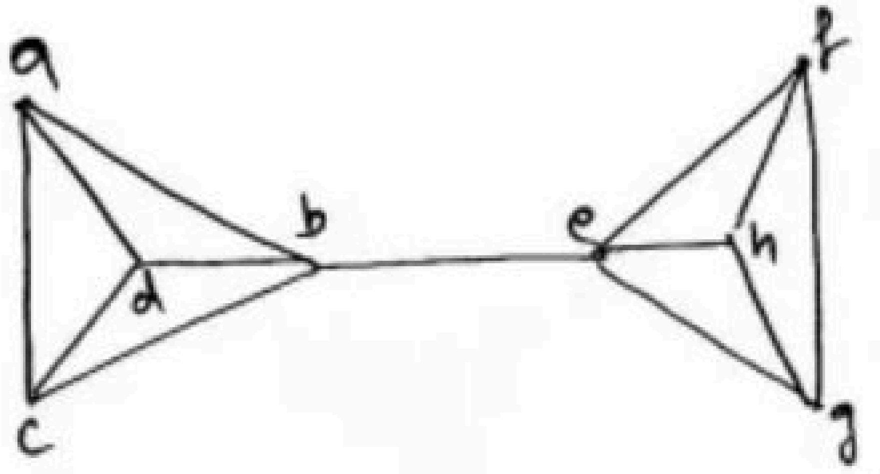


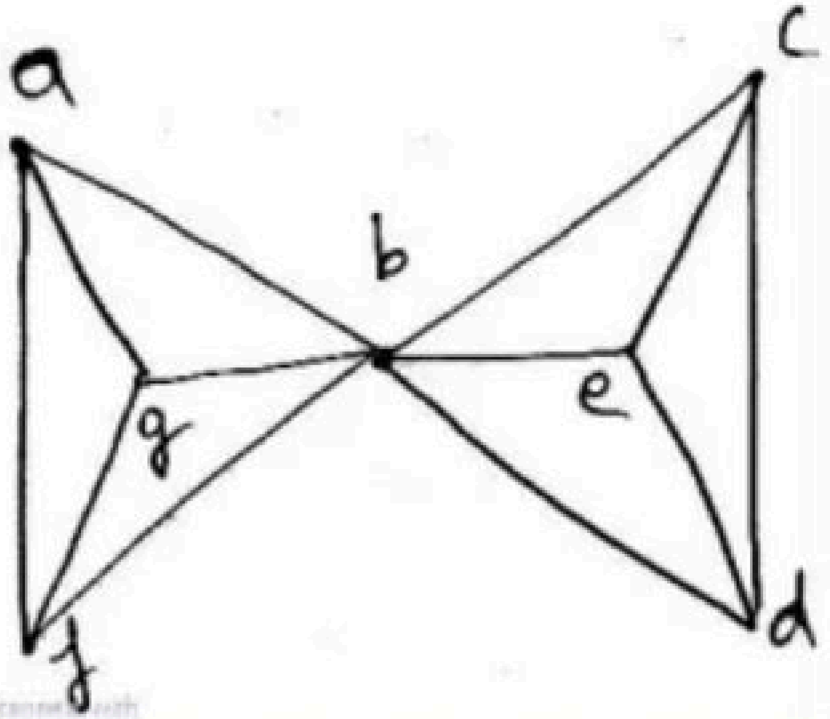
Vertex Connectivity: - Each cut-set of a connected graph G consist of a certain number of vertices. The number of vertices in the smallest cut-set is defined as the vertex connectivity of G. It is denoted by k(G).

- A connected graph is said to be separable of its vertex connectivity is one.
- If the vertex connectivity of a graph is one, then that vertex how removal disconnects a graph is called articulation point.









k(G)				
λ(G)				
δ(G)				

Q The maximum number of possible edges in an undirected graph with ‘a’ vertices and ‘k’ components is ______. **(GATE-1991) (2 Marks)**

Q G is a graph on n vertices and 2n – 2 edges. The edges of G can be partitioned into two edge-disjoint spanning trees. Which of the following is NOT true for G? **(GATE-2008) (2 Marks)**

(A) For every subset of k vertices, the induced subgraph has at most 2k-2 edges

(B) The minimum cut in G has at least two edges

(C) There are two edge-disjoint paths between every pair to vertices

(D) There are two vertex-disjoint paths between every pair of vertices

Q Let G be an arbitrary graph with n nodes and k components. If a vertex is removed from G, the number of components in the resultant graph must necessarily lie between (k-1)(n-1) **(GATE-2003) (1 Marks)**

(A) k and n

(B) $k - 1$ and $k + 1$

(C) $k - 1$ and $n - 1$

(D) $k + 1$ and $n - k$

Q Let G be a graph with 100! vertices, with each vertex labelled by a distinct permutation of the numbers 1, 2, …, 100. There is an edge between vertices u and v if and only if the label of u can be obtained by swapping two adjacent numbers in the label of v. Let y denote the degree of a vertex in G, and z denote the number of connected components in G. Then y + 10z = _______.

(GATE-2018) (2 Marks)

Q Let G = (V, E) be a directed graph where V is the set of vertices and E the set of edges. Then which one of the following graphs has the same strongly connected components as G? **(GATE-2014) (1 Marks)**

a) G1 = (V,E1) where E1={(u, v)|(u, v)∉E}

b) G2 = (V,E2) where E2={(u, v)|(v, u)∈E}

c) G3 = (V,E3) where E3={(u, v)| there is a path of length ≤2 from u to v in E}

d) G4 = (V4,E) where V4 is the set of vertices in G which are not isolated