AI2101: Convex Optimization

Assignment - 2

Handed out: 07 - Feb - 2024 **Due : 19 - Feb - 2024 (before 5 PM)**

Instructions :

1. Please submit the solutions to the assignment problems to the course page (on the canvas platform). Solutions submitted to the course page will only be evaluated. Refer to the assignment guidelines mentioned on the course page.
2. Submissions received after the deadline will attract negative marking.
3. It is suggested that you attempt all the problems. However, it is sufficient to submit solutions for problems that total 10 points.
4. Please note that you should use the template provided along with the assignment for submitting the programming part. The submission must be named in the following format: RollNo-Assignment-1.ipynb (for programming) or RollNo-Assignment-X.pdf (for non-programming submissions).

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1. (5 Points) Using a programming language of choice, implement the Jarvis march algorithm (the basic idea is covered in the supplement for the assignment) that finds the convex hull of a given finite set of vectors in $\mathcal{R}^2$.
   (a) Plot the points in $\mathcal{R}^2$ plane.
   (b) Color the obtained convex hull.
2. (5 Points) Design an algorithm to plot and fill the convex hull for a set of points in $\mathcal{R}^2$. You are allowed to implement any algorithm of your choice. Please provide the reference corresponding to the pseudo-code you are using.

   If you want, you can also use the rough pseudo code below to implement your algorithm:
   - Find the equation of all possible lines formed by choosing a pair of points from the given set.
   - These lines will act like a hyperplane in the 2-dimensional space. Thus, each line will divide the hyperplane into two half spaces [1]
   - Find the subset of lines such that each one of them divides the space into two half spaces where all the points lie in exactly one-half space.
   - These sets of lines will define the boundary of the convex hull. Alternatively, all the given points that are lying on these lines would be the boundary points of the convex hull.
3. (BONUS) The student whose algorithm runs in the smallest possible time for the second question will get a bonus mark.

[1] refer to the practice question for more information on half spaces

Practice Questions (Need Not be Submitted)

1. Consider the n-dimensional hypercube with a m-dimensional sphere inscribed inside it, such that the sphere is tangent to only 1 surface of the cube. Does such a system satisfy the definition of a convex set?

2. A *half space* is the set of points on one side of a hyperplane. Formally, for a hyperplane defined by the equation $ax + by + cz = d$, the half-space corresponding to the side where $ax + by + cz < d$ is located is denoted as $\{(x, y, z) \mid ax + by + cz < d\}$. We now extend this idea to n-dimensional spaces. A half space is defined as:

$$\mathbf{H} : \{x : a^T x \leq b\}$$

   Half spaces divide the whole space into 2 halves. The boundary of the half-space is a separating hyperplane. The complement of a half-space is known as an open half-space.

   (a) Prove that a half-space is a convex set

   (b) If a half-space is closed, should it also be compact? (refer to your Calculus 1 notes for closed and compact sets)

3. Show that any affine set in $\mathbb{R}^n$ that contains $\mathbf{0}$ is a subspace.

4. In each of the following cases, express the subspace spanned by the given set of vectors in $\mathbb{R}^3$ in the form $A\mathbf{x} = \mathbf{0}$:

   (a) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \\ 0 \end{bmatrix} \right\}$

   (b) $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix} \right\}$

   Let $\mathbb{U}$ denote a convex set. Show that the set of vectors $(\mathbf{x})$ that satisfy the condition $\mathbf{A}\mathbf{x} + \mathbf{b} \in \mathbb{U}$ constitute a convex set i.e., show that $\mathbb{X} = \{\mathbf{x} : \mathbf{A}\mathbf{x} + \mathbf{b} \in \mathbb{U}\}$ where $\mathbf{A} \in \mathcal{R}^{m \times n}$ and $\mathbf{b} \in \mathcal{R}^m$ is convex.

5. Let $\mathbb{X} = \{\mathbf{x} : \|\mathbf{x} - \mathbf{x}_0\|_2 \leq r\}$ where $\mathbf{x}, \mathbf{x}_0 \in \mathcal{R}^2$. Show that $\mathbb{X}$ is Convex. Indicate the region corresponding to the $\mathbb{X}$ for any choice of r.

6. Let $\mathbb{X} = \left\{ \mathbf{x} : (\mathbf{x} - \mathbf{x}_0)^T \mathbf{D} (\mathbf{x} - \mathbf{x}_0) \leq r \right\}$ where $\mathbf{x}, \mathbf{x}_0 \in \mathcal{R}^2$ and $\mathbf{D} \in \mathcal{R}^{2 \times 2}$ is a diagonal matrix whose diagonal entries can be positive or negative.

   (a) Indicate/represent the regions (in graphical form) that correspond to $\mathbb{X}$ for different scenarios of $\mathbf{D}$.

   (b) Determine (with appropriate proof) if $\mathbb{X}$ for different scenarios of $\mathbf{D}$ constitutes a convex set or not.

7. Let $\mathbb{X} = \{(\mathbf{x}, t) : \|\mathbf{x}\|_2 \leq t, t \geq 0\}$ where $\mathbf{x} \in \mathcal{R}^2$, $\mathbf{t} \in \mathcal{R}$ and $\mathbb{X} \subset \mathcal{R}^3$. Show that $\mathbb{X}$ is Convex. Indicate the region corresponding to the $\mathbb{X}$.