AI2101/MA2101: Convex Optimization

Assignment - 5[1]

Handed out: 23 - Feb - 2024 **Due : 04 - Mar - 2024 (before 5 PM)**

Instructions :

1. Please submit the solutions to the assignment problems to the course page (on the canvas platform). Solutions submitted to the course page will only be evaluated. Refer to the assignment guidelines mentioned on the course page.
2. Submissions received after the deadline will attract negative marking.
3. It is suggested that you attempt all the problems. However, it is sufficient to submit solutions for problems that total 10 points.
4. The submission must be named in the following format: RollNo-Assignment-X.pdf .

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1. (5 Points) Prove $f(\mathbf{x}) : \mathcal{R}^2 \to \mathcal{R} = \sqrt{x_1^2 + x_2^2}$ is convex. Is $f(\mathbf{x}) : \mathcal{R}^2 \to \mathcal{R} = \sqrt{x_1^n + x_2^n}$ convex $\forall n \in \mathbb{N}$?

2. (5 Points) Consider the following function :

$$f(x, \lambda) = \lambda e^{-\lambda x} \quad \text{for } x \in \mathcal{R}, \lambda \in \mathcal{R}$$

Calculate the Gradient and Hessian of the function considering both x and λ as variables. (You might have seen this function used in Probability before)

3. (10 Points) Prove/disprove the following result: If $f(\cdot)$ is convex then, $g(\mathbf{x}) := f(\mathbf{A}\mathbf{x} + \mathbf{b})$ is also convex $\forall \mathbf{A} \in \mathcal{R}^{m \times n}$ and $\mathbf{b} \in \mathcal{R}^m$. (Consider all possible cases, including the scenarios $m = n$ and $m = 1$ while stating your result). Use this result to show that the following function is convex:

$$J(\beta) = \sum_{i=1}^{N} (\log(1 + e^{-y_i \beta^T x_i})) + \lambda \|\beta\|^2$$

where $\lambda \in \mathbb{R}_+, y_i \in \{-1, 1\}, \mathbf{x_i} \in \mathbb{R}^n$ for i = 1, 2, ...N

4. (5 Points) Let $f : \mathcal{R}^n \to \mathcal{R}$ be a function. Defining $gt : \mathcal{R} \to \mathcal{R}$ as

$$g(t) := f(\mathbf{x}_1 + t\mathbf{x}_2)$$

where $t \in \mathcal{R}$ and $x_1, x_2 \in \mathcal{R}^n$. Then prove that f is convex if and only if $g(t)$ is convex $\forall \mathbf{x}_1, \mathbf{x}_2 \in \mathcal{R}^n$. [Hint: this is the converse of the result covered in the class].

[1] Good resource for Practice Problems: Chong and Zak, "An Introduction to Optimization", Wiley, 2017

5. (5 Points) [2] [Taken from Zak and Chong] Let $\mathbb{D} = \{\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}\}$. Show that at any point $\mathbf{x}^* \in \mathbb{D}$, the direction $\mathbf{d}$ is a feasible direction if and only if $\mathbf{A}\mathbf{d} = \mathbf{0}$.

6. (4 Points) Consider the optimization problem

$$\begin{aligned} \min \quad f(\mathbf{x}) &= x_1^2 + x_2^2 \\ \text{Sub. to} \quad \mathbf{x} &\in \mathcal{R}^2 \end{aligned} \tag{1}$$

   (a) (2 Points) Identify the point/s ($\mathbf{x}^*$) where the first order necessary condition for minima is/are satisfied.

   (b) (2 Points) Check if all the point/s that satisfy the first-order necessary condition for minima also satisfy the second-order necessary condition.

7. (5 Points) Consider the optimization problem

$$\begin{aligned} \min \quad f(\mathbf{x}) &= x_1^2 - x_2^2 \\ \text{Sub. to} \quad \mathbf{x} &\in \mathcal{R}^2 \end{aligned} \tag{2}$$

   (a) (2 Points) Identify the point/s ($\mathbf{x}^*$) where the first order necessary condition for minima is/are satisfied.

   (b) (2 Points) Check if all the point/s that satisfy the first-order necessary condition for minima also satisfy the second-order necessary condition.

8. (4 Points) Let $f(\mathbf{x})$ denote the objective function of an unconstrained optimization problem. Consider,

$$f(\mathbf{x}) = \mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{b}^T\mathbf{x} + 3 \text{ where } \mathbf{Q} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} \text{ and } \mathbf{b} = [1, 3]^T. \tag{3}$$

   (a) (2 Points) Identify the point/s ($\mathbf{x}^*$) where the first order necessary condition for minima is/are satisfied.

   (b) (2 Points) Check if all the point/s that satisfy the first-order necessary condition for minima also satisfy the second-order necessary condition.

9. (4 Points) Consider the optimization problem

$$\begin{aligned} \min \quad f(\mathbf{x}) &= 2x_1 + 3x_2 \\ \text{Sub. to} \quad \mathbf{x} &\in \mathbb{D} \subseteq \mathcal{R}^2 \end{aligned} \quad \text{where } \mathbb{D} = \{\mathbf{x} : \mathbf{A}\mathbf{x} \leq \mathbf{b}\} \text{ with } \mathbf{A} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \\ 1 & 5 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \tag{4}$$

   (a) (2 Points) Identify the point/s ($\mathbf{x}^*$) where the first order necessary condition for minima is/are satisfied.

   (b) (2 Points) Check if all the point/s that satisfy the first-order necessary condition for minima also satisfy the second-order necessary condition.

[2] Problems 5-9 are from the assignment/s in the previous year

10. (5 Points) Prove the following result: Let $\mathbf{f} : \mathcal{R}^n \to \mathcal{R}^m$ and $\mathbf{g} : \mathcal{R}^n \to \mathcal{R}^m$ be differentiable. Then, the derivative of $\mathbf{f}(\mathbf{x})^T\mathbf{g}(\mathbf{x})$ is given by

$$\mathbf{D}\left(\mathbf{f}(\mathbf{x})^T\mathbf{g}(\mathbf{x})\right) = \mathbf{f}(\mathbf{x})^T\mathbf{D}\mathbf{g}(\mathbf{x}) + \mathbf{g}(\mathbf{x})^T\mathbf{D}\mathbf{f}(\mathbf{x}) \tag{5}$$

where $\mathbf{D}\mathbf{g}(\mathbf{x}), \mathbf{D}\mathbf{f}(\mathbf{x}) \in \mathcal{R}^{m\times n}$.