Convex Optimization Assignment 3 Solutions

General Instructions

The following document contains the solutions to the questions for Assignment 3. Please note that the solutions provided may not be the only possible way to solve the questions. They indicate only one of the many (possibly) valid solutions. The solutions provided are relatively crisp and do not include all the steps that you must have. Your solution should be logical and contain all supporting arguments. Feel free to contact any of the TAs via email in case of any discrepancy you find in the solutions provided.

Question 1

We need to check for the following two set of vectors, if the affine space generated by them is a subspace or not. We know that every Affine space which has a null vector is a subspace. So if we can show that the null vector can be generated by the following set of vectors, the affine space will be a subspace.

$a)$

$$A = \left\{ \begin{bmatrix} 5 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 7 \end{bmatrix} \right\}$$

Let U_A be the affine space generated by A. Then,

$$U_A = \left\{ \mathbf{v} \in \mathbb{R}^3 : \mathbf{v} = \sum_{i=1}^{3} \alpha_i \mathbf{u}_i, \text{ where } \alpha_i \in \mathbb{R}, \sum_{i=1}^{3} \alpha_i = 1 \right\}$$

So, let $\mathbf{v} = \mathbf{0}$, then we get,

$$\alpha_1 \begin{bmatrix} 5 \\ 0 \\ 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + \alpha_3 \begin{bmatrix} 5 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We have three equations and three variables, with a condition that $\sum_i \alpha_i = 1$

The equations become

$$\begin{aligned} 5\alpha_1 + 2\alpha_2 + 5\alpha_3 &= 0 \\ 0\alpha_1 + 1\alpha_2 + 5\alpha_3 &= 0 \\ 3\alpha_1 + 2\alpha_2 + 7\alpha_3 &= 0 \\ \alpha_1 + \alpha_2 + \alpha_3 &= 1 \end{aligned}$$

On solving these equations, we get, $\alpha_1 = -\frac{1}{3}, \alpha_2 = \frac{5}{3}, \alpha_3 = -\frac{1}{3}$ which gives us

$$\alpha_1 \begin{bmatrix} 5 \\ 0 \\ 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + \alpha_3 \begin{bmatrix} 5 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Therefore, U_A is a subspace.

$b)$

$$B = \left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ 3 \end{bmatrix} \right\}$$

Let U_B be the affine space generated by A. Then,

$$U_B = \left\{ \mathbf{v} \in \mathbb{R}^3 : \mathbf{v} = \sum_{i=1}^{3} \alpha_i \mathbf{u}_i, \text{ where } \alpha_i \in \mathbb{R}, \sum_{i=1}^{3} \alpha_i = 1 \right\}$$

So, let $\mathbf{v} = \mathbf{0}$, then we get,

$$\alpha_1 \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 7 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We have three equations and three variables, with a condition that $\sum_i \alpha_i = 1$

The equations become

$$\begin{aligned} 2\alpha_1 - 2\alpha_2 + 7\alpha_3 &= 0 \\ 0\alpha_1 + 3\alpha_2 + 0\alpha_3 &= 0 \\ 0\alpha_1 + 0\alpha_2 + 3\alpha_3 &= 0 \\ \alpha_1 + \alpha_2 + \alpha_3 &= 1 \end{aligned}$$

From the equations, we get that $\alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0$ which doesn't satisfy the last equation that their sum be equal to 1. Therefore, there is no such element $\mathbf{v} \in \mathbb{R}^3$ such that $\mathbf{v} = 0$.

Hence, U_B is not a subspace.

Question 2

Writing the standard basis in three dimensions as e_x, e_y, e_z

Checking whether the given vectors are linearly independent of not.

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0,\ \alpha_i \in \mathbb{R}\,\forall\, i \in 1, 2, 3$$

Further,

$$\begin{aligned}\alpha_1 - \alpha_2 &= 0\\ 2\alpha_1 + \alpha_2 - \alpha_3 &= 0\\ 3\alpha_2 + 2\alpha_3 &= 0\\ \implies \alpha_i &= 0\,\forall\, i \in 1, 2, 3\end{aligned}$$

As, all $\alpha_i = 0$, hence, the vectors form a basis.

Let $T(v_1) = x_1,\ T(v_2) = x_2,\ T(v_3) = x_3$

Then,

$$\begin{aligned}T(e_x + 2e_y) = x_1 &= \begin{bmatrix}1\\2\end{bmatrix}\\ T(-e_x - e_y + 3e_z) = x_2 &= \begin{bmatrix}2\\2\end{bmatrix}\\ T(-e_y + 2e_z) = x_3 &= \begin{bmatrix}3\\2\end{bmatrix}\end{aligned}$$

$$\begin{aligned}\implies e_x &= \frac{5v_1 - 4v_2 + 6v_3}{9}\\ e_y &= \frac{2v_1 + 2v_2 - 3v_3}{9}\\ e_z &= \frac{v_1 + v_2 + 3v_3}{9}\end{aligned}$$

Thus,

$$\begin{aligned}T(e_x) &= \frac{1}{9}\begin{bmatrix}15\\14\end{bmatrix}\\ T(e_y) &= \frac{1}{9}\begin{bmatrix}-3\\2\end{bmatrix}\\ T(e_z) &= \frac{1}{9}\begin{bmatrix}12\\10\end{bmatrix}\end{aligned}$$

Thus, the matrix corresponding to the linear transformation T is

$$\mathbf{A} = \frac{1}{9}\begin{bmatrix}15 & -3 & 12\\ 14 & 2 & 10\end{bmatrix}$$

Question 3

$a)$ Let $\mathbf{p} = [p_1\ p_2\ p_3] \in S, \mathbf{q} = [q_1\ q_2\ q_3] \in S$

Let $0 < \alpha < 1$

$$\begin{aligned}\mathbf{v} &:= \alpha\mathbf{p} + (1-\alpha)\mathbf{q} \\ &= (\alpha p_1 + (1-\alpha)q_1, \alpha p_2 + (1-\alpha)q_2, \alpha p_3 + (1-\alpha)q_3)\end{aligned}$$

Using the fact that $p_3, \geq 0, q_3 \geq 0, \alpha \geq 0, (1-\alpha) \geq 0$, we can write that, $0 \leq \alpha p_3 + (1-\alpha)q_3$

Similarly, $0 \leq \alpha p_1 + (1-\alpha)q_1$. Without loss of generality assume $q_1 \leq p_1$

$$0 \leq \alpha p_1 + (1-\alpha)q_1 \leq \alpha p_1 + (1-\alpha)p_1 = p_1 \leq 5$$

Similarly, $0 \leq \alpha p_2 + (1-\alpha)q_2 \leq 5$. Thus, $\mathbf{v} \in S$ and so S is a convex set.

b) No. Proving this using contradiction.

Suppose set S can be expressed as the convex hull of a set V of finitely many vectors. Let

$$\begin{aligned}x_1 &= \max\left\{|v_1| \middle| \begin{bmatrix}v_1 & v_2 & v_3\end{bmatrix}^T \in V\right\} \\ x_2 &= \max\left\{|v_2| \middle| \begin{bmatrix}v_1 & v_2 & v_3\end{bmatrix}^T \in V\right\} \\ x_3 &= \max\left\{|v_3| \middle| \begin{bmatrix}v_1 & v_2 & v_3\end{bmatrix}^T \in V\right\}\end{aligned}$$

Then, the convex hull is clearly contained in the set X defined as

$$X := \left\{\begin{bmatrix}\alpha_1 & \alpha_2 & \alpha_3\end{bmatrix}^T \middle| \alpha_1 \in [-x_1, x_1], \alpha_2 \in [-x_2, x_2], \alpha_3 \in [-x_3, x_3]\right\}$$

As $S \subset X$ where X is a bounded set it implies that S is also a bounded set. But this is a contradiction to the fact that set S is unbounded.

Question 4

a) Checking P_m for Convexity, Affinity, and Subspace

$$P_m = f(x) = a_0 + a_1 x + \ldots + a_m x^m \mid a_i \in \mathbb{R}, 0 \leq i \leq m$$

Take two functions $f(x)$ and $g(x)$ in P_m. For P_m. For any $\lambda \in \mathbb{R}$, checking whether the function $\lambda f(x) + (1-\lambda)g(x) \in P_m$.

$$\begin{aligned}\lambda f(x) + (1-\lambda)g(x) &= \sum_{k=0}^{m}(\lambda f_k + (1-\lambda)g_k)x^k \\ &= \sum_{k=0}^{m} h_k x^k\end{aligned}$$

Let $h_k = \lambda f_k + (1-\lambda)g_k$, each h_k where $0 \leq i \leq m$ will be in $\mathbb{R}$. Therefore $h(x) \in P_m$. Therefore, P_m is affine. As every affine set is convex, P_m is convex.

Check if P_m is a Subspace. For P_m to be a subspace, the null vector should be in the affine set P_m. Let $a_i = 0 \leq i \leq m$, then $f(x) = 0 \in P_m$. Therefore, we can say that P_m is a subspace.

Thus, P_m is convex, affine, and a subspace.

b) Checking $P(m)_{++}$ for Convexity, Affinity and Subspace

$$P(m)_{++} = \{f(x) \in P_m \mid f(x) > 0 \, \forall x \in \mathbb{R}\}$$

Considering two functions have $f(x), g(x) \in P(m)_{++}$, then both $f(x), g(x) \geq 0$. So, the line segment joining $f(x)$ and $g(x)$, i.e., for any $\lambda \in [0, 1]$, the function h defined as

$$\begin{aligned} h(x) &:= \lambda f(x) + (1 - \lambda) g(x) \\ &\geq 0 \end{aligned}$$

will also be greater that 0 because all the terms are non-negative. Therefore, $P(m)_{++}$ is convex.

Checking if $P(m)_{++}$ is Affine

For $P(m)_{++}$ to be affine, for every two functions $f(x), g(x) \in P(m)_{++}$, $\forall \alpha \in \mathbb{R}$, the point $\alpha f(x) + (1 - \alpha) g(x) \in P(m)_{++}$.

Defining the function $h(x) := \alpha f(x) + (1 - \alpha) g(x)$. If $h(x) \in P(m)_{++}$, then

$$\begin{aligned} h(x) = \alpha f(x) + (1 - \alpha) g(x) > 0 \qquad \forall x \in \mathbb{R}^n \\ \implies \alpha(f(x) - g(x)) > -g(x) \\ \implies \alpha > \frac{-g(x)}{f(x) - g(x)} \end{aligned}$$

As the above inequality will not necessarily be true for all $\alpha \in \mathbb{R}^n$, hence, $h(x) \notin P(m)_{++} \; \forall \alpha \in \mathbb{R}$. Hence, $P(m)_{++}$ is not affine.

Thus, $P(m)_{++}$ is convex but not affine.

Question 5

Let S be the set of all periodic functions with period 2. Then, S is defined as

$$S := \{f \big| f(x) = f(x+2) \quad \forall x \in \text{domain}(f), f : \text{domain}(f) \rightarrow \mathbb{R}\}$$

let $f, g \in S$ and $\alpha \in \mathbb{R}$. Defining $h : \mathbb{R}^n \rightarrow \mathbb{R}$ as

$$h(x) := \alpha f(\mathbf{x}) + (1 - \alpha) g(\mathbf{x}) \quad \forall x \in \text{domain}(f) \cap \text{domain}(g)$$

As,

$$
\begin{aligned}
h(\mathbf{x}+2) &= \alpha f(\mathbf{x}+2) + (1-\alpha) g(\mathbf{x}+2) \\
&= \alpha f(\mathbf{x}) + (1-\alpha) g(\mathbf{x}) \\
&= h(\mathbf{x}) \quad \forall x \in \mathrm{domain}(h)
\end{aligned}
$$

Thus, $h \in S$. Hence S is an affine set.