Vector Calculus Assignment 3 Solutions

General Instructions

The following document contains the solutions to the questions for Assignment 3. Please note that the solutions provided may not be the only possible way to solve the questions. They indicate only one of the many (possibly) valid solutions. The solutions provided are relatively crisp and do not include all the steps that you must have. Your solution should be logical and contain all supporting arguments. Feel free to contact any of the TAs via email in case of any discrepancy you find in the solutions provided.

Question 1

$$f(x, y, z) = e^x \cos(yz)$$

$$v = 2\hat{x} + \hat{y} - 2\hat{z}$$

$$\implies \|v\| = 3$$

$$\implies \hat{v} = \frac{2}{3}\hat{x} + \frac{1}{3}\hat{y} - \frac{2}{3}\hat{z}$$

$$\implies \hat{v} = a\hat{x} + b\hat{y} - c\hat{z}$$

$$D_v(f(x, y, z)) = f_x(x, y, z)a + f_y(x, y, z)b + f_z(x, y, z)c$$

$$\begin{aligned} f_x(x, y, z) &= e^x \cos(yz) \\ f_y(x, y, z) &= -ze^x \sin(yz) \\ f_y(x, y, z) &= -ye^x \sin(yz) \end{aligned}$$

$$\implies D_v(f(x, y, z)) = \frac{2}{3}e^x \cos(yz) + \frac{1}{3}(-ze^x \sin(yz)) - \frac{2}{3}(-ye^x sin(yz))$$

Question 2

Given $g : \mathbb{R} \to \mathbb{R}^n$ a function satisying $\|g(\tau)|_2 = 2 \quad \forall \tau \in \mathbb{R}$.

$a)$ Proving that $\left\langle g(\tau), \frac{\partial g(\tau)}{d\tau} \right\rangle = 0 \quad \forall \tau \in \mathbb{R}$.

We can consider that $g(\tau) = [g_1(\tau)\ g_2(\tau)\ \cdots\ g_n(\tau)]^T$ which implies that

$$\frac{\partial g}{\partial \tau} = \begin{bmatrix} \frac{\partial g_1(\tau)}{\partial \tau} & \frac{\partial g_2(\tau)}{\partial \tau} & \cdots & \frac{\partial g_n(\tau)}{\partial \tau} \end{bmatrix}^T$$

Using these we can write

$$
\begin{aligned}
\|g(\tau)\|_2 &= 2 \\
\implies \sum_{i=1}^{n} g_i(\tau)^2 &= 4 \\
\implies \frac{\partial}{\partial \tau}\left(\sum_{i=1}^{n} g_i(\tau)^2\right) &= 0 \\
\implies \sum_{i=1}^{n} 2g_i(\tau)\frac{\partial g_i(\tau)}{\partial \tau} &= 0 \\
\implies \sum_{i=1}^{n} g_i(\tau)\frac{\partial g_i(\tau)}{\partial \tau} &= 0 \\
\implies \left\langle g(\tau), \frac{\partial g(\tau)}{d\tau}\right\rangle &= 0
\end{aligned}
$$

b) As the previous result is true for all $\tau \in \mathbb{R}$, we can replace the equality by an inequality and the statement is still true i.e.

$$
\left\langle g(\tau), \frac{\partial g(\tau)}{d\tau}\right\rangle \leq 0 \quad \forall \tau \in \mathbb{R}
$$

c) The normal vector of any curve is the derivative of the tangent vector of the curve at a given point. The normal is directed towards the center of curvature. The second derivative of the curve is thus radial along the center of curvature. Therefore, if the vector field is radial, the given cross product is 0. Else, the cross product is not 0. Thus, the overall statement is false.

Question 3

We derive the gradient in spherical coordinates by starting off with the Cartesian basis and converting it into spherical coordinates to circumvent the reliance of the basis on the position vector in the Spherical coordinate system.

$$
\begin{aligned}
x &= r\sin\theta\cos\phi, \\
y &= r\sin\theta\sin\phi, \\
z &= r\cos\theta
\end{aligned}
$$

To change the basis from cartesian to spherical coordinates, we apply the following normalized transformation

$$
\begin{bmatrix} \hat{e}_r \\ \hat{e}_\theta \\ \hat{e}_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} \hat{e}_x \\ \hat{e}_y \\ \hat{e}_z \end{bmatrix}
$$

Now, taking the derivatives w.r.t the cartesian coordinate system,

$$
\begin{aligned}
\frac{\partial r}{\partial x} &= \sin\theta\cos\phi, \frac{\partial r}{\partial y} = \sin\theta\sin\phi, \frac{\partial r}{\partial z} = \cos\theta, \\
\frac{\partial \theta}{\partial x} &= \frac{\cos\theta\cos\phi}{r}, \frac{\partial \theta}{\partial y} = \frac{\cos\theta\sin\phi}{r}, \frac{\partial \theta}{\partial z} = -\frac{\sin\phi}{r}, \\
\frac{\partial \phi}{\partial x} &= -\frac{\sin\phi}{r\sin\theta}, \frac{\partial \phi}{\partial y} = \frac{\cos\phi}{r\sin\theta}, \frac{\partial \phi}{\partial z} = 0
\end{aligned}
$$

Using the chain rule, we can express the gradient as

$$
\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{bmatrix} = \frac{\partial f}{\partial x}\hat{e}_x + \frac{\partial f}{\partial y}\hat{e}_y + \frac{\partial f}{\partial z}\hat{e}_z
$$

$$
\implies \nabla f = \begin{bmatrix} \frac{\partial f}{\partial r} & \frac{\partial f}{\partial \theta} & \frac{\partial f}{\partial \phi} \end{bmatrix} \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} & \frac{\partial r}{\partial z} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} & \frac{\partial \theta}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial r} & \frac{\partial f}{\partial \theta} & \frac{\partial f}{\partial \phi} \end{bmatrix} \mathbf{J}
$$

Thus, we can express the gradient in spherical coordinates by taking the dot product of the Jacobian matrix in the spherical coordinate system.

$$
\mathbf{J} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \frac{\cos\theta\cos\phi}{r} & \frac{\cos\theta\sin\phi}{r} & -\frac{\sin\theta}{r} & -\frac{\sin\phi}{r\sin\theta} & \frac{\cos\phi}{r\sin\theta} & 0 \end{bmatrix}
$$

We can now compute the gradient in spherical coordinates by taking the components of the overall gradient over $\hat{e}_r, \hat{e}_\theta$ and $\hat{e}_\phi$ respectively. On doing so, we get

$$
\nabla f = \frac{\partial f}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{e}_\phi
$$

Question 4

We are given a sphere centred at origin and curve:

$$
\begin{aligned}
S &: x^2 + y^2 + z^2 = 9 \\
C &: x^2 + y^2 - z - 3 = 0
\end{aligned}
$$

We first find the point(s) of intersection.

$$
\begin{aligned}
& x^2 + y^2 + z^2 = 9 \\
& \implies x^2 + y^2 = 9 - z^2
\end{aligned}
$$

$$
\begin{aligned}
& x^2 + y^2 - z - 3 = 0 \\
& \implies 9 - z^2 - z - 3 = 0 \\
& \implies z^2 + z - 6 = 0 \\
& \implies z = 2, -3
\end{aligned}
$$

$$
\begin{aligned}
& z = 2 \implies x^2 + y^2 = 9 - 2^2 \\
& \implies x^2 + y^2 = 5 \\
& \implies (x, y, z) = \left(\sqrt{5}\cos\theta, \sqrt{5}\sin\theta, 2\right)
\end{aligned}
$$

$$\begin{aligned} z = -3 &\implies x^2 + y^2 = 9 - (-3)^2 \\ &\implies x^2 + y^2 = 0 \\ &\implies x = y = 0 \\ &\implies (x, y, z) = (0, 0, -3) \end{aligned}$$

We now compute the gradients, hence, the normals, denoted by ∇S or n_S and ∇C or n_C repsectively.

$$\begin{aligned} \nabla S = n_S &= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(x^2 + y^2 + z^2 - 9\right) \\ &= 2x\hat{i} + 2y\hat{j} + 2z\hat{k} \\ \nabla C = n_C &= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(x^2 + y^2 - z - 3\right) \\ &= 2x\hat{i} + 2y\hat{j} - 1\hat{k} \end{aligned}$$

The angle between the surfaces is the same as the angle between their normals.

NOTE: As there are multiple points of intersection, there exist multiple answers to this problem. The angle can be computed at any of the points of intersection.

$$\begin{aligned} \implies \cos\alpha &= \frac{\langle n_S, n_C\rangle}{\|n_S\|\|n_S\|} \\ &= \frac{4x^2 + 4y^2 - 2z}{\sqrt{4x^2 + 4y^2 + 4z^2}\sqrt{4x^2 + 4y^2 + 1}} \\ &= \frac{4x^2 + 4y^2 - 2z}{2\sqrt{x^2 + y^2 + z^2}\sqrt{4x^2 + 4y^2 + 1}} \\ &= \frac{4x^2 + 4y^2 - 2z}{6\sqrt{4x^2 + 4y^2 + 1}} \end{aligned}$$

Note that the angle is computed only at points of intersection. This means that the points (x, y, z) lie on both S and C and hence, satisfy their equations. This is the reason the $\|n_S\|$ can be simplified with ease.

Consider $(x, y, z) = \left(\sqrt{5}\cos\theta, \sqrt{5}\sin\theta, 2\right)$

$$\begin{aligned} \cos\alpha &= \frac{4\left(\sqrt{5}\cos\theta\right)^2 + 4\left(\sqrt{5}\sin\theta\right)^2 - 2(2)}{6\sqrt{4\left(\sqrt{5}\cos\theta\right)^2 + 4\left(\sqrt{5}\sin\theta\right)^2 + 1}} \\ &= \frac{20\cos^2\theta + 20\sin^2\theta - 4}{6\sqrt{20\cos^2\theta + 20\sin^2\theta + 1}} \\ &= \frac{16}{6\sqrt{21}} = \frac{8}{3\sqrt{21}} \\ \implies \alpha &= \arccos\left(\frac{8}{3\sqrt{21}}\right) \end{aligned}$$

Consider $(x, y, z) = (0, 0, -3)$

$$
\begin{aligned}
\cos\alpha &= \frac{4(0)^2+4(0)^2-2(-3)}{6\sqrt{4(0)^2+4(0)^2+1}} \\
&= \frac{6}{6\sqrt{1}} = 1 \\
\implies \alpha &= \arccos(1) = 0^{\cdot}
\end{aligned}
$$

We can conclude that there are two answers to this problem.

Question 5

a) We need to evaluate a cascade of cross products here. We will solve the inner one first, followed by the outer one.

$$
\begin{aligned}
g &= x^2y\hat{x} - 2xz\hat{y} + 2yz\hat{z} \\
\nabla \times g &= \begin{bmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & -2xz & 2yz \end{bmatrix} \\
&= (2x+2z)\hat{x} - (x^2+2z)\hat{z}
\end{aligned}
$$

$$
\implies \nabla \times \begin{bmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & -2xz & 2yz \end{bmatrix}
$$

$$
\begin{aligned}
&= \nabla \times [(2x+2z)\hat{x} - (x^2+2z)\hat{z}] \\
&= \begin{bmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (2x+2z) & 0 & -(x^2+2z) \end{bmatrix} \\
&= 2(x+1)\hat{y}
\end{aligned}
$$

b) Let's solve this for a general case:

$$
\begin{aligned}
\nabla \cdot (\nabla \times g) &= \nabla \cdot \begin{bmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ g_x & g_y & g_z \end{bmatrix} \\
&= \nabla \cdot \left[\left(\frac{\partial g_z}{\partial y} - \frac{\partial g_y}{\partial z}\right)\hat{x} + \left(\frac{\partial g_x}{\partial z} - \frac{\partial g_z}{\partial x}\right)\hat{y} + \left(\frac{\partial g_y}{\partial x} - \frac{\partial g_x}{\partial y}\right)\hat{z}\right] \\
&= \frac{\partial}{\partial x}\left(\frac{\partial g_z}{\partial y} - \frac{\partial g_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial g_x}{\partial z} - \frac{\partial g_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial g_y}{\partial x} - \frac{\partial g_x}{\partial y}\right) \\
&= 0
\end{aligned}
$$

Thus, irrespective of the function, as long as its second partial derivatives are continuous, the answer is 0.

Question 6

We have been given the function $g(x, y) = x^3 + y^3 - 3xy$, and we need to find its minimum value. We can do this by finding the critical points of the function by setting the partial derivatives to 0.

$$\begin{aligned} \frac{\partial g}{\partial x} &= 3x^2 - 3y = 0 \\ \frac{\partial g}{\partial y} &= 3y^2 - 3x = 0 \end{aligned}$$

Solving the above equations

$$x^2 = y \quad \text{and} \quad y^2 = x \implies x^4 = x \implies x = 0, 1$$

Thus, the critical points are $(0, 0)$ and $(1, 1)$. We can now use the second derivative test to determine the nature of the critical points.

$$\begin{aligned} \mathcal{H} &= \begin{bmatrix} \frac{\partial^2 g}{\partial x^2} & \frac{\partial^2 g}{\partial x \partial y} \\ \frac{\partial^2 g}{\partial y \partial x} & \frac{\partial^2 g}{\partial y^2} \end{bmatrix} \\ &= \begin{bmatrix} 6x & -3 \\ -3 & 6y \end{bmatrix} \end{aligned}$$

For the point $(0, 0)$, we have

$$\mathcal{H} = \begin{bmatrix} 0 & -3 \\ -3 & 0 \end{bmatrix}$$

The determinant of the Hessian is negative and the trace is 0. Thus, the point $(0, 0)$ is a saddle point.

For the point $(1, 1)$, we have

$$\mathcal{H} = \begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix}$$

The determinant of the Hessian is positive and the trace is positive. Thus, the point $(1, 1)$ is a local minimum.

Thus, the minimum value of the function is $g(1, 1) = 1^3 + 1^3 - 3(1)(1) = -1$.

Let's repeat a similar procedure for the next function: $h(x, y) = \frac{1}{3}x^3 + y^2 + 2xy - 6x - 3y + 4$. We can find the minimum by finding the critical points of the function by setting the partial derivatives to 0.

$$\frac{\partial h}{\partial x} = x^2 + 2y - 6 = 0$$
$$\frac{\partial h}{\partial y} = 2x + 2y - 3 = 0$$

Subtracting the first equation from the second, we get

$$\begin{aligned} 2x + 2y - 3 - x^2 - 2y + 6 = 0 \\ \implies x^2 - 2x - 3 = 0 \\ \implies x = -1, 3 \end{aligned}$$

Substituting the value of x into the first equation, we get the critical points as $(-1, 5/2)$ and $(3, -3/2)$. We can now use the second derivative test to determine the nature of the critical points.

$$\begin{aligned} \mathcal{H} &= \begin{bmatrix} \frac{\partial^2 h}{\partial x^2} & \frac{\partial^2 h}{\partial x \partial y} \\ \frac{\partial^2 h}{\partial y \partial x} & \frac{\partial^2 h}{\partial y^2} \end{bmatrix} \\ &= \begin{bmatrix} 2x & 2 \\ 2 & 2 \end{bmatrix} \end{aligned}$$

For the point $(-1, 5/2)$, we have

$$\mathcal{H} = \begin{bmatrix} -2 & 2 \\ 2 & 2 \end{bmatrix}$$

The determinant of the Hessian is negative and the trace is 0. Thus

For the point $(3, -3/2)$, we have

$$\mathcal{H} = \begin{bmatrix} 6 & 2 \\ 2 & 2 \end{bmatrix}$$

The determinant of the Hessian is positive and the trace is positive. Thus, the point $(3, -3/2)$ is a local minimum.

Thus, the minimum value of the function is $h(3, -3/2) = \frac{1}{3}(3)^3 + (-3/2)^2 + 2(3)(-3/2) - 6(3) - 3(-3/2) + 4 = -29/4$.

a) Checking whether the following equality holds true:

$$\min_{x,y} g(x, y) + h(x, y) = g_0 + h_0$$

where g_0 and h_0 are the minimum values of $g(x, y)$ and $h(x, y)$ respectively.

This statement is not always true. For example, consider $g(x, y) = x^2 + y^2$ and $h(x, y) = (x - 2)^2 + (y - 2)^2$. The minimum value of both $g(x, y)$ and $h(x, y)$ is 0, but the minimum value of $g(x, y) + h(x, y)$ is 4, and is obtained at $(1, 1)$.

This statement is not always false either. If both $g(x, y)$ and $h(x, y)$ have the same arguments that minimize them, then the statement is true. For example, consider $g(x, y) = x^2 + y^2$ and $h(x, y) = x^4 + y^4$. The minimum value of each function is 0 and is attained at $(0, 0)$ and their summation i.e. $g(x, y) + h(x, y) = x^4 + x^2 + y^4 + y^2$ also has its minimum at $(0, 0)$ resulting in the minimum being 0 and hence, the condition is fulfilled.

b) Checing whether the following equality holds true:

$$\min_{x,y} g(x, y)h(x, y) = \left(\min_{x,y} g(x, y)\right) \left(\min_{x,y} h(x, y)\right)$$

This statement is not always true. For example, consider $g(x, y) = (x - 2)^2 + y^2 + 1$ and $h(x, y) = x^2 + 1$. Their minimum values are 1 and 2 respectively, but the minimum value of $g(x, y)h(x, y)$ is 4, and is obtained at $(1, 0)$.

This statement is not always false either. If both $g(x, y)$ and $h(x, y)$ have the same arguments that minimize them, then the statement is true. For example, consider $g(x, y) = x^2y^2$ and $h(x, y) = x^2 + y^2$. The minimum value of each function is 0 and is attained at $(0, 0)$ and their multiplication i.e. $g(x, y)h(x, y) = x^4y^2 + x^2y^4$ also has its minimum at $(0, 0)$ resulting in the minimum being 0 and hence the condition is fulfilled.