PWARI-G Atomic Structure Lab Notes
Volume II: Deriving the Field Equations

1. Purpose

This volume derives, from the Lagrangian constructed in Volume I, the full Euler–Lagrange equations governing the dynamics of the PWARI-G atomic system:

- The breathing soliton field ϕ
- The internal twist phase field θ
- The gravitational redshift field g

Our goal is to derive all equations of motion directly from variational principles, verify internal consistency, and prepare for numerical or analytical solution in hydrogen and beyond.

2. Euler–Lagrange Formalism

For a Lagrangian density $\mathcal{L}(\psi, \partial_t \psi, \nabla \psi)$, the Euler–Lagrange equation is:

$$\frac{\partial \mathcal{L}}{\partial \psi} - \partial_t \left(\frac{\partial \mathcal{L}}{\partial(\partial_t \psi)} \right) - \nabla \cdot \left(\frac{\partial \mathcal{L}}{\partial(\nabla \psi)} \right) = 0 \tag{1}$$

This will be applied separately to ϕ, θ, and g.

3. Deriving the Equation for ϕ

Total Lagrangian terms involving ϕ:

$$\mathcal{L}_\phi = \frac{1}{2g}(\partial_t \phi)^2 - \frac{g}{2}|\nabla \phi|^2 - \frac{\lambda}{4}\phi^4 \tag{2}$$

$$\mathcal{L}_\theta = \frac{1}{2}\phi^2 \left(\frac{1}{g}(\partial_t \theta)^2 - g|\nabla \theta|^2 \right) \tag{3}$$

Contributions to $\delta\mathcal{L}/\delta\phi$:

$$\frac{\partial\mathcal{L}}{\partial\phi} = -\lambda\phi^3 + \phi\left(\frac{1}{g}(\partial_t\theta)^2 - g|\nabla\theta|^2\right) \tag{4}$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_t\phi)} = \frac{1}{g}\partial_t\phi \tag{5}$$

$$\frac{\partial\mathcal{L}}{\partial(\nabla\phi)} = -g\nabla\phi \tag{6}$$

Putting it all together:

$$\ddot{\phi} = g^2\nabla^2\phi - \lambda g^2\phi^3 + g\phi(\partial_t\theta)^2 - g^3\phi|\nabla\theta|^2 \tag{7}$$

3.1 Soliton Equation Physics

Eq. (7):

$$\begin{aligned}\ddot{\phi} = \quad & \underbrace{g^2\nabla^2\phi}_{\text{Curvature pressure}} \quad - \quad \underbrace{\lambda g^2\phi^3}_{\text{Self-interaction}} \\ + \quad & \underbrace{g\phi(\partial_t\theta)^2}_{\text{Twist kinetic coupling}} \quad - \quad \underbrace{g^3\phi|\nabla\theta|^2}_{\text{Twist strain coupling}}\end{aligned}$$

This describes how internal twist dynamics modulate the soliton's evolution.

4. Deriving the Equation for θ

We now vary $\mathcal{L}_\theta$:

$$\frac{\partial\mathcal{L}}{\partial\theta} = 0 \tag{8}$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_t\theta)} = \frac{\phi^2}{g}\partial_t\theta \tag{9}$$

$$\frac{\partial\mathcal{L}}{\partial(\nabla\theta)} = -g\phi^2\nabla\theta \tag{10}$$

Resulting equation:

$$\partial_t\left(\frac{\phi^2}{g}\partial_t\theta\right) - \nabla\cdot(g\phi^2\nabla\theta) = 0 \tag{11}$$

4.1 Twist Wave Dynamics

Eq. (11) describes a wave in a medium with:

- Effective mass density: ϕ^2/g
- Effective stiffness: $g\phi^2$

Thus, the local phase velocity scales as $v_\theta \sim g$. Light slows in stronger gravitational wells.

5. Gravitational Relaxation Equation

This is not derived from a traditional variational principle but modeled as a relaxation equation:

$$\partial_t g = -\alpha_g(\rho_\phi + W\rho_\theta) + \varepsilon_{\text{relax}} \tag{12}$$

Where:

$$\rho_\phi = \frac{1}{2g^2}(\partial_t\phi)^2 + \frac{1}{2}|\nabla\phi|^2 + \frac{\lambda}{4}\phi^4 \tag{13}$$

$$\rho_\theta = \frac{1}{2g^2}\phi^2(\partial_t\theta)^2 + \frac{1}{2}\phi^2|\nabla\theta|^2 \tag{14}$$

5.1 Variational Alternative

We could instead define:

$$\mathcal{L}_g = \frac{(\partial_t g)^2}{2\alpha_g} - \frac{g^2}{2}(\rho_\phi + W\rho_\theta) \tag{15}$$

yielding:

$$\partial_t^2 g = -\alpha_g g(\rho_\phi + W\rho_\theta) \tag{16}$$

This second-order wave-like form supports redshift propagation if needed.

5.2 Relaxation Timescale

For atomic transitions ($\tau \sim 10^{-16}$s):

$$\alpha_g \sim \frac{\phi_0^2}{g_0\tau} \approx 10^{42}\,\mathrm{kg}^{-1}\mathrm{m}^{-3}\mathrm{s}^{-1} \tag{17}$$

This scale ensures that g equilibrates fast enough to track twist-soliton energy loss.

6. Summary

6.1 Energy Conservation

For static g, the total energy:

$$E = \int \left(\frac{(\partial_t\phi)^2}{2g} + \frac{g}{2}|\nabla\phi|^2 + \frac{\lambda}{4}\phi^4 + \frac{\phi^2}{2g}(\partial_t\theta)^2 + \frac{g\phi^2}{2}|\nabla\theta|^2 \right) d^3x \tag{18}$$

is conserved when Eqs. (7) and (11) hold.

6.2 System Overview

We have now derived the coupled dynamical equations governing PWARI-G atoms:

- ϕ obeys a nonlinear, gravity-weighted Klein–Gordon–Yukawa equation
- θ obeys a weighted wave equation with dynamic mass
- g relaxes or propagates to track the local energy density

These equations are ready for numerical solution in Volume III.