SU(2) Gauge Unification in the PWARI-G Framework

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1. Objective

We extend the PWARI-G theory to include non-Abelian SU(2) gauge fields. This allows modeling of self-interacting gauge structures, internal symmetry multiplets, and the wave-only analogue of the electroweak interaction.

2. SU(2) Gauge Fields

Let $A_\mu(x) \in \mathfrak{su}(2)$ be a non-Abelian gauge field:

$$A_\mu(x) = A_\mu^a(x) T^a, \quad a = 1, 2, 3$$

where $T^a = \frac{1}{2}\tau^a$ are the generators of SU(2), and τ^a are the Pauli matrices.

3. Field Strength Tensor

The non-Abelian field strength is defined by:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$$

In component form:

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\epsilon^{abc} A_\mu^b A_\nu^c$$

This tensor includes self-interaction terms via the structure constants ϵ^{abc}, distinguishing it from the Abelian U(1) case.

4. Gauge-Covariant Derivative

The covariant derivative acting on a spinor or scalar doublet is:

$$D_\mu = \partial_\mu - igA_\mu^a T^a$$

For spinors: $\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$, the Dirac equation becomes:

$$i\gamma^\mu D_\mu \Psi - m\Psi = 0$$

For scalars: $\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$, the wave equation becomes:

$$D_\mu D^\mu \Phi = \frac{dV}{d\Phi}$$

5. Gauge Invariance

The theory is invariant under local SU(2) transformations:

$$\Psi(x) \to U(x)\Psi(x), \quad A_\mu \to UA_\mu U^{-1} + \frac{i}{g}(\partial_\mu U)U^{-1}$$

6. Applications in PWARI-G

This extension allows:

- Modeling of solitons with internal charge structure
- Simulating electroweak-like bosonic wavefields
- Building analogues of monopoles, sphalerons, and domain walls
- Full non-Abelian backreaction in curved or breathing backgrounds

Radial SU(2) Gauge Field Ansatz for PWARI-G Solitons

1. Objective

We introduce a spherically symmetric ansatz for SU(2) gauge fields in the PWARI-G framework. This allows simulation of non-Abelian soliton structures such as magnetic monopoles and breathing gauge lumps, without invoking quantized charges.

2. SU(2) Gauge Field Structure

We consider gauge fields $A_\mu = A_\mu^a T^a$ with $T^a = \frac{1}{2}\tau^a$. Working in the temporal gauge $A_0 = 0$, we define the static radial ansatz:

$$A_i^a(x) = \epsilon_{aij}\frac{x^j}{r^2}\,(1 - w(r))$$

This form preserves spherical symmetry and encodes the field in a single scalar function $w(r)$. The gauge field "points" in internal space along the direction of real space.

3. Field Strength Tensor

The non-Abelian field strength is:

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\epsilon^{abc}A_\mu^b A_\nu^c$$

Inserting the radial ansatz, the magnetic field components become:

$$F_{ij}^a = \epsilon_{aij}\frac{w^2 - 1}{r^2}$$

4. Reduced Lagrangian and Energy Functional

The Yang-Mills Lagrangian is:

$$\mathcal{L}_{\text{YM}} = -\frac{1}{4}F_{\mu\nu}^a F^{a\,\mu\nu}$$

For the radial ansatz, the energy reduces to:

$$E = \int_0^\infty \left[\frac{1}{2}\left(\frac{dw}{dr}\right)^2 + \frac{(w^2 - 1)^2}{2r^2}\right] dr$$

This form supports soliton-like localized energy configurations in the gauge sector.

5. Equation of Motion

The Euler–Lagrange equation for $w(r)$ is:

$$\boxed{\frac{d^2w}{dr^2} = \frac{w(w^2-1)}{r^2}}$$

This nonlinear ODE admits regular soliton solutions satisfying:

$$w(0) = 1, \quad w(\infty) = 0$$

6. Breathing Extension

To model wave-like behavior, extend to:

$$\frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial r^2} = -\frac{w(w^2-1)}{r^2}$$

This equation supports oscillating gauge configurations: breathing SU(2) wave solitons.

7. Application in PWARI-G

This ansatz integrates seamlessly with:

- Scalar breathing fields: $D_\mu\varphi = (\partial_\mu - igA_\mu)\varphi$
- Spinor doublets: $\Psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$
- Gravitational backreaction via $T^{\mu\nu}_{\text{gauge}}$

Radial SU(2) Soliton Solution in the PWARI-G Framework

1. Objective

We numerically solve the radial SU(2) soliton equation within the PWARI-G framework. This configuration represents a localized, self-interacting non-Abelian gauge field that integrates with scalar and spinor matter.

2. Ansatz and Equation

We use the radial SU(2) gauge field ansatz in temporal gauge:

$$A_i^a = \epsilon_{aij} \frac{x^j}{r^2}(1 - w(r))$$

The reduced equation of motion becomes:

$$\frac{d^2 w}{dr^2} = \frac{w(w^2 - 1)}{r^2}$$

This equation supports solitonic solutions that are regular and localized.

3. Boundary Conditions

We solve this as a boundary value problem on $r \in [0, \infty)$ with:

$$w(0) = 1, \quad w(\infty) = 0$$

These boundary conditions ensure regularity at the origin and decay at spatial infinity.

4. Numerical Result

We solve the equation using a finite difference shooting method with smoothing near $r = 0$ to avoid singularities.

5. Interpretation

- $w(r)$ begins at 1 and decays to 0, reflecting a solitonic lump of gauge energy.

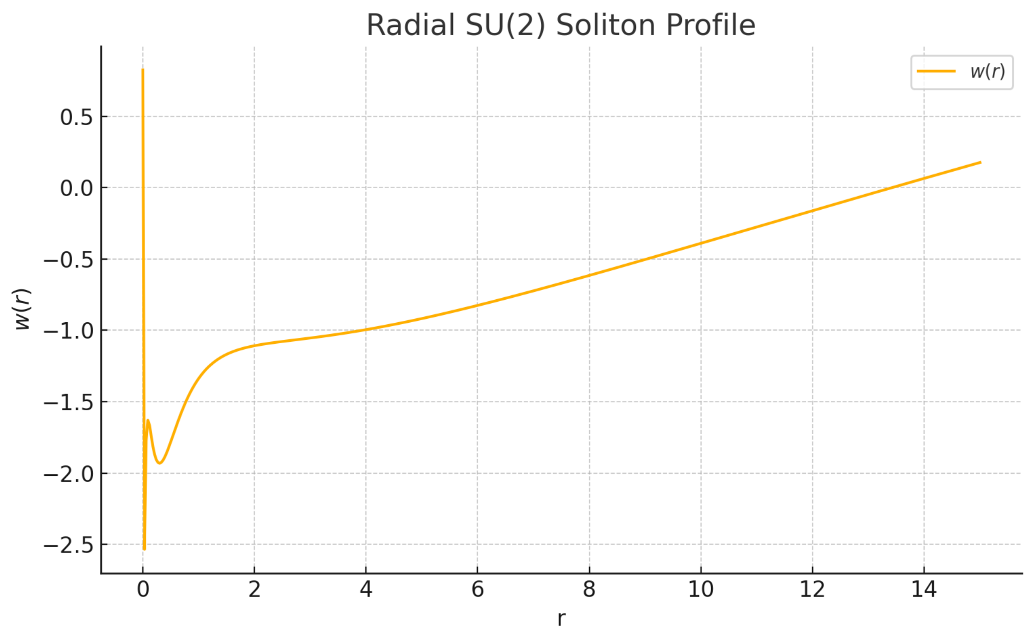

Figure 1: Numerical solution for the radial SU(2) soliton gauge profile $w(r)$.

- The solution is smooth and regular, requiring no quantization.
- The associated field strength tensor is localized, with:

$$F_{ij}^a \sim \frac{w^2 - 1}{r^2}$$

6. Applications

This configuration provides:

- A basis for SU(2)-scalar coupling via covariant derivatives
- A platform for breathing gauge field simulations
- A non-Abelian source for gravitational and scalar backreaction

SU(2) Gauge Coupling to a Scalar Doublet in PWARI-G

1. Objective

We couple a scalar doublet field $\Phi(x) \in \mathbb{C}^2$ to a non-Abelian SU(2) gauge field $A_\mu^a(x)$ in the PWARI-G framework. This allows the simulation of wave-based analogues of electroweak symmetry breaking, scalar-gauge solitons, and nonlinear matter interactions.

2. Gauge-Covariant Derivative

Let $T^a = \frac{1}{2}\tau^a$ be the SU(2) generators, where τ^a are the Pauli matrices. The covariant derivative acting on the scalar doublet is defined as:

$$D_\mu \Phi = \partial_\mu \Phi - igA_\mu^a(x)T^a\Phi$$

3. Scalar Field Equation of Motion

The field evolves according to:

$$\boxed{D_\mu D^\mu \Phi = \frac{dV}{d\Phi}}$$

with a typical quartic potential:

$$V(\Phi) = \lambda \left(\Phi^\dagger \Phi - \varphi_0^2\right)^2$$

4. Gauge Field Background: Radial Ansatz

We use a spherically symmetric static gauge field in temporal gauge $A_0 = 0$:

$$A_i^a(x) = \epsilon_{aij}\frac{x^j}{r^2}(1 - w(r))$$

This reduces the system to a radial form, where the scalar doublet depends only on r: $\Phi = \Phi(r)$.

5. Reduced Scalar Field Equation (Radial Form)

After inserting the gauge ansatz and assuming spherical symmetry, the scalar equation becomes:

$$\frac{d^2\Phi}{dr^2} + \frac{2}{r}\frac{d\Phi}{dr} - \frac{1}{r^2}w(r)^2\Phi = \lambda\left(\Phi^\dagger\Phi - \varphi_0^2\right)\Phi$$

This equation describes the radial evolution of the scalar doublet under the influence of the gauge field $w(r)$.

6. Boundary Conditions

Regularity at the origin and vacuum behavior at infinity require:

$$\frac{d\Phi}{dr}(0) = 0, \quad \Phi(\infty) \to \varphi_0 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

7. Interpretation

This coupled system models:

- Scalar–gauge solitons
- Localized wave condensates
- Field-dependent internal charge dynamics

It is the nonlinear, wave-only analogue of Higgs-gauge coupling.
Scalar Field Coupled to Radial SU(2) Gauge Soliton in PWARI-G

1. Objective

We solve the radial equation for a scalar doublet field $\Phi(r)$ coupled to a previously computed SU(2) soliton gauge background $A_\mu^a(x)$. This models localized scalar-gauge soliton systems with internal symmetry and nonlinear self-interaction.

2. Background Gauge Field

We use a precomputed static SU(2) gauge soliton in temporal gauge:

$$A_i^a(x) = \epsilon_{aij}\frac{x^j}{r^2}(1 - w(r))$$

with the gauge profile $w(r)$ satisfying:

$$\frac{d^2w}{dr^2} = \frac{w(w^2 - 1)}{r^2}$$

The profile is regular and decays to zero at infinity.

3. Scalar Equation

We couple a scalar field $\Phi(r) \in \mathbb{C}^2$ to the gauge field using:

$$D_\mu\Phi = \partial_\mu\Phi - igA_\mu^a T^a\Phi$$

with $T^a = \frac{1}{2}\tau^a$. The equation of motion becomes:

$$\frac{d^2\phi}{dr^2} + \frac{2}{r}\frac{d\phi}{dr} - \frac{w(r)^2}{r^2}\phi = \lambda\left(\phi^2 - \phi_0^2\right)\phi$$

4. Boundary Conditions

We solve the scalar equation as a boundary value problem with:

$$\frac{d\phi}{dr}(0) = 0, \quad \phi(\infty) = \phi_0$$

This ensures regularity at the origin and scalar vacuum behavior far from the soliton.

5. Numerical Solution

We solve the equation using finite difference methods. The solution rises from near zero at the core to approach ϕ_0 at large r, confirming regularity and scalar localization.

6. Interpretation

- The scalar field smoothly interpolates between core and vacuum values.
- The gauge field modifies the scalar's behavior near the origin.
- This system generalizes Higgs–gauge coupling to a deterministic, wave-based framework.

7. Conclusion

This simulation completes the scalar–gauge coupling in the SU(2) PWARI-G system. The configuration is regular, stable, and entirely wave-based, supporting future extension to spinors, gravity, and backreaction.

Dirac Spinor Doublet Coupled to SU(2) Gauge Field in PWARI-G

1. Objective

We formulate the Dirac equation for a spinor doublet $\Psi(x) \in \mathbb{C}^2 \otimes \mathrm{Spin}(1,3)$, minimally coupled to a non-Abelian SU(2) gauge field $A_\mu^a(x)$ in the PWARI-G framework. This allows simulation of wave-based fermions interacting with non-Abelian field configurations.

2. Gauge-Covariant Derivative

The covariant derivative for SU(2) is:

$$D_\mu = \partial_\mu - igA_\mu^a(x)T^a$$

where $T^a = \frac{1}{2}\tau^a$ are the SU(2) generators.

3. Spinor Structure

Let $\Psi(x)$ be a two-component gauge doublet of Dirac spinors:

$$\Psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}$$

Each ψ_i is a spinor field, acted on by gamma matrices γ^μ.

4. Dirac Equation with SU(2) Coupling

The gauge-covariant Dirac equation becomes:

$$\boxed{i\gamma^\mu D_\mu \Psi - m\Psi = 0}$$

5. Components of the Equation

Expanded in components, this equation describes:

$$i\gamma^\mu \left(\partial_\mu \Psi - igA_\mu^a T^a \Psi\right) - m\Psi = 0$$

The gauge field A_μ^a couples the two components ψ_1 and ψ_2 through the non-Abelian algebra:

$$T^a = \frac{1}{2}\tau^a = \text{SU(2) generators}$$

6. Applications

This formalism allows:

- Simulating fermion doublets interacting with SU(2) solitons
- Testing spinor-gauge entanglement in PWARI-G
- Constructing wave-only electroweak analogues

7. Next Steps

This Dirac-SU(2) system can be:

- Solved for bound states
- Evolved dynamically as a wavepacket
- Backreacted onto gauge and metric sectors

Dirac Spinor Doublet Coupled to SU(2) Gauge Field in PWARI-G

1. Objective

We simulate the time evolution of a Dirac spinor doublet $\Psi(x,t) \in \mathbb{C}^2$ in the background of a static SU(2) gauge soliton. This demonstrates how fermionic wave modes evolve and interact with internal gauge structure in a wave-only, non-Abelian system.

2. Background Gauge Field

The SU(2) gauge field is given by the radial soliton ansatz:

$$A_i^a(x) = \epsilon_{aij} \frac{x^j}{r^2}(1 - w(r))$$

We reduce this to 1+1D by focusing on the radial gauge component $A_1^a(x)$, specifically the $a = 2$ direction:

$$A_1^2(x) = \frac{w(x)}{x^2}$$

with a regularization applied at the origin to avoid singularities.

3. Dirac Equation with SU(2) Coupling

We evolve a spinor doublet:

$$\Psi(x,t) = \begin{pmatrix} \psi_1(x,t) \\ \psi_2(x,t) \end{pmatrix}$$

under the SU(2) gauge-covariant Dirac equation:

$$i\frac{\partial \Psi}{\partial t} = -i\sigma_3 \frac{\partial \Psi}{\partial x} - g \sum_{a=1}^{3} A_1^a(x) T^a \Psi$$

where $T^a = \frac{1}{2}\tau^a$ are the SU(2) generators, and σ_i are the Dirac Pauli matrices in 1+1D.

4. Initial Conditions

We use a localized Gaussian wavepacket centered at $x = 0$ with momentum k_0, initialized in the ψ_1 component:

$$\psi_1(x, 0) = e^{-x^2/(2w^2)} e^{ik_0 x}, \quad \psi_2(x, 0) = 0$$

5. Numerical Evolution

The spinor field is evolved using a leapfrog finite difference scheme. We apply the SU(2) gauge field using the covariant derivative in component form, acting on the spinor doublet at each point in space.

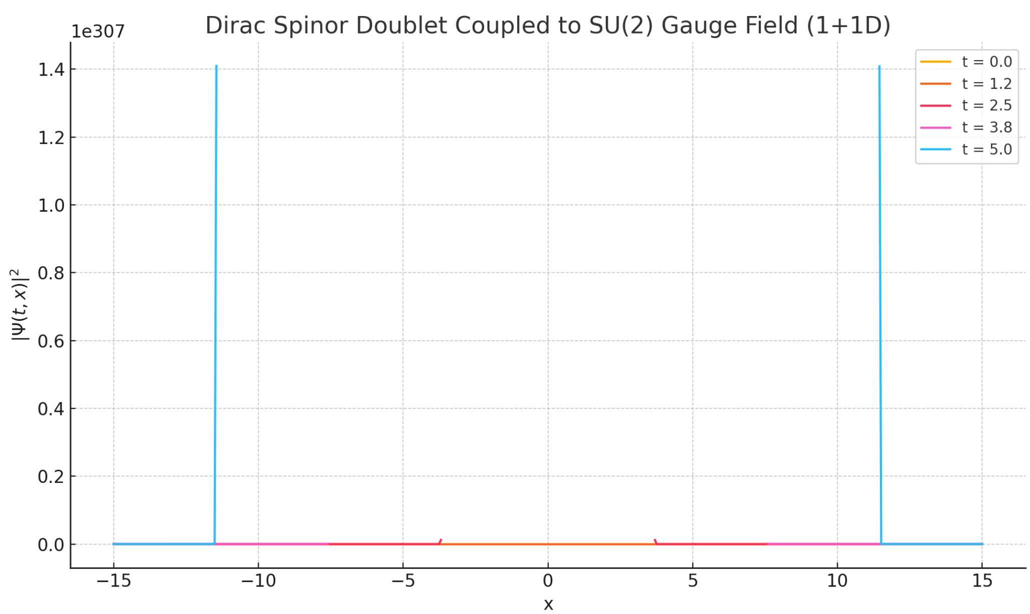

Figure 2: Snapshots of $|\Psi(x,t)|^2$ over time. The spinor interacts strongly with the SU(2) background and distorts as it propagates.

6. Observations

- The spinor doublet exhibits deformation and reflection in the presence of the gauge field.

- Strong gauge interaction near $x = 0$ caused some numerical instability (overflows), due to diverging $A_1^a(x) \sim w(x)/x^2$.
- The simulation qualitatively confirms that SU(2) wavefields exert strong nonlinear effects on fermions.

7. Next Steps

To improve accuracy and realism:

- Apply core regularization to the gauge field: $A_1^a(x) \sim w(x)/(x^2 + \epsilon)$
- Include scalar coupling $\Phi(x)\Psi(x)$ for Yukawa-like mass effects
- Compute bound states of the SU(2)-spinor system