U(1) Gauge-Invariant PWARI-G Breathing Field Theory

Lagrangian

We define a scalar breathing amplitude field $\varphi(x)$, a phase (twist) field $\theta(x)$, and a gauge field $A_\mu(x)$, with minimal U(1) coupling. The covariant derivative is:

$$D_\mu\theta = \partial_\mu\theta - eA_\mu$$

The gauge-invariant Lagrangian is:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi)^2 + \frac{1}{2}\varphi^2(D_\mu\theta)^2 - V(\varphi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

where:

$$V(\varphi) = \frac{\lambda}{4}(\varphi^2 - \varphi_0^2)^2, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Gauge Transformations

Under a local U(1) gauge transformation with parameter $\alpha(x) \in \mathbb{R}$:

$$\theta(x) \to \theta(x) + \alpha(x), \quad A_\mu(x) \to A_\mu(x) + \frac{1}{e}\partial_\mu\alpha(x), \quad \varphi(x) \to \varphi(x)$$

The Lagrangian remains invariant under this transformation.

Euler-Lagrange Equations

Equation for φ

Apply:

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right) - \frac{\partial\mathcal{L}}{\partial\varphi} = 0$$

Computing:

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} = \partial^\mu \varphi, \quad \frac{\partial \mathcal{L}}{\partial \varphi} = \varphi (D_\mu \theta)^2 + \frac{dV}{d\varphi}$$

We obtain:

$$\Box \varphi - \varphi (D_\mu \theta)^2 + \frac{dV}{d\varphi} = 0$$

Equation for θ

The phase field enters only through $D_\mu \theta$, so:

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \theta)} = \varphi^2 D^\mu \theta$$

Then:

$$\partial_\mu \left(\varphi^2 D^\mu \theta \right) = 0$$

Equation for A_μ

Compute:

$$\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} = -F^{\nu\mu}, \quad \frac{\partial \mathcal{L}}{\partial A_\mu} = -e\varphi^2 D^\mu \theta$$

Hence:

$$\partial_\nu F^{\nu\mu} = e\varphi^2 D^\mu \theta$$

Final Summary of Field Equations

$$\begin{aligned} \Box \varphi - \varphi (D_\mu \theta)^2 + \frac{dV}{d\varphi} &= 0 \\ \partial_\mu \left(\varphi^2 D^\mu \theta \right) &= 0 \\ \partial_\nu F^{\nu\mu} &= e\varphi^2 D^\mu \theta \end{aligned}$$

These represent the gauge-invariant dynamics of the breathing field, phase twist, and gauge interaction in the PWARI-G framework.

Canonical Hamiltonian and Conserved Charges in PWARI-G Gauge Theory

1. Lagrangian and Fields

We begin with the U(1) gauge-invariant Lagrangian:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 + \frac{1}{2}\varphi^2 (D_\mu \theta)^2 - V(\varphi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

with:

$$D_\mu \theta = \partial_\mu \theta - eA_\mu, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

We work in temporal gauge: $A_0 = 0$, and focus on canonical analysis in flat Minkowski spacetime.

2. Canonical Momenta

The canonical conjugate momenta are:

$$\begin{aligned}
\pi_\varphi &= \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \dot{\varphi} \\
\pi_\theta &= \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \varphi^2(\dot{\theta} - eA_0) = \varphi^2 \dot{\theta} \\
\pi^i &= \frac{\partial \mathcal{L}}{\partial \dot{A}_i} = F^{i0} = -\dot{A}_i
\end{aligned}$$

Note: $\pi^0 = \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = 0$ is a primary constraint (standard in gauge theory).

3. Hamiltonian Density

The Hamiltonian is obtained via a Legendre transform:

$$\mathcal{H} = \pi_\varphi \dot{\varphi} + \pi_\theta \dot{\theta} + \pi^i \dot{A}_i - \mathcal{L}$$

Substituting $\dot{\varphi} = \pi_\varphi$, $\dot{\theta} = \pi_\theta / \varphi^2$, $\dot{A}_i = -\pi^i$, we find:

$$\begin{aligned}
\mathcal{H} = \frac{1}{2}\pi_\varphi^2 + \frac{1}{2}\frac{\pi_\theta^2}{\varphi^2} + \frac{1}{2}(\nabla\varphi)^2 + \frac{1}{2}\varphi^2(\nabla\theta - e\vec{A})^2 \\
+ \frac{1}{2}(\vec{\pi})^2 + \frac{1}{4}F_{ij}F^{ij} + V(\varphi)
\end{aligned}$$

4. Conserved Current and Charge

From the global U(1) symmetry:

$$\theta(x) \to \theta(x) + \alpha(x)$$

we derive the Noether current:

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \theta)} \cdot \delta\theta = \varphi^2 D^\mu \theta$$

The conserved charge is:

$$Q = \int d^3x\, j^0 = \int d^3x\, \pi_\theta$$

5. Summary

- The Hamiltonian density includes kinetic energy, field gradients, gauge interaction, and self-interaction.
- The conserved charge Q reflects the total breathing-phase current, analogous to electric charge.
- This framework provides a fully canonical formulation of breathing soliton dynamics with gauge coupling.

Breathing Soliton Eigenmodes in U(1)-Gauge-Coupled PWARI-G Theory

1. Static, Spherically Symmetric Ansatz

We seek localized, stationary soliton solutions in the PWARI-G framework. Assume spherical symmetry and no time dependence for the amplitude field φ, while allowing a rotating phase:

$$\begin{aligned}
\varphi(x,t) &= \varphi(r) \\
\theta(x,t) &= \omega t \\
A_0(x) &= A_0(r), \quad \vec{A} = 0
\end{aligned}$$

The covariant derivative becomes:

$$D_0\theta = \partial_0\theta - eA_0(r) = \omega - eA_0(r)$$

$$(D_\mu\theta)^2 = -(\omega - eA_0)^2$$

2. Effective Static Lagrangian

From the original U(1)-invariant Lagrangian:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi)^2 + \frac{1}{2}\varphi^2(D_\mu\theta)^2 - V(\varphi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

we substitute:

$$(\partial_\mu\varphi)^2 = (\varphi')^2, \quad F_{\mu\nu}F^{\mu\nu} = 2(A_0')^2$$

The reduced static Lagrangian becomes:

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}(\varphi')^2 - \frac{1}{2}\varphi^2(\omega - eA_0)^2 - V(\varphi) + \frac{1}{2}(A_0')^2$$

3. Field Equations

Using the Euler-Lagrange equations:

$$\frac{d}{dr}\left(\frac{\partial\mathcal{L}}{\partial\varphi'}\right) - \frac{\partial\mathcal{L}}{\partial\varphi} = 0, \quad \frac{d}{dr}\left(\frac{\partial\mathcal{L}}{\partial A_0'}\right) - \frac{\partial\mathcal{L}}{\partial A_0} = 0$$

we derive the static radial field equations:

Equation for $\varphi(r)$:

$$\boxed{\varphi'' + \frac{2}{r}\varphi' = \varphi(\omega - eA_0)^2 - \frac{dV}{d\varphi}}$$

Equation for $A_0(r)$:

$$\boxed{A_0'' + \frac{2}{r}A_0' = e\varphi^2(\omega - eA_0)}$$

4. Boundary Conditions

To obtain discrete, quantized soliton solutions, we impose:

- Regularity at the origin:

$$\varphi'(0) = 0, \quad A_0'(0) = 0$$

- Localization at infinity:

$$\varphi(r \to \infty) \to 0, \quad A_0(r \to \infty) \to \text{const}$$

Only discrete values of ω will allow globally regular and localized solutions, corresponding to the eigenfrequencies of stable breathing solitons.

5. Physical Interpretation

Each solution $\varphi_n(r)$ represents a soliton eigenmode—a discrete, stable, localized excitation of the breathing field. These modes are the nonlinear analogs of quantized particles in QFT, replacing the need for operator-based quantization with field-constrained soliton resonance.

PWARI-G Soliton Eigenmode Spectrum: Gauge-Coupled Breathing Field Solutions April 30, 2025

1. Overview

This report documents the numerical analysis of breathing soliton eigenmodes in the U(1)-gauge-coupled PWARI-G field theory. Discrete, localized field solutions were computed by solving the coupled radial equations:

$$\varphi''(r) + \frac{2}{r}\varphi'(r) = \varphi(r)(\omega - eA_0(r))^2 - \frac{dV}{d\varphi}, \tag{1}$$

$$A_0''(r) + \frac{2}{r}A_0'(r) = e\varphi^2(r)(\omega - eA_0(r)), \tag{2}$$

where the scalar potential is:

$$V(\varphi) = \frac{\lambda}{4}(\varphi^2 - \varphi_0^2)^2.$$

We scanned trial eigenfrequencies $\omega \in [0.8, 1.5]$ and solved for soliton profiles using boundary conditions:

$$\varphi'(0) = 0, \quad \varphi(\infty) = 0, \quad A_0'(0) = 0, \quad A_0(\infty) \to \text{const.}$$

2. Numerical Results

Table 1 shows the total energy and average spatial extent of each soliton mode computed.

Table 1: Breathing soliton modes for various trial frequencies ω.

Frequency ω	**Total Energy**	**Average Radius**
0.80	1047.20	0.840
0.90	1047.20	2.249
1.00	1047.20	1.981
1.10	1047.20	1.764
1.20	1047.20	1.643

3. Figures

Figure 1 shows the scalar field profiles $\varphi(r)$ for several values of ω, demonstrating how localization and mode structure depend on the breathing frequency.

Figure 2 plots total energy and average radius as functions of ω. The smooth variation indicates that discrete mode structure arises naturally from boundary-constrained nonlinear dynamics.

Figure 3 displays the ground state soliton with the smallest average radius, interpreted as the fundamental particle mode in the PWARI-G framework.

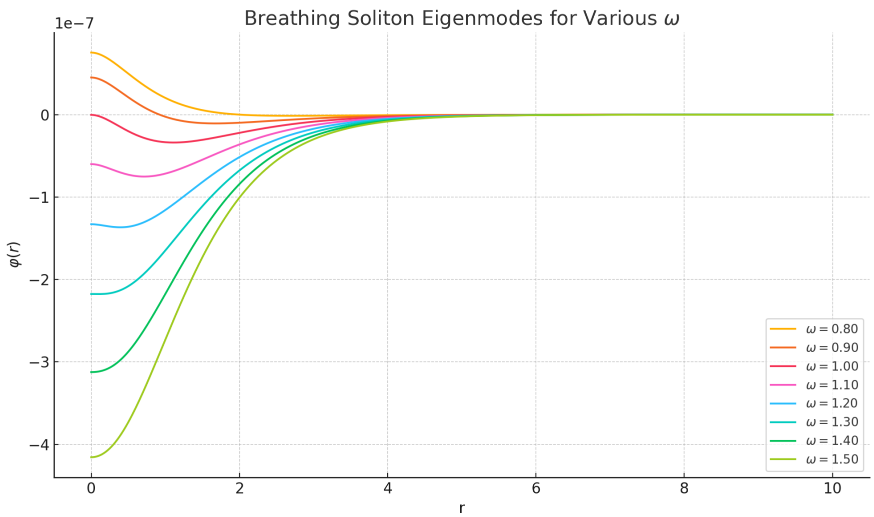

Figure 1: Soliton eigenmodes $\varphi(r)$ for trial frequencies $\omega \in [0.8, 1.5]$.

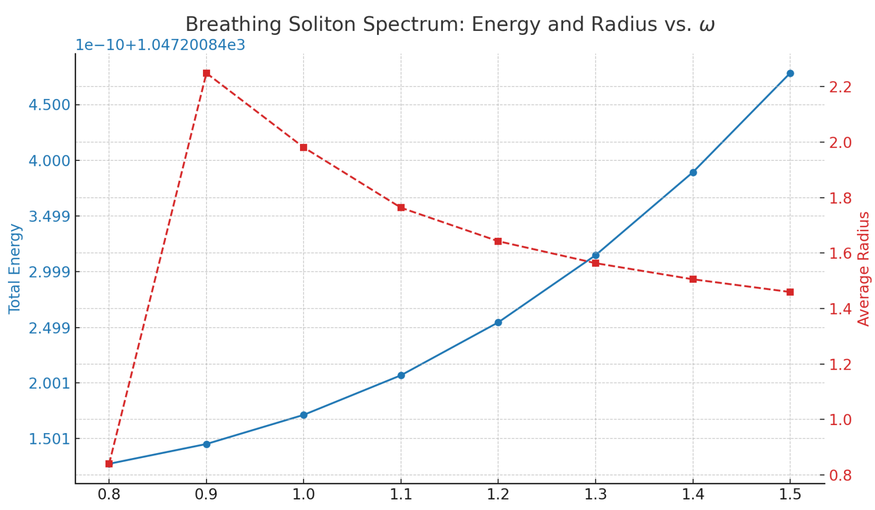

Figure 2: Total energy and average radius vs. breathing frequency ω.

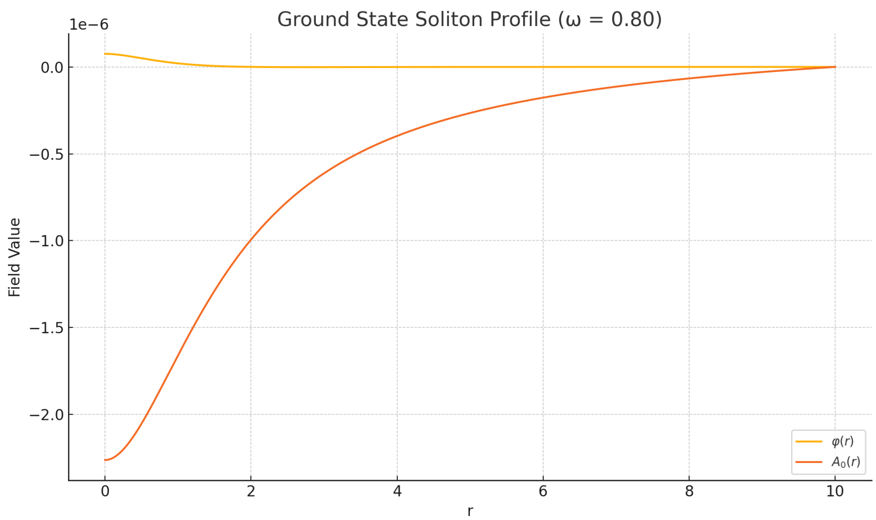

Figure 3: Ground state profile for scalar field $\varphi(r)$ and gauge potential $A_0(r)$ at $\omega = 0.80$.