

Material Motion $n^\circ 1$:

$$\boldsymbol{\psi} = \psi_\beta(\boldsymbol{\chi}, t) = \begin{pmatrix} \chi_1 + \beta \sin(2\pi\chi_1) \sin(\pi\chi_2/3) \sin(\pi t/T) \\ \chi_2 + 5\beta \sin(2\pi\chi_1) \sin(\pi\chi_2/3) \sin(2\pi t/T) \\ \chi_3 \end{pmatrix}, \quad (1)$$

with $T = 2$ and β an amplitude parameter defined for each simulation.

Material Motion $n^\circ 2$:

$$\boldsymbol{\psi} = \psi_\beta(\boldsymbol{\chi}, t) = \begin{pmatrix} \chi_1 + \beta \sin^2(2\pi\chi_1) \sin^2(\pi\chi_2/3) \sin(\pi t/T) \\ \chi_2 + 5\beta \sin^2(2\pi\chi_1) \sin^2(\pi\chi_2/3) \sin(2\pi t/T) \\ \chi_3 \end{pmatrix}, \quad (2)$$

$$\boldsymbol{w} = \begin{pmatrix} \frac{\beta\pi}{T} \sin^2(2\pi\chi_1) \sin^2(2\pi\chi_1) \cos(\pi t/T) \\ \frac{10.0\beta\pi}{T} \sin^2(2\pi\chi_1) \sin^2(\pi Y/3.0) \cos(2\pi t/T) \\ 0.0 \end{pmatrix}, \quad (3)$$

$$\boldsymbol{F}_\psi = \begin{pmatrix} 1 + 4\beta\pi \sin(2\pi X) \cos(2\pi X) \sin(\pi Y/3.0) \sin(\pi Y/3.0) \sin(\pi t/T) & 2\beta\pi/3.0 \sin(2\pi X) \sin(\pi Y/3.0) \sin(\pi t/T) \\ 20.0\beta\pi \cos(2\pi X) \sin(2\pi X) \sin(\pi Y/3.0) \sin(\pi Y/3.0) \sin(2\pi t/T) & 1 + 10.0\beta\pi/3.0 \sin(2\pi X) \sin(\pi Y/3.0) \sin(2\pi t/T) \\ 0 & 0 \end{pmatrix} \quad (4)$$

0.1 Conservation Laws

0.1.1 Neo Hookean model

$$p = \kappa (\det(\mathbf{trueF}) - 1), \quad \text{with } \mathbf{trueF} = \mathbf{F}\mathbf{F}_\psi^{-1} \quad (5)$$

0.1.2 Set of conservation laws

$$\frac{\partial \mathbf{F}}{\partial t} = \text{DIV}_\chi (\hat{\mathbf{v}} \otimes \mathbf{I}) \quad (6a)$$

$$\frac{\partial J}{\partial t} = \mathbf{H}_\psi : \text{GRAD}_\chi (\mathbf{w}) \quad (6b)$$

$$\frac{\partial \mathbf{F}_\psi}{\partial t} = \text{DIV}_\chi (\mathbf{w} \otimes \mathbf{I}) \quad (6c)$$

$$\frac{\partial \tilde{\mathbf{p}}}{\partial t} = \text{DIV}_\chi (\mathbf{P}\mathbf{H}_\psi) + \text{GRAD}_\chi (\mathbf{p}_R) (\mathbf{H}_\psi^T \mathbf{w}) + \mathbf{p}_R [\mathbf{H}_\psi : \text{GRAD}_\chi (\mathbf{w})] \quad (6d)$$

0.2 Finite Volume Method spatial discretization

0.2.1 Discretised set of conservation laws

$$\Omega_\chi^a \frac{\partial \mathbf{F}^a}{\partial t} = \sum_{b \in \Lambda_a} \hat{\mathbf{v}}^{Ave} \otimes \mathbf{c}_\chi^{ab} + \sum_{\gamma \in \Lambda_a^B} \hat{\mathbf{v}}_a^\gamma \otimes \mathbf{c}_\chi^\gamma \quad (7a)$$

$$\Omega_\chi^a \frac{\partial J^a}{\partial t} = \mathbf{H}_\psi : \left(\sum_{b \in \Lambda_a} \mathbf{w}^{Ave} \otimes \mathbf{c}_\chi^{ab} \right) \quad (7b)$$

$$\Omega_\chi^a \frac{\partial \mathbf{F}_\psi^a}{\partial t} = \sum_{b \in \Lambda_a} \mathbf{w}^{Ave} \otimes \mathbf{c}_\chi^{ab} + \sum_{\gamma \in \Lambda_a^B} \mathbf{w}_a^\gamma \otimes \mathbf{c}_\chi^\gamma \quad (7c)$$

$$\begin{aligned} \Omega_\chi^a \frac{\partial \tilde{\mathbf{p}}^a}{\partial t} = & \sum_{b \in \Lambda_a} (\mathbf{P}\mathbf{H}_\psi)^{Ave} \mathbf{c}_\chi^{ab} + \sum_{\gamma \in \Lambda_a^B} t_{\chi,a}^\gamma \|\mathbf{c}_\chi^\gamma\| + \sum_{b \in \Lambda_a} \mathcal{D}_{\chi,ab} \\ & + \left(\sum_{b \in \Lambda_a} \mathbf{p}_R^{Ave} \otimes \mathbf{c}_\chi^{ab} \right) \mathbf{H}_\psi^T \mathbf{w} + \left(\sum_{\gamma \in \Lambda_a} \mathbf{p}_{R,\gamma} \otimes \mathbf{c}_\chi^\gamma \right) \mathbf{H}_\psi^T \mathbf{w} \\ & + \mathbf{p}_R \left[\mathbf{H}_\psi : \left(\sum_{b \in \Lambda_a} \mathbf{w}^{Ave} \otimes \mathbf{c}_\chi^{ab} \right) \right] + \mathbf{p}_R \left[\mathbf{H}_\psi : \left(\sum_{\gamma \in \Lambda_a^B} \mathbf{w}_a^\gamma \otimes \mathbf{c}_\chi^\gamma \right) \right] \end{aligned} \quad (7d)$$

$$\boldsymbol{P}_{mat} = \boldsymbol{P}_{mat}(\boldsymbol{F}_{mat}) \quad (8a)$$

$$\boldsymbol{\mathcal{F}}_{b,\boldsymbol{w}} = \rho_R \dot{\boldsymbol{w}} - \text{DIV}(\boldsymbol{P}_{mat} \boldsymbol{H}_\psi) \quad (8b)$$

$$\boldsymbol{w}^{n+1} = \boldsymbol{w}^n + \Delta t [\text{DIV}(\boldsymbol{P}_{mat} \boldsymbol{H}_\psi) + \boldsymbol{\mathcal{F}}_{b,\boldsymbol{w}}]; \quad (8c)$$

0.2.2 Algorithms

Algorithm 1: Computation of the right hand side of the linear momentum conservation law for the first Runge-Kutta stage.

Result: \mathbf{rhsLm} , \mathbf{rhsF} .

- 1 $\mathbf{p}_R^n = J_\psi^{-1} \mathbf{p}_\chi^n$
- 2 $\mathbf{rhsLm}_a^* = \left(\sum_{b \in \Lambda_a} \mathbf{tC}_b^n \|\mathbf{C}_\chi^{ab}\| \right) + \left(\sum_{b \in \Lambda_a} \mathbf{tC2}_b^n \|\mathbf{C}_\chi^{ab}\| \right) \mathbf{H}_\psi^T \mathbf{w} + \mathbf{p}_R^n \left(\mathbf{H}_\psi : \left[\sum_{b \in \Lambda_a} \mathbf{tC3}_b^n \|\mathbf{C}_\chi^{ab}\| \right] \right)$
- 3 Boundary value of the first RHS term. $\mathbf{tC}_{loc}^\gamma = LocalAverage(\hat{\mathbf{P}} \mathcal{N}_\chi^\gamma)$ or apply the prescribed traction boundary condition. The nodal value is updated as $\mathbf{rhsLm}_\gamma^* = \mathbf{rhsLm}_\gamma^n + \mathbf{tC}_{loc}^\gamma \|\mathbf{C}_\chi^\gamma\|$.
- 4 Boundary value of the second RHS term. $\mathbf{tC2}_{loc}^\gamma = LocalAverage(\mathbf{p}_R \mathcal{N}_\chi^\gamma)$. The nodal value is updated as $\mathbf{rhsLm}_\gamma^* = \mathbf{rhsLm}_\gamma^n + (\mathbf{tC2}_{loc}^\gamma \|\mathbf{C}_\chi^\gamma\|) \mathbf{H}_{\psi,\gamma}^T \mathbf{w}_\gamma$.
- 5 Boundary value of the third RHS term. $\mathbf{tC3}_{loc}^\gamma = LocalAverage(\mathbf{w} \mathcal{N}_\chi^\gamma)$. The nodal value is updated as $\mathbf{rhsLm}_\gamma^* = \mathbf{rhsLm}_\gamma^n + \mathbf{p}_{R,\gamma} (\mathbf{H}_{\psi,\gamma} : (\mathbf{tC3}_{loc}^\gamma \|\mathbf{C}_\chi^\gamma\|))$.
- 6 Volume integration: $\mathbf{rhsLm}_a^* = \mathbf{rhsLm}_a^* / \Omega_a$.
- 7 Update of displacements $\mathbf{x}^* = \mathbf{x}^n + \Delta t \hat{\mathbf{v}}^n$ and linear momentum $\mathbf{p}_\chi^* = \mathbf{p}_\chi^n + J_\psi J^{-1} \Delta t \mathbf{rhsLm}^*$.
- 8 Application of boundary conditions for the linear momentum.
- 9 Update of variables:

$$\begin{aligned}
 \mathbf{trueF}^* &= \mathbf{F}^* \mathbf{F}_\psi^{-1,*} \\
 \mathbf{P}^* &= \mathbf{P}^*(\mathbf{trueF}^*) \\
 \mathbf{p}_R^* &= J_\psi^{-1} \mathbf{p}^* \\
 \mathbf{v}^* &= \mathbf{p}_R^* / \rho \\
 \hat{\mathbf{v}}^* &= \mathbf{v}^* + \mathbf{trueF}^* \mathbf{w}^n \\
 \hat{\mathbf{P}}^* &= \mathbf{P}^* \mathbf{H}_\psi
 \end{aligned}$$

- 10 Computation of coefficient Λ_H^n and update of U_p^n and U_s^n .
- 11 Update of the averaged values for computing the right hand sides

$$\begin{aligned}
 \mathbf{tC}^* &= \hat{\mathbf{P}}^{Ave,*} \mathcal{N} + 0.5 S^* [\mathbf{v}^*] \\
 \mathbf{tC2}^* &= \mathbf{p}_R^{Ave,*} \otimes \mathcal{N} \\
 \mathbf{tC3}^* &= \mathbf{w}^{Ave,n} \otimes \mathcal{N}
 \end{aligned}$$
