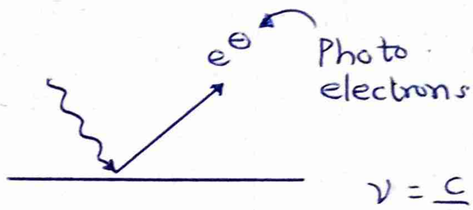
Date: 18.01.2023 Class: 01

① Photoelectric Effect:

Ejection of $e^{\ominus}$ from metal Surface on the application of photon.



$\nu = \frac{c}{\lambda}$

Lenard's observation:

(i) whenever light with frequency greater than a particular frequency falls on a metal Surface, $e^{\ominus}$ are emitted from the Surface.
(The minimum is thresold frequency)

$E_{min} = h\nu_0$ $\nu > \nu_0$ = PEE occurs

Energy of Single photon. $E = h\nu$

Power. $P = \frac{E}{t} = \frac{nh\nu}{t}$

Intensity. $I = \frac{P}{A} = \frac{E}{tA} = \frac{nh\nu}{tA}$

λ, ν depends on Coulor of light

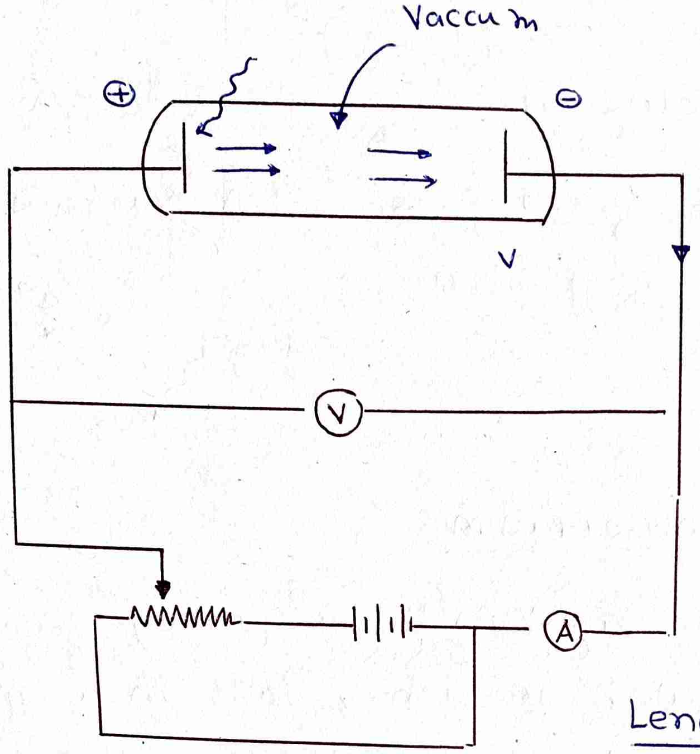
(ii) The ejection of photoelectrons (when frequency of photons are less than thresold frequency) then donot depends on the intensity of light falling on it. Non-Relativistic effect

(iii) The ejection of $e^{\ominus}$ increases (when frequency of photons are greater than thresold frequency) as the intensity of light increases.

$\nu > \nu_0$ intensity ↑ ejection ↑

> Einsteins Explanation

$\nu > \nu_0$



V ↑ electron is repealed. Current decreas

$eV_s = KE$

Lenard's setup

Electron absorbs Photons (energy of photon)

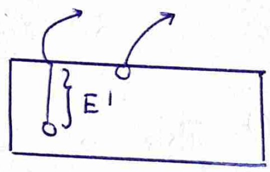
Energy of Photon > Binding Energy $\Rightarrow$ $h\nu > BE$ (work function)

$h\nu_0 = w$ (work function) $\quad \nu_0 = \frac{w}{h}$

$h\nu = w + E' + KE$

w: Same in all $e^{\ominus}$ in a metal

E': Different for all $e^{\ominus}$



By assuming $E' = 0$

$KE = KE_{max}$

KE_{max} is Constant but KE Variable

$$\boxed{h\nu = w + KE_{max}}$$

w changes if metal changes.

$h\nu_0 = w \Rightarrow \nu_0 = \frac{w}{h}$

Cutt of freq $\quad$ Thresold frequency $\quad KE_{max} = h[\nu - \nu_0]$

$w = h\nu_0$

$$KE_{max} = hc\left[\frac{1}{\lambda} - \frac{1}{\lambda_0}\right] \qquad hc = 1240 \text{ eV-nm}$$

stopping potential, $V_s = \frac{KE_{max}}{e}$

$$V_s = \frac{hc}{e}\left[\frac{1}{\lambda} - \frac{1}{\lambda_0}\right]$$

At stopping potential Current will be zero

<u>class 02</u>

For electron, $m_0c^2 = 0.511$ MeV

Proton, $m_0c^2 = 931$ MeV

neutron $m_0c^2 = 938$ MeV

STR is valid when $v \sim c$

<u>Photon</u>

$E = h\nu = \frac{hc}{\lambda}$

$P = \frac{E}{c} = \frac{h}{\lambda}$

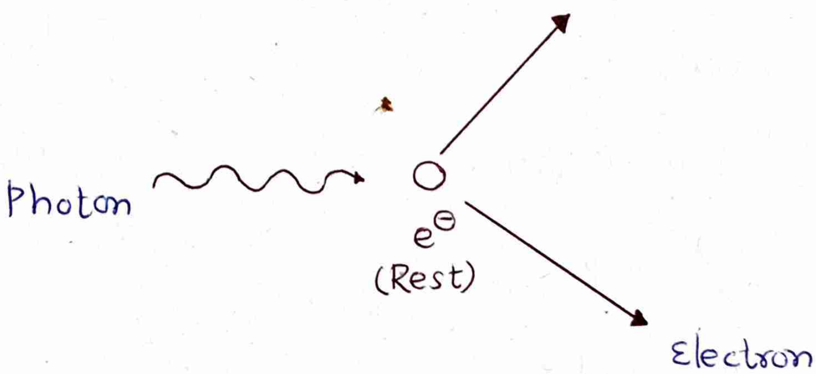
<u>Other Particle.</u>

Rest. $E = m_0c^2 \quad P = 0$

Moving $E = \sqrt{p^2c^2 + m_0^2c^4}$

$P \neq 0$

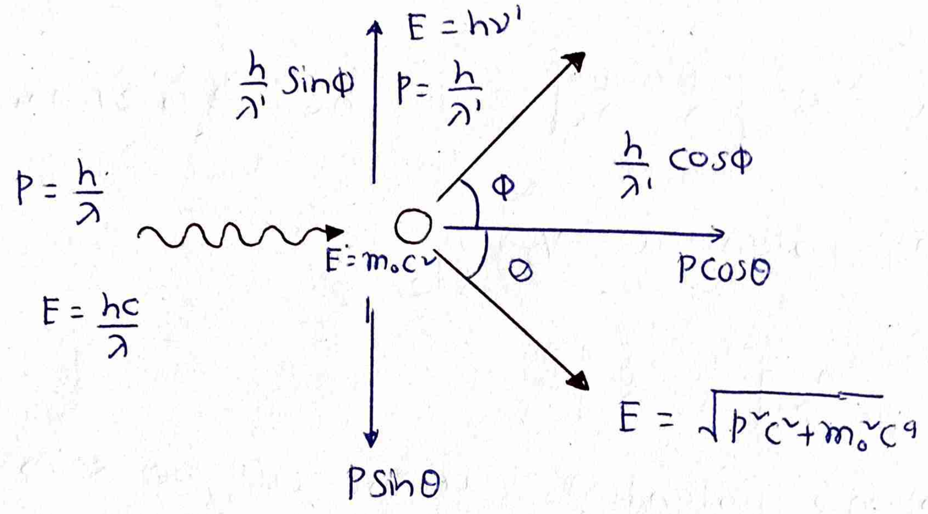
② <u>Compton Effect:</u> (Relativistic)



speed of moving $e^{\ominus}$ in atom is 10^6 m/s

and photon is 3×10^8 m/s

so, Comparitively electron is at rest and photon strikes



It is an inelastic scattering of a photon by an $e^{\ominus}$ (free charge particle). this results decrement in the energy of photon and hence it's wavelength increases. The shift in wavelength of photon is called Compton shift

$E' < E \qquad \lambda' > \lambda \qquad \lambda' - \lambda = \text{Compton shift}$

By Conservation of momentum,

$$(P_x)_i = (P_x)_f$$

$$\Rightarrow \frac{h\nu}{c} + 0 = \frac{h\nu'}{c}\cos\phi + P\cos\theta$$

$$\Rightarrow P\cos\theta = \frac{h\nu}{c} - \frac{h\nu'}{c}\cos\phi \quad \text{— (i)}$$

$$(P_y)_i = (P_y)_f$$

$$\Rightarrow 0 = \frac{h\nu'}{c}\sin\phi - P\sin\theta$$

$$\Rightarrow P\sin\theta = \frac{h\nu'}{c}\sin\phi \quad \text{— (ii)}$$

By Squaring and adding

$$p^2\cos^2\theta + p^2\sin^2\theta = \left(\frac{h\nu}{c} - \frac{h\nu'}{c}\cos\phi\right)^2 + \left(\frac{h\nu'}{c}\sin\phi\right)^2$$

$$\Rightarrow p^2 = \frac{h^2\nu^2}{c^2} - \frac{2h^2\nu\nu'}{c^2}\cos\phi + \frac{h^2\nu'^2}{c^2}$$

$$\Rightarrow p^2c^2 = (h\nu)^2 + (h\nu')^2 - 2(h\nu)(h\nu')\cos\phi \quad \text{— (iii)}$$

Conservation of energy

$$h\nu + m_0c^2 = h\nu' + \sqrt{p^2c^2 + m_0^2c^4}$$

$$\Rightarrow (h\nu - h\nu') + m_0c^2 = \sqrt{p^2c^2 + m_0^2c^4}$$

$$\Rightarrow (h\nu - h\nu')^2 + m_0^2c^4 + 2(h\nu - h\nu')m_0c^2 = p^2c^2 + m_0^2c^4$$

$$\Rightarrow p^2c^2 = (h\nu)^2 + (h\nu')^2 - 2h\nu \cdot h\nu' + 2h(\nu - \nu')m_0c^2 \quad - (4)$$

From equation (iii) and (iv) we get

$$-2(h\nu)(h\nu')\cos\phi = -2(h\nu)(h\nu') + 2h(\nu - \nu')m_0c^2$$

$$\Rightarrow h\nu\nu' - h\nu\nu'\cos\phi = (\nu - \nu')m_0c^2$$

$$\Rightarrow h\frac{c}{\lambda}\frac{c}{\lambda'} - h\frac{c}{\lambda}\frac{c}{\lambda'}\cos\phi = c\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right)m_0c^2$$

$$\Rightarrow \frac{h}{\lambda\lambda'} - \frac{h}{\lambda\lambda'}\cos\phi = m_0c\left(\frac{\lambda' - \lambda}{\lambda\lambda'}\right)$$

$$\Rightarrow \frac{h}{m_0c}(1 - \cos\phi) = \lambda' - \lambda$$

Compton shift. $\Delta\lambda = \frac{h}{m_0c}(1 - \cos\phi)$

λ' = wavelength of photon after striking
λ = " " before striking

$$\frac{h}{m_0c} = 2.424 \times 10^{-12} m = 2.42 pm$$

$\underline{\phi = 180^\circ}$ $\cos\phi = -1$ Shift is maximum

$$\Delta\lambda = \frac{2h}{m_0c} = 4.84 pm$$

$\underline{\phi = 0}$ $\cos\phi = 1$ $\lambda' = \lambda$ Shift is minimum

$$\Delta\lambda = 0$$

Decrease in energy of photon

$$\Delta E = h(\nu - \nu') = KE \text{ of electron}$$

$$h\nu - h\nu' = \sqrt{p^2c^2 + m_0^2c^4} - m_0c^2$$

Relation between θ and ϕ

$$P\cos\theta = \frac{h\nu}{c} - \frac{h\nu'}{c}\cos\phi$$

$$P\sin\theta = \frac{h\nu'}{c}\sin\phi$$

$$\Rightarrow \tan\theta = \frac{h\nu'/_c \sin\phi}{\frac{h\nu}{c} - \frac{h\nu'}{c}\cos\phi}$$

$$\Rightarrow \tan\theta = \frac{\nu'\sin\phi}{\nu - \nu'\cos\phi}$$

$$\Rightarrow \tan\theta = \frac{\lambda}{\lambda + \lambda_c}\cot\phi/_2$$

① Prove that Kinetic energy of $e^\ominus$ is equal to

$$KE_{max} = \frac{h\nu}{1 + \frac{m_0c^2}{2h\nu}}$$

$\Rightarrow$ we have, $\lambda' - \lambda = \frac{h}{m_0c}[1 - \cos\phi]$

$$\Rightarrow \frac{c}{\nu'} - \frac{c}{\nu} = \frac{h}{m_0c}(1 - \cos\phi)$$

$$\Rightarrow \frac{1}{\nu'} - \frac{1}{\nu} = \frac{h}{m_0c^2}(1 - \cos\phi)$$

For Kinetic energy to be maximum $\phi = 180$

$$\frac{1}{\nu'} - \frac{1}{\nu} = \frac{h}{m_0c^2}(1+1)$$

$$\frac{1}{\nu'} = \frac{2h}{m_0c^2} + \frac{1}{\nu}$$

$$\Rightarrow \frac{1}{\nu'} = \frac{2h\nu + m_0c^2}{m_0c^2\nu}$$

$$\Rightarrow \nu' = \frac{m_0c^2\nu}{2h\nu + m_0c^2}$$

change in Kinetic energy

$$KE = h(\nu' - \nu) = h\left(\frac{m_0c^2\nu}{2h\nu + m_0c^2} - \nu\right)$$

$$KE = \frac{h\nu}{1 + \frac{m_0c^2}{2h\nu}}$$

Range of vissible light → 400 - 700 nm

X- Rays 21.01.2023

EM Sepctrum: They all have Same Speed c

γ Rays	x Ray's (10–100 Å)	UV Rays	VIBGYOR Visible (4000 - 7000 Å)	IR	Micro	Radio wave

→ ν decrease

→ λ Increase

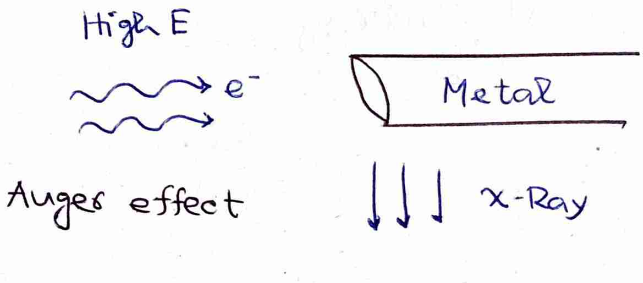
→ E decrease

wavelength of x-ray, $\lambda \sim 0.1Å - 10Å$ } Range
$\nu = 10^{18}$ Hz

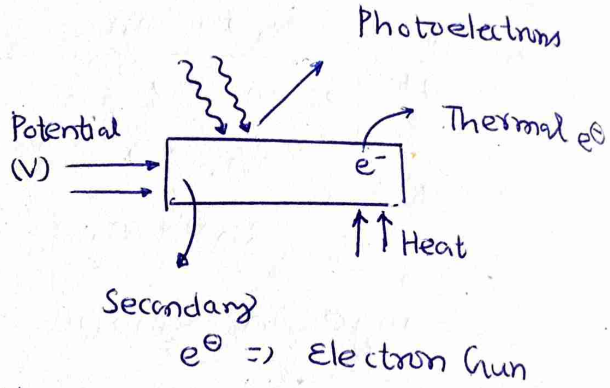
Lenard wave very close in x-ray discovery but disovered by Ronetgen

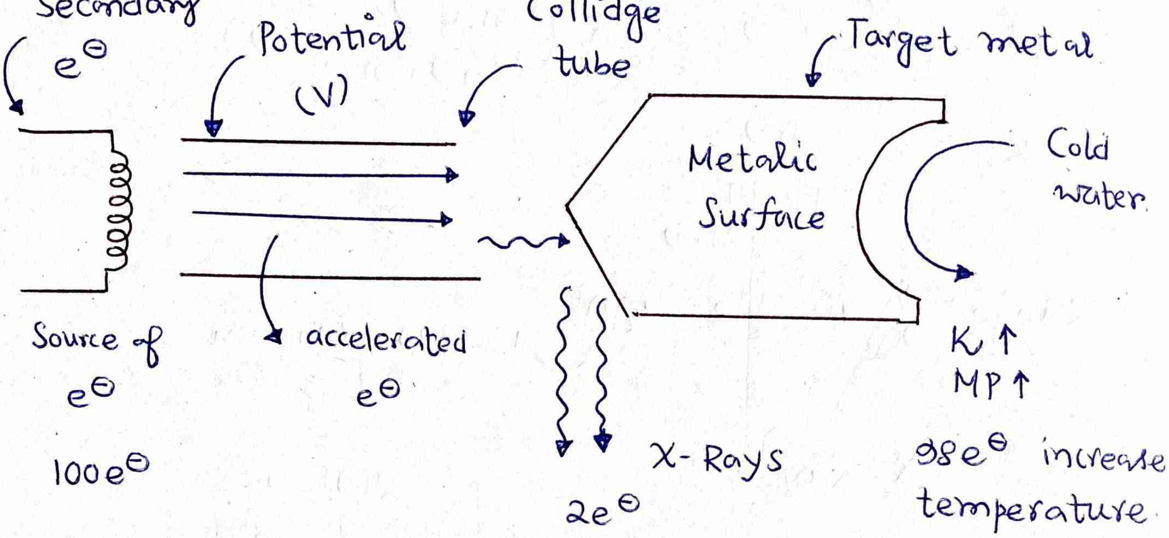
Generation of x-Ray:

Tungsten has more thermal conductivity and melting point



Secondary e^-:





Most of KE is Converted into heat energy

> **Properties of Target Metal:**

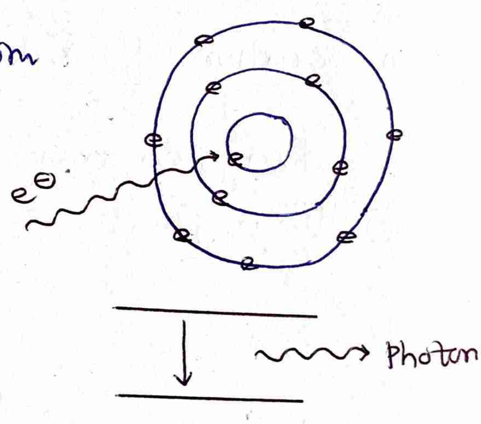
① High melting point

② High thermal Conductivity so that it became Cool down very fast

③ Atomic no should be very high so that it Can produce hard x-rays Mo. Platinum, Tungsten

④ 98% of e⊖ are just Conveiting Kinetic energy into heat energy.

Most of Space of an atom is empty so most e⊖ wasted and makes heat Energy



Energy of photon, $0 \rightarrow KE$

Kinetic energy of $e^{\ominus}$, $KE = eV$

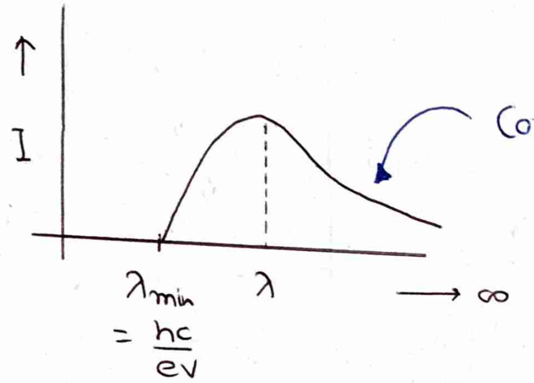
$KE_{max} = eV$

$$\lambda = \frac{h}{\sqrt{2mE}}$$

$\therefore KE \sim 0$ to eV

$$\lambda = \frac{hc}{\lambda_{min}} = E_{max} \Rightarrow \frac{hc}{\lambda_{min}} = eV \Rightarrow \lambda_{min} = \frac{hc}{eV}$$

$\lambda_{max} = \infty$



only energy transfers but orbit does't changes.

Different body part $\equiv$ different X Ray

$=$ different $\lambda \equiv$ different Potential (V)

$$\lambda_{min} = \frac{hc}{eV}$$

Mosley made the modern periodic table (Z)

$e^{\ominus}$ Knocks out an $e^{\ominus}$ from the inner orbit of target metal

$$\lambda_{min} = \frac{1.24 \times 10^{-6}}{V} \text{ V.m}$$

L-shell to K Shell $\Rightarrow K_{\alpha}$ x-rays

M-Shell to K shell $\Rightarrow K_{\beta}$ x-rays

N-shell to K shell $\Rightarrow K_{\gamma}$ x-rays

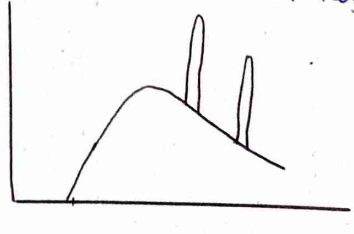
M shell to L shell $\Rightarrow L_{\alpha}$ x rays

N shell to L shell $\Rightarrow L_{\beta}$ x-rays

$$\lambda_{K_{\alpha}} = \frac{hc}{E_L - E_K}$$

$$\lambda_{M_{\beta}} = \frac{hc}{E_O - E_M}$$

Characteristic X-Ray



Mosley's Law:

From Bohr atomic model. $\frac{1}{\lambda} = RZ^2\left[\frac{1}{n_1^2} - \frac{1}{n_2^2}\right]$

$$\frac{hc}{\lambda} = RhcZ^2\left[\frac{1}{n_1^2} - \frac{1}{n_2^2}\right]$$

$R = 1.1 \times 10^7 m^{-1}$

$$\frac{hc}{\lambda} = Rhc\left[\frac{1}{n_1^2} - \frac{1}{n_2^2}\right]$$

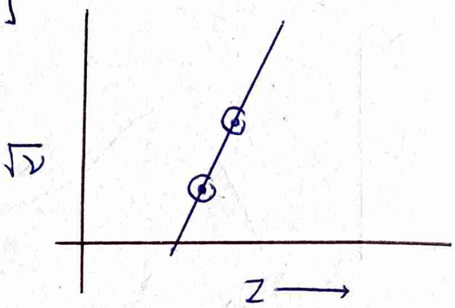
$n_1 = 1$

$n_2 = 2$

$$\nu = RC(Z-b)^2\left[\frac{1}{1^2} - \frac{1}{2^2}\right]$$

or $\nu = \frac{3}{4}RC(Z-b)^2$

or $\sqrt{\nu} = \sqrt{\frac{3RC}{4}}(Z-b)$



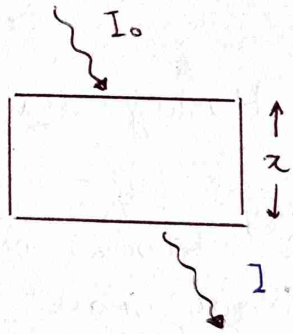
$\boxed{\sqrt{\nu} = a(Z-b)}$ Mosley's law

b is Known as Screnning Constant

a, b depends on element

If b is not given then b = 1

Absorption of X-Rays:



$I = I_0 e^{-\mu x}$

μ = Absorption Coefficient

x = Thickness of material

~~Heard~~ Hard X-Ray means λ is very Small. which Can more penetrate

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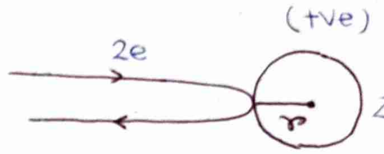
Bohr's Atomic Model

> **Dalton's Model:** Atoms are the smallest unit which is indivisible

> **Thomson's Model:** Discovery of $e^{\ominus}$ from cathode ray

> **Rutherford's α- Scattering Experiment:**

① Most of α- particle were seen going pass the gold foil undeflected



$KE = \frac{1}{4\pi\epsilon_0} \frac{(2e)(2e)}{r}$

r = distance of closest approach

$r = \frac{1}{4\pi\epsilon_0} \frac{(2e)(2e)}{KE}$

$\alpha \rightarrow {}^{4}_{2}He \Rightarrow 2e^{\oplus}$

we can estimate radius of orbit

Most of the space in an atom is empty.

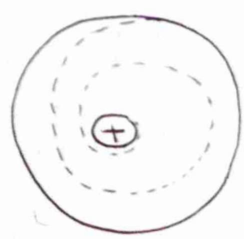
② centre which is positively charged is very small compare to the size of the atom. $\frac{R_N}{R_A} = 10^{-5}$

$R_N \sim 10^{-15} m$ $R_A \sim 10^{-10} m$

③ Electron revolve around the nucleus in random orbits

Accelerated charged particles always emit radiation (Energy) ⇒ Maxwell's statement

Not stabilize the atoms



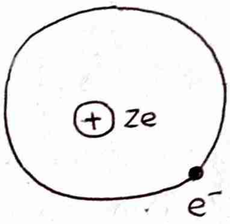
$\text{Radius} \propto \frac{1}{\text{Energy}}$

Bohr's Model (H- Like atom)

H, He^+, Li^{2+}

Valid for Single e^- atom

Centripetal and Electrostatic force balanced each other



$$\frac{1}{4\pi\epsilon_0}\frac{(Ze)e}{r^2} = \frac{mv^2}{r} \quad -①$$

$$mvr = \frac{nh}{2\pi} = n\hbar \quad -②$$

Velocity, $v = \dfrac{nh}{2\pi mr}$

$$\frac{1}{4\pi\epsilon_0}\frac{Ze^2}{r^2} = \frac{m}{r}\cdot\frac{n^2h^2}{4\pi^2m^2r^2}$$

$$r = \frac{n^2h^2\epsilon_0}{\pi mZe^2} = \frac{n^2}{Z}\left[\frac{\epsilon_0 h^2}{\pi me^2}\right]$$

$$r = \frac{n^2}{Z}\times 0.53\times10^{-10}\,m$$

$$\boxed{r = 0.53\left(\frac{n^2}{Z}\right)\text{Å}}$$

$r \propto n^2$

$r \propto \frac{1}{Z}$

H-atom $\quad Z=1,\ n=1 \quad r=0.53\text{Å}$

first orbit

$$v = \frac{nh}{2\pi mr} = \frac{nh}{2\pi m}\cdot\frac{Z\pi me^2}{n^2\epsilon_0 h^2} = \left(\frac{Z}{n}\right)\frac{e^2c}{2\epsilon_0 hc}$$

Fine structure constant, $\alpha = \dfrac{e^2}{2\epsilon_0 hc} = \dfrac{1}{137}$

$$v = \frac{c}{137}\left(\frac{Z}{n}\right) = \alpha c\left(\frac{Z}{n}\right)$$

$v \propto Z$

$v \propto \frac{1}{n}$

H-atom $\quad Z=1,\ n=1 \quad \boxed{v=\alpha c}$

Time period, $T = \dfrac{2\pi r_n}{v_n} = 2\pi\dfrac{n^2}{Z}\dfrac{\epsilon_0 h^2\ n\,2\epsilon_0 h}{\pi me^2\times Ze^2}$

$$T = \left(\frac{4\epsilon_0^2h^3}{me^4}\right)\frac{n^3}{Z^2}$$

$T \propto n^3$

$T \propto \frac{1}{Z^2}$

Current $I = \frac{e}{T} = \frac{e \times me^4 z^2}{4\epsilon_0^2 h^3 n^3}$

$$I = \frac{e^5 m}{4\epsilon_0^2 h^3} \frac{z^2}{n^3}$$

Magnetic field, $B = \frac{\mu_0 i}{2R}$

$$B = \frac{\mu_0}{2} \times \frac{e^5 m}{4\epsilon_0^2 h^3} \times \frac{z^2}{n^3} \times \frac{\pi m z e^2}{n^2 h^2 \epsilon_0}$$

Angular frequency, $\omega = \frac{2\pi}{T} = \frac{2\pi \times me^4 z^2}{4\epsilon_0^2 h^3 n^3}$

> Potential energy

$$PE = -\frac{1}{4\pi\epsilon_0} \frac{(ze)e}{r}$$

$$PE = -\frac{1}{4\pi\epsilon_0} \frac{ze^2 z \pi m e^2}{n^2 \epsilon_0 h^2}$$

$$PE = -\frac{1}{4\pi\epsilon_0} \frac{z^2 e^4 m \pi}{\epsilon_0 h^2 n^2}$$

$$PE = \left(\frac{-me^4}{4\epsilon_0^2 h^2}\right) \frac{z^2}{n^2}$$

$PE \propto z^2$

$PE \propto \frac{1}{n^2}$

> Kinetic Energy

$$\frac{1}{2} \frac{mv^2}{r} = \frac{1}{2} \frac{1}{4\pi\epsilon_0} \frac{ze^2}{r^2}$$

$$\frac{1}{2} mv^2 = \frac{1}{2}\left(\frac{1}{4\pi\epsilon_0} \frac{ze^2}{r}\right) \Rightarrow KE = -\frac{PE}{2}$$

$$KE = \frac{1}{2} \cdot \left(\frac{me^4}{4\epsilon_0^2 h^2}\right) \frac{z^2}{n^2}$$

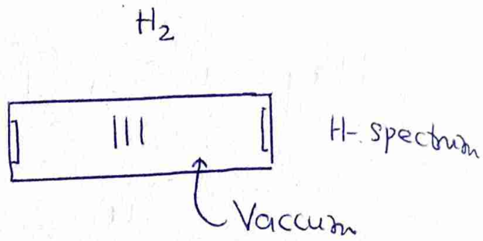
Total energy, $E = KE + PE = -\frac{PE}{2} + PE = \frac{1}{2} PE$

$$E = -\frac{1}{2}\left[\frac{me^4}{4\epsilon_0^2 h^2}\right] \frac{z^2}{n^2}$$

Energy of nth orbit

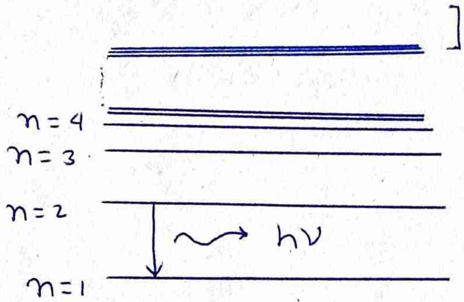
$$E_n = \left(\frac{-me^4}{8\epsilon_0^2 h^2}\right)\frac{Z^2}{n^2} = Rhc\,\frac{Z^2}{n^2} = -13.6\,\frac{Z^2}{n^2}\ \text{eV}$$

Balmer Series:



$$\frac{1}{\lambda} = R\left[\frac{1}{2^2} - \frac{1}{n^2}\right]$$

So we can see that, $E_n \propto \frac{1}{n^2}$



$$E_n = \frac{-me^4}{8\epsilon_0^2 h^2}\frac{Z^2}{n^2}$$

$$E_{n_2} - E_{n_1} = \frac{-me^4}{8\epsilon_0^2 h^2}\left[\frac{Z^2}{n_2^2} - \frac{Z^2}{n_1^2}\right]$$

$h\nu = E_{n_2} - E_{n_1}$

$$h\nu = \frac{me^4}{8\epsilon_0^2 h^2} Z^2 \left[\frac{1}{n_1^2} - \frac{1}{n_2^2}\right]$$

$$\frac{1}{\lambda} = \left(\frac{me^4}{8\epsilon_0^2 h^3 c}\right) Z^2 \left[\frac{1}{n_1^2} - \frac{1}{n_2^2}\right]$$

Infinite nucleus mass approximation

$$\frac{me^4}{8\epsilon_0^2 h^3 c} = R_\infty$$

Rydberg Constant

$$R_\infty = \frac{9.1\times10^{-31}\times(1.6\times10^{-19})^4}{8(8.85\times10^{-12})^2(6.63\times10^{-34})^3\times3\times10^8}$$

$$R_\infty = 10886366.59\ m^{-1} = \frac{me^4}{8\epsilon_0^2 h^3 c}$$

For two body system $\mu = \frac{m_1 m_2}{m_1 + m_2}$ $\qquad \frac{m_e}{m_N} = \frac{1}{1836}$

Here $\mu = \frac{m_e m_N}{m_e + m_N} = \frac{m_e \cancel{m_N}}{\cancel{m_N}\left[1 + \frac{m_e}{m_N}\right]} = m_e$

$\boxed{\mu = m_e}$

$R_\infty = 10886366.59\ m^{-1}$ $\qquad \frac{m_e}{m_N} << 1$

$$R_H = \frac{\mu e^4}{8\epsilon_0^2 h^3 c}$$

$$\mu = \frac{m_e}{1 + \frac{m_e}{m_p}}$$

$$R_H = 10900157.129 \, m^{-1}$$

$$R_\infty = 10886366.59 \, m^{-1}$$

$$\mu = \frac{9.1\times10^{-31}}{1+\frac{1}{1836}}$$

$$\mu = 9.095046274\times10^{-31}$$

R for Helium:

$$\mu_{He} = \frac{4m_p m_e}{4m_p + m_e}$$

$$\mu_{He} = \frac{7344}{7345} m_e$$

$$\mu_{He} = \frac{m_e}{1+\frac{m_e}{4}}$$

$$R_{He} = \frac{7344}{7345} \times R_\infty$$

$$\mu_{He} = \frac{9.1\times10^{-31}}{1+\frac{1}{7344}}$$

$$\mu = 9.098761058\times10^{-31}$$

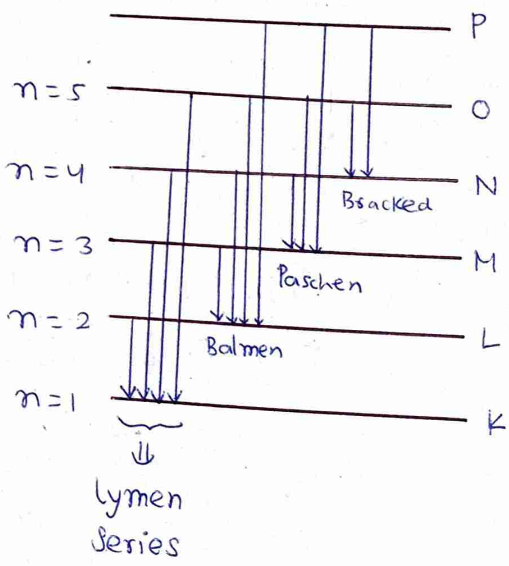
$$\frac{1}{\lambda} = R_\infty \left[\frac{1}{n_1^2} - \frac{1}{n_2^2}\right] Z^2$$ Here, $n_2 > n_1$

> Lymen Series:

when, $n_1 = 1, n_2 > 1$

Lymen Series $(Z=1)$

$n_2 = 2, 3, 4, 5 \, \cdots\cdots \, \infty$



$$\frac{1}{\lambda_\alpha} = R_H \left[\frac{1}{1^2} - \frac{1}{2^2}\right] \Rightarrow \lambda_\alpha = \frac{4}{3R_H}$$

First line

$n_2 \to \infty$

$$\frac{1}{\lambda_1} = R_H$$ Limited line

$\lambda_1 \leq \lambda \leq \lambda_\alpha$ – Lymen Series Range

Positronium atom,

Positron — Antiparticle of $e^{\ominus}$

$e^- \Rightarrow -1.6\times10^{-19}$

$e^+ \Rightarrow +1.6\times10^{-19}$

$m = 9.1\times10^{-31}$ Same

$$\mu = \frac{m_e m_e}{m_e+m_e} = \frac{m_e}{2}$$

$$R_p = \frac{1}{2}\left[\frac{me^4}{8\epsilon_0^2h^3c}\right] = \frac{1}{2}R_\infty$$

Muoniac atom: e^- replaced by μ^-

$m_{\mu^-} = 207\, m_e$

$$R = \frac{\mu e^4}{8\epsilon_0^2h^3c}$$

$$\mu = \frac{m_\mu m_p}{m_\mu+m_p}$$

$R = 186.02\, R_\infty$

$$\mu = \frac{m_\mu}{1+\frac{m_\mu}{m_p}}$$

$$\mu = \frac{207 m_e}{1+\frac{207}{1836}} = 186.02 m_e$$

Ionization Energy:

Energy required to remove the electron from it's orbit

$$\Delta E = Z^2Rhc\left[\frac{1}{n_1^2}-\frac{1}{n_2^2}\right]$$

$E_z = Z^2Rhc$

$n_1 = 1,\quad n_2 = \infty$

$\Delta E = Rhc$

① For Positronium. wavelength of first Balmer Line

$\Rightarrow$ Rydberg Constant $R_\infty = \frac{1}{2}R$

$$\frac{1}{\lambda} = R_\infty\left(\frac{1}{n_1^2}-\frac{1}{n_2^2}\right) \Rightarrow \frac{1}{\lambda} = \frac{1}{2}R\left(\frac{1}{4}-\frac{1}{9}\right)$$

$$\Rightarrow \frac{1}{\lambda} = \frac{R}{2}\times\frac{5}{36}$$

$$\lambda = \frac{72}{5R} = \frac{14.4}{1.1}\times10^{-7}$$

$\lambda = 13126.\mathring{A}$

Nuclear Physics — class:07

Nucleus

↳ By Rutherford — α- Scattering experiment

x → Protons (Z)
x → Neutrons (N)

Mass No, $A = Z + N$

Notation is, ${}^{A}_{Z}X_{N}$, ${}_{Z}X_{A}$, ${}_{Z}X^{A}$.

Nuclear Radius:

Emperical Formula (Experimants)

Nucleus is Spherical in shape,

$R_0 = 1.2\,fm$

$R_0 = 1.2 \times 10^{-15}\,m$

$R \propto A^{1/3} \Rightarrow R = R_0 A^{1/3}$

$V = \frac{4}{3}\pi R^3 \Rightarrow V = \frac{4}{3}\pi R_0^3 A \Rightarrow V \propto A$

$\rho = \frac{M}{V}$

$M_p = 1.672 \times 10^{-27}\,Kg$

$M_N = 1.675 \times 10^{-27}\,Kg$

$M_e = 9.1 \times 10^{-31}\,Kg$

$\Rightarrow \rho = \dfrac{A \cdot M_p}{\frac{4}{3}\pi R_0^3 A}$

$\Rightarrow \rho = \dfrac{3 M_p}{4\pi R_0^3} = \dfrac{3 \times 1.67 \times 10^{-27}}{4 \times 3.14 \times (1.2 \times 10^{-15})^3} = 2.21 \times 10^{17}\,Kg/m^3$

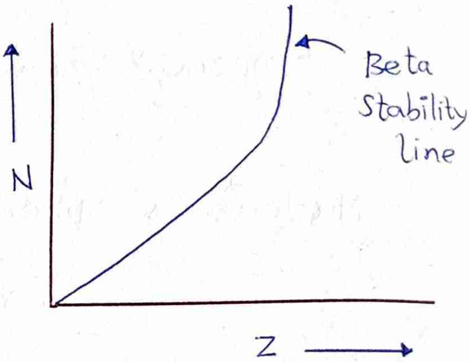
Most of the part of an atom is always empty. that's why H_2. He is in air

> Distance between two proton in a nucleus is less than ($r > fm$) so Coloumb's law is not valid but due to strong Nuclear force the nucleus become stabilize

> Properties of strong Force:

(1) Strongest force found in nature
(2) Always attractive in nature
(3) Non central force ($\vec{\nabla} \times \vec{F} \neq 0$).
(4) They depends on spin and velocity.
(5) They are charge independentent.

> At Higher value of Z the radius become, $r > fm$
So replusion between two proton or Colombic force increase. And to balance or stabilize the nucleus no of neutron became increased



> Isotopes:

Same Z but difference A

> Isobars:

Same A but difference Z

> Isotons: Same Neutrons.

> Mirror Nuceli:

$^{13}_{6}C$ $^{13}_{7}N$

	C	N
A	13	13
P	6	7
N	7	6

Radioactivity:

Spontaneous emission of α, β & γ - particles from a nucleus is called Radioactivity
(Temp, pressure, Heat independent)

Kinetic energy generates within in, no external factor is responsible here.

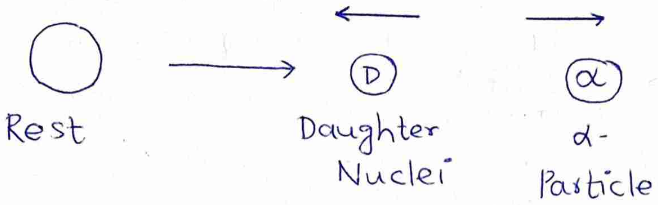
> Radioactivity is a nuclear phenomeon.

> ## α- decay:

$\alpha \rightarrow$ He nuclei $\quad {}^{4}_{2}He$

$${}^{A}_{Z}X \xrightarrow{\alpha\text{- decay}} {}^{A-4}_{Z-2}Y + {}^{4}_{2}He$$

Daughter Nuclei $\qquad$ α-particle



Before decay

$P_i = 0$

After decay

$P_f = 0$

$\vec{P}_D + \vec{P}_\alpha = 0 \qquad |\vec{P}_D| = |\vec{P}_\alpha|$

$\Rightarrow \vec{P}_D = -\vec{P}_\alpha$

From energy Conservation,

$$E_D + E_\alpha = Q$$

or, $\frac{1}{2} m_D V_D^2 + \frac{1}{2} m_\alpha V_\alpha^2 = Q$

or, $\frac{P_D^2}{2m_D} + \frac{P_\alpha^2}{2m_\alpha} = Q \qquad (\because P_\alpha = P_D)$

$m_D = (A-4) m_p$

$m_\alpha = 4 m_p$

or, $\frac{P_\alpha^2}{2m_\alpha}\left[1 + \frac{m_\alpha}{m_D}\right] = Q$

or, $KE_\alpha \left[\frac{m_D + m_\alpha}{m_D}\right] = Q$

or, $KE_\alpha \left[\frac{(A-4)m_p + 4m_p}{(A-4)m_p}\right] = Q$

or, $KE_\alpha = \frac{(A-4)}{A} Q$

$$KE_\alpha + KE_D = Q$$

or $\left(\frac{A-4}{A}\right) Q + KE_D = Q$

$$KE_D = Q - \left(\frac{A-4}{A}\right) Q$$

$$\Rightarrow KE_D = Q\left[1 - \frac{A-4}{A}\right]$$

$$\Rightarrow KE_D = \left(\frac{4}{A}\right) Q$$

$$KE_\alpha = \left(\frac{A-4}{A}\right) Q$$

$$KE_D = \left(\frac{4}{A}\right) Q$$

$$^{238}_{92}U \longrightarrow {}^{234}_{90}Th + {}^{4}_{2}He + Q$$

$$KE_\alpha = \left(\frac{A-4}{A}\right) Q = \left(\frac{234}{238}\right) Q = 0.983 Q$$

$$KE_D = \left(\frac{4}{A}\right) Q = \left(\frac{4}{238}\right) Q = 0.017 Q$$

$$^{A}_{Z}X \longrightarrow {}^{A-4}_{Z-2}Y + {}^{4}_{2}He + Q \text{ value}$$

Energy of the reaction

Q value = Energy of Reactants − Energy of products

β - particle

β^+ emit (e^+)

$$P \rightarrow n + e^+ + \nu$$

$$^{A}_{Z}X \longrightarrow {}^{A}_{Z-1}Y + e^+ + \nu$$

β^- emites (e^-)

$$n \rightarrow P + e^- + \bar{\nu}$$

$$^{A}_{Z}X \rightarrow {}^{A}_{Z+1}Y + e^- + \bar{\nu}$$

$\nu, \bar{\nu}$ (spin $\frac{1}{2}$) helps to Conserve angular momentum

$$P \rightarrow n + e^+ + \nu$$

$$\frac{1}{2} = \frac{1}{2} \quad \frac{1}{2} - \frac{1}{2}$$

Spin is a vector quantity

$$^{A}_{Z}X \xrightarrow[\text{decay}]{\beta^+} {}^{A}_{Z-1}Y + e^+ + \nu$$

$A = n + Z$

$A = (n+1) + (Z-1)$

$A = n + Z$

A is unchanged

$$^{A}_{Z}X \xrightarrow[\text{decay}]{\beta^-} {}^{A}_{Z+1}Y + e^- + \bar{\nu}$$

$A = n + Z$ = unchanged

25.01.2024

Law of Radioactive Decay:

$$^{238}_{92}U \xrightarrow{\alpha\text{-decay}} {}^{234}_{90}Th$$

$$^{238}_{92}U \xrightarrow{\beta^+} {}^{238}_{91} + \beta^+ \qquad {}^{238}_{92}U \xrightarrow{\beta^-} {}^{238}_{93} + \beta^-$$

(i) This is spontaneous emission of α, β, γ particles and these particles are emitted simultaneously

(ii) Decay rate, $-\frac{dN}{dt} \propto N$ N = number of atom present at the moment

$$\Rightarrow \frac{dN}{dt} = -\lambda N \quad (\lambda \rightarrow \text{decay Constant})$$

$$\Rightarrow \int \frac{dN}{N} = -\lambda \int dt$$

$$\Rightarrow \ln N = -\lambda t + C$$

At $t=0$, $N = N_0$, $\ln N_0 = C$

$$\ln N = -\lambda t + \ln N_0 \Rightarrow \boxed{N = N_0 e^{-\lambda t}}$$

when, $N = \frac{N_0}{2}$, $t = T_{1/2} \rightarrow$ Half life Time

$$\frac{N_0}{N} = 2 \qquad \ln 2 = \lambda T_{1/2} \qquad \boxed{T_{1/2} = \frac{\ln 2}{\lambda}}$$

$$\Rightarrow T_{1/2} = \frac{0.693}{\lambda}$$

$$N_0 \xrightarrow{T_{1/2}} \frac{N_0}{2} \xrightarrow{T_{1/2}} \frac{N_0}{4} \xrightarrow{T_{1/2}} \frac{N_0}{8} \xrightarrow{T_{1/2}} \frac{N_0}{16}$$

$N = N_0 e^{-\lambda t}$ [NB] $N = \frac{N_0}{2^n}$

$\Rightarrow \frac{N_0}{4} = N_0 e^{-\lambda t}$ decay time, $\boxed{t = nT_{1/2}}$

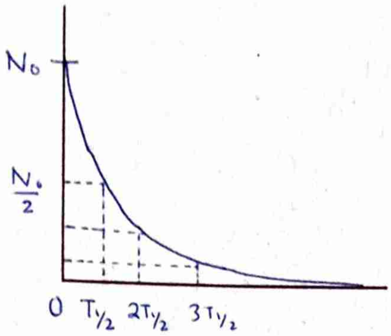
$$\Rightarrow T = 2\left[\frac{\ln 2}{\lambda}\right]$$

$$\Rightarrow T = 2T_{1/2}$$

$$N = N_0 e^{-\lambda t}$$

Mean life time

at; $t = \langle T \rangle$



$$N = \frac{N_0}{e}$$

$$N = N_0 e^{-\lambda t}$$

$$\Rightarrow N_0 e^{-1} = N_0 e^{-\lambda \langle T \rangle}$$

$$\Rightarrow \lambda \langle T \rangle = 1 \Rightarrow \langle T \rangle = \frac{1}{\lambda}$$

Average or mean life time, $\boxed{\langle T \rangle = \frac{1}{\lambda}}$

$$\langle t \rangle = \frac{\int_0^\infty t\,dN}{\int_0^\infty dN}$$

$$N = N_0 e^{-\lambda t}$$

$$\Rightarrow dN = -N_0 e^{-\lambda t} \lambda\,dt$$

$$\Rightarrow \langle t \rangle = \frac{-\int_0^\infty t N_0 e^{-\lambda t} \lambda\,dt}{-\int_0^\infty \lambda N_0 e^{-\lambda t}\,dt} = \frac{\lambda N_0 \int_0^\infty e^{-\lambda t}\,t\,dt}{\lambda N_0 \int_0^\infty e^{-\lambda t}\,dt}$$

$$\Rightarrow \langle t \rangle = \frac{\Gamma 2/\lambda}{e^{-\lambda t}\Big|_\infty^0} = \frac{\Gamma 2}{\lambda} = \frac{1}{\lambda} \Rightarrow \langle T \rangle = \frac{1}{\lambda}$$

> <u>Succesive Radioactive Decay:</u>

	unstable		unstable		Stable
	A	$\xrightarrow{\lambda_A}$	B	$\xrightarrow{\lambda_B}$	C
$t=0$,	N_0		$N_B = 0$		$N_c = 0$
$t=t$	N_A		N_B		N_c

$$N_A = N_0 e^{-\lambda_A t}$$

$$\frac{dN_B}{dt} = \lambda_A N_A - \lambda_B N_B$$

$$\Rightarrow \frac{dN_B}{dt} = \lambda_A N_0 e^{-\lambda_A t} - \lambda_B N_B$$

$$\Rightarrow \frac{dN_B}{dt} + \lambda_B N_B = \lambda_A N_0 e^{-\lambda_A t}$$

$$\Rightarrow \frac{dy}{dx} + Px = Q$$

Integrating factor.

$$IF = e^{\int P dx} = e^{\int \lambda_B dt} = e^{\lambda_B t}$$

$$y(IF) = \int (IF)\, Q\, dx + c$$

$$\Rightarrow \quad N_B (e^{\lambda_B t}) = \int e^{\lambda_B t} \lambda_A N_0 e^{-\lambda_A t} dt + c$$

$$\Rightarrow \quad N_B e^{\lambda_B t} = \frac{\lambda_A N_0 \, e^{(\lambda_B - \lambda_A)t}}{\lambda_B - \lambda_A} + c$$

$$\Rightarrow \quad N_B = \left(\frac{\lambda_A N_0}{\lambda_B - \lambda_A} \right) e^{-\lambda_A t} + c e^{-\lambda_B t}$$

At $t = 0$. $N_B = 0$

$$0 = \frac{\lambda_A N_0}{\lambda_B - \lambda_A} + c \Rightarrow c = \frac{-\lambda_A N_0}{\lambda_B - \lambda_A}$$

$$N_B = \frac{N_0 \lambda_A}{\lambda_B - \lambda_A} e^{-\lambda_A t} - \frac{\lambda_A N_0}{\lambda_B - \lambda_A} e^{-\lambda_B t}$$

$$\Rightarrow \quad N_B(t) = \frac{N_0 \lambda_A}{\lambda_B - \lambda_A} \left[e^{-\lambda_A t} - e^{-\lambda_B t} \right]$$

Also, $\frac{dN_c}{dt} = \lambda_B N_B$

$$\Rightarrow \quad \frac{dN_c}{dt} = \frac{\lambda_B \lambda_A N_0}{\lambda_B - \lambda_A} \left[e^{-\lambda_A t} - e^{-\lambda_B t} \right]$$

$$\Rightarrow \quad N_c = \frac{\lambda_B \lambda_A N_0}{\lambda_B - \lambda_A} \left[\frac{e^{-\lambda_A t}}{-\lambda_A} + \frac{e^{-\lambda_B t}}{\lambda_B} \right] + c$$

at $t=0$, $N_c = 0 = \left[-\frac{1}{\lambda_A} + \frac{1}{\lambda_B} \right] \frac{\lambda_B \lambda_A N_0}{\lambda_B - \lambda_A} + c$

so, $c = N_0$

$$N_c(t) = \frac{\lambda_B \lambda_A}{\lambda_B - \lambda_A} N_0 \left[\frac{e^{-\lambda_B t}}{\lambda_B} - \frac{e^{-\lambda_A t}}{\lambda_A} \right] + N_0$$

$$\Rightarrow \quad N_c(t) = \frac{N_0}{\lambda_B - \lambda_A} \left[\lambda_A e^{-\lambda_B t} - \lambda_B e^{-\lambda_A t} \right] + N_0$$

Binding Energy: 27.01.2023

Einstein's formula, $E = \Delta m c^2$

n = no of neutron

In Nucleus

Total mass of neutron, $M_N = n m_n$

Total mass of proton, $M_p = Z M_p$

Total mass of nucleus. M_N^*

So, it should be $M_N^* = n m_n + Z M_p$ ✗

$M_N^* < n m_n + Z M_p$

Mass defeact,

$$\Delta m = (Z M_p + n m_n) - M_N^*$$

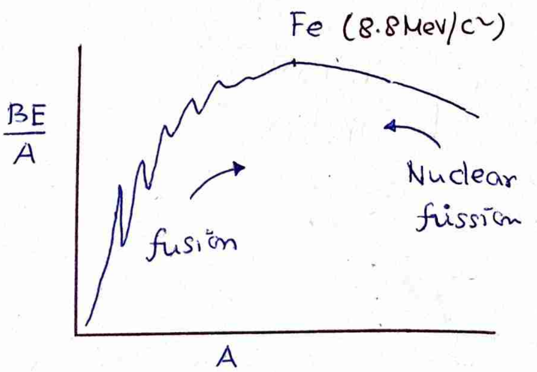
The energy is in Nucleus Binding energy

$$BE = \Delta m c^2 = \left[(Z M_p + n m_n) - M_N^*\right] c^2$$

Binding energy per nucleon,

$$\frac{BE}{A} = \left[(Z M_p + n m_n) - M_N^*\right] \frac{c^2}{A}$$

Proton	1.6726×10^{-27} Kg	1.007276	938.28 Mev/c^2
Neutron	1.6750×10^{-27} Kg	1.008665	939.57 Mev/c^2
$_1H^1$	1.6736×10^{-27} Kg	1.007825	938.79 Mev/c^2



Fe has maximum binding energy per nucleon That's why Fe is most stable

By fission and they trends to stable and try goes to Fe

Q value of a Reaction:

Q = Energy of Reactants − Energy of Products

Q > 0 Exo-ergic (Exothermic)

Q < 0 Endo ergic (Endothermic)

Topic	α- Particle	β- Particle	γ - Particle
Charge & mass	2e, 4Mp	me, (−e)	charge and mass less
Velocity	less	$V_\alpha < V_\beta < V_\gamma$	$V = c = 3\times10^8$ m/s
Ionization Energy	Maximum	Middle	Minimum
Pentration Power	Less (as mass high)	Mid of them	Maximum Penetration power
Electric & Magnetic field	Get deflected	Get deflected	doest deflected.

Young's Double Slit Experiment with Electron

Dual Nature of Radiation

28.01.2023

Light:

Corpuscular theory of light (By Newton)

Can explain Reflection, Refraction

But not Interference & diffraction

So, Interference and diffraction by wave Nature.

Photoelectric Effect
Compton Effect (Collision by particle) } Particle Nature $\Downarrow$ Photon

Some Phenomenon,

Particle Nature
wave Nature] Dual Nature

Energy Packet: Quanta

Dual Nature of Matter

Matter → Particle Nature
Matter → Wave Nature → Davison Gerrman Exp. Verified with Bragg's law

So $e^{\ominus}$ also have wave Nature

de-Broglie wavelength:

wavelength, $\lambda = \frac{h}{p} = \frac{h}{mv}$

Free particle, PE, V = 0

$E = T + PE = T$

$KE = \frac{1}{2}mv^2 = \frac{p^2}{2m} = E$

$p = \sqrt{2mE}$

So, $\lambda = \frac{h}{\sqrt{2mE}}$

> For a charged Particle, $E = qV$ (V = potential)

$$\lambda = \frac{h}{\sqrt{2mqV}} = \sqrt{\frac{150}{V}}\ \mathring{A} = \frac{12.27}{\sqrt{V}}\ \mathring{A} = \frac{0.1227}{\sqrt{V}}\ nm$$

> Thermal Neutrons get energy from Heat Energy

$E = f(\frac{1}{2}KT)$ But generally, $E = K_B T$

de-Broglie wavelength, $\lambda = \frac{h}{\sqrt{3mKT}}$ (3-D)

$$\lambda = \frac{h}{\sqrt{2mKT}}$$

> $E = \sqrt{p^2c^2 + m_0^2c^4}$

$\Rightarrow (T + m_0c^2) = \sqrt{p^2c^2 + m_0^2c^4}$

$\lambda = \frac{hc}{\sqrt{T(T+2m_0c^2)}}$

$\Rightarrow T^2 + 2m_0c^2T + m_0^2c^4 = p^2c^2 + m_0^2c^4$

$\Rightarrow p = \frac{1}{c}\sqrt{T(T+2m_0c^2)}$

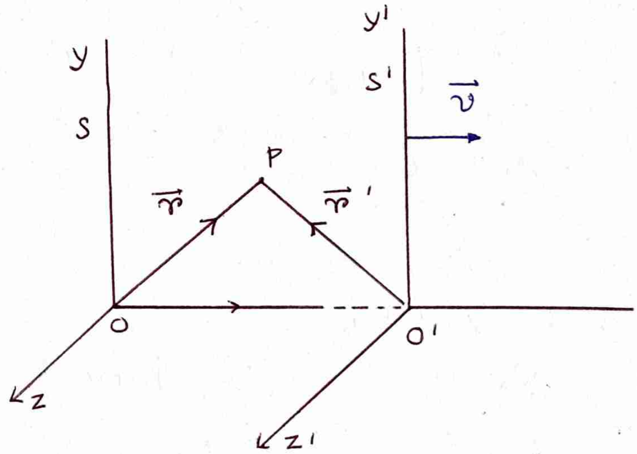
For photons,

$p = \frac{E}{c} = \frac{h}{\lambda}$

Special Theory of Relativity 29.01.2023

At $t=0$,

O and O' coincide



From $\Delta OO'P$

$\vec{OO'} + \vec{O'P} = \vec{OP}$

$\vec{v}t + \vec{r}' = \vec{r}$

$\vec{r}' = \vec{r} - \vec{v}t$

$\vec{v} = v\hat{i}$

$x' = x - vt$
$y' = y$
$z' = z$

Galilian Transformation

So, $\vec{r}' = \vec{r} - \vec{v}t$

$\frac{d\vec{r}'}{dt} = \frac{d\vec{r}}{dt} - \vec{v}$

$\vec{V}_{s'} = \vec{V}_s - \vec{v}$

V_s' = Velocity of P wrt s' frame

V_s = Velocity of P wrt s frame

v = Velocity of s' frame

$\frac{d\vec{V}_{s'}}{dt} = \frac{d\vec{V}_s}{dt} - 0$

$\vec{a}' = \vec{a}$

$m\vec{a}' = m\vec{a}$

$\vec{F}' = \vec{F}$

Inverse Galilian Transformation

$x = x' + vt'$
$y = y'$
$z = z'$

Lorentz Transformation Equation

when $v \sim c$

$x' = ax + bt$

$y' = y$

$z' = z$

$t' = dx + ft$

They are based on 2 postulates of Einstein. These are –

(i) Every physics laws are same in all frame of reference

(ii) Velocity of light is constant $(v = c)$ and frame independent

Here, $a = \frac{1}{\sqrt{1 - v^2/c^2}} = f$ and $b = \frac{-v}{\sqrt{1 - v^2/c^2}}$. $d = \frac{-v}{c^2\sqrt{1 - v^2/c^2}}$

$$x' = ax + bt = \frac{1}{\sqrt{1 - v^2/c^2}}x - \frac{v}{\sqrt{1 - v^2/c^2}}t = \frac{x - vt}{\sqrt{1 - v^2/c^2}}$$

$$y' = y, \quad z' = z, \quad t' = \frac{-vx}{c^2\sqrt{1 - v^2/c^2}} + \frac{t}{\sqrt{1 - v^2/c^2}} = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}$$

$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$ and $\frac{v}{c} = \beta$

So,
$x' = \gamma(x - vt)$
$y' = y$
$z' = z$
$t' = \gamma\left(t - \frac{vx}{c^2}\right)$

Lorentz Transformation

$x = \gamma(x' + vt')$
$y = y'$
$z = z'$
$t = \gamma\left(t' + \frac{vx'}{c^2}\right)$

Inverse Lorentz Transformation.

As we have, $\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$

or, $\gamma^2 = \frac{1}{1 - v^2/c^2}$

or $\gamma^2 = \frac{1}{1 - \beta^2}$

or $\gamma^2(1 - \beta^2) = 1$

or $\gamma^2 - \gamma^2\beta^2 = 1$

Relativistic Mass

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

m = Moving mass
m_0 = Rest mass
v = Velocity of mass m

Length element,

$$c^2t'^2 - x'^2 - y'^2 - z'^2$$

$$= c^2\left\{\gamma^2\left(t - \frac{vx}{c^2}\right)^2\right\} - \gamma^2(x-vt)^2 - y^2 - z^2$$

$$= c^2\gamma^2\left(t^2 + \frac{x^2v^2}{c^4} - \frac{2txv}{c^2}\right) - \gamma^2\left(x^2+v^2t^2 - 2xvt\right) - y^2 - z^2$$

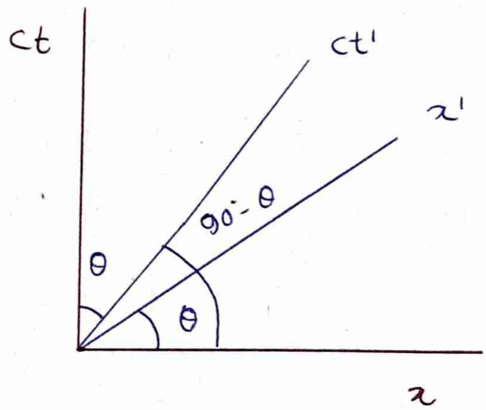
$$= c^2t^2\left(\gamma^2 - \frac{\gamma^2v^2}{c^2}\right) - x^2\left(\gamma^2 - \gamma^2\frac{v^2}{c^2}\right) - \gamma^2(\cancel{2xvt}) + \gamma^2(\cancel{2xvt}) - y^2 - z^2$$

$$= c^2t^2(\gamma^2 - \gamma^2\beta^2) - x^2(\gamma^2 - \gamma^2\beta^2) - y^2 - z^2$$

$$= c^2t^2 - x^2 - y^2 - z^2$$

So we get the length element is same in both frame of reference. So it is called Lorentz invarient quantity. ⇒ Length element.

TIFR 2023



$x' = \gamma(x - vt)$

or $x' = \gamma\left(x - \frac{v}{c}(ct)\right)$

and, $t' = \gamma\left(x - \frac{xv}{c^2}\right)$

$ct' = \gamma\left[ct - x\left(\frac{v}{c}\right)\right]$

At x' axis

$ct' = 0$ $\quad 0 = ct - x\left(\frac{v}{c}\right)$

$ct = \left(\frac{v}{c}\right)x \equiv y = mx$

At ct' axis $\quad x - \frac{v}{c}(ct) = 0$

$x' = 0$

$ct = \left(\frac{c}{v}\right)x$

$y = m'x$

$m = \tan\theta = \frac{v}{c}$

$m' = \tan\theta' = \frac{c}{v}$

$\tan\theta' = \frac{1}{\tan\theta}$

$\tan\theta' = \cot\theta$

$\theta' = 90° - \theta$

Hyperpolic Function:

$$\operatorname{Sinh}\theta = \frac{e^{\theta}-e^{-\theta}}{2} \qquad \tanh\theta = \frac{e^{\theta}-e^{-\theta}}{e^{\theta}+e^{-\theta}}$$

$$\operatorname{Cosh}\theta = \frac{1}{2}(e^{\theta}+e^{-\theta}) \qquad \operatorname{Sech}\theta = \frac{2}{e^{\theta}+e^{-\theta}}$$

$$\operatorname{Cosh}^2\theta - \operatorname{Sinh}^2\theta = 1 \qquad \operatorname{Sech}^2\theta + \tanh^2\theta = 1$$

$$\operatorname{Coth}^2\theta - \operatorname{Cosech}^2\theta = 1$$

Now we have to Consider, $\frac{v}{c} = \tanh\theta$

$$1-\frac{v^2}{c^2} = 1-\tanh^2\theta = \operatorname{Sech}^2\theta$$

$$\frac{1}{\sqrt{1-v^2/c^2}} = \frac{1}{\operatorname{Sech}\theta} = \operatorname{Cosh}\theta$$

According to Lorentz transformation,

$$x' = \gamma\left[x - \frac{v}{c}(ct)\right] \qquad y' = y$$

$$\Rightarrow x' = \operatorname{Cosh}\theta\left[x - \tanh\theta(ct)\right] \qquad z' = z$$

$$\Rightarrow x' = (\operatorname{Cosh}\theta)x - (\operatorname{Sinh}\theta)ct$$

$$ct' = \gamma\left(ct - x\frac{v}{c}\right)$$

$$\Rightarrow ct' = \operatorname{Cosh}\theta\left[ct - x\tanh\theta\right]$$

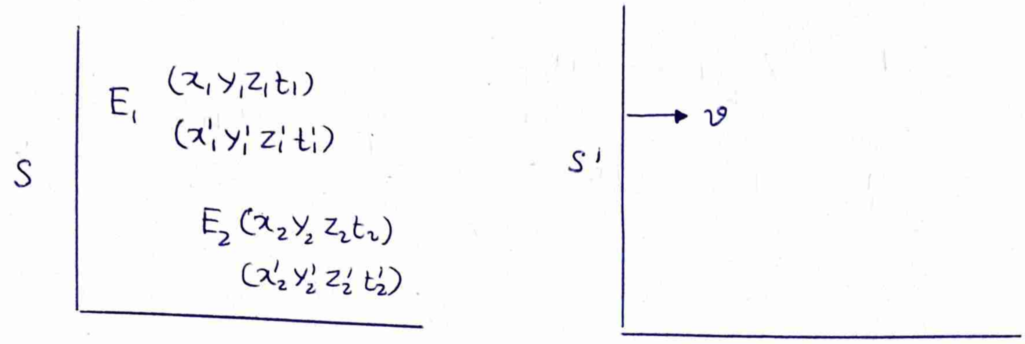
$$\Rightarrow ct' = (\operatorname{Cosh}\theta)ct - (\operatorname{Sinh}\theta)x$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ ct' \end{bmatrix} = \begin{bmatrix} \operatorname{Cosh}\theta & 0 & 0 & -\operatorname{Sinh}\theta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\operatorname{Sinh}\theta & 0 & 0 & \operatorname{Cosh}\theta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ ct \end{bmatrix}$$

$\Downarrow$

Transformation Matrix.

About Event:



$x_1' = \gamma(x_1 - vt_1)$

$y_1' = y_1$

$z_1' = z_1$

$t_1' = \gamma\left[t_1 - \frac{vx_1}{c^2}\right]$

$x_2' = \gamma(x_2 - vt_2)$

$y_2' = y_2$

$z_2' = z_2$

$t_2' = \gamma\left[t_2 - \frac{x_2 v}{c^2}\right]$

Now we get,

$$x_2' - x_1' = \gamma\left[(x_2 - x_1) - v(t_2 - t_1)\right]$$

$$\Rightarrow \Delta x' = \gamma\left[\Delta x - v\Delta t\right] \quad \text{--- (i)}$$

$$t_2' - t_1' = \gamma\left[(t_2 - t_1) - \frac{v}{c^2}(x_2 - x_1)\right]$$

$$\Rightarrow \Delta t' = \gamma\left[\Delta t - \frac{v}{c^2}\Delta x\right] \quad \text{--- (2)}$$

$\Delta y' = \Delta y$

$\Delta z' = \Delta z$

Case 1 Both the events are taking place at same place in S frame.

So $\Delta x = 0$ $\quad \Delta x' = -\gamma v \Delta t \neq 0$

$\Delta t' = \gamma \Delta t$

$$t' = \frac{t}{\sqrt{1 - v^2/c^2}} \Leftarrow \text{Time dilation}$$

Case 2 Simultaneous in S frame

$\Delta t = 0$ $\quad \Delta t' = t_2' - t_1' = -\frac{\gamma \Delta x v}{c^2}$

$$\Rightarrow -t_2' + t_1' = +\frac{v}{c^2}\frac{x_2 - x_1}{\sqrt{1 - v^2/c^2}}$$

Case ③ Simultaneous in S' frame

$$\Delta t' = t_2' - t_1' = 0 \Rightarrow t_2' = t_1'$$

$$\Delta t' = \gamma\left[\Delta t - \frac{V\Delta x}{c^2}\right]$$

$$\Rightarrow 0 = \Delta t - \frac{V\Delta x}{c^2} \qquad \Delta t = \frac{V\Delta x}{c^2}$$

Relativistic Doppler Effect

Apparant frequency,

α is the angle between Source & receiver.

$$\nu' = \nu\left[\frac{1 - \frac{V}{c}\cos\alpha}{\sqrt{1 - V^2/c^2}}\right]$$

V is the velocity of Source wrt observer

Case ①: If $\alpha = 0$. $\cos\alpha = 1$

$$\nu' = \nu\left[\frac{1 - V/c}{\sqrt{1 - V^2/c^2}}\right] = \nu\sqrt{\frac{c-V}{c+V}}$$

$$\nu' = \nu\sqrt{\frac{c-V}{c+V}} \quad \text{and} \quad \lambda' = \lambda\sqrt{\frac{c+V}{c-V}} \qquad \left(\nu = \frac{c}{\lambda}\right)$$

$\nu' < \nu$ $\qquad$ $\lambda' > \lambda$

object moving away $\qquad$ wavelength increase

Case ② Source is moving toward observer

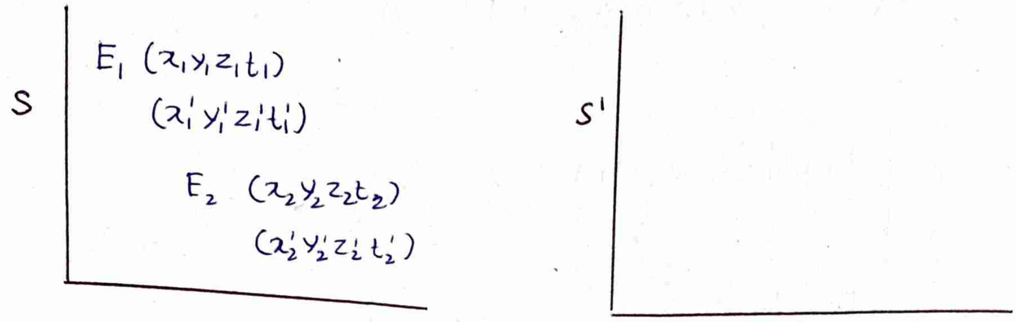
$\alpha = \pi \qquad \cos\alpha = -1$

$$\nu' = \nu\sqrt{\frac{c+V}{c-V}} \qquad \lambda' = \lambda\sqrt{\frac{c-V}{c+V}} \qquad \lambda' < \lambda$$

Case③ They are perpendicular to each other

$\alpha = \pi/2 \qquad \cos\pi/2 = 0$

$$\nu' = \nu\sqrt{1 - V^2/c^2} \qquad \lambda' = \frac{\lambda}{\sqrt{1 - V^2/c^2}}$$



So, $\Delta t' = \gamma \left[\Delta t - \frac{\Delta x v}{c^2}\right]$

$\Delta t' = t_2' - t_1'$
$\Delta t = t_2 - t_1$
$\Delta x = x_2 - x_1$

$$\Delta t' = \frac{\Delta t}{\sqrt{1 - v^2/c^2}}$$

$\Delta t'$ = Time measured by clock in moving frame

$\Delta t' > \Delta t \quad \because \gamma > 1$

Δt = time measured by clock in rest frame

Time: Time is quantity which measured by clock.

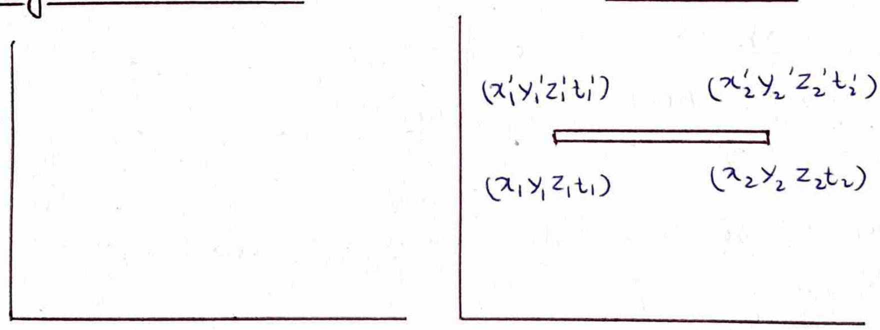
clock
- Aging
- Atomic clock
- Vibration of atom

Time dilation

Since. $\gamma > 1 \quad \Delta t' > \Delta t$

so Moving clock slows down $\qquad \Delta t' = \frac{\Delta t}{\sqrt{1 - v^2/c^2}}$

Length Contraction: $\qquad$ 30.01.2024



$$x_2' - x_1' = \gamma \left[(x_2 - x_1) - v(t_2 - t_1)\right]$$

Since this is a Simultaneous event, $t_2 - t_1 = 0$

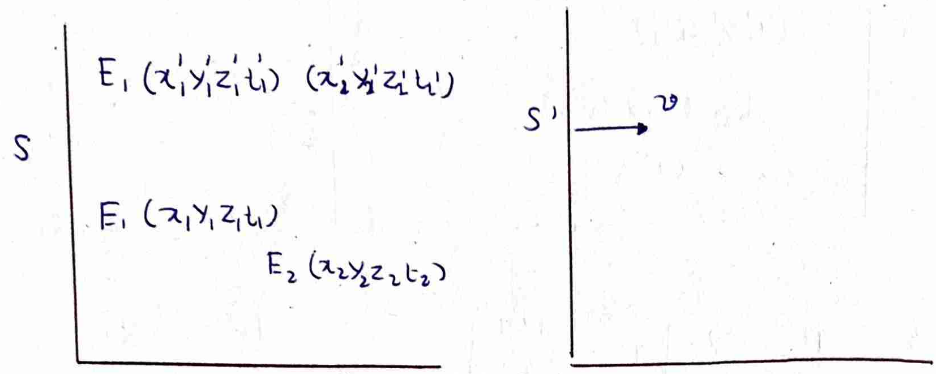
$\Delta x' = \gamma \Delta x \Rightarrow \Delta x = \frac{\Delta x'}{\gamma}$

$\Delta x' = l_0$ = Rest length $\qquad l = l_0 \sqrt{1 - v^2/c^2}$

$\Delta x = l$ = moving length

$l_0 > l$

Relativistic velocity addition:



$$x_2' - x_1' = \gamma\left[(x_2 - x_1) - v(t_2 - t_1)\right]$$

$$u_x' = \frac{x_2' - x_1'}{t_2' - t_1'} = \frac{\gamma\left[x_2 - vt_2\right] - \gamma\left[x_1 - vt_1\right]}{\gamma\left[t_2 - \frac{vx_2}{c^2}\right] - \gamma\left[t_1 - \frac{vx_1}{c^2}\right]}$$

$$\Rightarrow u_x' = \frac{(x_2 - x_1) - v(t_2 - t_1)}{(t_2 - t_1) - \frac{v}{c^2}(x_2 - x_1)}$$

$$\Rightarrow u_x' = \frac{u_x - v}{1 - \frac{vu_x}{c^2}} \qquad \text{Here, } u_x = \frac{x_2 - x_1}{t_2 - t_1}$$

Now for the motion along y direction we have

$$u_y' = \frac{y_2' - y_1'}{t_2' - t_1'} = \frac{y_2 - y_1}{\gamma\left[(t_2 - t_1) - \frac{v}{c^2}(x_2 - x_1)\right]}$$

$$u_y' = \frac{u_y}{\gamma\left[1 - \frac{vu_x}{c^2}\right]} = \frac{u_y\sqrt{1 - v^2/c^2}}{1 - \frac{vu_x}{c^2}}$$

Similarly for the motion along z direction

$$u_z' = \frac{u_z\sqrt{1 - v^2/c^2}}{1 - \frac{u_x v}{c^2}}$$

So, velocity addition

$$u_x' = \frac{u_x - v}{1 - \frac{vu_x}{c^2}}$$

$$u_y' = \frac{u_y\sqrt{1 - v^2/c^2}}{1 - \frac{u_x v}{c^2}}$$

$$u_z' = \frac{u_z\sqrt{1 - v^2/c^2}}{1 - \frac{u_x v}{c^2}}$$

Relativistic Dynamics

$$m = \frac{m_0}{\sqrt{1-v^2/c^2}}$$

or $m^2\left(1-\frac{v^2}{c^2}\right) = m_0^2$

or $m^2\left(\frac{c^2-v^2}{c^2}\right) = m_0^2$

or $m^2c^2 - m^2v^2 = m_0^2c^2$

or $c^2(2m\,dm) - m^2(2v\,dv) - v^2(2m\,dm) = 0$

or $c^2 dm = mv\,dv + v^2 dm$

or $c^2 dm - v^2 dm = mv\,dv$

Rest mass energy

$E = m_0c^2$

$$KE = \frac{1}{2}mv^2$$

$\Rightarrow$ $dK = \frac{1}{2}\left[v^2 dm + 2v\,dv\,m\right]$

From Newton's second law,

$$F = \frac{dP}{dt} = \frac{d}{dt}(mv) = m\frac{dv}{dt} + v\frac{dm}{dt}$$

or $F\,dx = m\frac{dv}{dt}dx + v\frac{dm}{dt}dx$

or $dW = m\frac{dx}{dt}dv + v\frac{dx}{dt}dm$

or $dW = mv\,dv + v^2 dm = dKE$ — (i)

We also have $c^2 dm = mv\,dv + v^2 dm$ — (ii)

$$\int_{m_0}^{m} c^2 dm = \int d(KE)$$

$\Rightarrow$ $KE = (mc^2 - m_0c^2)$

$\Rightarrow$ $KE = \frac{m_0}{\sqrt{1-v^2/c^2}}c^2 - m_0c^2$

$\Rightarrow$ $KE = (\gamma-1)m_0c^2$

$$E = KE + RME = (\gamma-1)m_0c^2 + m_0c^2$$

$$E = \gamma m_0c^2 = \frac{m_0}{\sqrt{1-v^2/c^2}}c^2 \Rightarrow \boxed{E = mc^2}$$

Momentum of the particle

$$p = mv = \frac{m_0 v}{\sqrt{1-v^2/c^2}} \Rightarrow p^2 = \frac{m_0^2 v^2}{1-v^2/c^2}$$

Total energy $\quad E = \dfrac{m_0 c^2}{\sqrt{1-v^2/c^2}}$

$$\Rightarrow E^2 = \frac{m_0^2 c^4}{1-v^2/c^2}$$

$$\Rightarrow E^2 = \frac{m_0^2 c^2 (c^2)}{1-v^2/c^2}$$

$$\Rightarrow E^2 = \frac{m_0^2 c^2 (c^2 - v^2 + v^2)}{1-v^2/c^2}$$

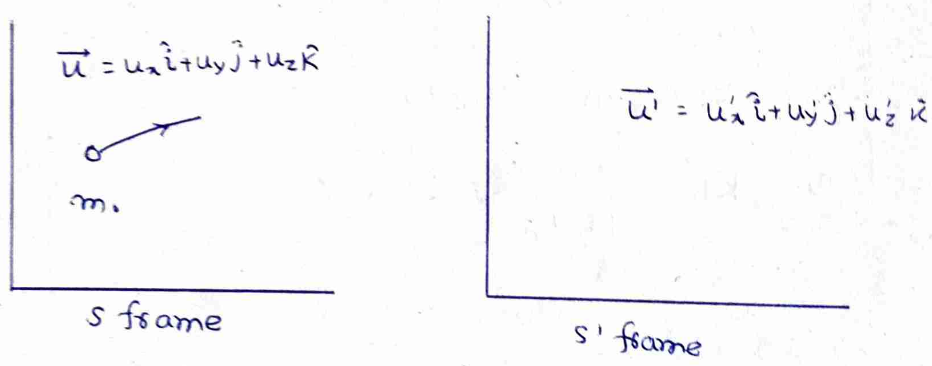
$$\Rightarrow E^2 = \frac{m_0^2 c^4}{1-v^2/c^2} - \frac{m_0^2 c^2 v^2}{1-v^2/c^2} + \frac{m_0^2 c^2 v^2}{1-v^2/c^2}$$

$$\Rightarrow E^2 = \left[\frac{m_0 v}{\sqrt{1-v^2/c^2}}\right]^2 c^2 + \frac{m_0^2 c^2 (c^2-v^2) c^2}{c^2-v^2}$$

$$\Rightarrow E^2 = p^2c^2 + m_0^2 c^4 \Rightarrow E = \sqrt{p^2c^2 + m_0^2c^4}$$

31.01.2024

Energy and Momentum Transformation:



$$p' = \frac{m_0 u'}{\sqrt{1-\frac{u'^2}{c^2}}}$$

$$u'^2 = u'^2_x + u'^2_y + u'^2_z$$

$$u'^2 = \frac{(u_x - v)^2 + \left(u_y\sqrt{1-v^2/c^2}\right)^2 + \left(u_z\sqrt{1-v^2/c^2}\right)^2}{\left(1-\frac{u_x v}{c^2}\right)^2}$$

$$u_x'^2 = \frac{(u_x-v)^2 + (u_y^2+u_z^2)\left(1-\frac{v^2}{c^2}\right)}{\left(1-\frac{u_x v}{c^2}\right)^2}$$

or $$\frac{u_x'^2}{c^2} = \frac{\left(\frac{u_x}{c}-\frac{v}{c}\right)^2 + \left(\frac{u_y^2}{c^2}+\frac{u_z^2}{c^2}\right)\left(1-\frac{v^2}{c^2}\right)}{\left(1-\frac{u_x v}{c^2}\right)^2}$$

or. $$1-\frac{u'^2}{c^2} = \frac{\left(1-\frac{u_x v}{c^2}\right)^2 - \left(\frac{u_x}{c}-\frac{v}{c}\right)^2 - \left(\frac{u_y^2}{v^2}+\frac{u_z^2}{c^2}\right)\left(1-\frac{v^2}{c^2}\right)}{\left(1-\frac{u_x v}{c^2}\right)^2}$$

or $$1-\frac{u'^2}{c^2} = \frac{\left(1-\frac{u_x v}{c^2}\right) - \frac{v^2}{c^2}\left(1-\frac{u_x^2}{c^2}\right) - \left(\frac{u_y^2}{c^2}+\frac{u_z^2}{c^2}\right)\left(1-v^2/c^2\right)}{\left(1-\frac{u_x v}{c^2}\right)^2}$$

or $$1-\frac{u'^2}{c^2} = \frac{\left(1-\frac{u_x^2}{c^2}\right)\left(1-\frac{v^2}{c^2}\right) - \left(1-\frac{v^2}{c^2}\right)\left(\frac{u_y^2}{c^2}+\frac{u_z^2}{c^2}\right)}{\left(1-\frac{u_x v}{c^2}\right)^2}$$

or $$1-\frac{u'^2}{c^2} = \frac{\left(1-\frac{v^2}{c^2}\right)\left(1-\frac{u_x^2}{c^2}-\frac{u_y^2}{c^2}-\frac{u_z^2}{c^2}\right)}{\left(1-\frac{u_x v}{c^2}\right)^2}$$

or $$1-\frac{u'^2}{c^2} = \frac{\left(1-\frac{v^2}{c^2}\right)\left(1-\frac{u^2}{c^2}\right)}{\left(1-\frac{u_x v}{c^2}\right)^2}$$

x component of momentum

$$P_x = \frac{m_0 u_x}{\sqrt{1-\frac{u^2}{c^2}}} \qquad P_y = \frac{m_0 u_y}{\sqrt{1-u^2/c^2}} \qquad P_z = \frac{m u_z}{\sqrt{1-u^2/c^2}}$$

$$P_x' = \frac{m_0 u_x'}{\sqrt{1-\frac{u'^2}{c^2}}} = \frac{m_0\left[\frac{u_x-v}{1-u_x v/c^2}\right]}{\frac{\sqrt{1-v^2/c^2}\,\sqrt{1-u^2/c^2}}{\left(1-\frac{u_x v}{c^2}\right)}} = \frac{m_0(u_x-v)}{\sqrt{1-\frac{v^2}{c^2}}\,\sqrt{1-u^2/c^2}}$$

$$P_x' = \frac{m u_x}{\sqrt{1-v^2/c^2}\,\sqrt{1-u^2/c^2}} - \frac{m_0 v c^2}{c^2\sqrt{1-v^2/c^2}\,\sqrt{1-\frac{u^2}{c^2}}}$$

$$P_x' = \frac{P_x}{\sqrt{1-v^2/c^2}} - \frac{Ev/c^2}{\sqrt{1-v^2/c^2}} = \frac{1}{\sqrt{1-v^2/c^2}}\left(P_x - \frac{Ev}{c^2}\right)$$

$$P_x' = \gamma\left(P_x - \frac{Ev}{c^2}\right)$$

$$P_y' = \frac{m_0 u_y'}{\sqrt{1-\frac{u'^2}{c^2}}} = \frac{m\,\dfrac{u_y\sqrt{1-v^2/c^2}}{1-\frac{u_x V}{c^2}}}{\dfrac{\sqrt{1-v^2/c^2}\,\sqrt{1-u^2/c^2}}{\left(1-\frac{u_x V}{c^2}\right)}} = \frac{m u_y}{\sqrt{1-u^2/c^2}} = P_y$$

Simmilarly $P_z' = P_z$

Energy $E = \dfrac{m_0 c^2}{\sqrt{1-u'^2/c^2}}$

$$E' = \frac{m_0 c^2}{\dfrac{\sqrt{1-v^2/c^2}\sqrt{1-u^2/c^2}}{1-\frac{u_x V}{c^2}}} = \frac{m_0 c^2\left(1-\frac{u_x V}{c^2}\right)}{\sqrt{1-v^2/c^2}\,\sqrt{1-u^2/c^2}}$$

$$E' = \frac{m_0 c^2}{\sqrt{1-v^2/c^2}\sqrt{1-u^2/c^2}} - \frac{P_x \frac{V}{c^2}\times c^2}{\sqrt{1-v^2/c^2}}$$

$$E' = \frac{E - P_x V}{\sqrt{1-v^2/c^2}} = \gamma(E - P_x V)$$

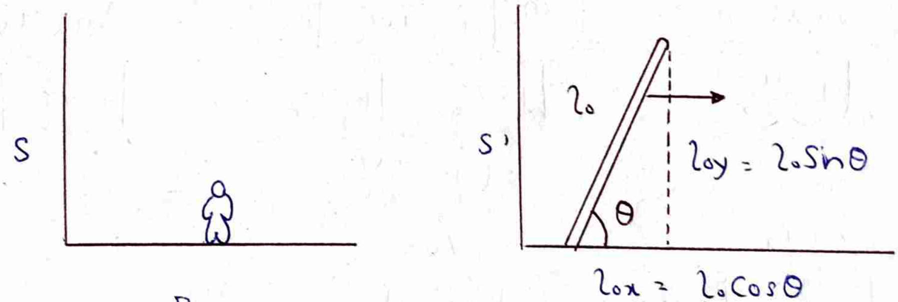
So finally we get $\quad t' = \gamma\left(t - \frac{V u_x}{c^2}\right)$

$P_x' = \gamma\left[P_x - \frac{EV}{c^2}\right] \qquad E' = \gamma\left[E - P_x V\right]$

$P_y' = P_y \,, \; P_z' = P_z \qquad x' = \gamma(x - Vt)$

γ comes from the velocity of frame

<u>Motion of an inclined Rod:</u>



$\tan\theta = \dfrac{l_{oy}}{l_{ox}}$

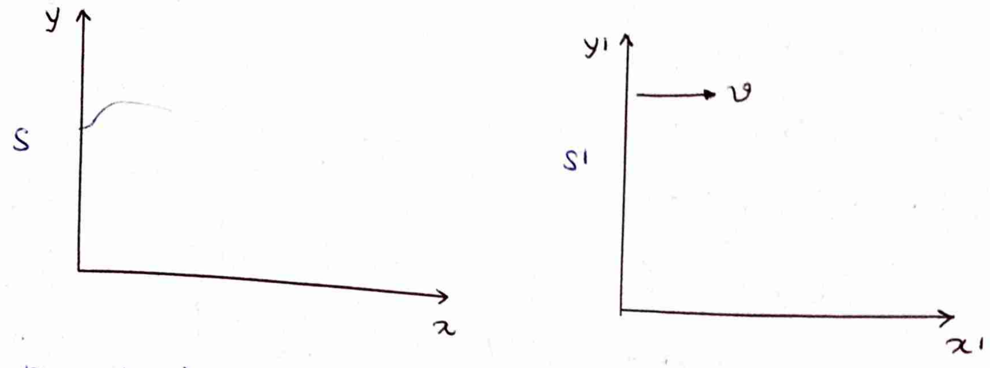
$\tan\theta' = \dfrac{l_y}{l_x} = \dfrac{l_{oy}}{l_{ox}\sqrt{1-v^2/c^2}}$

$l_y = l_{oy}$

$l_x = l_{ox}\sqrt{1-v^2/c^2}$

$\tan\theta' = \gamma \tan\theta$

Variation of E and B in different frame



Perendicular Component of E. $E_\perp = E_y\hat{j} + E_z\hat{k}$

Parallel Component of F $E_{\parallel} = E_x$

Simmilarly for magnetic field. $B_\perp = B_y\hat{j} + B_z\hat{k}$ $B_{\parallel} = B_x$

Parallel Component remains Same $E_{\parallel} = E'_{\parallel}$

Now for Perpendicular component we have

$$E'_\perp = \frac{\vec{E_\perp} + \vec{v}\times\vec{B}}{\sqrt{1-v^2/c^2}} \qquad B'_\perp = \frac{\vec{B_\perp} - \frac{\vec{v}\times\vec{E}}{c^2}}{\sqrt{1-v^2/c^2}}$$

$$E_y'\hat{j} + E_z'\hat{k} = \frac{(E_y\hat{j}+E_z\hat{k}) + [v\hat{i}\times(B_x\hat{i}+B_y\hat{j}+B_z\hat{k})]}{\sqrt{1-v^2/c^2}}$$

$$\Rightarrow E_y'\hat{j} + E_z'\hat{k} = \gamma\left[(E_y\hat{j}+E_z\hat{k}) + \hat{k}(vB_y) - \hat{j}(vB_z)\right]$$

By Compairing the Components we have

$$E_y' = \gamma[E_y - vB_z] \qquad E_z' = \gamma[E_z + vB_y] \qquad E_x' = E_x$$

— This is the transformation equation for electric field.

Since $B'_\perp = \gamma\left[B_\perp - \frac{\vec{v}\times\vec{E}}{c^2}\right]$

$$B_y'\hat{j} + B_z'\hat{k} = \gamma\left[(B_y\hat{j}+B_z\hat{k}) - \frac{1}{c^2}\{v\hat{i}\times(E_x\hat{i}+E_y\hat{j}+E_z\hat{k})\}\right]$$

$$B_y'\hat{j} + B_z'\hat{k} = \gamma\left[(B_y\hat{j}+B_z\hat{k}) - \left(\frac{E_y v}{c^2}\hat{k} - \frac{E_z v}{c^2}\hat{j}\right)\right]$$

So, $B_x' = B_x$ $\quad B_y' = \gamma\left(B_y + \frac{E_z v}{c^2}\right)$ $\quad B_z' = \gamma\left(B_z - \frac{E_y v}{c^2}\right)$

Transformation equation of Magnetic field

$$\vec{E} = E_x\hat{i} + E_y\hat{j} + E_z\hat{k}$$
$$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$

$$\vec{E}\cdot\vec{B} = E_xB_x + E_yB_y + E_zB_z$$
$$\vec{E}'\cdot\vec{B}' = E_x'B_x' + E_y'B_y' + E_z'B_z'$$
$$= E_xB_x + \gamma^2\left[(E_y - VB_z)(B_y + \frac{V}{c^2}E_z)\right]$$
$$+ \gamma^2\left[(E_z + VB_y)(B_z - \frac{V}{c^2}E_y)\right]$$

So, $$\vec{E}'\cdot\vec{B}' = E_xB_x + \gamma^2\left[E_yB_y + \cancel{\frac{V}{c^2}E_yE_z} - \cancel{VB_zB_y} - \frac{V^2}{c^2}B_zE_z\right]$$
$$+ \gamma^2\left[E_zB_z - \cancel{\frac{V}{c^2}E_zE_y} + \cancel{VB_yB_z} - \frac{V^2}{c^2}B_yE_y\right]$$

$$\Rightarrow \vec{E}'\cdot\vec{B}' = E_xB_x + E_yB_y(\gamma^2 - \gamma^2\beta^2) + E_zB_z(\gamma^2 - \gamma^2\beta^2)$$
$$\Rightarrow \vec{E}'\cdot\vec{B}' = E_xB_x + E_yB_y + E_zB_z = \vec{E}\cdot\vec{B}$$

$\vec{E}'\cdot\vec{B}' = \vec{E}\cdot\vec{B} \Rightarrow$ Lorentz invarient Quantity

<u>Quantity $E'^2 - B'^2c^2$</u>

$$E_x'^2 + E_y'^2 + E_z'^2 - c^2(B_x'^2 + B_y'^2 + B_z'^2)$$
$$= E_x^2 + \gamma^2\left[(E_y - \frac{VB_z}{c^2})^2\right] + \gamma^2\left[(E_z + \frac{VB_y}{c^2})^2\right] - c^2B_x^2$$
$$- c^2\gamma^2(B_y + \frac{V}{c^2}E_z)^2 - c^2\gamma^2(B_z - \frac{V}{c^2}E_y)^2$$

$$= (E_x^2 - c^2B_x^2) + \gamma^2\left[E_y^2 + \frac{V^2B_z^2}{c^4} - \frac{\cancel{2E_yB_zV}}{c^2}\right] + \gamma^2\left[E_z^2 + \frac{V^2B_y^2}{c^4} + \frac{\cancel{2E_zVB_y}}{c^2}\right]$$
$$- c^2\gamma^2\left(B_y^2 + \frac{V^2}{c^4}B_z^2 + \frac{\cancel{2E_zB_yV}}{c^2}\right) - c^2\gamma^2\left(B_z^2 + \frac{V^2E_y^2}{c^4} - \frac{\cancel{2B_zVE_y}}{c^2}\right)$$

$$= (E_x^2 - c^2B_x^2) + E_y^2(\gamma^2 - \gamma^2\beta^2) + E_z^2(\gamma^2 - \gamma^2\beta^2) - c^2(B_x^2 + B_y^2 + B_z^2)$$
$$= (E_x^2 - c^2B_x^2) + (E_y^2 - c^2B_y^2) + (E_z^2 - c^2B_z^2)$$
$$= E_x^2 + E_y^2 + E_z^2 - c^2(B_x^2 + B_y^2 + B_z^2)$$

So, finally. $E'^2 - B'^2c^2 = E^2 - B^2c^2$

So it is a Lorentz invarient Quantity

For massless particle like photon, neutrino.

$m_0 = 0$ velocity $= c$

Momentum $p = \frac{E}{c}$ Energy $E = pc$

Problems based on Relativistic Collision

① A particle of mass M initially at rest breaks up into a particle of mass m and another particle of zero mass (rest). Calculate speed of the particle whose rest mass is m

=> Initially $m_0 c^2$ Finally

$M,\ P = 0$ $P_1,\ P_2$

From momentum conservation $\vec{P_1} + \vec{P_2} = 0$

$\Rightarrow \vec{P_1} = -\vec{P_2}$

$P_2 = \frac{E_2}{c}$ (Massless) $P_1 = \frac{mv}{\sqrt{1-v^2/c^2}}$

$\frac{mv}{\sqrt{1-v^2/c^2}} = \frac{E_2}{c}$ $E_1 = \frac{mc^2}{\sqrt{1-v^2/c^2}}$

$E_2 = \frac{mvc}{\sqrt{1-v^2/c^2}}$

From Energy Conservation

$$Mc^2 = \frac{mc^2}{\sqrt{1-v^2/c^2}} + \frac{mvc}{\sqrt{1-v^2/c^2}}$$

$$\Rightarrow Mc = \frac{mc}{\sqrt{1-v^2/c^2}} + \frac{mv}{\sqrt{1-v^2/c^2}}$$

$$\Rightarrow \left(\frac{M}{m}\right)c = \frac{c+v}{\sqrt{1-v^2/c^2}}$$

$$\Rightarrow \frac{v}{c} = \frac{M^2-m^2}{M^2+m^2}$$

$$\Rightarrow v = \left(\frac{M^2-m^2}{M^2+m^2}\right)c$$

② Kinetic energy of a relativistic particle of rest mass m_0 is equal to four times the rest energy of the particle. Calculate the momentum of the Particle

$\Rightarrow$

$$(\gamma-1)m_0c^2 = 4m_0c^2$$

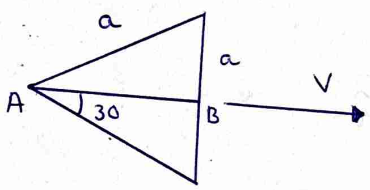
$$\Rightarrow \quad \gamma = 5 = \frac{1}{\sqrt{1-v^2/c^2}}$$

$$\Rightarrow \quad \frac{1}{25} = 1-\frac{v^2}{c^2} \Rightarrow \frac{v^2}{c^2} = \frac{24}{25} \Rightarrow v = \sqrt{\frac{24}{25}}\,c$$

Momentum $p = \dfrac{m_0 v}{\sqrt{1-v^2/c^2}} = 5m\dfrac{\sqrt{24}}{5}c = \sqrt{24}\,mc$

③ An equilateral triangle of side a is moved with velocity v parallel to bisector of one of its angle. Calculate apparent perimeter of the triangle

$\Rightarrow$



$$AB = \frac{\sqrt{3}}{2}a$$

$$a_x = a\cos 30 = \frac{\sqrt{3}}{2}a$$

$$a_y = a\sin 30 = \frac{a}{2}$$

$$a'_x = \frac{\sqrt{3}a}{2}\sqrt{1-v^2/c^2}$$

$$a' = \sqrt{a_x'^2 + a_y'^2}$$

$$a'_y = a_y = \frac{a}{2}$$

$$a' = \left[\frac{3a^2}{4}\left(1-\frac{v^2}{c^2}\right) + \frac{a^2}{4}\right]^{1/2}$$

Perimeter $= 3a'$

④ Two events A and B occur at places separated by 2×10^6 Km at time interval 10 sec then

$\Rightarrow$

$$\Delta x = 2\times10^6\ \text{Km}$$
$$= 2\times10^9\ \text{Km}$$

$$\Delta t = 10\ \text{sec}$$

For Same place

$$\Delta x' = 0$$

$$\Delta x' = \gamma[\Delta x - v\Delta t]$$

$$v = \frac{\Delta x}{\Delta t} = \frac{2\times10^9}{10} = \frac{2}{3}c$$

03.02.2024

① Wave Functions & operators:

> Quantum Mechanics:

Mathematical notation of wave is called wave function. The wave must be progressive wave. means $\psi = \psi(x,t)$. wave function is required due to matter wave i.e de-Broglie wavelength

$$\psi(x,t) = \psi_0 e^{i(Kx \pm \omega t)}$$

Here, ψ_0 = Amplitude

K = wave number $= \frac{2\pi}{\lambda}$

$$\psi(x,t) = \psi_0 \sin(\pm Kx \pm \omega t)$$

$\omega = \frac{2\pi}{T} = 2\pi\nu$

$$\psi(x,t) = \psi_0 \cos(\pm Kx \pm \omega t)$$

$V = \frac{\omega}{K}$ = Phase velocity

Phase is $\left.\begin{array}{l} +Kx+\omega t \\ -Kx-\omega t \end{array}\right\}$ wave is travelling in negative x direction.

$\left.\begin{array}{l} Kx-\omega t \\ -Kx+\omega t \end{array}\right\}$ wave is travelling in positive x direction

In classical mechanics we need to know the position of a particle to calculate it's variable. like

$$P = m\frac{dr}{dt}, \quad a = \frac{d^2r}{dt^2} \dots$$

Simmilarly, in QM we need the wave function to calculate any physical variable associated with it.

$\psi(x,t)$ represents the state of a system and it contains all the information related to the object

Physical Significance of $\psi(x,t)$:

There is no physical significance of $\psi(x,t)$

Probability density, $\psi^*(x,t) * \psi(x,t) = \rho$

$\psi\psi^*$ is a real quantity

ψ^* is the complete conjugate of ψ

Probability $P = \int \rho \, d^3x = \int \psi\psi^* d^3x$

$\int_{x_1}^{x_2} \psi^* \psi \, dx \Rightarrow$ Probability of finding in the range x_1 to x_2

> **Mean Value/ Expectation value:**

$$\langle x \rangle = \int P(x)\, x\, dx \quad \Leftarrow \text{Probability density is } P(x)$$

$$\langle x^2 \rangle = \int P(x)\, x^2\, dx$$

> **Eigen Value:**

$$E = h\nu = \frac{h\omega}{2\pi}$$

$$E = \hbar\omega$$

$$\hat{A}\psi = a\psi$$

Eigen value equation

ψ should remain same

a is Eigen value.

ψ is Eigen function

Position $\hat{x}$

Momentum $\hat{p}$

Energy | Hamiltonian $\hat{H}$

$$p = \frac{h}{\lambda} = \frac{h}{\lambda} \times \frac{2\pi}{2\pi}$$

$$p = \frac{h}{2\pi} \times \frac{2\pi}{\lambda} \Rightarrow p = \hbar K$$

$$\psi = e^{iKx}$$

$$\frac{d\psi}{dx} = iKe^{iKx} = (iK)\psi$$

$$a = iK$$

$\hat{A}$ is operator.

② **Momentum Operator:**

$$\psi = \psi_0 e^{i(Kx - \omega t)}$$

or $$\frac{\partial \psi}{\partial x} = iK\psi_0 e^{i(Kx-\omega t)}$$

or $$\frac{\partial \psi}{\partial x} = i\frac{p}{\hbar}\psi$$

or $$\frac{\hbar}{i}\frac{d\psi}{dx} = p\psi$$

or $$-\hbar i \frac{\partial \psi}{\partial x} = p\psi$$

$$\boxed{\hat{p} = -i\hbar \frac{\partial}{\partial x}}$$

$$\boxed{\hat{p} = -i\hbar\vec{\nabla}}$$

③ **Hamiltonian operator:**

$$\psi = \psi_0 e^{i(Kx-\omega t)}$$

or $$\frac{\partial \psi}{\partial t} = (-i\omega)\psi_0 e^{i(Kx-\omega t)}$$

or $$\frac{\partial \psi}{\partial t} = \frac{1}{i}\frac{E}{\hbar}\psi$$

or $$i\hbar\frac{\partial \psi}{\partial t} = E\psi$$

So, Energy operator

$$\boxed{\hat{E}\psi = i\hbar\frac{\partial \psi}{\partial t}}$$

Time dependent Hamiltonian

Expectation value of p

$$\langle \hat{p} \rangle = \frac{\int \psi^* \hat{p} \psi \, dx}{\int \psi^* \psi \, dx} = \frac{\int \psi^* \left(-i\hbar \frac{\partial}{\partial x}\right) \psi \, dx}{\int \psi^* \psi \, dx}$$

$-i\hbar \frac{\partial}{\partial x} (\psi^* \psi) \Rightarrow$ differentiation of Probability not wave function

$f(-x) = -f(x) \Rightarrow$ odd

$f(-x) = f(x) \Rightarrow$ even

e^{-x} is neither odd nor even

> $$\int_{-\infty}^{+\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}$$

$\alpha x^2 = t$

$dx = \frac{dt}{2\sqrt{\alpha}\sqrt{t}}$

$$= 2\int_0^{\infty} e^{-t} \frac{dt}{2\sqrt{\alpha}} t^{-1/2} = \frac{2}{2\sqrt{\alpha}} \int_0^{\infty} e^{-t} t^{-1/2} dt = \frac{1}{\sqrt{\alpha}} \Gamma\tfrac{1}{2} = \sqrt{\frac{\pi}{\alpha}}$$

> $$\int_{-\infty}^{+\infty} e^{-x^2} x \, dx = 0$$

$e^{-x^2} \Rightarrow$ Even

$x \Rightarrow$ odd

$x e^{-x^2}$ is odd

$\Downarrow$

For Symmetric limit value is 0

> $$\int_{-\infty}^{\infty} e^{-x^2} x^2 dx = 2\int_0^{\infty} e^{-x^2} x \cdot x \, dx$$

$$= 2\int_0^{\infty} e^{-t} t^{1/2} \frac{dt}{2} = \int_0^{\infty} e^{-t} \cdot t^{3/2 - 1} = \Gamma\left(\tfrac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$$

> $$\int_{-\infty}^{+\infty} e^{-\alpha x^2 - \beta x} dx = \int_{-\infty}^{+\infty} e^{-\alpha\left[x^2 + \frac{\beta}{\alpha}x + \frac{\beta^2}{4\alpha^2} - \frac{\beta^2}{4\alpha^2}\right]} dx$$

$$= e^{\frac{\beta^2}{4\alpha}} \int_{-\infty}^{+\infty} e^{-\alpha\left(x + \frac{\beta}{2\alpha}\right)^2} dx = e^{\beta^2/2\alpha} \int_{-\infty}^{\infty} e^{-\alpha y^2} dy = \sqrt{\frac{\pi}{\alpha}}\, e^{\beta^2/4\alpha}$$

$$\int_{-\infty}^{+\infty} e^{-\alpha x^2 - \beta x} dx = e^{\beta^2/4\alpha} \sqrt{\frac{\pi}{\alpha}}$$

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$$\langle \hat{A} \rangle = \frac{\int \psi^* \hat{A} \psi \, dx}{\int \psi^* \psi \, dx}$$

$$\langle \hat{A}^2 \rangle = \frac{\int \psi^* \hat{A}^2 \, dx}{\int \psi^* \psi \, dx}$$

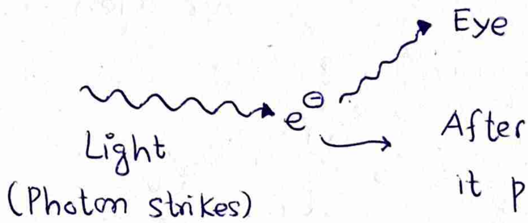
Hesienberg Relation

$$\Delta x \, \Delta P_x \geq \frac{\hbar}{2}$$

$$\Delta E \, \Delta t \geq \frac{\hbar}{2}$$

uncertainity in measurment of a function A

$$\Delta A = \sqrt{\langle A^2 \rangle - \langle A \rangle^2} \Rightarrow \text{Standard deviation}$$



After strike, the e^- changes it position so cannot be determined exactly. So it also gains some momentum

If $\Delta x = 0$. $\quad \Delta P_x \geq \frac{\hbar}{2\Delta x} \quad \Delta P \to \infty$

If $\Delta P = 0 \quad \Delta x \geq \frac{\hbar}{2 P_x} \quad \Delta x \to \infty$

$\hat{A}\psi = \lambda \psi'$

WF changes so uncertainity is present

$\Delta A \neq 0$

does't follow Eigen value equation

$\hat{A}\psi = \lambda \psi$

WF does't changes we can measure A precisely. $\Delta A = 0$

It follow eigen value equation

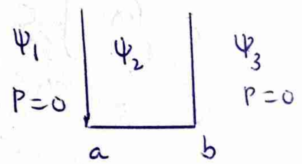
> Acceptable wave function:

(1) In this interval, the wave function will be continous and should be continous. $[a,b]$ finite

(2) $\psi_1 |_{x=a} = \psi_2 |_{x=a}$

$\psi_2 |_{x=b} = \psi_3 |_{x=b}$

wave function will be continous in boundary



(3) $\left.\frac{d\psi_1}{dx}\right|_{x=a} = \left.\frac{d\psi_2}{dx}\right|_{x=a}$

$\left.\frac{d\psi_1}{dx}\right|_{x=b} = \left.\frac{d\psi_2}{dx}\right|_{x=b}$

First order derivative should be continuous in boundary

> An electron is in space (x: $-\infty$ to ∞). Is the wave function $\psi(x) = Ae^{-\alpha x}$ acceptable?

=) $\psi(\infty) = Ae^{-\infty} = 0$ (finite)

$\psi(-\infty) = Ae^{\infty} = \infty$ (Not finite)

Since in interval $x = -\infty$ ψ is not finite so $\psi(x) = Ae^{-\alpha x}$ is not acceptable.

> In the range x: 0 to ∞ $\psi(x) = Ae^{-\frac{\alpha^2 x^2}{2}}$ acceptable

> Dimension of wave function

Two dimension

$\int \psi^* \psi \, dA = 1$

$[\psi^2][L^2] = 1$

$[\psi] = L^{-1}$

One dimensional

$\int \psi^* \psi \, dx = 1$

$[\psi^2][L] = [L^0]$

$[\psi] = [L^{-1/2}]$

n- dimension. $[\psi] = [L^{-n/2}]$

> Probability Current Density

One dimension $\quad J = \frac{\hbar i}{2m}\left[\psi \frac{d\psi^*}{dx} - \psi^* \frac{d\psi}{dx}\right]$

three dimension $\quad J = \frac{\hbar i}{2m}\left[\psi \vec{\nabla}\psi^* - \psi^* \vec{\nabla}\psi\right]$

It follows $\quad \vec{\nabla}.\vec{J} + \frac{\partial \rho}{\partial t} = 0$

(Equation of continuity)

For any real function $J = 0$

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① $\left(\frac{d}{dx}-x\right)\left(\frac{d}{dx}+x\right)$

$$= \left(\frac{d}{dx}-x\right)\left(\frac{d\psi}{dx}+x\psi\right)$$

$$= \frac{d}{dx}\left(\frac{d\psi}{dx}+x\psi\right) - x\frac{d\psi}{dx} - x^2\psi$$

$$= \frac{d^2\psi}{dx^2} + x\cancel{\frac{d\psi}{dx}} + \psi - x\cancel{\frac{d\psi}{dx}} - x^2\psi$$

$$= \frac{d^2}{dx^2} - x^2 + 1$$

② $\psi(x) = \sqrt{\frac{15}{16}}(1-x^2)$ where $-1 \leq x \leq 1$ and 0 every where

The uncertainity in measurment ΔP is

$\Rightarrow$

$$\langle P^2 \rangle = \int \psi^* \hat{p}^2 \psi dx$$

$$= -\hbar^2 \int \frac{\sqrt{15}}{4}(1-x^2)\frac{d^2}{dx^2}\left[\frac{\sqrt{15}}{4}(1-x^2)\right]dx$$

$$= 2\hbar^2 \frac{15}{16}\int_{-1}^{+1}(1-x^2)dx$$

$$= \frac{15\hbar^2}{8}\left[x-\frac{x^3}{3}\right]_{-1}^{+1}$$

$$= \frac{15\hbar^2}{8}\left[1-\frac{1}{3}+1-\frac{1}{3}\right] = \frac{5}{2}\hbar^2$$

$$\langle P \rangle = \int \psi^* \hat{p}\psi dx = \int \sqrt{\frac{15}{16}}(1-x^2)\left(-i\hbar\frac{\partial}{\partial x}\right)(1-x^2)dx = 0$$

Since, ψ is a real function

$\langle P \rangle = 0$

$$\Delta P = \sqrt{\langle P^2 \rangle - \langle P \rangle^2} = \sqrt{\frac{5}{2}}\hbar$$

Schrödinger Equation

Energy operator is Hamiltonian Operator

$\hat{H}\psi = E\psi$

E = Eigen value

$\hat{H}$ = Hamiltonian operator

Hamiltonian,

$\hat{p} = -\hbar i \frac{\partial}{\partial x}$

$$H = KE + PE$$

$\hat{p}^2 = -\hbar^2 \frac{\partial^2}{\partial x^2}$

or $$H = \frac{\hat{p}^2}{2m} + V(x)$$

or $$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \hat{V}(x)$$

Now, $\hat{H}\psi = E\psi \Rightarrow \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]\psi = E\psi$

$$\boxed{\Rightarrow -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + \hat{V}(x)\psi = E\psi}$$

– Time independent Schrödinger Equation

In 3-dimension.

$$\hat{H}\psi = E\psi$$

$$\Rightarrow (KE + PE)\psi = E\psi$$

$$\Rightarrow \left[\frac{\hat{p}^2}{2m} + V(r)\right]\psi = E\psi$$

$$\Rightarrow \left[-\frac{\hbar^2}{2m}\nabla^2 + \hat{V}(r)\right]\psi = E\psi$$

$$\boxed{-\frac{\hbar^2}{2m}\nabla^2\psi + \hat{V}(r)\psi = E\psi}$$

3-D Schrödinger equation

> **Free Particle:** Have only $E = KE$ $V(x) = 0$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{dx^2} = E\psi \Rightarrow \frac{d^2\psi}{dx} + \frac{2mE}{\hbar^2}\psi = 0$$

> **SHO:** $PE = \frac{1}{2}Kx^2 = \frac{1}{2}m\omega^2x^2$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{dx^2} + \frac{1}{2}m\omega^2x^2\psi = E\psi$$

> Infinite Potential:

Potential $V(x) \to \infty$ $KE = 0$

$\lambda = \frac{h}{\sqrt{2m(KE)}} \to \infty$ Straight line no crest or tough

So wave function is not exist.

> $\frac{d^2y}{dx^2} + a^2 y = 0$ If Boundary is define or for finite Limit

$y = C_1 \sin ax + C_2 \cos ax$

If Boundary is not finite, then the Solution will be

$y = C_1 e^{iax} + C_2 e^{-iax}$

> $\frac{d^2\psi}{dx^2} + \cancel{K} \frac{2mE}{\hbar} \psi = 0$ $\frac{1}{K} = \frac{\hbar}{\sqrt{2mE}}$

$\frac{d^2\psi}{dx^2} + K^2\psi = 0$ $P = \hbar K$

$\psi = C_1 e^{-iKx}, C_2 e^{+iKx}, C_1 e^{iKx} + C_2 e^{-iKx}$

$\psi = A \sin Kx + B \cos Kx$

These are the Solution if the particle is free or $V(x) = 0$ Vice-Versa

Q If the wave function $\psi = e^{\beta x} e^{iKx}$ then potential of the (GATE 2024)

:, $\psi = a e^{iKx}$ Since β is Constant

$e^{\beta} = a$ is Constant

So,

$V(x) = 0$ $\frac{d^2\psi}{dx^2} + K^2\psi = 0$

$K = \frac{\sqrt{2mE}}{\hbar}$

The wave function of a particle of mass m is in a unidemsionar $V(x) = \frac{Kx^2}{2}$. has ground state $\psi(x) = Ae^{-\alpha x^2}$. The Value of α is

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}(Ae^{-\alpha x^2}) + V(x)Ae^{-\alpha x^2} = \frac{\hbar\omega}{2}Ae^{-\alpha x^2}$$

$$\Rightarrow -\frac{\hbar^2}{2m}\left[4\alpha^2x^2Ae^{-\alpha x^2} - 2\alpha Ae^{-\alpha x^2}\right] + V(x)Ae^{-\alpha x^2} = \frac{\hbar\omega}{2}Ae^{-\alpha x^2}$$

$$\Rightarrow -\frac{\hbar^2}{2m}\left[4\alpha^2x^2 - 2\alpha\right]Ae^{-\alpha x^2} + \frac{1}{2}Kx^2Ae^{-\alpha x^2} = \frac{\hbar\omega}{2}Ae^{-\alpha x^2}$$

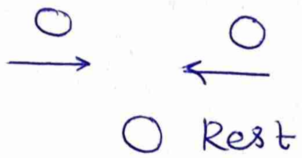
$$\Rightarrow -\frac{\hbar^2}{2m}\left[4\alpha^2x^2 - 2\alpha\right] + \frac{1}{2}Kx^2 = \frac{\hbar\omega}{2}$$

$$\Rightarrow -\frac{\hbar^2}{2m}4\alpha^2x^2 + \frac{\hbar^2}{2m}\cdot 2\alpha + \frac{1}{2}Kx^2 = \frac{\hbar\omega}{2}$$

$$\Rightarrow \underbrace{-\frac{\hbar^2}{2m}4\alpha^2x^2} + \frac{\hbar^2}{2m}2\alpha + \underbrace{\frac{1}{2}Kx^2} = \frac{\hbar\omega}{2}$$

$$\frac{1}{2}K = \frac{\hbar^2}{2m}4\alpha^2 \Rightarrow \alpha = \frac{m\omega}{2\hbar}$$

Q A proton and an antiproton having rest mass 1 GeV/c^2 travelling with the same speed Collide with another from opposite direction. If the rest mass of the Particle created is 4 GeV/c^2 the Speed of one of the original particles in the frame of other is



$\Rightarrow$ From Energy conservation

$$\frac{2mc^2}{\sqrt{1-v^2/c^2}} = Mc^2 \Rightarrow \frac{2m}{\sqrt{1-v^2/c^2}} = M \Rightarrow \frac{2\times 1}{\sqrt{1-v^2/c^2}} = 4$$

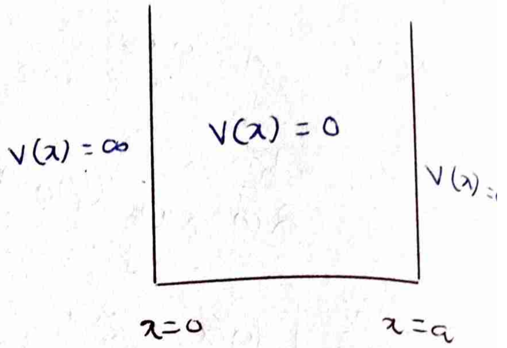
$$1 - v^2/c^2 = \frac{1}{4}$$

$$\Rightarrow v = \frac{\sqrt{3}}{2}c$$

Particle in a Box

Infinite Potential Well

Potential Profile will be



Rigid walls.

$$V(x) = 0 \quad \text{when} \quad 0 \leq x \leq a$$
$$= \infty \quad \text{elsewhere}$$

Schrodinger Equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \cancel{V}^{\,0}\psi = E\psi$$

$$\Rightarrow \frac{d^2\psi}{dx^2} + \frac{2mE}{\hbar^2}\psi = 0$$

So here

$$K^2 = \frac{2mE}{\hbar^2}$$

$$E = \frac{\hbar^2}{2m}K^2$$

$$\Rightarrow \frac{d^2\psi}{dx^2} + K^2\psi = 0$$

$$\Rightarrow \psi(x) = A\cos Kx + B\sin Kx$$

From Boundary value Condition

$$\psi(x=0) = \psi(x=a) = 0$$

$$\psi(x=0) = A\cos 0 + B \times 0 = 0 \Rightarrow A = 0$$

So $\psi(x) = B\sin Kx$

Since $B \neq 0$, $\psi(a) = B\sin Ka = 0$

$$\sin Ka = \sin n\pi$$
$$K = \frac{n\pi}{a}$$

So the wave function

$$\psi(x) = B\sin\frac{n\pi x}{a}$$

The Constant B can be determined by Normalization.

So. $\int_0^a \psi\psi^* dx = 1$

or $B^2 \int_0^a \sin^2 \frac{n\pi x}{a} dx = 1$

or $B^2 \times \frac{a}{2} = 1 \Rightarrow B = \sqrt{\frac{2}{a}}$

So the required wave function is

$\psi(x) = \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a}$

Also we have, $K^2 = \frac{2mE}{\hbar^2}$

$E = \frac{\hbar^2}{2m} K^2 = \frac{\hbar^2}{2m} \frac{n^2\pi^2}{a^2}$

Energy. $E = \frac{\hbar^2}{2m} \cdot \frac{n^2\pi^2}{a^2}$

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wave function. $\psi(x) = \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a}$

Energy $E_n = \frac{\hbar^2}{2m} \frac{n^2\pi^2}{a^2}$

Different Value of n.

(i) $n=0$: At $n=0$, $\psi_0(x) = 0$ So wave function is not exist. but it is not possible. So it should not be considered.

(ii) $n=1$: $E_1 = \frac{\hbar^2}{2m} \frac{\pi^2}{a^2} \Rightarrow$ Ground state Energy

$\psi_1(x) = \sqrt{\frac{2}{a}} \sin \frac{\pi x}{a} \Rightarrow$ Ground state wave function

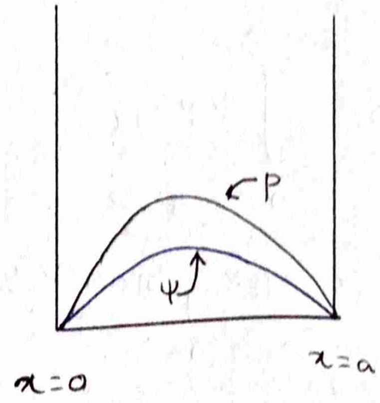
(3) $n=2$: $E_2 = \frac{\hbar^2}{2m} \frac{4\pi^2}{a^2} \Rightarrow$ First excited state energy

$\psi_2(x) = \sqrt{\frac{2}{a}} \sin \frac{2\pi x}{a} \Rightarrow$ First excited state wave function

- $\psi_1(x) = \sqrt{\frac{2}{a}} \sin \frac{\pi x}{a}$

At $x = 0$ $\psi_1(0) = 0$ Nodes : 0

$x = a$, $\psi_1(a) = 0$



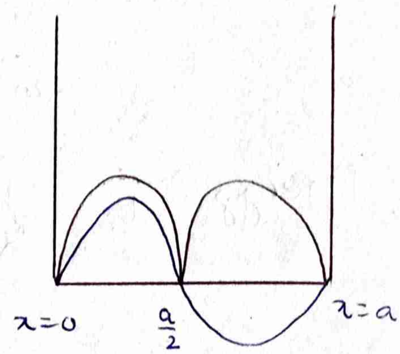
<u>Nodes:</u> Point where wave function vanishes and the Points should not be the extreme point.

- $\psi_2(x) = \sqrt{\frac{2}{a}} \sin \frac{2\pi x}{a}$

At $x=0$ $\psi_2(0) = 0$ Nodes: 1

$x = a$ $\psi_2(a) = 0$

$x = a/2$ $\psi_2(a/2) = 0$



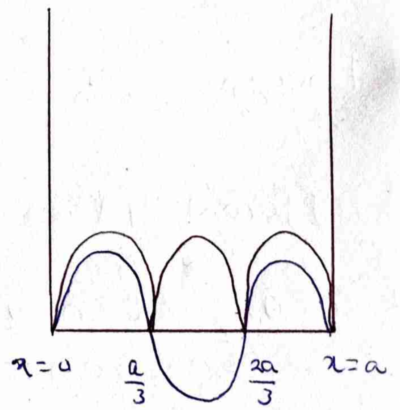
- $\psi_3(x) = \sqrt{\frac{2}{a}} \sin \frac{3\pi x}{a}$

At $x=0$ $\psi_3(a) = 0$

$x = a$ $\psi_3(a) = 0$ Nodes. 2

$x = \frac{a}{3}$ $\psi_3(\frac{a}{3}) = 0$

$x = \frac{2a}{3}$ $\psi_3(\frac{2a}{3}) = 0$



> No of nodes $\Rightarrow n = 1$

wave number,

$$K = \frac{n\pi}{a}$$

$$\frac{2\pi}{\lambda} = \frac{n\pi}{a}$$

$$\lambda = \frac{2a}{n}$$

$$\boxed{\lambda = \frac{2a}{n}}$$

Expectation value of x:

$$\langle x \rangle = \int_0^a \psi^* x \psi \, dx = \int_0^a \frac{2}{a} x \sin^2 \frac{n\pi x}{a} dx = \frac{2}{a}\int_0^a x \sin^2 \frac{n\pi x}{a} dx$$

$$= \frac{1}{a}\int_0^a \left[x\left(1-\cos\frac{2n\pi x}{a}\right) dx\right.$$

$$= \frac{1}{a}\left\{\int_0^a x\,dx - \int_0^a x \cos\frac{2n\pi x}{a} dx\right\}$$

$$= \frac{1}{a}\left\{\frac{a^2}{2} - \left[x \frac{\sin 2n\pi x}{2n\pi a}\right|_0^a - \int_0^a\left[\frac{d}{dx}(x)\int_0^a \cos\frac{2n\pi x}{a}dx\right]dx\right.$$

$$= \frac{1}{a}\times\frac{a^2}{2} = \frac{a}{2} \qquad \langle x \rangle = a/2$$

Expectation Value of x^2:

$$\langle x^2 \rangle = \int_0^a \psi^* x^2 \psi \, dx = \frac{2}{a}\int_0^a \sin^2 \frac{n\pi x}{a} dx$$

$$= \frac{1}{a}\int_0^a \left[x^2\left(1-\cos\frac{2n\pi x}{a}\right)dx\right. \qquad \langle x^2 \rangle = a^2/3\left[1-\frac{3}{2n^2\pi^2}\right]$$

$$= \frac{1}{a}\left\{\int_0^a x^2 dx - \int_0^a x^2 \cos\frac{2n\pi x}{a} dx\right\} = a^2/3\left[1-\frac{3}{2n^2\pi^2}\right]$$

Expectation value of P:

$$\langle P \rangle = \int \psi^* \hat{p}\psi \, dx = \frac{2}{a}\int_0^a \sin\frac{n\pi x}{a}\left(-i\hbar\frac{\partial}{\partial x}\right)\sin\frac{n\pi x}{a}dx = 0$$

As real function. so $\langle P \rangle = 0$

Expectation value of p^2:

$$\langle p^2 \rangle = \frac{2}{a}\int_0^a \sin\frac{n\pi x}{a}\left(-\hbar^2\frac{\partial^2}{\partial x^2}\sin\frac{n\pi x}{a}\right)dx$$

$$= \frac{2}{a}\frac{n^2\pi^2}{a^2}\hbar^2\int_0^a \sin^2\frac{n\pi x}{a}dx = \frac{n^2\pi^2}{a^2}\hbar^2$$

Ground State ($n=1$)

$\langle x \rangle = \frac{a}{2}$ $\quad\quad$ $\langle x^2 \rangle = a^2 \left[\frac{1}{3} - \frac{1}{2\pi^2} \right]$

$\langle p \rangle = 0$ $\quad\quad$ $\langle p^2 \rangle = \frac{\pi^2 \hbar^2}{a^2}$

uncertainity in Position. $\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = 0.1806\,a$

$$\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \frac{\pi \hbar}{a}$$

$$\Delta x \Delta p = (0.1806)a\,(3.14)\frac{\hbar}{a} = 0.56\hbar$$

So, $\Delta x \Delta p = 0.56\hbar$

From Heisenberg uncertainity.

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

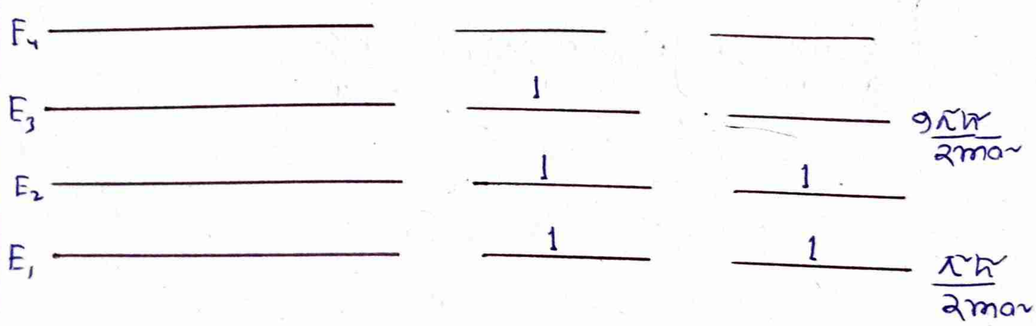
They satisfies each other

Fermions	Bosons
(1) Particle having Spin odd half integral ($n = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}$)	(1) Particle having integral Spin ($n = \pm 1, \pm 2, \pm 3$)
(2) Two Ferminon Can'nt exist in a Single state ⇓ Pauli Exclusion Principle	(2) They Can Stay together (Bother hood)
3) Degenercy $g = 2s+1$	

We have to Put $5e^{\ominus}$ in Particle in a Box:

$s = \frac{1}{2}$ $\quad$ $g = 2s+1 = 2$

Energy of state is Same



Energy of the System,

$$= 2 \times \frac{\pi^2\hbar^2}{2ma^2} + 2 \times \frac{4\pi^2\hbar^2}{2ma^2} + 1 \times \frac{9\pi^2\hbar^2}{2ma^2} = 19\,\frac{\pi^2\hbar^2}{2ma^2}$$

② <u>If there are 5 Basons in P in a Box :</u>

Here degenerecy is not required

So 5 of them in same state

Energy. $E = 5 \times \frac{\pi^2\hbar^2}{2ma^2}$

③ <u>4 Particle with spin 3 :</u>

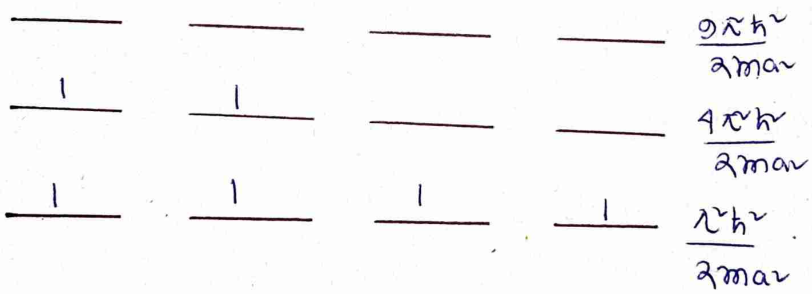
Here n = integral.

So degenerecy is not required

The four particle will remain in same state.

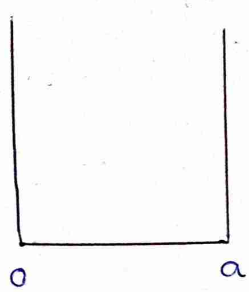
④ <u>6 Particle with spin $3/2$:</u>

Degenrecy $g = 2s+1 = (2 \times \frac{3}{2}) + 1 = 4$

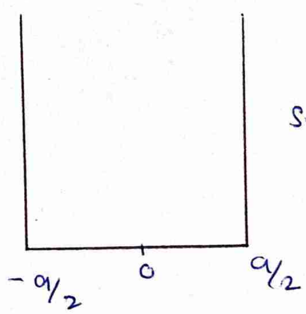


Energy eigen.

$$E = (4+8)\frac{\hbar^2\pi^2}{2ma^2} = \frac{6\hbar^2\pi^2}{ma^2}$$



Asymmetric



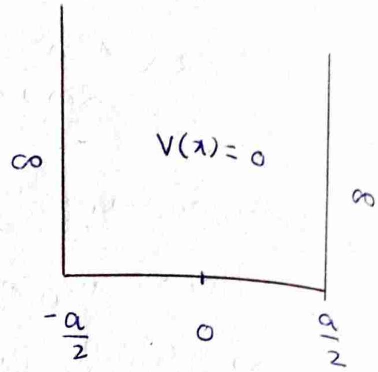
Symmetric

Symmetric Box (Origin at middle):

wave function will be dissappeared at

$\psi(-\frac{a}{2}) = 0$

$\psi(\frac{a}{2}) = 0$



From Schrodinger equation

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = E\psi \Rightarrow \frac{d^2\psi}{dx^2} + \frac{2mE}{\hbar^2}\psi = 0$$

$$\Rightarrow \frac{d^2\psi}{dx^2} + K^2\psi = 0$$

So, the solution of the differential equation will be

$$\psi(x) = A\sin Kx + B\cos Kx$$

$\psi(-\frac{a}{2}) = 0$

$\Rightarrow A\sin K(-\frac{a}{2}) + B\cos\frac{Ka}{2} = 0$

$\Rightarrow -A\sin\frac{Ka}{2} + B\cos\frac{Ka}{2} = 0$

$\psi(\frac{a}{2}) = 0$

$A\sin\frac{Ka}{2} + B\cos\frac{Ka}{2} = 0$

By adding them

$B\cos\frac{Ka}{2} = 0$

$B = 0$

$\cos\frac{Ka}{2} = 0$

Substract them

$A\sin\frac{Ka}{2} = 0$

$A = 0$

$\sin\frac{Ka}{2} = 0$

A, B Cannot be zero Simultaneously) because here wave function doest exist when. $A = 0$, $B \neq 0$ $\cos\frac{Ka}{2} = 0$. $\sin\frac{Ka}{2} \neq 0$

$$\cos\frac{Ka}{2} = 0$$

or $\cos\frac{Ka}{2} = \cos\frac{(2n+1)\pi}{2}$

or $K = \frac{(2n+1)\pi}{a}$

or $K = n'\frac{\pi}{a}$

(n' is odd number)

when $A \neq 0$ $B = 0$ $\cos\frac{Ka}{2} \neq 0$ $\sin\frac{Ka}{2} = 0$

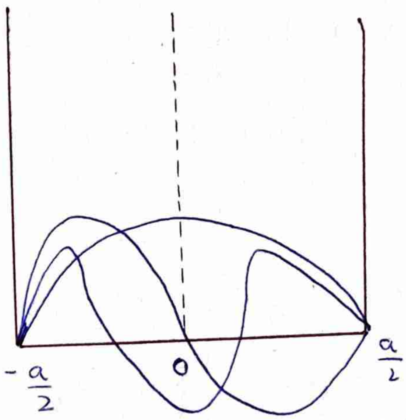
$$\sin\frac{Ka}{2} = 0$$

. $\sin\frac{Ka}{2} = \sin\frac{n\pi}{a}$

a $K = \frac{2n\pi}{a}$

or $K = \frac{n'\pi}{a}$

(n' is even number)



Even Symmetry

$$f(-x) = f(x)$$

$\psi(x) = A\sin Kx + B\cos Kx$

$-A\sin Kx + B\cos Kx = A\sin Kx + B\cos Kx$

$\Rightarrow$ $\cancel{B\cos Kx} = 0$

$\Rightarrow$ $A\sin Kx = 0$

So $A = 0$

$\psi(x) = B\cos Kx$

$\frac{Kx}{2} = 0$

$K = \frac{n\pi}{a}$ (n = odd)

odd symmetry

$$f(-x) = -f(x)$$

$B\cos Kx = 0$

$B = 0$

$\psi(x) = A\sin Kx$

$A\sin\frac{Ka}{2} = 0$

$K = \frac{n\pi}{a}$

(even number)

when $A = 0$

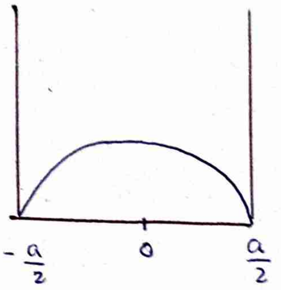
$$\psi(x) = B \cos \frac{n\pi}{a} x$$

$x = a/2 \Rightarrow \psi = 0$

$x = -a/2 \Rightarrow \psi = 0$

$x = 0 \Rightarrow \psi = 1$

$$\psi(x) = \begin{cases} \sqrt{\frac{2}{a}} \cos \frac{n\pi x}{a} & n = \text{odd} \\ \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a} & n = \text{even}) \end{cases}$$



$$\psi(x) = \sqrt{\frac{2}{a}} \cos \frac{n\pi x}{a}$$

$n = \text{odd}$, $\psi = \text{Even}$

$$K = \frac{n\pi}{a}$$

a : width of box

$$K^2 = \frac{n^2\pi^2}{a^2}$$

$$\frac{2mE}{\hbar^2} = \frac{n^2\pi^2}{a^2}$$

$$E_n = \frac{\hbar^2}{2m} \frac{n^2\pi^2}{a}$$

Particle in a 2D asymmetric box:

$x : 0 \longrightarrow a$

$y : 0 \longrightarrow b$

$$\psi(x,y) = \sqrt{\frac{2}{a}}\sqrt{\frac{2}{b}} \sin \frac{n_x \pi x}{a} \sin \frac{n_y \pi y}{b}$$

$$E_{n_x n_y} = \frac{\hbar^2}{2m} \pi^2 \left[\frac{n_x^2}{a^2} + \frac{n_y^2}{b^2} \right]$$

Particle in a 2D square Box:

$$\psi(x,y) = \sqrt{\frac{2}{a}}\sqrt{\frac{2}{a}} \sin \frac{n_x \pi x}{a} \sin \frac{n_y \pi y}{a}$$

$x : 0 \longrightarrow a$

$y : 0 \longrightarrow a$

$$E_{n_x n_y} = \frac{\hbar^2}{2m} \frac{\pi^2}{a^2} \left[n_x^2 + n_y^2 \right]$$

n_x and n_y can never be zero because then $\psi = 0$

Ground state: $n_x = 1$ $n_y = 1$

First excited state:

$n_x = 2, \; n_y = 1$

$$E_{21} = 5 \frac{\pi^2 \hbar^2}{2ma^2}$$

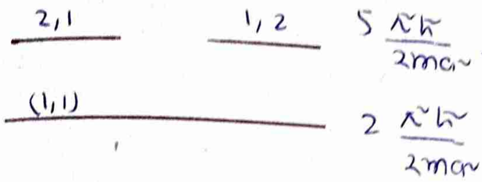
$$\psi = \frac{2}{a} \sin \frac{2\pi x}{a} \sin \frac{\pi y}{a}$$

$n_x = 1 \quad n_y = 2$

$$E_{12} = 5 \frac{\pi^2 \hbar^2}{2ma^2}$$

$$\psi = \frac{2}{a} \sin \frac{\pi x}{a} \sin \frac{2\pi y}{a}$$

Same energy but different ~~energy~~ wave function is called degenercy



2 fold degenerate

Second excited state.

$n_x = 2 \quad n_y = 2$

$$\psi = \frac{2}{a} \sin \frac{2\pi x}{a} \sin \frac{2\pi y}{a}$$

$$E_{2,2} = 8 \frac{\pi^2 \hbar^2}{2ma^2}$$

Particle in a 2D Box.

x : $-a$ to a

y : 0 to $2a$

$$\psi_x = \sqrt{\frac{2}{2a}} \sin \frac{n\pi x}{2a} \quad (n = \text{even})$$

$$\sqrt{\frac{2}{2a}} \cos \frac{n\pi x}{2a} \quad (n = \text{odd})$$

$$\psi_y = \sqrt{\frac{2}{2a}} \sin \frac{n\pi y}{2a}$$

$$\psi(x,y) = \frac{1}{a} \sin \frac{n_x \pi x}{2a} \sin \frac{n_y \pi y}{2a} \quad (n_x = \text{even}$$

$$= \frac{1}{a} \cos \frac{n_x \pi x}{2a} \sin \frac{n_y \pi y}{2a} \quad n_x = \text{odd})$$

Particle in a 3D Box:

$x : 0 \rightarrow a$

$y : 0 \rightarrow b$

$z : 0 \rightarrow c$

$$\psi = \sqrt{\frac{2}{a}}\sqrt{\frac{2}{b}}\sqrt{\frac{2}{c}} \sin\frac{n_x \pi x}{a} \sin\frac{n_y \pi y}{b} \sin\frac{n_z \pi z}{c}$$

$$E_{n_x n_y n_z} = \left(\frac{n_x^2}{a^2} + \frac{n_y^2}{b^2} + \frac{n_z^2}{c^2}\right)\frac{\hbar^2 \pi^2}{2m}$$

Cuboidal:

$$\psi(x,y,z) = \left(\frac{2}{a}\right)^{3/2} \sin\frac{n_x \pi x}{a} \sin\frac{n_y \pi y}{a} \sin\frac{n_z \pi z}{a}$$

$$E_{n_x n_y n_z} = (n_x^2 + n_y^2 + n_z^2)\frac{\pi^2}{a^2}\frac{\hbar^2}{2m}$$

GS: $E_{111} = \dfrac{3\pi^2\hbar^2}{2ma^2}$

FES $E = \dfrac{6\pi^2\hbar^2}{2ma^2}$ 112, 121, 2,1,1

Triply degenerate

SES $E = \dfrac{9\pi^2\hbar^2}{2ma^2}$ 221, 212, 122

222

311 313 133 $\dfrac{9\hbar^2\pi^2}{2ma^2}$

221 212 122

211 121 112

111 $\dfrac{3\hbar^2\pi^2}{2ma^2}$

10.02.2024

Simple Harmonic Oscillator

Hamilton operator. $\hat{H}\Psi = E\Psi$

Potential energy $V(x) = \frac{1}{2}Kx^2 = \frac{1}{2}m\omega^2x^2$

From Schrodinger equation.

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega^2x^2\,\psi(x) = E\,\psi(x)$$

If potential is $V(x) = \frac{1}{2}m\omega^2x^2$

Energy. $E_n = (n+\frac{1}{2})\hbar\omega$

Let. Potential, $V(x) = 4m\omega^2x^2 = \frac{1}{2} \times m\,8\omega^2x^2$

$$V(x) = \frac{1}{2} \times m \times (2\sqrt{2}\omega)^2 x^2$$

$$E_n = (n+\frac{1}{2})\hbar\,(2\sqrt{2}\omega) = (2\sqrt{2}n + \sqrt{2})\hbar\omega$$

Energy state:

For Simple Harmonic Oscillator $n = 0, 1, 2, 3, 4 \ldots$

Particle in a box $n = 1, 2, 3.$

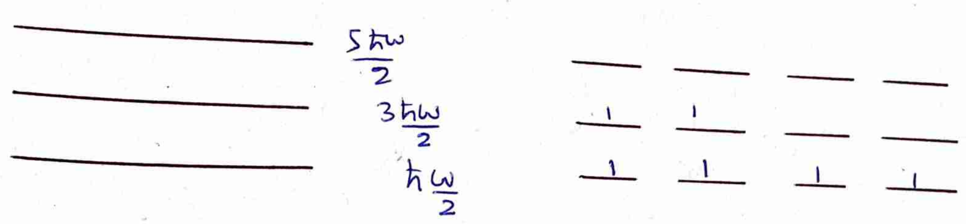
$n=0$, $E_0 = \frac{\hbar\omega}{2}$ => Ground state energy

$n=1$. $E_1 = \frac{3}{2}\hbar\omega$ => First excited state

$n=2$ $E_2 = \frac{5}{2}\hbar\omega$ => Second excited state

Energy gap is $\hbar\omega$

> 6 Spin $\frac{3}{2}$ Particle $g = 2s+1 = 4$



2D SHO: $V(x,y) = \frac{1}{2} m \omega_x^2 x^2 + \frac{1}{2} m \omega_y^2 y^2$

$$E_{n_x n_y} = (n_x + \tfrac{1}{2}) \hbar \omega_x + (n_y + \tfrac{1}{2}) \hbar \omega_y$$

> $V(x) = \frac{1}{2} m \omega^2 x^2 + 2 m \omega^2 y^2$

$= \frac{1}{2} m \omega^2 x^2 + \frac{1}{2} m (2\omega)^2 y^2$

$\omega_x = \omega$
$\omega_y = 2\omega$

$E = (n_x + \frac{1}{2}) \hbar \omega + (n_y + \frac{1}{2}) \hbar \omega \times 2$

$E = (n_x + 2n_y + 3/2) \hbar \omega$

(0,1) (2,0) ——— $\frac{7}{2} \hbar \omega$ SES

(1,0) ——— $\frac{5}{2} \hbar \omega$ FES

(0,0) ——— $\frac{3}{2} \hbar \omega$ Ground state

Given $V(x) = \frac{1}{2} m \omega^2 x^2$ $E_n = (n + \frac{1}{2}) \hbar \omega$

=) $E = \frac{17}{2} \hbar \omega$ find degenecy

$\frac{17}{2} = n + \frac{1}{2} \Rightarrow n = 8$

Since $V(x)$ is 1D so no degenaracy

Given: $V(x) = 2 m \omega^2 x^2 + \frac{1}{2} m \omega^2 y^2$

$E = \frac{19}{2} \hbar \omega$ degenery?

=) $\omega_x = 2\omega$ $\omega_y = \omega$

$E = (n_x + \frac{1}{2}) 2\omega + (n_y + \frac{1}{2}) \hbar \omega$

$= (2n_x + 1 + n_y + \frac{1}{2}) \hbar \omega = (2n_x + n_y + \frac{3}{2}) \hbar \omega$

$2n_x + n_y = \frac{19}{2} - \frac{3}{2} = 8$

0,8 ... 4 0
2,4 1,6
~~[illegible]~~
3, 2

So $g = 5$

3D SHO:

$$V(x,y,z) = \frac{1}{2}m\omega_x^2 x^2 + \frac{1}{2}m\omega_y^2 y^2 + \frac{1}{2}m\omega_z^2 z^2$$

$$E_{n_x n_y n_z} = (n_x+\tfrac{1}{2})\hbar\omega_x + (n_y+\tfrac{1}{2})\hbar\omega_y + (n_z+\tfrac{1}{2})\hbar\omega_z$$

> Potential.

$$V(x) = \frac{1}{2}m\omega^2 x^2 + \frac{1}{2}m\omega^2 y^2 + \frac{1}{2}m\omega^2 z^2$$

$$E = (n_x+n_y+n_z+\tfrac{3}{2})\hbar\omega$$

> Potential.

$$V(x,y,z) = 2m\omega^2 x^2 + \frac{1}{2}m\omega^2 y^2 + \frac{9}{2}m\omega^2 z^2$$

$\omega_x = 2\omega \qquad \omega_y = \omega \qquad \omega_z = 3\omega$

$$E = (n_x+\tfrac{1}{2})2\omega + (n_y+\tfrac{1}{2})\hbar\omega + (n_z+\tfrac{1}{2})3\hbar\omega$$

$$= (2n_x+n_y+3n_z+1+\tfrac{1}{2}+\tfrac{3}{2})\hbar\omega$$

$$= (2n_x+n_y+3n_z+3)\hbar\omega$$

$6\hbar\omega$ ———— (0,0,1) (0,3,0) (110)

$5\hbar\omega$ ———— (0,2,0) (1,0,0)

$4\hbar\omega$ ———— (0,1,0)

$3\hbar\omega$ ———— (0,0,0)

Special Case: Potential. $V(x) = \frac{1}{2}m\omega^2x^2 + ax$

$$V(x) = \frac{1}{2}m\omega^2\left[x^2 + \frac{2}{m\omega^2}ax\right]$$

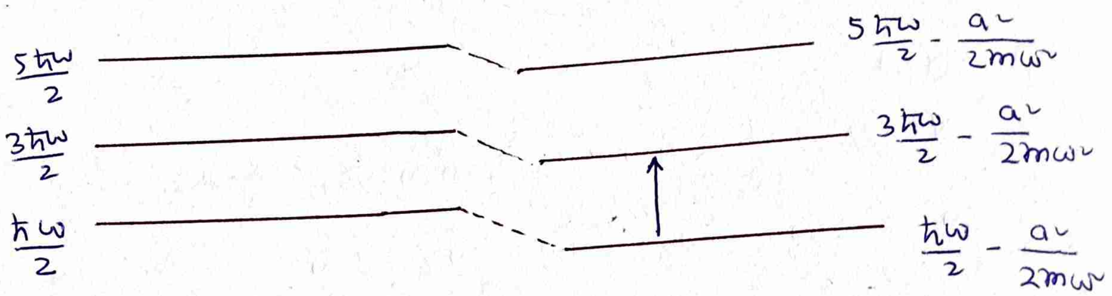
$$= \frac{1}{2}m\omega^2\left[x^2 + 2\left(\frac{a}{m\omega^2}\right)x + \frac{a^2}{m^2\omega^4} - \frac{a^2}{m^2\omega^4}\right]$$

$$= \frac{1}{2}m\omega^2\left[\left(x+\frac{a}{m\omega^2}\right)^2\right] - \frac{1}{2}m\omega^2 \cdot \frac{a^2}{m^2\omega^4}$$

$$V(x) = \frac{1}{2}m\omega^2\left[\left(x+\frac{a}{m\omega^2}\right)^2\right] - \frac{a^2}{2m\omega^2}$$

$$V(x) = \frac{1}{2}m\omega^2 y^2 - \frac{a^2}{2m\omega^2}$$

Energy. $E_n = (n+\frac{1}{2})\hbar\omega - \frac{a^2}{2m\omega^2}$



Energy state goes down but gaps are same

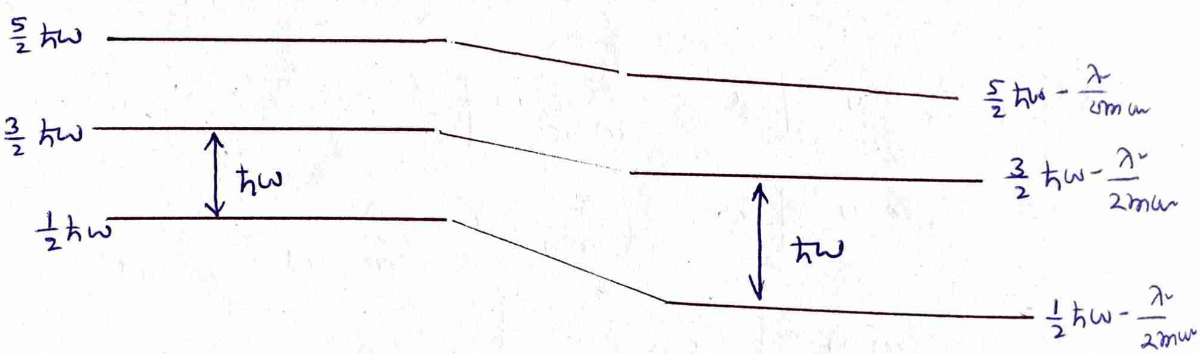
Case ② $V(x) = \frac{1}{2}m\omega^2 x^2 - \lambda x$

$$V(x) = \frac{1}{2}m\omega^2\left[x^2 - \frac{2}{m\omega^2}\lambda x\right]$$

$$V(x) = \frac{1}{2}m\omega^2\left[x^2 - 2\frac{\lambda}{m\omega^2}x + \frac{\lambda^2}{m^2\omega^4} - \frac{\lambda^2}{m^2\omega^4}\right]$$

$$V(x) = \frac{1}{2}m\omega^2\left[\left(x-\frac{\lambda}{m\omega^2}\right)^2 - \frac{\lambda^2}{m^2\omega^4}\right]$$

$$V(x) = \frac{1}{2}m\omega^2\left[\left(x-\frac{\lambda}{m\omega^2}\right)^2\right] - \frac{\lambda^2}{2m\omega^2}$$



$$\psi(x) = Ae^{iKx} + Be^{-iKx}$$ (Oscillating in Nature)

(+x dir) (-x dir)

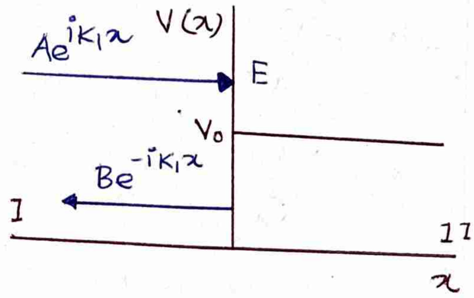
$$K = \sqrt{\frac{2mE}{\hbar^2}} \qquad K = \sqrt{\frac{2m(E-V)}{\hbar^2}} \qquad K = \sqrt{\frac{2m(V-E)}{\hbar^2}}$$

wave Number

① <u>Potential step:</u>

<u>14.02.2024</u>

Potential Profile



$$V(x) = \begin{cases} 0 & \text{at } x<0 \\ V_0 & \text{at } x \geq 0 \end{cases}$$

<u>Region I:</u> Here potential, $V(x) = 0$ $\boxed{E > V_0}$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_1}{dx^2} + 0\psi(x) = E\psi_1$$

or $$\frac{d^2\psi_1}{dx^2} + \frac{2mE}{\hbar^2}\psi_1 = 0$$ or $$\frac{d^2\psi_1}{dx^2} + K_1^2\psi_1 = 0$$

and here $K_1 = \sqrt{\frac{2mE}{\hbar^2}}$

$$\psi_1(x) = Ae^{iK_1x} + Be^{-iK_1x}$$

Now for incident wave, $\psi = Ae^{iK_1x}$

$$J_I = -\frac{\hbar i}{2m}\left[\psi^*\frac{\partial\psi}{\partial x} - \psi\frac{\partial\psi^*}{\partial x}\right]$$

$$J_1 = \frac{\hbar K_1 A^2}{m}$$

for Reflected wave $\psi = Be^{-iK_1x}$

$$J_R = \frac{-\hbar i}{2m}\left[\psi^*\frac{\partial\psi}{\partial x} - \psi\frac{\partial\psi^*}{\partial x}\right] = \frac{K_1\hbar B^2}{m}$$

Reflection coefficient $R = \frac{J_R}{J_I} = \frac{B^2}{A^2}$

Transmission coefficient $T = \frac{J_T}{J_I}$ and $T+R=1$

Region II:

Potential, $V(x) = V_0$ and $E > V_0$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_2}{dx^2} + V_0\psi_2 = E\psi_2$$

or $$\frac{d^2\psi_2}{dx^2} + \frac{2m(E-V_0)}{\hbar^2}\psi_2 = 0$$

or $$\frac{d^2\psi_2}{dx^2} + K_2^2\psi_2 = 0$$

But the part De^{-iK_2x} is to be ignored.

$$\psi_2 = Ce^{iK_2x} + De^{-iK_2x}$$

Transmitted wave, $\psi_2 = Ce^{iK_2x}$

$$J_T = \frac{-\hbar i}{2m}\left[\psi^*\frac{\partial\psi}{\partial x} - \psi\frac{\partial\psi^*}{\partial x}\right] = \frac{\hbar C^2}{m}K_2$$

Transmission Coefficient

$$T = \frac{J_T}{J_I} = \frac{C^2K_2}{A^2K_1} = T$$

$$R = \frac{B^2}{A^2}$$

At Boundary: $\psi_1(x) = \psi_2(x)$ at $x=0$

$$Ae^{iK_1x} + Be^{-iK_1x} = Ce^{iK_2x}$$

$$A+B = C \quad \text{— (i)}$$

$$\left.\frac{d\psi_1}{dx}\right|_{x=0} = \left.\frac{d\psi_2}{dx}\right|_{x=0}$$

or $$K_1A - K_2B = K_2C \quad \text{— (ii)}$$

So the equations are. $A+B=C \quad -(i)\times K_2$

$K_1A - K_1B = K_2C \quad -(ii)\times 1$

$$K_2A + K_2B = K_2C$$
$$\underline{-\;K_1A - K_1B = K_2C}$$
$$A(K_2-K_1) = -B(K_1+K_2)$$

$$\frac{B}{A} = \frac{(K_1-K_2)}{(K_1+K_2)}$$

Also we have. $A+B=C \quad -(i)\times K_1$

$K_1A-K_1B=K_2C \quad -(ii)$

$$K_1A+K_1B = K_1C$$
$$\underline{K_1A-K_1B = K_2C}$$
$$2K_1A = C(K_1+K_2)$$

$$\frac{C}{A} = \frac{2K_1}{K_1+K_2}$$

$$K_2 = \sqrt{\frac{2m}{\hbar^2}(E-V_0)}$$

$$K_1 = \sqrt{\frac{2m}{\hbar^2}E}$$

$$\frac{K_2}{K_1} = \sqrt{\frac{E-V_0}{E}} = \sqrt{1-\frac{V_0}{E}}$$

Since we know.

$$R = \frac{B^2}{A^2}$$

$$R = \left(\frac{K_1-K_2}{K_1+K_2}\right)^2$$

$$R = \frac{K_1^2}{K_1^2}\left(\frac{1-\frac{K_2}{K_1}}{1+\frac{K_2}{K_1}}\right)^2$$

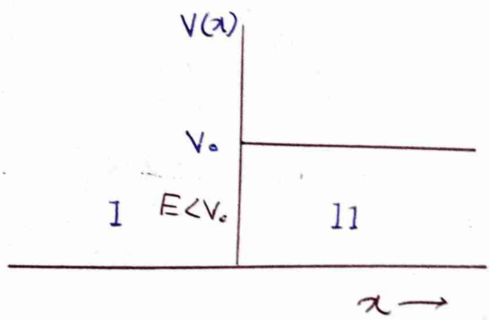
$$R = \left[\frac{1-\sqrt{1-V_0/E}}{1+\sqrt{1-V_0/E}}\right]^2$$

and we have $T = 1-R = 1-\left[\frac{1-\sqrt{1-V_0/E}}{1+\sqrt{1-V_0/E}}\right]^2$

Case ②: $E < V_0$

Potential Profile

$$V(x) = \begin{cases} 0 & x<0 \\ V_0 & x \geq 0 \end{cases}$$



Region I:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_1}{dx^2} + 0\psi = E\psi_1$$

or $$\frac{d^2\psi_1}{dx^2} + \frac{2mE}{\hbar^2}\psi_1 = 0$$

$$\psi_1 = \underbrace{Ae^{ik_1x}}_{\text{Incident}} + \underbrace{Be^{-ik_1x}}_{\text{Reflected.}}$$

$$J_1 = \frac{A^2k_1\hbar}{m} \qquad J_R = \frac{B^2k_1\hbar}{m} \qquad R = \frac{B^2}{A^2}.$$

Second Region: $(x>0)$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_2}{dx^2} + V_0\psi = E\psi$$

or $$\frac{d^2\psi_2}{dx^2} - \frac{2m}{\hbar^2}(V_0-E)\psi_2 = 0.$$

Since Kinetic energy is negative so wave can't be progress $K^2 = \dfrac{2m(V_0-E)}{\hbar^2}$

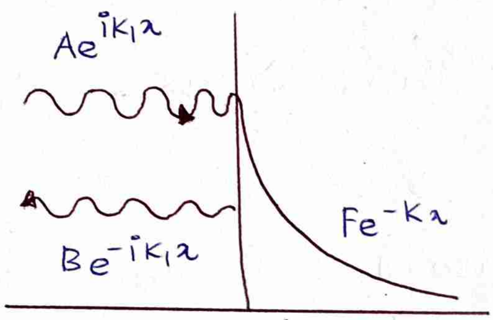
$$\frac{d^2\psi_2}{dx^2} - K^2\psi_2 = 0$$

$$\psi_2 = Ee^{Kx} + Fe^{-Kx}$$

But at infinity Ee^{Kx} became diverge

$$\psi_2 = Fe^{-Kx}$$

Transmitted Current density. $J_T = 0$.



$$T = \frac{J_T}{J_I} = 0$$

$$T+R=1 \Rightarrow R=1$$

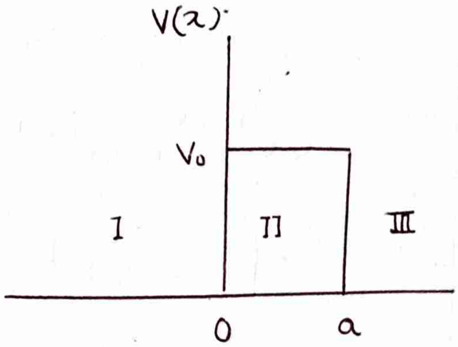
$T=0$
$R=1$

17.02.2024

② Potential Barrier:

Potential Profile

$$V(x) = \begin{cases} 0 & x < 0 \\ V_0 & 0 \le x \le a \\ 0 & x > a \end{cases}$$



(a) when $E > V_0$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_1}{dx^2} = E\psi_1 \qquad K_1^2 = \frac{2mE}{\hbar^2}$$

or $$\frac{d^2\psi_1}{dx^2} + \frac{2mE}{\hbar^2}\psi_1 = 0 \qquad \psi_1(x) = Ae^{iK_1x} + Be^{-iK_1x}$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_3}{dx^2} = E\psi_3 \qquad K_1^2 = \frac{2mE}{\hbar^2}$$

or $$\frac{d^2\psi_3}{dx^2} + \frac{2mE}{\hbar^2}\psi_3 = 0 \qquad \psi_3(x) = Ee^{iK_1x} + Fe^{-iK_1x}$$

$$\psi_3(x) = Ee^{iK_1x}$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_2}{dx^2} + V_0\psi_2 = E\psi_2 \qquad K_2^2 = \frac{2m}{\hbar^2}(E - V_0)$$

or $$\frac{d^2\psi_2}{dx^2} + \frac{2m}{\hbar^2}(E - V_0)\psi_3 = 0$$

$$\psi_2(x) = Ce^{iK_2x} + De^{-iK_2x}$$

Boundary Conditions:

> $\psi_1(x=0) = \psi_2(x=0)$ $\qquad$ $\left.\frac{d\psi_1}{dx}\right|_{x=0} = \left.\frac{d\psi_2}{dx}\right|_{x=0}$

$$A + B = C + D \qquad K_1A - K_1B = K_2C - K_2D$$

> $\psi_2(x=a) = \psi_3(x=a)$ $\qquad$ $\left.\frac{d\psi_2}{dx}\right|_{x=a} = \left.\frac{d\psi_3}{dx}\right|_{x=a}$

$$Ce^{iK_2a} + De^{-iK_2a} = Ee^{iK_1a} \qquad K_2Ce^{iK_2a} - K_2De^{-iK_2a} = K_1Fe^{iK_1a}$$

$$A+B = C+D \quad \text{—— (i)}$$

$$K_1(A-B) = K_2(C-D) \quad \text{—— (ii)}$$

$$Ce^{iK_2a} + De^{-iK_2a} = Ee^{iK_1a} \quad \text{—— (iii)}$$

$$K_2\left[Ce^{iK_2a} - De^{-iK_2a}\right] = K_1Ee^{iK_1a} \quad \text{—— (iv)}$$

$$K_2(Ce^{iK_2a} + De^{-iK_2a}) = K_2Ee^{iK_1a}$$

$$K_2Ce^{iK_2a} + K_2De^{-iK_2a} = K_2Ee^{iK_1a}$$

$$K_2Ce^{iK_2a} - K_2De^{-iK_2a} = K_1Ee^{iK_1a}$$

$$C(2K_2e^{iK_2a}) = Ee^{iK_1a}(K_2+K_1)$$

$$C = Ee^{iK_1a}\,\frac{(K_1+K_2)}{2K_2}$$

$$D = Ee^{i(K_1+K_2)a}\,\frac{K_2-K_1}{2K_2}$$

$$A = C\,\frac{(K_1+K_2)}{2K_1} + \frac{D(K_1-K_2)}{2K_1}$$

or $$A = Ee^{i(K_1-K_2)a}\,\frac{(K_1+K_2)^2}{4K_2K_1} - Ee^{i(K_1+K_2)a}\,\frac{(K_2-K_1)^2}{4K_1K_2}$$

or $$A = E\left[\frac{e^{i(K_1-K_2)a}(K_1+K_2)^2 - e^{i(K_1+K_2)a}(K_2-K_1)^2}{4K_1K_2}\right]$$

or $$\frac{E}{A} = \left[\frac{4K_1K_2}{(K_1+K_2)^2e^{i(K_1-K_2)a} - (K_1-K_2)^2e^{i(K_1+K_2)a}}\right]$$

or $$\frac{E}{A} = \frac{4K_1K_2\,e^{i(K_2-K_1)a}}{(K_1+K_2)^2 - (K_1-K_2)^2e^{2iK_2a}}$$

$$T = \left|\frac{E}{A}\right|^2 = \left(\frac{E}{A}\right)^* \left(\frac{E}{A}\right)$$

or $$T = \frac{(4K_1K_2)^2}{\left[(K_1+K_2)^2 - (K_1-K_2)^2 e^{-2iK_2a}\right]\left[(K_1+K_2)^2 - (K_1-K_2)^2 e^{2iK_2a}\right]}$$

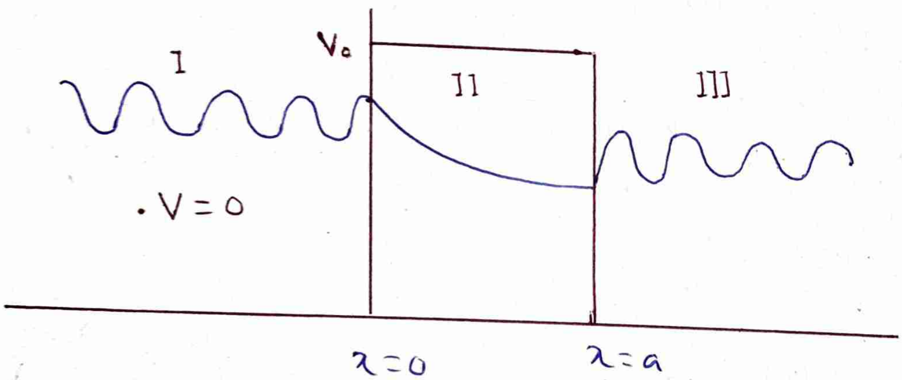
or $$T = \frac{(4K_1K_2)^2}{(K_1+K_2)^4 - (K_1+K_2)^2(K_1-K_2)^2 e^{2iK_2a} - (K_1-K_2)^2(K_1+K_2)^2 e^{-2iK_2a} + (K_1-K_2)^4}$$

or $$T = \frac{(4K_1K_2)^2}{\left[(K_1+K_2)^2 - (K_1-K_2)^2\right]^2 + 4(K_1+K_2)^2(K_1-K_2)^2 \sin^2 K_2a}$$

or $$T = \left[1 + \frac{(K_2^2-K_1^2)^2}{4K_1^2K_2^2} \sin^2 K_2a\right]^{-1}$$

<u>when $E < V_0$:</u>

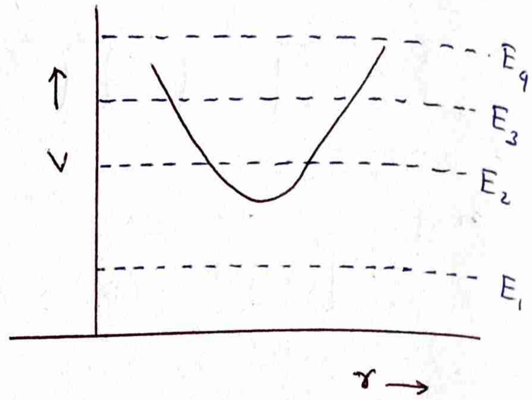
$$T = \left[1 + \frac{(K^2+K_1^2)^2}{4K_1^2K^2} \sinh^2 Ka\right]^{-1}$$



(1) $E_1 < V$

K.E. is (-)ve

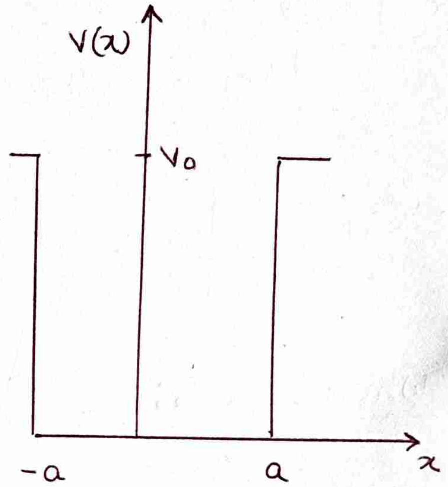
classically Forbidden region



③ Finite Potential well:

Potential Profile

$$V(x) = \begin{cases} V_0 & |x| > a \\ 0 & |x| < a \end{cases}$$



For $|x| < a$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + 0\psi = E\psi$$

or $$\frac{d^2\psi}{dx^2} + \frac{2mE}{\hbar^2}\psi = 0$$

$$\frac{d^2\psi}{dx^2} + K^2\psi = 0$$

$$\psi(x) = A\sin Kx + B\cos Kx$$

$x < -a$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V_0\psi = E\psi$$

$$\Rightarrow \frac{d^2\psi}{dx^2} - \frac{2m}{\hbar^2}(V_0 - E)\psi = 0$$

$$\frac{d^2\psi}{dx^2} - K^2\psi = 0$$

$$K^2 = \frac{2m}{\hbar^2}(V_0 - E)$$

Also for $x > +a$ it is also same

$$\psi(x) = Ce^{Kx} + De^{-Kx}$$

For. $x < -a$

$\psi(x) = Ce^{Kx}$

For $x > a$

$\psi(x) = De^{-Kx}$

wave function.

$$\psi(x) = \begin{cases} Ce^{Kx} & x < -a \\ A\sin Kx + B\cos Kx & -a \leq x \leq a \\ De^{-Kx} & x > a \end{cases}$$

Now, $\psi(x) = A\sin Kx + B\cos Kx$

Even function

$\psi(x) = B\cos Kx$

Odd function

$\psi(x) = A\sin Kx$

Boundary Condition:

$$\text{I} \quad \psi(x) = \begin{cases} 1 \quad Ce^{Kx} & x < -a \\ 2 \quad B\cos Kx & -a < x < a \\ 3 \quad De^{-Kx} & x > a \end{cases}$$

$$\text{II} \quad \psi(x) = \begin{cases} Ce^{Kx} & x < -a \\ A\sin Kx & -a < x < a \\ De^{-Kx} & x > a \end{cases}$$

Case I

$Ce^{-Ka} = B\cos Ka$ — ①

$\left.\frac{d\psi_1}{dx}\right|_{x=-a} = \left.\frac{d\psi_2}{dx}\right|_{x=-a}$

$De^{-Ka} = B\cos Ka$ — ③

$KCe^{-Ka} = BK\sin Ka$ — ②

$\left.\frac{d\psi_2}{dx}\right|_{x=a} = \left.\frac{d\psi_3}{dx}\right|_{x=a}$

$-BK\sin Ka = KDe^{-Ka}$ — ④

$(2) \div (1)$

$$\frac{\kappa C e^{-\kappa a}}{C e^{-\kappa a}} = \frac{B K \sin Ka}{B \cos Ka}$$

$$K^2 = \frac{2mE}{\hbar^2}$$

$$\kappa^2 = \frac{2m(V_0 - E)}{\hbar^2}$$

$$\kappa = K \tan Ka$$

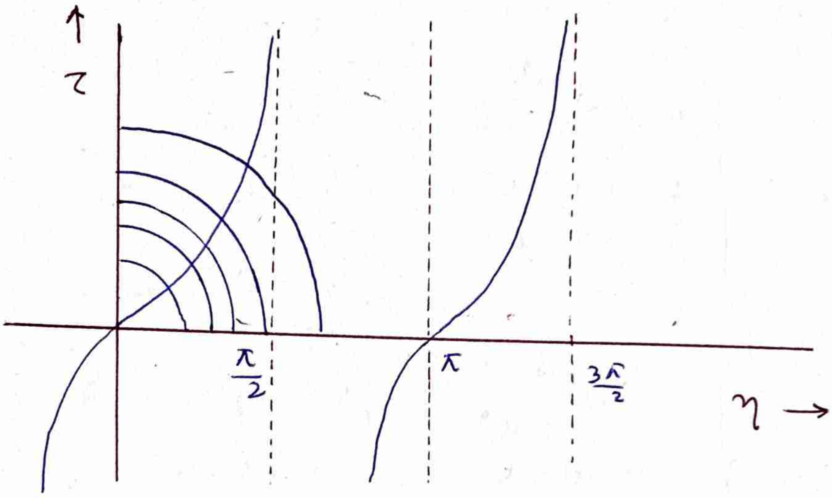
$$K^2 + \kappa^2 = \frac{2mV_0}{\hbar^2}$$

$$a\kappa = aK \tan Ka$$

$$a^2(K^2 + \kappa^2) = \frac{2mV_0}{\hbar^2} a^2$$

Let $Ka = \eta$ and $\kappa a = \tau$

$$\tau = \eta \tan \eta \qquad \eta^2 + \tau^2 = \frac{2mV_0}{\hbar^2} a^2$$



Case II

$$\psi(x) = \begin{cases} C e^{\kappa x} & x < -a \\ A \sin Kx & -a < x < a \\ D e^{-\kappa x} & x > a \end{cases}$$

$$C e^{-\kappa a} = -A \sin Ka$$

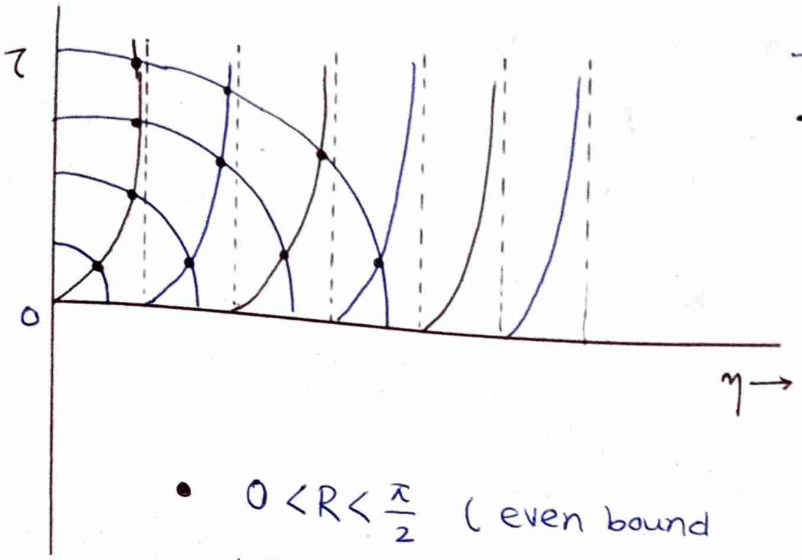
$$\kappa C e^{-\kappa a} = AK \cos Ka$$

$$\kappa = -K \cot Ka$$

or $\tau = -\eta \cot \eta$

Simmilarly we get

$$\eta^2 + \tau^2 = \frac{2mV_0}{\hbar^2} a^2$$



$\tau = -\eta \cot\eta$

$\tau = \eta \tan\eta$

- $0 < R < \frac{\pi}{2}$ (even bound state)

1 Bound state

$0 < R^2 < \frac{\pi^2}{4} \Rightarrow 0 < \frac{2mV_0}{\hbar^2} a^2 < \frac{\pi^2}{4}$

$\Rightarrow 0 < V_0 < \frac{\pi^2 \hbar^2 V_0}{4(2mV_0 a^2)}$

$\Rightarrow 0 < V_0 < \frac{\hbar^2}{2m} \frac{\pi^2}{4a^2}$

$\Rightarrow 0 < V_0 < \frac{\hbar^2}{2m} \frac{\pi^2}{l^2}$ $(l = 2a)$

- $\frac{\pi}{2} < R < \pi$

2 Bound state

1 even, 1 odd

$\frac{\pi^2}{4} < R^2 < \frac{\hbar^2}{2m} \pi^2$

$\frac{\pi^2 \hbar^2}{2ml^2} < V_0 < \frac{\hbar^2}{2m} \frac{4\pi^2}{l^2}$

- $\pi < R < \frac{3\pi}{2}$

2 even, 1 odd

① A particle of rest mass m whose KE is twice its rest mass energy collides with a particle of equal mass at rest. These two particle compile into a single new particle. Rest mass of new particle is

$\Rightarrow$ $KE = 2m_0c^2$

or $(\gamma - 1) m_0 c^2 = 2m_0 c^2$

or $\gamma - 1 = 2$

$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = 3$

$v = \frac{2\sqrt{2}}{3} c$

$3mc^2 \; m \; \longrightarrow \; mc^2 \; m \Rightarrow M$

$v = \frac{2\sqrt{2}}{3} c$ $\qquad V = 0 \qquad V_1 = ?$

$P_i = mv\gamma$

$= 3m \frac{2\sqrt{2}}{3} c$

$= 2\sqrt{2}\, mc$

$P_f = \frac{MV_1}{\sqrt{1 - \frac{V_1^2}{c^2}}}$

$E_f = \frac{Mc^2}{\sqrt{1 - V_1^2/c^2}}$

Now from momentum Conservation

$$2\sqrt{2}mc = \frac{MV_1}{\sqrt{1-V_1^2/c^2}}$$

From energy Conservation

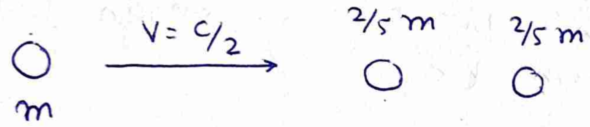
$$3mc^2 + mc^2 = \frac{Mc^2}{\sqrt{1-V_1^2/c^2}} = 4mc^2$$

$$\frac{4mc^2}{2\sqrt{2}mc} = \frac{Mc^2}{MV_1} \Rightarrow V_1 = c/\sqrt{2}$$

$$2\sqrt{2}mc = \frac{M}{\sqrt{1-\frac{1}{2}}}\frac{c}{\sqrt{2}} \Rightarrow M = 2\sqrt{2}m$$

② A particle of rest mass m moving with Speed c/2 decays into two particles of rest masses 2/5 m each. The daughter particles move in the same line as the direction of motion of the original particle. The velocities of the daughter particle is

$\Rightarrow$



$$p_i = \frac{m\,c/2}{\sqrt{1-\left(\frac{c}{2c}\right)^2}} \qquad p_f = \frac{2m}{5}\frac{V_1}{\sqrt{1-V_1^2/c^2}} + \frac{2m}{5}\frac{V_2}{\sqrt{1-V_2^2/c^2}}$$

$$\frac{mc/2}{\frac{\sqrt{3}}{2}} = \frac{2m}{5}\frac{V_1}{\sqrt{1-V_1^2/c^2}} + \frac{2m}{5}\frac{V_2}{\sqrt{1-V_2^2/c^2}}$$

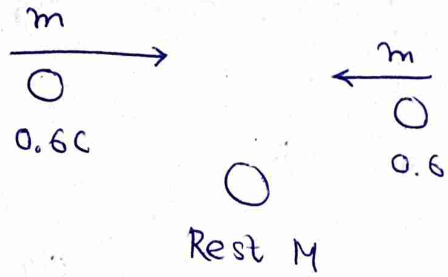
$$\frac{mc^2}{\sqrt{1-\left(\frac{c}{2c}\right)^2}} = \frac{2m}{5}\frac{c^2}{\sqrt{1-V_1^2/c^2}} + \frac{2m}{5}\frac{c^2}{\sqrt{1-V_2^2/c^2}}$$

Now Assuming $V_1 = V_2$

Velocity of daughter nucleus = 0.72 c

(3) Two particles each of rest mass m Collides head on and stick together. Before Collision the speed of each mass was 0.6 times the speed of light in free Space. Mass of final entity is

=)



$$\gamma mc^2 + \gamma mc^2 = Mc^2 \Rightarrow M = 2\gamma m$$

$$\Rightarrow M = \frac{2m}{\sqrt{1-(0.6)^2}} = \frac{2m}{0.8} = \frac{m}{0.4}$$

(4) Two particles each of rest mass m_0 move with speed V w.r.t an initial frame in opposite direction. The energy of one particle in the rest frame of other is

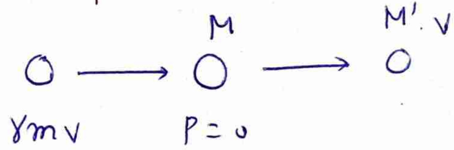
=) $$u_x' = \frac{u_x - V}{1 - \frac{u_x V}{c^2}}$$ $\quad u_x = V$, $V = -V$

$$u_x' = \frac{V+V}{1+V^2/c^2} = \frac{2Vc^2}{(c^2+V^2)} \Rightarrow u_x'^2 = \frac{4V^2c^4}{(c^2+V^2)^2}$$

$$E = \frac{1}{\sqrt{1-\frac{u_x'^2}{c^2}}} mc^2 = \frac{m_0c^2}{\sqrt{1-\frac{4V^2c^2}{(c^2+V^2)^2}}} = m_0c^2\left(\frac{c^2+V^2}{c^2-V^2}\right)$$

(5) A particle of rest mass m and speed V Collides and Sticks to a stationary particle of mass M. The final Speed of the Comile particle is

=)



$$\gamma mV + 0 = \gamma' M' V' \quad —(1)$$

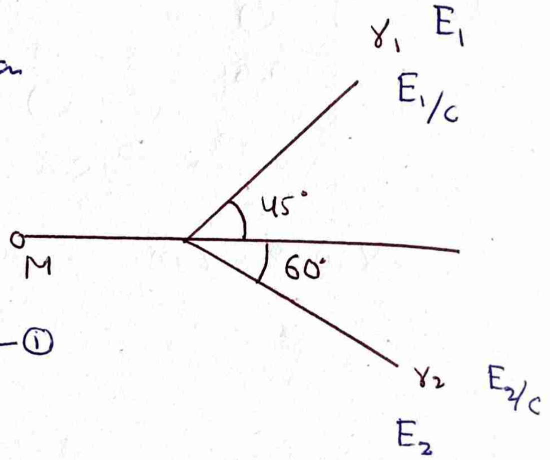
$$\gamma mc^2 + Mc^2 = \gamma' M' c^2 \quad —(2)$$

(6) A particle of rest mass M is moving along +x direction. If decays into two particles γ_1 and γ_2

$E_{\gamma_1} = 1\,GeV, \quad E_{\gamma_2} = 0.82\,GeV$

what is the value of M — (in GeV/c^2)

=) Momentum along x direction



$$\frac{MV}{\sqrt{1-v^2/c^2}} = \frac{E_1}{c}\cos 45 + \frac{E_2}{c}\cos 60$$

$$\frac{MV}{\sqrt{1-v^2/c^2}} = \frac{1}{c\sqrt{2}}\left(E_1 + \frac{E_2}{\sqrt{2}}\right) \quad -①$$

Along y direction

$$P_2 \sin 60 = P_1 \sin 45 \Rightarrow \frac{E_2}{c}\sin 60 = \frac{E_1}{c}\sin 45$$

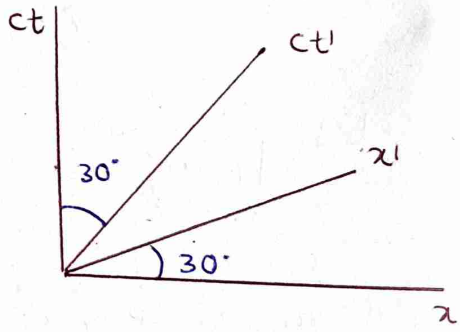
From energy conservation we have

$$\frac{Mc^2}{\sqrt{1-v^2/c^2}} = E_1 + E_2 \quad -⑪$$

$$\frac{V}{c^2} = \frac{1}{c\sqrt{2}}\left(E_1 + \frac{E_2}{\sqrt{2}}\right)\frac{1}{(E_1+E_2)}$$

$V = \_\_ \quad \Leftarrow \quad \frac{V}{c} = \frac{1}{\sqrt{2}}\left(1 + \frac{0.82}{\sqrt{2}}\right)\frac{1}{1+0.82}$

(7) If a particle of rest mass m_0 is at rest in frame S'. total energy in S frame



=) $\tan 30 = \frac{V}{c} \Rightarrow V = c/\sqrt{3}$

$$E = \frac{m_0c^2}{\sqrt{1-v^2/c^2}} = \sqrt{\frac{3}{2}}\,m_0c^2$$

Q A one dimensional Harmonic oscillator carrying a charge $-q$ is placed in a uniform field $\vec{E}$ along positive x-axis. The Hamilton operator (GATE-2005)

$\Rightarrow$ potential, $V = -Ex$

Energy $U = -qV = qEx$

so the schrodinger equation will

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}Kx^2 + qEx$$

> Ladder operator:

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left[a + a^{\dagger}\right]$$

$$\hat{p} = i\sqrt{\frac{m\omega\hbar}{2}}\left[a^{\dagger} - a\right]$$

$aa^{\dagger} = \hat{N}$

$\Downarrow$

Number operator

$$a = \sqrt{\frac{1}{2m\hbar\omega}}\left[m\omega\hat{x} + i\hat{p}\right]$$

$$a^{\dagger} = \sqrt{\frac{1}{2m\hbar\omega}}\left[m\omega\hat{x} - i\hat{p}\right]$$

$$a|n\rangle = \sqrt{n}\,|n-1\rangle$$

$$a^{\dagger}|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

$[a, a^{\dagger}] = 1$

they are commute

$$a|0\rangle = 0$$

Q Eigen value of $a^{\dagger}a^{\dagger}aa|n\rangle$ is

$\Rightarrow$ $a^{\dagger}a^{\dagger}aa|n\rangle = a^{\dagger}a^{\dagger}a\sqrt{n}\,|n-1\rangle = \sqrt{n}\,a^{\dagger}a^{\dagger}a|n-1\rangle$

$= \sqrt{n}\,a^{\dagger}a^{\dagger}\sqrt{n-1}\,|n-2\rangle = \sqrt{n(n-1)}\,a^{\dagger}a^{\dagger}|n-2\rangle$

$= \sqrt{n(n-1)}\,a^{\dagger}\sqrt{n-1}\,|n-1\rangle = (n-1)\sqrt{n}\,a^{\dagger}|n-1\rangle$

$= (n-1)\sqrt{n}\sqrt{n}\,|n\rangle = n(n-1)|n\rangle$

(2)
$$V(x) = \begin{cases} \frac{1}{2} m\omega^2 x^2 & x > 0 \\ \infty & x \leq 0 \end{cases}$$

$$|\psi\rangle = -\frac{1}{\sqrt{5}}|\psi_0\rangle + \frac{2}{\sqrt{5}}|\psi_1\rangle$$

The expectation value of energy

$\Rightarrow$ Since the wave function is asymmetric only odd numbers will allowed so the energy eigen $E_n = \left[(2n+1)+\frac{1}{2}\right]\hbar\omega$

$E_0 = \frac{3}{2}\hbar\omega$

$E_1 = \frac{7}{2}\hbar\omega$

Expectation value of energy

$$\langle E\rangle = P_1E_0 + P_2E_1$$
$$= \frac{1}{5}\times\frac{3}{2}\hbar\omega + \frac{4}{5}\times\frac{7}{2}\hbar\omega$$
$$= \frac{31}{10}\hbar\omega$$

(3) Let $|0\rangle$ and $|1\rangle$ denote the normalized eignstate the uncertainity ΔP in the state $\frac{1}{\sqrt{2}}[|0\rangle + |1\rangle]$ is

$\Rightarrow$
$$\hat{p} = i\sqrt{\frac{m\omega\hbar}{2}}\left[a^{\dagger} - a\right]$$
$$\hat{p}^2 = -\frac{m\omega\hbar}{2}\left[a^{\dagger 2} + a^2 - a^{\dagger}a - aa^{\dagger}\right]$$

$$\langle\hat{p}\rangle = \langle\psi|\hat{p}|\psi\rangle = \frac{i}{\sqrt{2}}\sqrt{\frac{m\omega\hbar}{2}}\left[(\langle 0| + \langle 1|)(a^{\dagger} - a)(|0\rangle + |1\rangle)\right]$$

$$= \frac{i}{\sqrt{2}}\sqrt{\frac{m\omega\hbar}{2}}\Big[\langle 0|\cancel{a}^{\dagger}|0\rangle + \langle 0|\cancel{a}^{\dagger}|1\rangle + \langle 0|\cancel{a}|0\rangle$$
$$+ \langle 0|a|1\rangle + \langle 1|a^{\dagger}|0\rangle\Big]$$

$$= \frac{i}{2}\sqrt{\frac{m\omega\hbar}{2}}\left[-1+1\right] = i\sqrt{m\omega\hbar - m\omega\hbar} = 0$$

So, $\langle\hat{p}\rangle = 0$

$$\hat{p}^2 = -\frac{m\omega\hbar}{2}\left[a^{\dagger 2} + a^2 - a^\dagger a - a a^\dagger\right]$$

$$\cdot \langle \hat{p}^2 \rangle = -\frac{m\omega\hbar}{2}\left[(\langle 0| + \langle 1|)(a^{\dagger 2} + a^2 - a^\dagger a - a a^\dagger)(|0\rangle + |1\rangle)\right]$$

$$= -\frac{m\omega\hbar}{2}\Big[-\langle 0|a^\dagger a|0\rangle - \langle 1|a^\dagger a|1\rangle - \langle 0|a a^\dagger|0\rangle$$

$$- \langle 1|a a^\dagger|1\rangle\Big] \cdot m\omega\hbar$$

⑥ Let the wavefunction of the Particle is SHO

$$\psi(x) = -\frac{1}{\sqrt{5}}\psi_0 + \frac{2}{\sqrt{5}}\psi_1 \qquad \langle E \rangle = ?$$

$$\Rightarrow \quad E_0 = \frac{\hbar\omega}{2} \qquad P_0 = \frac{1}{5} \quad P_1 = \frac{4}{5}$$

$$E_1 = \frac{3\hbar\omega}{2} \qquad \langle E \rangle = \frac{1}{5}\frac{\hbar\omega}{2} + \frac{4}{5}\frac{3\hbar\omega}{2} = \frac{13}{10}\hbar\omega$$

⑤ A particle of mass m in one dimensional Potential box

$$V(x) = \begin{cases} 0, & 0 < x < L \\ \infty, & \text{Otherwise} \end{cases}$$

$$\psi(x) = \frac{1}{\sqrt{3}}\psi_1(x) + i\sqrt{\frac{2}{3}}\psi_2(x) \qquad \begin{array}{l}\psi_1 = GS \\ \psi_2 = FES.\end{array}$$

Find the Value of $\langle E \rangle$ and $\langle X \rangle$

$$\Rightarrow \quad E = \frac{\hbar^2}{2m}\frac{n^2\pi^2}{a^2} \qquad E_0 = \frac{\hbar^2}{2m}\frac{\pi^2}{a^2} \qquad E_1 = \frac{\hbar^2}{2m}\frac{4\pi^2}{a^2} = 4E_0$$

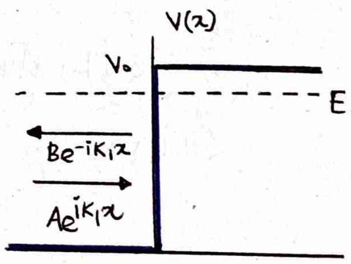
$$\langle E \rangle = \frac{\frac{1}{3}\times E_0 + \frac{2}{3}\times 4E_0}{\frac{1}{3} + \frac{2}{3}} = 3E_0 = \frac{3\hbar^2}{2m}\frac{\pi^2}{a^2}$$

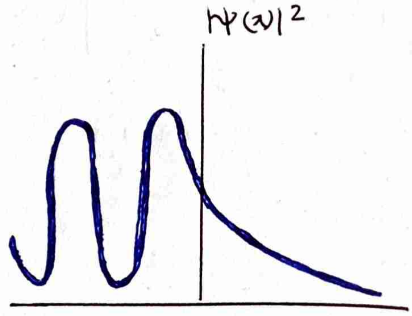
$$\langle X \rangle = \frac{1}{3}\langle \psi_1|x|\psi_1\rangle + \frac{2}{3}\langle \psi_2|x|\psi_1\rangle + \frac{i}{\sqrt{3}}\sqrt{\frac{2}{3}}\langle \psi_1|x|\psi_2\rangle - \frac{i}{\sqrt{3}}\sqrt{\frac{2}{3}}\langle \psi_2|x|\psi_1\rangle$$

$$= \left(\frac{1}{3}\times\frac{L}{2}\right) + \left(\frac{2}{3}\times\frac{L}{2}\right) = \frac{L}{2}$$

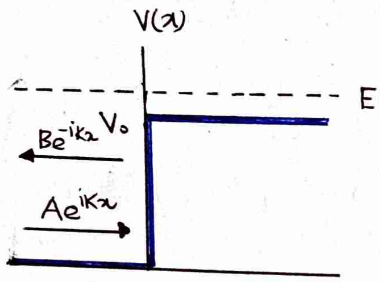
① Potential step (step Potential)

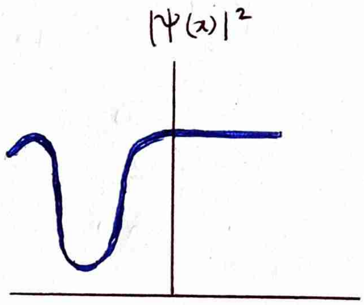
① when $E<V_0$



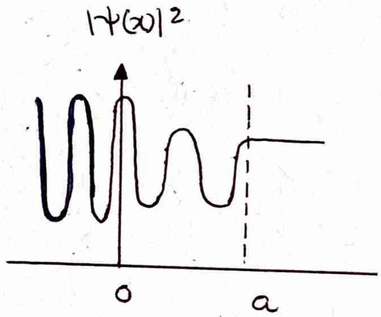


② when $E>V_0$

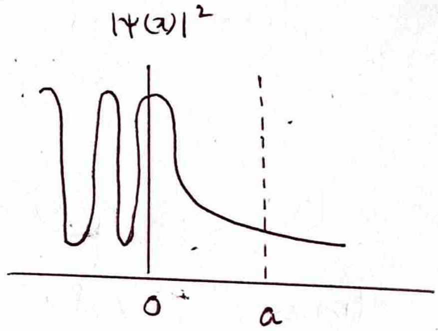




② Potential Barrier or Well



$E>V_0$



$E<V_0$

① Potential step:

① when $E<V_0$:

$R=1$

$T=0$

② when $E>V_0$

$$R=\left[\frac{1-\sqrt{1-V_0/E}}{1+\sqrt{1-V_0/E}}\right]^2$$

$T=1-R$

23.05.2024

Dirac - Delta Potential

Schrödinger Equation $\qquad \hat{p} = -i\hbar \frac{\partial}{\partial x}$

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]\psi = E\psi \Rightarrow H\psi = E\psi$$

Harmonic Oscillator $V(x) = \frac{1}{2}m\omega^2 x^2$

Free Particle. $V(x) = 0$

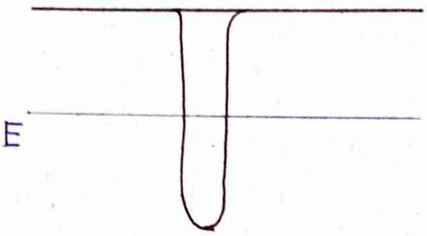
step potential, $V(x) = V_0 \quad x \geq 0$

$\qquad = 0 \quad x < 0$

Attractive Dirac Delta Potential

Potential Profile $V(x) = -V_0\,\delta(x)$

so, $\left.\begin{array}{l} V(x) = 0 \text{ at } x \neq 0 \\ \quad = -\infty \text{ at } x = 0 \end{array}\right\}$



Schrödinger Equation

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}[E - V(x)]\psi = 0$$

At $x \neq 0$ $V(x) = 0$ $\quad \frac{d^2\psi}{dx^2} + \frac{2mE}{\hbar^2}\psi = 0$

$K^2 = \frac{-2mE}{\hbar^2}$

$K = \sqrt{\frac{-2mE}{\hbar^2}}$

K is real because E is negative.

or $\frac{d^2\psi}{dx^2} - \frac{2m}{\hbar^2}(-E)\psi = 0$

or $\frac{d^2\psi}{dx^2} - K^2\psi = 0$

$\psi(x) = c_1 e^{Kx} + c_2 e^{-Kx}$

If $x > 0$ at $x \to \infty$

$\psi(x) = 0$ so $c_1 = 0$

If $x < 0$ at $x \to -\infty$

$\psi(x) = 0$ so $c_2 = 0$

At $x > 0$

$\psi_1(x) = c_2 e^{-Kx}$

At $x < 0$

$\psi_2(x) = c_1 e^{Kx}$

At $x = 0$ $\psi_1 = \psi_2$ so $c_1 = c_2 = c$

so $\psi(x) = c\,e^{-K|x|}$ $\qquad |x| = +x,\ x > 0$

$\qquad\qquad = -x,\ x < 0$

So the wavefunction became

$$\psi(x) = Ce^{-K|x|}$$

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}[E - V(x)]\psi = 0$$

Now $V(x) = -V_0\,\delta(x)$

or, $$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}[E + V_0\delta(x)]\psi = 0$$

or $$\frac{d}{dx}\left(\frac{d\psi}{dx}\right) + \frac{2m}{\hbar^2}E\psi(x) + \frac{2m}{\hbar^2}V_0\delta(x)\psi(x) = 0$$

or $$\int_{-\infty}^{+\infty}\frac{d}{dx}\left(\frac{d\psi}{dx}\right)dx + \frac{2m}{\hbar^2}E\int_{-\infty}^{+\infty}\psi(x)dx + \frac{2m}{\hbar^2}V_0\int_{-\infty}^{+\infty}\delta(x)\psi(x)dx = 0$$

or $$\left[\frac{d\psi}{dx}\right]_{-\infty}^{+\infty} + \frac{2mE}{\hbar^2}\left[\int_{-\infty}^{0}\psi(x)dx + \int_{0}^{\infty}\psi(x)dx\right] + \frac{2m}{\hbar^2}V_0\psi(0) = 0$$

$$\frac{2mE}{\hbar^2}\left[c\int_{-\infty}^{0}e^{Kx}dx + c\int_{0}^{\infty}e^{-Kx}dx\right] + \frac{2mV_0}{\hbar^2}\psi(0) = 0$$

$$\frac{2mE}{\hbar^2}\left[\frac{c}{K}(1-0) + \frac{c}{(-K)}(0-1)\right] + \frac{2mV_0}{\hbar^2}c = 0$$

$$\frac{2mE}{\hbar^2}\left[\frac{c}{K} + \frac{c}{K}\right] + \frac{2mV_0}{\hbar^2}c = 0$$

or $$\frac{2mE}{\hbar^2}\times\frac{2}{K} + \frac{2mV_0}{\hbar^2} = 0 \Rightarrow E = \frac{-V_0K}{2}$$

So Energy $\Rightarrow E^2 = \left(\frac{V_0}{2}\right)^2K^2$

$$E = -\frac{mV_0^2}{2\hbar^2} \qquad \Rightarrow E = \frac{-2mV_0^2}{4\hbar^2}$$

(only one Energy state Exist)

$$\psi(x) = Ce^{-K|x|} \qquad \psi(x) = \sqrt{K}e^{-K|x|}$$

$$\int_{-\infty}^{+\infty}|\psi|^2dx = c^2\int_{-\infty}^{+\infty}e^{-2K|x|}dx \Rightarrow C = \sqrt{K}$$

$$\Delta x\Delta P = \frac{\hbar}{\sqrt{2}}$$

Operators in Quantum Mechanics

· Equation of Plane wave,

wavefunction. $\psi(x,t) = \psi_0 e^{i(Kx-\omega t)}$

$$\frac{\partial\psi}{\partial x} = (iK)\psi_0 e^{i(Kx-\omega t)}$$

$K = \frac{2\pi}{\lambda}$

$\lambda = \frac{h}{p} \Rightarrow p = \hbar K$

or, $-i\frac{\partial\psi}{\partial x} = K\psi$

or, $-i\frac{\partial\psi}{\partial x} = \frac{p}{\hbar}\psi \Rightarrow \boxed{-i\hbar\frac{\partial\psi}{\partial x} = \hat{p}\psi}$ $\hat{p}$ is momentum operator

$\hat{p}\psi = p\psi$ is also eigen value equation

Also $\psi(x,t) = \psi_0 e^{i(Kx-\omega t)}$ $E = \hbar\omega$

or $\frac{\partial\psi}{\partial t} = (-i\omega)\psi$

or $\frac{\partial\psi}{\partial t} = -\frac{i}{\hbar}E\psi \Rightarrow \boxed{i\hbar\frac{\partial\psi}{\partial t} = E\psi}$

This is energy operator or time dependent Hamiltonian operator..

The Schrodinger Equation

Hamiltonian operator

$KE = \frac{p^2}{2m}$ $\hat{p} = -i\hbar\frac{\partial}{\partial x}$

$\hat{H}\psi = E\psi$

$= -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$

$$i\hbar\frac{\partial\psi}{\partial t} = H\psi$$

$$i\hbar\frac{\partial\psi}{\partial t} = \left[-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi\right]$$

Also $\boxed{i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V(r)\psi}$

Probability density. $\rho = \psi^*\psi$

Probability current density

$$\vec{j} = \frac{\hbar i}{2m}\left[\psi\vec{\nabla}\psi^* - \psi^*\vec{\nabla}\psi\right]$$

Eqn of Continuity. $\boxed{\vec{\nabla}\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0}$

Any dynamical Variable A, depends on (q_i, P_i, t)

> operators are used in quantum mechanics Corresponding to a dynamical Variable in classical. This dynamical Variables are Called observables

> How observable process take place, then we have to introduce operators

> $\hat{p}\psi$ we simply measured the momentum

> $\hat{A}\psi = \lambda\psi$ is Eigen value equation and λ is Eigen value and $\hat{A}$ is operators

> $\hat{A}\psi = \lambda\psi'$ is not an eigen value equation as wave function become changed

wave function for particle in a box

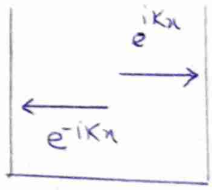
$$\psi = \sqrt{\frac{2}{l}} \sin\left(\frac{n\pi x}{l}\right)$$

$$\hat{p}\psi = -i\hbar \frac{\partial}{\partial x}\left[\sqrt{\frac{2}{l}} \sin\left(\frac{n\pi x}{l}\right)\right]$$

$$= -i\hbar \sqrt{\frac{2}{l}}\left(\frac{n\pi}{l}\right) \cos\left(\frac{n\pi x}{l}\right) = \lambda\left(\sqrt{\frac{2}{l}} \cos\frac{n\pi x}{l}\right)$$

Not an Eigen value equation

$\sin\frac{n\pi x}{l}$ is the combination of two wave function e^{ikx} and e^{-ikx}



The momentum cannot be measure accuratly as uncertainity present for both direction of wave function

> $\hat{p}\psi = p\psi$ (Schrodinger Notation)

$\hat{p}|\psi\rangle = p|\psi\rangle$ (Dirac Notation)

> Expectation value of $\hat{A}$ is

$$\langle \hat{A} \rangle = \int \psi^* \hat{A}\psi d\tau \quad (\psi \text{ must be normalized})$$

$$= \langle \psi | \hat{A} | \psi \rangle$$

So we have, $\langle \hat{A} \rangle = \int \psi^* \hat{A} \psi \, d\tau$

$$\frac{d}{dt}\langle \hat{A} \rangle = \frac{d}{dt}\int \psi^* \hat{A} \psi \, d\tau$$

or $$\frac{d\langle \hat{A} \rangle}{dt} = \int \frac{\partial \psi^*}{\partial t}\hat{A}\psi \, d\tau + \int \psi^* \frac{\partial \hat{A}}{\partial t}\psi \, d\tau + \int \psi^* \hat{A}\frac{\partial \psi}{\partial t} d\tau$$

or $$\frac{d\langle \hat{A} \rangle}{dt} = \left\langle \frac{\partial \hat{A}}{\partial t}\right\rangle + \frac{i}{\hbar}\int (\psi^* \hat{H}\hat{A}\psi - \psi^* \hat{A}\hat{H}\psi) d\tau$$

or $$\frac{d\langle \hat{A} \rangle}{dt} = \left\langle \frac{\partial \hat{A}}{\partial t}\right\rangle + \frac{i}{\hbar}\int \psi^* (\hat{H}\hat{A} - \hat{A}\hat{H})\psi \, d\tau$$

$$\boxed{\frac{d\langle \hat{A} \rangle}{dt} = \left\langle \frac{\partial \hat{A}}{\partial t}\right\rangle + \frac{i}{\hbar}\langle [\hat{H}, \hat{A}] \rangle}$$

($\hat{A}$ can be any operator)

Poission / Commutator Bracket

$$[\hat{x}, \hat{p}_x]\psi = [\hat{x}, -i\hbar \tfrac{\partial}{\partial x}]\psi$$

$$= x(-i\hbar \tfrac{\partial}{\partial x})\psi - (-i\hbar)\tfrac{\partial}{\partial x}(x\psi)$$

$$= -i\hbar x \frac{\partial \psi}{\partial x} + i\hbar \left(x\frac{\partial \psi}{\partial x} + \psi\right) = i\hbar\psi$$

So $[\hat{x}, \hat{p}_x]\psi = i\hbar\psi \Rightarrow [\hat{x}, \hat{p}_x] = i\hbar$

So $\boxed{[\hat{x}, \hat{p}_x] = i\hbar \{x, p_x\}}$ $\quad \boxed{[\hat{x}_i, \hat{p}_j] = i\hbar \delta_{ij}}$

Poission bracket of $[a, \cos x]$ $\quad a = x + \frac{d}{dx}$

$$[a, \cos x]\psi = [x + \tfrac{d}{dx}, \cos x]\psi$$

$$= \left(x + \frac{d}{dx}\right)\cos x \psi - \cos x\left(x + \frac{d}{dx}\right)\psi$$

$$= x\cos x\psi + \frac{d}{dx}(\psi \cos x) - x\cos x\psi - \cos x\frac{d\psi}{dx}$$

$$= \cancel{x\cos x\psi} - \sin x\psi + \cancel{\cos x\frac{d\psi}{dx}} - \cancel{x\cos x\psi} - \cancel{\cos x\frac{d\psi}{dx}}$$

$$= -\psi \sin x$$

$$[a, \cos x] = -\sin x$$

$$-i\hbar \frac{d}{dx} = \hat{P}_x$$

$$\frac{d}{dx} = \frac{1}{-i\hbar} \hat{P}_x$$

By an another method,

$$[a, \cos x] = [x + \frac{d}{dx}, \cos x]$$

$$= [x - \frac{1}{i\hbar} P_x, \cos x]$$

$$\{x - \frac{1}{i\hbar} P_x, \cos x\} = \{x, \cos x\} - \frac{1}{i\hbar}\{P_x, \cos x\}$$

$$= -\frac{1}{i\hbar}\left\{\frac{\partial P_x}{\partial x}\frac{\partial \cos x}{\partial P_x} - \frac{\partial P_x}{\partial P_x}\cdot\frac{\partial \cos x}{\partial x}\right\}$$

$$= \frac{1}{i\hbar}(-\sin x) \quad \text{so } [x - \frac{1}{i\hbar} P_x, \cos x] = i\hbar\left(\frac{-\sin x}{i\hbar}\right)$$

$$[a, \cos x] = -\sin x$$

■ Commutator bracket of $[x, P_x e^{-P_x}]$

$$\{x, P_x e^{-P_x}\} = \frac{\partial x}{\partial x}\frac{\partial}{\partial P_x}(P_x e^{-P_x}) - \frac{\partial x}{\partial P_x}\frac{\partial P_x e^{-P_x}}{\partial x}$$

$$= \frac{\partial}{\partial P_x}(P_x e^{-P_x}) = e^{-P_x} + P_x e^{-P_x}(-1)$$

$$\{x, P_x e^{-P_x}\} = e^{-P_x}(1 - P_x)$$

$$[x, P_x e^{-P_x}] = i\hbar e^{-P_x}(1 - P_x)$$

> Like to $A = \frac{A + A'}{2} + \frac{A - A'}{2}$ Simmilarly

$$\hat{x}\hat{P}_x = \frac{\hat{x}\hat{P}_x + \hat{P}_x\hat{x}}{2} \quad (\text{symmetrization})$$

> For Position operator

$$\{x, f(x)\} = 0$$

$$\frac{d}{dt}\langle\hat{x}\rangle = \frac{i}{\hbar}\langle[\hat{H}, \hat{x}]\rangle + \langle\cancel{\frac{\partial x}{\partial t}}^{\,0}\rangle$$

$$\frac{d\langle x\rangle}{dt} = -\frac{1}{i\hbar}\langle[\hat{x}, \hat{H}]\rangle$$

$$\{\hat{x}, \hat{H}\}$$

$$= \{x, \frac{P_x^2}{2m} + V(x)\}$$

$$= \{x, \frac{P_x^2}{2m}\} + \{x, V(x)\}$$

$$\{x, H\} = \frac{1}{2m}\left[\frac{\partial x}{\partial x}\frac{\partial P_x^2}{\partial P_x} - \frac{\partial x}{\partial P_x}\cdot\frac{\partial P_x^2}{\partial x}\right] = \frac{P_x}{m}$$

$$\langle[\hat{H}, \hat{x}]\rangle = -\langle[\hat{x}, H]\rangle = i\hbar\left\langle\frac{P_x}{m}\right\rangle$$

$$\frac{d\langle x\rangle}{dt} = \frac{1}{i\hbar}\, i\hbar\left\langle\frac{P_x}{m}\right\rangle \Rightarrow \boxed{\frac{d\langle x\rangle}{dt} = \left\langle\frac{P_x}{m}\right\rangle} \Rightarrow \langle P_x\rangle = m\langle\dot{x}\rangle$$

(classicaly true)

$$\left[\frac{P_x^2}{2m} + \beta x^2, \frac{P_x^2}{m} + \gamma x^2\right] = 0$$ Relation b/w γ, β

$$\left\{\frac{P_x^2}{2m} + \beta x^2, \frac{P_x^2}{m} + \gamma x^2\right\}$$

$$= \frac{\partial}{\partial x}\left(\frac{P_x^2}{2m} + \beta x^2\right)\frac{\partial}{\partial P_x}\left(\frac{P_x^2}{m} + \gamma x^2\right) - \frac{\partial}{\partial P_x}\left(\frac{P_x^2}{2m} + \beta x^2\right)\frac{\partial}{\partial x}\left(\frac{P_x^2}{m} + \gamma x^2\right)$$

$$= 2\beta x \cdot \frac{2P_x}{m} - \frac{P_x}{m}\cdot 2\gamma x = 0 \Rightarrow \gamma = 2\beta$$

$$\boxed{\text{Virial theorem. } \langle T\rangle = \frac{n}{2}\langle V\rangle} \qquad V(r) \propto r^n$$

For a Simple harmonic oscillator,

$$V(x) = \frac{1}{2}m\omega^2 x^2 \propto x^2 \quad (n=2)$$

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega$$

$\boxed{\langle P_x^2\rangle = ?}$

$$\langle T\rangle = \frac{n}{2}\langle V\rangle$$
$$(n = 2)$$
$$\langle T\rangle = \langle V\rangle$$

$$\langle E\rangle = \langle T\rangle + \langle V\rangle$$

$$\hbar\omega\left(n + \frac{1}{2}\right) = 2\langle T\rangle$$

$$\langle T\rangle = \frac{\hbar\omega}{2}\left(n + \frac{1}{2}\right)$$

$$\left\langle\frac{P_x^2}{2m}\right\rangle = \frac{\hbar\omega}{2}\left(n + \frac{1}{2}\right) \Rightarrow \langle P_x^2\rangle = m\hbar\omega\left(n + \frac{1}{2}\right)$$

$\boxed{\langle x^2\rangle = ?}$

$$\langle E\rangle = 2\langle V\rangle$$

$$\left\langle\frac{1}{2}m\omega^2 x^2\right\rangle = \frac{\hbar\omega}{2}\left(n + \frac{1}{2}\right)$$

$$\langle x^2\rangle = \frac{\hbar}{m\omega}\left(n + \frac{1}{2}\right)$$

Hermitian Operator

$A^{\dagger} = A$ (Hermitian matrix)

$\hat{A}^{\dagger} = A$ (Hermitian operator)

$A^{*} = A$ (Functional form)

Physical quantities

$\hat{P}_x, \hat{x}, \hat{L}$ (Real)

(They really exist)

For real number, its complex is itself.

$\hat{P}_x \hat{x}$ is not a physical quantity

(Operators corresponding to Real quantities is always Hermitian)

General defination

Condition to be Hermitian —

$$\boxed{\int \psi^{*} A \psi \, d\tau = \int \psi (A\psi)^{*} d\tau}$$

$$\langle A \rangle = \int \psi^{*} \hat{A} \psi \, d\tau$$

$$\langle A \rangle^{*} = \left(\int \psi^{*} A \psi \, d\tau \right)^{*}$$

$$= \int \psi (A\psi)^{*} d\tau$$

- check $\hat{x}\hat{P}_x$ is Hermitian or Not

$$\hat{a} = \hat{x}\hat{P}_x = \hat{x}\left(-i\hbar \frac{\partial}{\partial x}\right) = -i\hbar \left(x \frac{\partial}{\partial x}\right) = \hat{a}$$

$$\int \psi^{*} \hat{a} \psi \, d\tau = \int \psi^{*} \left(-i\hbar x \frac{\partial}{\partial x}\right) \psi \, d\tau = -i\hbar \int_{-\infty}^{+\infty} \psi^{*} \left(x \frac{\partial \psi}{\partial x}\right) d\tau$$

$$= -i\hbar \left[\psi^{*} x \psi \Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} \left[\frac{d}{dx}(\psi^{*} x) \int \frac{d\psi}{\partial x} dx \right] dx \right.$$

$$= i\hbar \int_{-\infty}^{+\infty} \psi \frac{\partial}{\partial x} (\psi^{*} x) \, dx$$

$$= i\hbar \int_{-\infty}^{+\infty} \psi \left(\frac{\partial \psi^{*}}{\partial x} x + \psi^{*} \frac{\partial x}{\partial x} \right) dx$$

$$= \int_{-\infty}^{+\infty} \psi \, i\hbar \frac{\partial}{\partial x} (\psi^{*} x) \, dx$$

So, $\int \psi^* \hat{a}\psi d\tau = \int \psi \, i\hbar \frac{\partial}{\partial x}(\psi^* x)\, dx$

$\int \psi (a\psi)^* d\tau = \int \psi \left(-i\hbar \frac{\partial \psi}{\partial x}\right)^* dx$

As the operators are not same $\hat{x}\hat{p}_x$ is not Hermitian operator.

■ check $\hat{A} = -i\hbar \frac{\partial}{\partial x}$ Hermitian or not

$$\int \psi^* \hat{A} \psi d\tau = \int \psi^* \left(-i\hbar \frac{\partial}{\partial x}\right)\psi dx = -i\hbar \int_{-\infty}^{+\infty} \psi^* \frac{\partial \psi}{\partial x} dx$$

$$= -i\hbar \left[\psi^* \psi \Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} \left[\frac{\partial \psi^*}{\partial x}\psi\right] dx \right]$$

$$= i\hbar \int_{-\infty}^{+\infty} \psi \frac{\partial \psi^*}{\partial x} dx = \int_{-\infty}^{+\infty} \psi \, i\hbar \frac{\partial \psi^*}{\partial x} dx = \int_{-\infty}^{+\infty} \psi \left(-i\hbar \frac{\partial \psi}{\partial x}\right)^* dx$$

So $\int_{-\infty}^{+\infty} \psi^* \left(-i\hbar \frac{\partial \psi}{\partial x}\right) dx = \int_{-\infty}^{+\infty} \psi \left(-i\hbar \frac{\partial \psi}{\partial x}\right)^* dx$

(So, $\hat{A} = -i\hbar \frac{\partial}{\partial x}$ is Hermitian operator)

> For any real operator $(\hat{p}_x)^\dagger = \hat{p}_x$

$$\left(-i\hbar \frac{\partial}{\partial x}\right)^* = -i\hbar \frac{\partial}{\partial x}$$

$$i\hbar \frac{\partial}{\partial x}^* = -i\hbar \frac{\partial}{\partial x} \Rightarrow \left(\frac{\partial}{\partial x}\right)^* = -\left(\frac{\partial}{\partial x}\right)$$

(Skew Hermitian)

> $\left(\frac{\partial^2}{\partial x^2}\right)^* = \left(\frac{\partial}{\partial x}\frac{\partial}{\partial x}\right)^* = \left(\left(\frac{\partial}{\partial x}\right)^*\left(\frac{\partial}{\partial x}\right)^*\right)$

$$= \left(\left(-\frac{\partial}{\partial x}\right)\left(-\frac{\partial}{\partial x}\right)\right) = \frac{\partial^2}{\partial x^2}$$

$\left(\frac{\partial^2}{\partial x^2}\right)^* = \frac{\partial^2}{\partial x^2}$ (Hermitian operator)

> A, B are Hermitian operator then $[A,B]^\dagger = ?$

$\Rightarrow$ As $\hat{A}, \hat{B}$ are Hermitian operator

$$\hat{A}^\dagger = \hat{A}$$

$$[A,B]^\dagger \qquad \hat{B}^\dagger = \hat{B}$$

$$= [\hat{A}\hat{B} - \hat{B}\hat{A}]^\dagger = (AB)^\dagger - (BA)^\dagger = B^\dagger A^\dagger - A^\dagger B^\dagger$$

$$= \hat{B}\hat{A} - \hat{A}\hat{B} = -(\hat{A}\hat{B} - \hat{B}\hat{A}) = -[A,B]$$

$$[\hat{A},\hat{B}]^\dagger = -[\hat{A},\hat{B}]$$ (Skew Hermitian)

> $\hat{A}$ is skew hermitian operator

$\Rightarrow$ $(i\hat{A})^\dagger = -i\hat{A}^\dagger = i\hat{A}$ $\qquad \hat{A}^\dagger = -\hat{A}$

$\hat{A}$ is skew hermitian then iA is Hermitian operator

If A be hermitian then (iA) is skew Hermitian

> check $i\hbar\left(x^2 \frac{\partial}{\partial x} - \frac{\partial}{\partial x} x^2\right)$ is Hermitian or not

$\Rightarrow$ $i\hbar\left[x^2, \frac{\partial}{\partial x}\right] =$ $\qquad$ (x^2 is real $\Rightarrow$ Hermitian

$i\hbar\frac{\partial}{\partial x}$ is Hermitian)

$= \left[x^2, i\hbar\frac{\partial}{\partial x}\right]$ $\qquad [A,B]^\dagger = -[A,B]$

$\hookrightarrow$ Skew Hermitian $\qquad$ Both are Hermitian

> $[\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$ is Commutator operator

$\{\hat{A},\hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A} \Rightarrow$ is Anticommutator

① If $\{A,B\} = 0$ then $[A,BC]$ is

$\Rightarrow$ $\{A,B\} = \hat{A}\hat{B} + \hat{B}\hat{A} = 0 \Rightarrow \hat{A}\hat{B} = -\hat{B}\hat{A}$

$$[\hat{A},\hat{B}\hat{C}] = B[\hat{A},\hat{C}] + [A,B]\hat{C}$$

$$= B(\hat{A}\hat{C} - \hat{C}\hat{A}) + ABC - BAC$$

$$= -BCA + ABC$$

$$= -BAC - BCA = -B\{AC + CA\} = -B\{A,C\}$$

② Given $\hat{A}\hat{B}-\hat{C}=\hat{I}$. I will unit operator

$A=\frac{d}{dx}$, $\hat{B}=\hat{x}$, $\hat{C}$ is

$\Rightarrow$ $(\hat{A}\hat{B}-\hat{C})\psi=\hat{I}\psi$

$(\frac{d}{dx}x-\hat{C})\psi=\psi$

$\Rightarrow$ $\frac{d}{dx}(x\psi)-\hat{C}\psi=\psi$

$\Rightarrow$ $x\frac{d\psi}{dx}+\psi-\hat{C}\psi=\psi$

$\Rightarrow$ $x\frac{d\psi}{dx}=C\psi$

$\Rightarrow$ $\hat{C}=x\frac{d}{dx}$

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③ If $[\hat{x},\hat{y}]=i\hbar I$ if $\hat{x}=\alpha_{11}\hat{Q}_1+\alpha_{12}\hat{Q}_2$

$\hat{y}=\alpha_{21}\hat{Q}_{21}+\alpha_{22}\hat{Q}_2$ and $[\hat{Q}_1,\hat{Q}_2]=z\hat{I}$

then $\alpha_{11}\alpha_{22}-\alpha_{12}\alpha_{21}=?$

$\Rightarrow$ $[\hat{x},\hat{y}]=i\hbar I$

$[\alpha_{11}\hat{Q}_1+\alpha_{12}\hat{Q}_2,\ \alpha_{21}\hat{Q}_1+\alpha_{22}\hat{Q}_2]=i\hbar I$

$(\alpha_{11}\hat{Q}_1+\alpha_{12}\hat{Q}_2)(\alpha_{21}\hat{Q}_1+\alpha_{22}\hat{Q}_2)-(\alpha_{21}\hat{Q}_1+\alpha_{22}\hat{Q}_2)(\alpha_{11}\hat{Q}_1+\alpha_{12}\hat{Q}_2)$

$\alpha_{11}\alpha_{21}\hat{Q}_1\hat{Q}_1+\alpha_{11}\alpha_{22}\hat{Q}_1\hat{Q}_2+\alpha_{12}\alpha_{21}\hat{Q}_2\hat{Q}_1+\cancel{\alpha_{12}\alpha_{22}\hat{Q}_2\hat{Q}_2}$
$-\alpha_{21}\alpha_{11}\hat{Q}_1\hat{Q}_1-\alpha_{21}\alpha_{12}\hat{Q}_1\hat{Q}_2-\alpha_{22}\alpha_{11}\hat{Q}_2\hat{Q}_1-\cancel{\alpha_{22}\alpha_{12}\hat{Q}_2\hat{Q}_2}=i\hbar I$

or $\alpha_{11}\alpha_{22}(\hat{Q}_1\hat{Q}_2-\hat{Q}_2\hat{Q}_1)+\alpha_{12}\alpha_{21}(\hat{Q}_2\hat{Q}_1-\hat{Q}_1\hat{Q}_2)=i\hbar I$

$\alpha_{11}\alpha_{22}[\hat{Q}_1,\hat{Q}_2]+\alpha_{12}\alpha_{21}[\hat{Q}_2,\hat{Q}_1]=i\hbar I$

$\alpha_{11}\alpha_{22}\,zI+\alpha_{12}\alpha_{21}\cdot(-zI)=i\hbar I$

$zI(\alpha_{11}\alpha_{22}-\alpha_{12}\alpha_{21})=i\hbar I$

$\cancel{z}\ (\alpha_{11}\alpha_{22}-\alpha_{12}\alpha_{21})=\frac{i\hbar}{z}$

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④ $\hat{A}\psi=x^3\psi$ and $\hat{B}\psi=x\frac{d\psi}{dx}$ Find $[\hat{A},\hat{B}]=-3\hbar x^3$

$\Rightarrow$ $\hat{A}=x^3$

$\hat{B}=x\frac{d}{dx}$

$[\hat{A},\hat{B}]=[AB-BA]$

$=(x^4\frac{d}{dx}-x\frac{d}{dx}x^3)\psi$

$=x^4\frac{d\psi}{dx}-x\frac{d}{dx}(\psi x^3)$

$=x^4\frac{d\psi}{dx}-\psi x\cdot 3x^2-x^4\frac{d\psi}{dx}=$

$=-3x^3\psi\ \cancel{\hbar}$

check $\hat{A} = -i\hbar \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}$ is Hermitian or not

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$$\int \psi (\hat{A}\psi)^* d\tau = \int \psi^* A \psi \, d\tau$$

RHS $\int \psi^* A \psi \, d\tau = \int \psi^* \left(-i\hbar \frac{1}{\sin\theta}\frac{d\psi}{d\theta}\right) r^2 \sin\theta \, dr \, d\theta \, d\phi$

$$= -i\hbar \int \psi^* \frac{d\psi}{d\theta} d\theta \, (r^2 dr \, d\phi)$$

$$= -i\hbar \left[\psi^* \psi \Big|_{-\infty}^{+\infty} - \int \frac{d\psi^*}{d\theta} \psi \, d\theta \right] r^2 dr \, d\phi$$

$$= i\hbar \int \psi \frac{d\psi^*}{d\theta} d\theta \, r^2 dr \, d\phi$$

$$= i\hbar \int \psi \frac{\sin\theta}{\sin\theta} \frac{d\psi^*}{dr} r^2 dr \, d\theta \, d\phi$$

$$= i\hbar \int \psi \frac{1}{\sin\theta} \frac{d}{d\theta} \psi^* \, r^2 \sin\theta \, d\theta \, dr \, d\phi$$

$$= \int \psi \left(\frac{i\hbar}{\sin\theta} \frac{d}{d\theta} \psi^* \right) r^2 \sin\theta \, d\theta \, d\phi \, dr$$

$$= \int \psi \left(-i\hbar \frac{1}{\sin\theta} \frac{d}{d\theta} \psi \right)^* r^2 \sin\theta \, dr \, d\theta \, d\phi$$

So $\hat{A} = \left(-i\hbar \frac{1}{\sin\theta}\frac{d}{d\theta}\right)$ is Hermitian operator

Simple Harmonic oscillator : Algebric Method

Hamiltonian of a simple Harmonic oscillator

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 \qquad \hat{p} = \frac{\hat{P}}{\sqrt{m\omega\hbar}}$$

$$= \frac{\hbar\omega}{2} (\hat{p}^2 + \hat{q}^2) \qquad \hat{q} = \sqrt{\frac{m\omega}{\hbar}}\, \hat{X} = \hat{x}$$

$$\hat{a} = \frac{1}{\sqrt{2}} (\hat{x} + i\hat{p}) \quad \text{and} \quad \hat{a}^\dagger = \frac{1}{\sqrt{2}} (\hat{x} - i\hat{p})$$

Now, $\hat{a}\hat{a}^\dagger = \frac{1}{2} (\hat{x}+i\hat{p})(\hat{x}-i\hat{p})$

$$= \frac{1}{2}\left[(x^2+p^2) + i(-\hat{x}\hat{p} + \hat{p}\hat{x})\right]$$

$$= \frac{1}{2}\left[(x^2+p^2) - i(\hat{x}\hat{p} - \hat{p}\hat{x})\right]$$

$$= \frac{1}{2}\left[x^2+p^2 - i[\hat{x},\hat{p}]\right]$$

$$[\hat{x},\hat{p}] = \left[\frac{1}{\sqrt{\frac{\hbar}{m\omega}}}\hat{X}, \frac{1}{\sqrt{m\omega\hbar}}\hat{P}\right] = \frac{1}{\hbar}[\hat{X},\hat{P}]$$

$$[\hat{x},\hat{p}] = \frac{1}{\hbar} i\hbar \Rightarrow [\hat{x},\hat{p}] = i$$

$$\hat{a}\hat{a}^\dagger = \frac{1}{2}(x^2+p^2) + \frac{1}{2}$$

Now $\hat{a}^\dagger\hat{a} = \frac{1}{2}(\hat{x}-i\hat{p})(\hat{x}+i\hat{p}) = \frac{1}{2}\left[(x^2+p^2) + i(\hat{x}\hat{p} - \hat{p}\hat{x})\right)$

$$\hat{a}^\dagger\hat{a} = \frac{1}{2}(x^2+p^2) - \frac{1}{2} \qquad \boxed{[\hat{a},\hat{a}^\dagger] = 1}$$

$$[\hat{a},\hat{a}^\dagger] = \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1$$

$$\frac{1}{2}(x^2+p^2) = \hat{a}^\dagger\hat{a} + \frac{1}{2} \Rightarrow \frac{\hbar\omega}{2}(x^2+p^2) = (\hat{a}^\dagger\hat{a}+\tfrac{1}{2})\hbar\omega$$

$$\Rightarrow \hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a}+\tfrac{1}{2}) = \hbar\omega(\hat{N}+\tfrac{1}{2})$$

($\hat{N} = \hat{a}^\dagger\hat{a}$ = Number operator) $\qquad \boxed{[\hat{a}^\dagger,\hat{a}] = -1}$

For number operator, $\hat{N}|n\rangle = n|n\rangle$

$$\hat{H}|n\rangle = E_n|n\rangle$$

Energy Eigen value. $E_n = (n+\frac{1}{2})\hbar\omega$

$$[\hat{a},\hat{H}] = [\hat{a}, \hbar\omega(\hat{a}^{\dagger}\hat{a}+\tfrac{1}{2})] = \hbar\omega[\hat{a}, \hat{a}^{\dagger}\hat{a}+\tfrac{1}{2}]$$

$$= \hbar\omega[\hat{a},\hat{a}^{\dagger}\hat{a}] = \hbar\omega\{\hat{a}^{\dagger}[\hat{a},\hat{a}] + [\hat{a},\hat{a}^{\dagger}]\hat{a}\} = \hbar\omega\hat{a}$$

$$[\hat{a}^{\dagger},\hat{H}] = [\hat{a}^{\dagger}, \hbar\omega(\hat{a}^{\dagger}\hat{a}+\tfrac{1}{2})] = \hbar\omega[\hat{a}^{\dagger},\hat{a}^{\dagger}\hat{a}]$$

$$= \hbar\omega\{\hat{a}^{\dagger}[\hat{a}^{\dagger},\hat{a}] + [\hat{a}^{\dagger},\hat{a}^{\dagger}]\hat{a}\} = -\hbar\omega\hat{a}^{\dagger}$$

Annihilation & Creation

$$[\hat{a}^{\dagger},\hat{H}] = \hat{a}^{\dagger}\hat{H} - \hat{H}\hat{a}^{\dagger} = -\hbar\omega\hat{a}^{\dagger}$$

$$[\hat{a},\hat{H}] = \hat{a}\hat{H} - \hat{H}\hat{a} = \hbar\omega\hat{a} \Rightarrow \hat{H}\hat{a} = \hat{a}\hat{H} - \hbar\omega\hat{a}$$

$$\hat{H}(\hat{a}|n\rangle) = (\hat{a}\hat{H} - \hbar\omega\hat{a})|n\rangle = (\hat{H}-\hbar\omega)(\hat{a}|n\rangle)$$

$$= (E_n - \hbar\omega)(\hat{a}|n\rangle)$$

$$[\hat{a}^{\dagger},\hat{H}] = \hat{a}^{\dagger}\hat{H} - \hat{H}\hat{a}^{\dagger} = -\hbar\omega\hat{a}^{\dagger} \Rightarrow \hat{H}\hat{a}^{\dagger} = \hat{a}^{\dagger}\hat{H} + \hbar\omega\hat{a}^{\dagger}$$

$$\hat{H}(\hat{a}^{\dagger}|n\rangle) = (\hat{a}^{\dagger}\hat{H} + \hbar\omega\hat{a}^{\dagger})|n\rangle = (E_n + \hbar\omega)(\hat{a}^{\dagger}|n\rangle)$$

$$\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle$$

$$\hat{a}^{\dagger}|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

For energy eigen state $|n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^{\dagger})^n|0\rangle$

Ground State

$$\hat{p} = \frac{\hat{P}}{\sqrt{m\omega\hbar}} = -i\sqrt{\frac{\hbar}{m\omega}}\frac{d}{dx} = -ix_0\frac{d}{dx}$$

$$\hat{x} = \sqrt{\frac{m\omega}{\hbar}}\hat{X} = \frac{\hat{X}}{x_0} \qquad \hat{a} = \frac{1}{\sqrt{2}}(\hat{x}+i\hat{p})$$

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\frac{\hat{X}}{x_0} + i(-i)x_0\frac{d}{dx}\right) = \frac{1}{\sqrt{2}x_0}\left(\hat{X} + x_0^2\frac{d}{dx}\right)$$

$$\langle x|\hat{a}|0\rangle = \frac{1}{\sqrt{2}x_0}\langle x|\hat{X} + x_0^2\frac{d}{dx}|0\rangle$$

$$= \frac{1}{\sqrt{2}x_0}\left[x\psi_0(x) + x_0^2\frac{d\psi_0(x)}{dx}\right] = 0$$

$$x\psi_0(x) = -x_0^2\frac{d\psi_0(x)}{dx}$$

$$x\psi_0(x) = -x_0^2 \frac{d\psi_0(x)}{dx}$$

or $$\int \frac{d\psi_0(x)}{\psi_0(x)} = -\int \frac{x}{x_0^2}\, dx$$

or $$\log \psi_0(x) = \log A e^{-\frac{x^2}{2x_0^2}}$$

$$\psi_0(x) = A e^{-x^2/2x_0^2}$$

From Normalization

$$\int_0^\infty |\psi_0(x)|^2 dx = 1$$

$$A^2 \int_0^\infty e^{-x^2/x_0^2} dx = 1$$

$$A^2 \sqrt{\pi}\, x_0 = 1$$

$$A^2 = \frac{1}{x_0 \sqrt{\pi}}$$

$$\psi_0(x) = \left(\frac{1}{x_0\sqrt{\pi}}\right)^{1/2} e^{-x^2/2x_0^2} \qquad x_0 = \sqrt{\frac{\hbar}{m\omega}}$$

$$\psi_0(x) = \left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}} e^{-\frac{x^2}{2\hbar} m\omega}$$

<u>First Excited state:</u>

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(\frac{\hat{x}}{x_0} - i(-ix_0)\frac{d}{dx}\right)$$

$$= \frac{1}{\sqrt{2}\,x_0}\left(\hat{x} - x_0^2 \frac{d}{dx}\right)$$

$$|n\rangle = \frac{1}{\sqrt{n!}} (\hat{a}^\dagger)^n |0\rangle$$

$$\langle x|1\rangle = \langle x|\hat{a}^\dagger|0\rangle =$$

$$= \frac{1}{\sqrt{2}\,x_0}\left(\hat{x} - x_0^2 \frac{d}{dx}\right)\langle x|0\rangle$$

$$= \frac{1}{\sqrt{2}\,x_0}\left(\hat{x} - x_0^2 \frac{d}{dx}\right)\psi_0(x)$$

$$= \frac{1}{\sqrt{2}\,x_0}\left[x - x_0^2\left(-\frac{x}{x_0^2}\right)\right]\psi_0(x) = \frac{\sqrt{2}}{x_0}\, x\psi_0(x)$$

$$\hat{a} = \frac{1}{\sqrt{2}}(\hat{x} + i\hat{p}) \qquad \hat{a} + \hat{a}^\dagger = \sqrt{2}\,\hat{x}$$

$$\hat{x} = \frac{1}{\sqrt{2}}(\hat{a} + \hat{a}^\dagger)$$

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}}(\hat{x} - i\hat{p})$$

$$\hat{p} = \frac{1}{i\sqrt{2}}(\hat{a} - \hat{a}^\dagger)$$

$$\hat{a} - \hat{a}^\dagger = i\sqrt{2}\,\hat{p}$$

Momentum operator $\boxed{\hat{p} = \frac{1}{i\sqrt{2}}(\hat{a} - \hat{a}^\dagger)}$

Position operator

$$\boxed{\hat{x} = \frac{1}{\sqrt{2}}(\hat{a} + \hat{a}^\dagger)}$$

Heisenberg uncertainity

$$\langle x\rangle = \langle n|\hat{x}|n\rangle = \langle n|\frac{1}{\sqrt{2}}(\hat{a}+\hat{a}^{+})|n\rangle = \frac{1}{\sqrt{2}}\langle n|\hat{a}+\hat{a}^{+}|n\rangle$$

$$= \frac{1}{\sqrt{2}}\left[\langle n|\hat{a}|n\rangle + \langle n|\hat{a}^{+}|n\rangle\right]$$

$$= \frac{1}{\sqrt{2}}\left[\sqrt{n}\,\langle n|n-1\rangle + \sqrt{n+1}\,\langle n|n+1\rangle\right] = 0$$

$$\langle p\rangle = \langle n|\hat{p}|n\rangle = \frac{1}{i\sqrt{2}}\langle n|\hat{a}+\hat{a}^{+}|n\rangle = \frac{1}{\sqrt{2}i}\langle n|\hat{a}-\hat{a}^{+}|n\rangle$$

$$= \frac{1}{i\sqrt{2}}\left[\langle n|\hat{a}|n\rangle - \langle n|\hat{a}^{+}|n\rangle\right]$$

$$= \frac{1}{i\sqrt{2}}\left[\sqrt{n}\,\langle n|n-1\rangle - \sqrt{n+1}\,\langle n|n+1\rangle\right] = 0$$

$$\langle \hat{x}^2\rangle = \langle n|\hat{x}^2|n\rangle = \frac{1}{2}\langle n|\hat{a}^2 + \hat{a}\hat{a}^{+} + \hat{a}^{+}\hat{a} + \hat{a}^{+2}|n\rangle$$

$$= \frac{1}{2}\left[\langle n|\hat{a}^2|n\rangle + \langle n|\hat{a}\hat{a}^{+}|n\rangle + \langle n|\hat{a}^{+}\hat{a}|n\rangle + \langle \hat{a}^{+}|n\rangle\right]$$

$$= \frac{1}{2}\left[\sqrt{n+1}\,\langle n|\hat{a}|n+1\rangle + \sqrt{n}\,\langle n|\hat{a}^{+}|n-1\rangle\right]$$

$$= \frac{1}{2}\left[\sqrt{n+1}\sqrt{n+1}\,\langle n|n\rangle + \sqrt{n}\cdot\sqrt{n}\,\langle n|n\rangle\right]$$

$$= \frac{1}{2}\left[n+1+n\right] = (2n+1)\frac{1}{2} = \left(n+\frac{1}{2}\right)\frac{\hbar}{m\omega}$$

$$\langle p^2\rangle = \langle n|\hat{p}^2|n\rangle = -\frac{1}{2}\langle n|\hat{a}^2 + \hat{a}^{+2} - \hat{a}\hat{a}^{+} - \hat{a}^{+}\hat{a}|n\rangle$$

$$= \frac{1}{2}\langle n|\hat{a}\hat{a}^{+} + \hat{a}^{+}\hat{a}|n\rangle$$

$$= \frac{1}{2}\left[\sqrt{n+1}\,\langle n|\hat{a}|n+1\rangle + \sqrt{n}\,\langle n|\hat{a}^{+}|n-1\rangle\right]$$

$$= \frac{1}{2}\left[\sqrt{n+1}\sqrt{n+1}\,\langle n|n\rangle + \sqrt{n}\sqrt{n}\,\langle n|n\rangle\right] = \left(n+\frac{1}{2}\right)m\omega\hbar$$

$$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\left(n+\frac{1}{2}\right)\frac{\hbar}{m\omega}}$$

$$\Delta p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{\left(n+\frac{1}{2}\right)m\omega\hbar} \qquad \boxed{\Delta x\cdot\Delta p_x = \left(n+\frac{1}{2}\right)\hbar}$$

$$\Delta x\cdot\Delta p_x = \left(n+\frac{1}{2}\right)\frac{\hbar\omega}{\omega} = \left(n+\frac{1}{2}\right)\hbar$$

Matrix form of $\hat{a}$ and $\hat{a}^{+}$

For a $n \times n$ matrix
$$\begin{bmatrix} a_{00} & a_{01} & a_{02} & \cdot & a_{0n} \\ a_{10} & a_{11} & a_{12} & & a_{1n} \\ \vdots & & & & \\ a_{n0} & a_{n1} & a_{n2} & & a_{nn} \end{bmatrix}$$

$\langle 0|a|0\rangle = a_{00}$

$a_{10} = \langle 1|\hat{a}|0\rangle$
$a_{20} = \langle 2|\hat{a}|0\rangle$
$a_{23} = \langle 2|\hat{a}|3\rangle$

$\langle 0|\hat{a}|0\rangle = 0 = a_{00}$; $a_{10} : \langle 1|\hat{a}|0\rangle = 0$.

~~Ea~~ $a_{10} = \langle 0|\hat{a}|1\rangle = \sqrt{1}\, \langle 0|0\rangle = \sqrt{1}$

$a_{12} = \langle 1|\hat{a}|2\rangle = \sqrt{2}\, \langle 1|1\rangle = \sqrt{2}$

$a_{23} = \langle 2|\hat{a}|3\rangle = \sqrt{3}\, \langle 2|2\rangle = \sqrt{3}$

$a_{nm} = \langle n|\hat{a}|m\rangle = \sqrt{m}\, \langle n|m-1\rangle = \sqrt{m}\, \delta_{n,m-1}$

$$\left\{ \begin{array}{ll} \delta_{n,m-1} = 0 & \text{when } n \neq m-1 \\ \quad = 1 & \text{when } n = m-1 \end{array} \right\}$$

$\boxed{\hat{a}_{nm} = \sqrt{m}\, \delta_{n,m-1}}$ $\quad \hat{a}_{56} = \sqrt{6}$, $\hat{a}_{25} = \sqrt{5}\, \delta_{2,4} = 0$

$$\hat{a} = \begin{bmatrix} 0 & \sqrt{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{5} \end{bmatrix}$$

Now $\langle n|\hat{a}^{+}|m\rangle = \sqrt{m+1}\, \langle n|m+1\rangle$
$= \sqrt{m+1}\, \delta_{n,m+1}$

$\boxed{\hat{a}^{+}_{nm} = \sqrt{m+1}\, \delta_{n,m+1}}$

$a_{00} = \langle 0|\hat{a}^\dagger|0\rangle = 0 \qquad a_{10} = \langle 1|\hat{a}^\dagger|0\rangle = \sqrt{1}$

$a_{02} = \langle 0|\hat{a}^\dagger|2\rangle = 0 \qquad a_{32} = \langle 3|\hat{a}^\dagger|2\rangle = \sqrt{3}$

$$\hat{a}^\dagger = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \sqrt{1} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{4} & 0 \end{bmatrix}$$

① Let $|0\rangle$ and $|1\rangle$ denote the normalized eigenstate the uncertainity ΔP, in the state $\frac{1}{\sqrt{2}}[|0\rangle + |1\rangle]$ is

$\Rightarrow$

$$\hat{p} = \sqrt{\frac{m\omega\hbar}{2}}\,\frac{1}{i}\times\frac{1}{2}\left\{[\langle 0| + \langle 1|](\hat{a}-\hat{a}^\dagger)[|0\rangle + |1\rangle]\right\}$$

$$= \frac{1}{2i}\sqrt{\frac{m\omega\hbar}{2}}\left\{\cancel{\langle 0|\hat{a}|0\rangle} + \langle 0|\hat{a}|1\rangle + \langle 0|\hat{a}^\dagger|0\rangle - \langle 0|\hat{a}^\dagger|1\rangle + \cancel{\langle 1|\hat{a}|0\rangle} + \cancel{\langle 1|\hat{a}|1\rangle} \mp \langle 1|\hat{a}^\dagger|0\rangle - \langle 1|\hat{a}^\dagger|1\rangle\right]$$

$$= \frac{1}{2i}\sqrt{\frac{m\omega\hbar}{2}}\left\{\sqrt{1}\,\langle 0|0\rangle - \sqrt{1}\,\langle 0|1\rangle - \sqrt{2}\,\langle 0|2\rangle - \sqrt{1}\,\langle 1|1\rangle\right] = 0$$

$$p^2 = -\frac{m\omega\hbar}{8}\times\frac{1}{2}\left\{[\langle 0| + \langle 1|](\hat{a}^2 - \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} + \hat{a}^{\dagger 2})[|0\rangle + |1\rangle]\right\}$$

$$= +\frac{m\omega\hbar}{16}\left\{[\langle 0| + \langle 1|](\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a})[|0\rangle + |1\rangle]\right\}$$

$$= \frac{m\omega\hbar}{16}\left\{\langle 0|\hat{a}\hat{a}^\dagger|0\rangle + \langle 0|\hat{a}\hat{a}^\dagger|1\rangle + \langle 1|\hat{a}\hat{a}^\dagger|0\rangle + \langle 1|\hat{a}^\dagger\hat{a}|1\rangle\right]$$

$$= \frac{m\omega\hbar}{16}\left\{\sqrt{1}\,\langle 0|\hat{a}|1\rangle + \sqrt{2}\,\langle 0|\hat{a}|2\rangle + \sqrt{1}\,\langle 1|\hat{a}|1\rangle + 1\,\langle 1|\hat{a}^\dagger|0\rangle\right\}$$

$$= \frac{m\omega\hbar}{16}\left\{\sqrt{1}\sqrt{1}\,\langle 0|0\rangle + \sqrt{1}\sqrt{1}\,\langle 1|1\rangle\right\} = \frac{m\omega\hbar}{8}$$

$$\Delta P = \sqrt{m\omega\hbar}$$

③ $x = \sqrt{\frac{\hbar}{2m\omega}}\,(\hat{a} + \hat{a}^{+})$ the matrix representation of first 3 rows and coloms

$\Rightarrow$

$$A = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix} = \sqrt{\frac{\hbar}{2m\omega}} \begin{bmatrix} 0 & \sqrt{1} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{bmatrix}$$

$a_{00} = \langle 0|\hat{a}+\hat{a}^{+}|0\rangle = \langle 0|\hat{a}|0\rangle + \langle 0|\hat{a}^{+}|0\rangle = 0$

$a_{01} = \langle 0|\hat{a}+\hat{a}^{+}|1\rangle = \langle 0|\hat{a}|1\rangle + \langle 0|\hat{a}^{+}|1\rangle$

$= \sqrt{1}\,\langle 0|0\rangle = \sqrt{1}$

$a_{02} = \langle 0|\hat{a}+\hat{a}^{+}|2\rangle = \langle 0|\hat{a}|2\rangle + \langle 0|\hat{a}^{+}|2\rangle$

$= 0$

$a_{10} = \langle 1|\hat{a}|0\rangle + \langle 1|\hat{a}^{+}|0\rangle = 0$

$a_{11} = \langle 1|\hat{a}|1\rangle + \langle 1|\hat{a}^{+}|1\rangle = 0$

$a_{12} = \langle 1|\hat{a}|2\rangle + \langle 1|\hat{a}^{+}|2\rangle = \sqrt{2}\,\langle 1|1\rangle + \sqrt{3}\,\langle 1|3\rangle = \sqrt{2}$

$a_{20} = \langle 2|\hat{a}|0\rangle + \langle 2|\hat{a}^{+}|0\rangle = 0$

$a_{21} = \langle 2|\hat{a}|1\rangle + \langle 2|\hat{a}^{+}|1\rangle = \sqrt{1}\,\langle 2|0\rangle + \sqrt{2}\,\langle 2|2\rangle = \sqrt{2}$

$a_{22} = \langle 2|\hat{a}|2\rangle + \langle 2|a^{+}|2\rangle = 0$

④ Find

$\hat{a}^{+}\hat{a}^{+}\hat{a}\hat{a}|n\rangle$

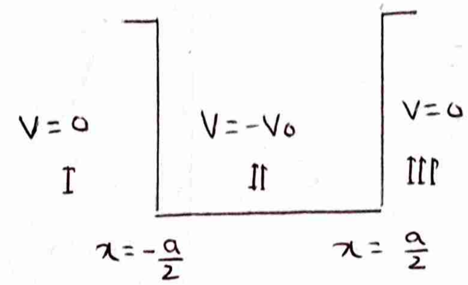
$= \sqrt{n}\,\hat{a}^{+}\hat{a}^{+}\hat{a}|n-1\rangle$

$= \sqrt{n}\sqrt{n-1}\,\hat{a}^{+}\hat{a}^{+}|n-2\rangle = \cancel{\sqrt{n}\sqrt{n-1}}\,\sqrt{n}\,\hat{a}^{+}$

$= \sqrt{n}\sqrt{n-1}\sqrt{n-1}\,\hat{a}^{+}|n-1\rangle = \sqrt{n}\sqrt{n-1}\sqrt{n-1}\sqrt{n}$

$= n(n-1)$

Finite Potential well



From Schrodinger's eqn

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi$$

$$\text{a} \quad -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - V_0\psi = E\psi$$

$$\text{a} \quad \frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + (V_0+E)\psi = 0$$

$$\frac{d^2\psi}{dx^2} + \frac{2m(V_0-E)}{\hbar^2}\psi = 0$$

Wavefunction of Harmonic oscillator

Potential, $V(x) = \frac{1}{2}m\omega^2x^2 = \frac{1}{2}Kx^2$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega^2x^2\psi = E\psi \qquad \alpha^2 = \frac{m\omega}{\hbar}$$

$$\frac{d^2\psi}{dx^2} - \left(\frac{1}{2}m\omega^2x^2 - E\right)\psi = 0$$

$$\psi_n(x) = \left(\frac{\alpha}{2^n\, n!\,\sqrt{\pi}}\right)^{1/2} e^{-\frac{\alpha^2x^2}{2}} H_n(x)$$

(Normalization constant)

$$\boxed{H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}\cdot e^{-x^2}}$$

$H_0(x) = 1$

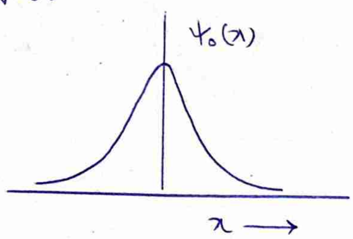
$H_1(x) = 2x$

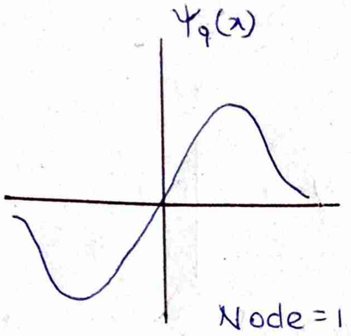
$H_2(x) = 4x^2-2$

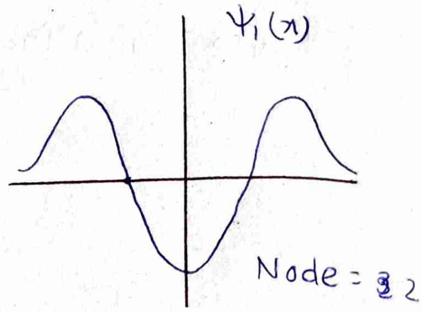
$H_3(x) = 8x^3-12x$

$H_4(x) = 16x^4-48x^2+12$

$$\psi_0(x) = \left(\frac{\alpha}{\sqrt{\pi}}\right)^{1/2} e^{-\frac{\alpha^2x^2}{2}}$$

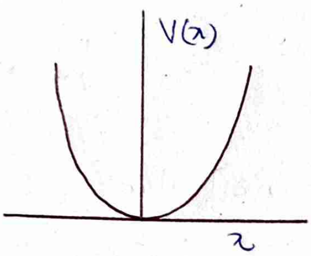






<u>n is odd</u>: Graph will pass through origin

<u>n is even</u>: Symmetric about y-axis
(will not pass through origin)



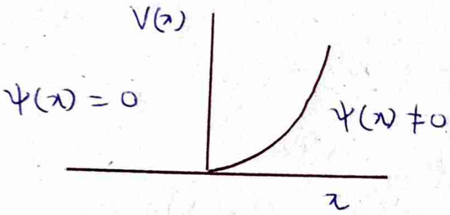
$V(x) = \frac{1}{2} m\omega^2 x^2 \quad -\infty < x < \infty$

$E_n = (n + \frac{1}{2})\hbar\omega \quad n = 0, 1, 2 \ldots$

(Symmetric potential).

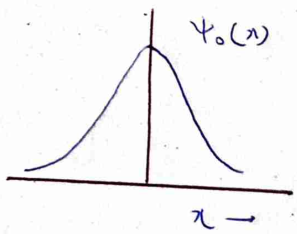
> $V(x) = \frac{1}{2} m\omega^2 x^2 \quad x > 0$
> $\quad\quad = \infty \quad\quad x \leq 0$

Asymmetric potential



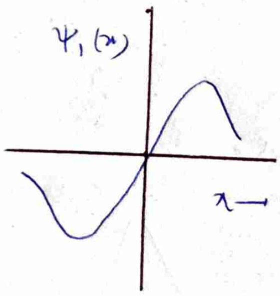
wave function is continuous at $x = 0$, $\psi = 0$

if $\psi(x) \neq 0$ at $x = 0$ the wf is not allowed



$\Rightarrow$ Not allowed

$\psi(0) \neq 0$



$\Rightarrow$ Allowed becaus

$\psi(0) = 0$

So only odd are allowed

Angular Momentum Operator

23.06.2024

$[L^2, L_z] = [L^2, L_x] = [L^2, L_y] = 0$

$[L, H] = 0$ Angular momentum Conserved

$$L_x = i\hbar\left[\cot\theta\cos\phi\frac{\partial}{\partial\phi} + \sin\phi\frac{\partial}{\partial\theta}\right]$$

$$L_y = i\hbar\left[-\cos\phi\frac{\partial}{\partial\theta} + \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right]$$

$$L_z = -i\hbar\frac{\partial}{\partial\phi}$$

$[J^2, J_z] = 0$

λ give EV of J^2

m give EV of J_z

$|\lambda, m\rangle$

$\Downarrow$

Eigen state of both J^2 & J_z

If $[A, B] = 0$

$A|\psi\rangle = a|\psi\rangle$

$B|\psi\rangle = b|\psi\rangle$

A, B has Simultaneous Eigen state

But Eigen value is not same may be different

$J^2|\lambda, m\rangle = \lambda^2\hbar^2|\lambda, m\rangle$

$J_z|\lambda, m\rangle = m\hbar|\lambda, m\rangle$

$J_+ = J_x + iJ_y$

$J_- = J_x - iJ_y$

So, $J_+^\dagger = J_-$

$\boxed{[J_z, J_+] = \hbar J_+}$

$$\begin{aligned}[J_z, J_+] &= [J_z, J_x + iJ_y] \\ &= [J_z, J_x] + i[J_z, J_y] \\ &= i\hbar J_y + \hbar J_x \\ &= \hbar(J_x + iJ_y) = \hbar J_+\end{aligned}$$

$\boxed{[J_z, J_-] = -\hbar J_-}$

$$\begin{aligned}[J_z, J_-] &= [J_z, J_x - iJ_y] \\ &= [J_z, J_x] - i[J_z, J_y] = -\hbar J_-\end{aligned}$$

$[J_+, J_-] = 2\hbar J_z$

$$\begin{aligned}J_+J_- &= (J_x + iJ_y)(J_x - iJ_y) \\ &= J_x^2 - iJ_xJ_y + iJ_yJ_x + J_y^2 = J_x^2 + J_y^2 + \hbar J_z\end{aligned}$$

$$\begin{aligned}J_-J_+ &= (J_x - iJ_y)(J_x + iJ_y) \\ &= J_x^2 + iJ_xJ_y - iJ_yJ_x + J_y^2 \\ &= J_x^2 + J_y^2 + i(J_xJ_y - J_yJ_x) = J_x^2 + J_y^2 - \hbar J_z\end{aligned}$$

$J_x^2 + J_y^2 + J_z^2 = J^2$

$J_x^2 + J_y^2 = J^2 - J_z^2$

$[J_+, J_-] = J_+J_- - J_-J_z$

$= \hbar J_z + \hbar J_z = 2\hbar J_z$

so $\boxed{[J_+ J_-] = 2\hbar J_z}$

$J_+J_- = J_x^2 + J_y^2 + \hbar J_z = J^2 - J_z^2 + \hbar J_z$

$J_- J_+ = J_x^2 + J_y^2 - \hbar J_z = J^2 - J_z^2 - \hbar J_z$

$J_+J_- + J_-J_+ = 2(J_x^2 + J_y^2)$

$$\boxed{(J_x^2 + J_y^2) = \frac{1}{2}(J_+J_- + J_-J_+) = (J^2 - J_z^2)}$$

$(J_x^2 + J_y^2)|\lambda, m\rangle \geq 0$

$(J^2 - J_z^2)|\lambda, m\rangle \geq 0$

$\Rightarrow \quad J^2|\lambda, m\rangle - J_z^2|\lambda, m\rangle \geq 0$

$\Rightarrow \quad \lambda^2\hbar^2|\lambda, m\rangle - m^2\hbar^2|\lambda, m\rangle \geq 0$

$\Rightarrow \quad (\lambda^2\hbar^2 - m^2\hbar^2)|\lambda, m\rangle \geq 0 \Rightarrow \lambda^2 \geq m^2 \Rightarrow \boxed{\lambda \geq m}$

$J_z|\lambda, m\rangle = m\hbar|\lambda, m\rangle$

$[J_z, J_+] = \hbar J_+$

or $J_zJ_+ - J_+J_z = \hbar J_+$

or $J_zJ_+ = \hbar J_+ + J_+J_z$

or

$J_zJ_+|\lambda, m\rangle$

$= (\hbar J_+ + J_+J_z)|\lambda, m\rangle$

$= J_+[\hbar + J_z]|\lambda, m\rangle$

$= J_+[\hbar + m\hbar]|\lambda, m\rangle$

$= J_+\hbar(m+1)|\lambda, m\rangle$

$$\boxed{J_zJ_+|\lambda, m\rangle = J_+\hbar(m+1)|\lambda, m\rangle}$$

$$\boxed{J_+|\lambda, m\rangle = c|\lambda, m+1\rangle}$$

$$\boxed{J_-|\lambda, m\rangle = c_1|\lambda, m-1\rangle}$$

$J_+ |\lambda, m\rangle = c|\lambda, m+1\rangle$ Ket $|m\rangle^*$ has Complex

$\langle \lambda, m| J_- = c^* \langle \lambda, m+1|$ Conjugate $\langle m|$ (Bra)

$\langle \lambda, m| J_- J_+ |\lambda, m\rangle = |c|^2 \langle \lambda, m+1|\lambda, m+1\rangle$

or $\langle \lambda, m| J^2 - J_z^2 - \hbar J_z |\lambda, m\rangle = |c|^2$

or $\langle \lambda, m| J^2 |\lambda, m\rangle - \langle \lambda, m| J_z^2 |\lambda, m\rangle - \langle \lambda, m| \hbar J_z |\lambda, m\rangle = |c|^2$

or $\lambda^2 \hbar^2 \langle \lambda, m|\lambda, m\rangle - m^2\hbar^2 \langle \lambda, m|\lambda, m\rangle - \hbar m \hbar \langle \lambda, m|\lambda, m\rangle = |c|^2$

or $\lambda^2\hbar^2 - m^2\hbar^2 - \hbar^2 m = |c|^2$

or $c = \hbar\sqrt{\lambda^2 - m(m+1)}$ $\boxed{c = \hbar\sqrt{\lambda^2 - m(m+1)}}$

$J_- |\lambda, m\rangle = c_1 |\lambda, m-1\rangle$

$\langle \lambda, m| J_+ = c_1^* |\lambda, m-1|$

$\langle \lambda, m| J_+ J_+ |\lambda, m\rangle = |c_1|^2 \langle \lambda, m-1|\lambda, m-1\rangle$

$\langle \lambda, m| J^2 - J_z^2 + \hbar J_z |\lambda, m\rangle = |c_1|^2$

$\langle \lambda, m| J^2 |\lambda, m\rangle - \langle \lambda, m| J_z^2 |\lambda, m\rangle + \hbar \langle \lambda, m| J_z |\lambda, m\rangle = |c_1|^2$

$(\lambda^2\hbar^2 - m^2\hbar^2 + m\hbar^2) \langle \lambda, m|\lambda, m\rangle = |c_1|^2$

$(\lambda^2 - m^2 + m)\hbar^2 = |c_1|^2$

$\lambda^2 - m(m-1)\hbar^2 = |c_1|^2$ $\boxed{c_1 = \hbar\sqrt{\lambda^2 - m(m-1)}}$

As we have, $\lambda \geq m$

Let J be the max value of m

$J_+ |\lambda, J\rangle = 0$ (Not $|\lambda, J+1\rangle$)

$J_z J_+ |\lambda, J\rangle = 0$ $J_- J_+ |\lambda, J\rangle = 0$

$J^2 - J_z^2 - \hbar J_z |\lambda, J\rangle = 0$

So $J^2 |\lambda, J\rangle - J_z^2 |\lambda, J\rangle - \hbar J_z |\lambda, J\rangle = 0$

or $(\lambda^2\hbar^2 - J^2\hbar^2 - \hbar^2 J)|\lambda, J\rangle = 0$

or $\lambda^2 - J^2 - J = 0$

or $\lambda^2 = J(J+1)$

Here J is max value of m

$\boxed{\lambda^2 = j(j+1)}$

$$J_+|\lambda, m\rangle = \sqrt{\lambda^2 - m(m+1)}\,\hbar\,|\lambda, m+1\rangle$$

$$\boxed{J_+|\lambda, m\rangle = \sqrt{J(J+1) - m(m+1)}\,\hbar\,|J, m+1\rangle}$$

$$\boxed{J_-|\lambda, m\rangle = \sqrt{J(J+1) - m(m-1)}\,\hbar\,|J, m+1\rangle}$$

Let J' be the minimum value of m

so $\quad J_-|\lambda, J'\rangle = 0 \qquad J_+J_-|\lambda, J'\rangle = 0$

$$(J^2 - J_z^2 + \hbar J_z)|\lambda, J'\rangle = 0$$

or $\quad J^2|\lambda, J'\rangle - J_z^2|\lambda, J'\rangle + \hbar J_z|\lambda, J'\rangle = 0$

or $\quad \lambda^2\hbar^2|\lambda, J'\rangle - J'^2\hbar^2|\lambda, J'\rangle + \hbar^2 J'|\lambda, J'\rangle = 0$

or $\quad (\lambda^2\hbar^2 - J'^2\hbar^2 + \hbar^2 J')|\lambda, J'\rangle = 0$

or $\quad \lambda^2 = J'(J'-1) \qquad \boxed{\lambda^2 = J'(J'-1)}$

But λ is a constant so,

$$J(J+1) = J'(J'-1)$$

By solving $\quad J' = -J, \quad J' = J+1$

so $\quad \boxed{J' = -J}$

Min can't be greater than maximum so $J' \neq J+1$

so m can take value from $-J$ to $+J$

$(2J+1)$ No of values

$$\boxed{\hat{J}^2|j, m\rangle = j(j+1)\hbar^2|j, m\rangle} \qquad -j \leq m \leq j$$

$$\boxed{\hat{J}_z|j.m\rangle = m\hbar|j.m\rangle}$$

$$J_x|J, m\rangle = \frac{1}{2}(J_+ + J_-)|J, m\rangle$$

$$J_y|J, m\rangle = \frac{1}{2i}(J_+ - J_-)|J, m\rangle$$

$$j = \frac{1}{2} \qquad m = -\frac{1}{2}, \frac{1}{2}$$

$$|1\rangle = \left|\frac{1}{2}, -\frac{1}{2}\right\rangle \qquad J_x = \begin{bmatrix} \langle 1|J_x|1\rangle & \langle 1|J_x|2\rangle \\ \langle 2|J_x|1\rangle & \langle 2|J_x|2\rangle \end{bmatrix}$$

$$|2\rangle = \left|\frac{1}{2}, \frac{1}{2}\right\rangle$$

$$\langle 1|J_x|1\rangle = \frac{1}{2}\left\langle \frac{1}{2}, -\frac{1}{2}\right| J_+ + J_- \left|\frac{1}{2}, -\frac{1}{2}\right\rangle$$

$$= \frac{1}{2}\left[\left\langle \frac{1}{2}, -\frac{1}{2}\right| J_+ \left|\frac{1}{2}, -\frac{1}{2}\right\rangle + \left\langle \frac{1}{2} -\frac{1}{2}\right| J_- \left|\frac{1}{2}, -\frac{1}{2}\right\rangle\right] = 0$$

$$\langle 1|J_x|2\rangle = \frac{1}{2}\left\langle \frac{1}{2} -\frac{1}{2}\right| J_+ + J_- \left|\frac{1}{2} \frac{1}{2}\right\rangle$$

$$= \frac{1}{2}\left\langle \frac{1}{2} -\frac{1}{2}\right| J_- \left|\frac{1}{2} \frac{1}{2}\right\rangle$$

$$= \frac{1}{2}\left[\left\langle \frac{1}{2} -\frac{1}{2}\right| \hbar\sqrt{\frac{1}{2}\left(\frac{1}{2}+1\right) - \frac{1}{2}\left(\frac{1}{2}-1\right)} \left|\frac{1}{2} -\frac{1}{2}\right\rangle\right]$$

$$= \frac{1}{2}\hbar \left\langle \frac{1}{2} -\frac{1}{2} \middle| \frac{1}{2} -\frac{1}{2}\right\rangle = \frac{\hbar}{2}$$

$$\langle 2|J_x|1\rangle = \frac{1}{2}\left\langle \frac{1}{2} \frac{1}{2}\right| J_+ + J_- \left|\frac{1}{2} -\frac{1}{2}\right\rangle$$

$$= \frac{1}{2}\left\langle \frac{1}{2} \frac{1}{2}\right| J_+ \left|\frac{1}{2}, -\frac{1}{2}\right\rangle$$

$$= \frac{1}{2}\left\langle \frac{1}{2} \frac{1}{2}\right| \hbar\sqrt{\frac{1}{2}\left(\frac{1}{2}+1\right) - \left(-\frac{1}{2}\right)\left(-\frac{1}{2}+1\right)} \left|\frac{1}{2} \frac{1}{2}\right\rangle$$

$$= \frac{\hbar}{2}$$

So $$J_x = \begin{bmatrix} 0 & \hbar/2 \\ \hbar/2 & 0 \end{bmatrix} = \hbar/2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$J_y = \begin{bmatrix} \langle 1|J_y|1\rangle & \langle 1|J_y|2\rangle \\ \langle 2|J_y|1\rangle & \langle 2|J_y|2\rangle \end{bmatrix}$$

$$\langle 1|J_y|2\rangle = \frac{1}{2i}\left\langle \frac{1}{2} -\frac{1}{2}\right| J_+ - J_- \left|\frac{1}{2} \frac{1}{2}\right\rangle \qquad J_y = \begin{bmatrix} 0 & i\hbar/2 \\ i\hbar/2 & 0 \end{bmatrix}$$

$$= -\frac{1}{2i}\left\langle \frac{1}{2} -\frac{1}{2}\right| J_- \left|\frac{1}{2} \frac{1}{2}\right\rangle$$

$$= -\frac{\hbar}{2i}\left[\left\langle \frac{1}{2} -\frac{1}{2}\right| \sqrt{\frac{1}{2}\left(\frac{1}{2}+1\right) - \frac{1}{2}\left(\frac{1}{2}-1\right)} \left|\frac{1}{2} -\frac{1}{2}\right\rangle\right] = \frac{i\hbar}{2}$$

$$\langle 2|J_y|1\rangle = \frac{1}{2i}\left\langle \frac{1}{2} \frac{1}{2}\right| J_+ - J_- \left|\frac{1}{2} -\frac{1}{2}\right\rangle$$

$$= -\frac{1}{2i}\left\langle \frac{1}{2} \frac{1}{2}\right| J_+ \left|\frac{1}{2} -\frac{1}{2}\right\rangle$$

$$= -\frac{1}{2i}\left[\left\langle \frac{1}{2} \frac{1}{2}\right| \hbar\sqrt{\frac{1}{2}\left(\frac{1}{2}+1\right) - \left(-\frac{1}{2}\right)\left(-\frac{1}{2}+1\right)} \left|\frac{1}{2} \frac{1}{2}\right\rangle = \frac{i\hbar}{2}\right.$$

$$J_z = \begin{bmatrix} \langle 1|J_z|1\rangle & \langle 1|J_z|2\rangle \\ \langle 2|J_z|1\rangle & \langle 2|J_z|2\rangle \end{bmatrix} = \begin{bmatrix} -\hbar/2 & 0 \\ 0 & \hbar/2 \end{bmatrix}$$

$$J_x = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \frac{\hbar}{2}\sigma_x$$

$$J_y = \frac{\hbar}{2}\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = \frac{\hbar}{2}\sigma_y$$

$$J_z = \frac{\hbar}{2}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{\hbar}{2}\sigma_z$$

05.04.2024

Black-body Radiation

① **white body:** Every radiation gets reflected.

② **Black body:** Every radiation get absorbed.

Substances at all finite temperature radiate energy and the amount increases with increases increase temperature and not affected by surrounding objects.

> **Theory of Exchange:**

All objects emit radiation at all temperature amount of radiation depends on material & temperature

All object absorb radiation at all temperature It depends on material and temperature.

Surrounding temperature > Body temperature

So, Absorption > Emission

In Equillibrium absorption = Emission

Q is the emitted heat then

$$Q = EAt$$

t is time

A is area

E is emissive power

$$\boxed{E = Q/At}$$

Spectral emissive Power is $E_\lambda d\lambda$ & $E = \int_0^\infty E_\lambda d\lambda$

> **Absorptive Power:**

$$a = \frac{\text{Heat absorbed by body}}{\text{Heat incident on the body}}$$

$$(0 \leq a \leq 1)$$

For ideal black body $a = 1$

"Good absorbers are always good emitters"

Absorptivity + Refectivity + Transmitivity = 1

Stefan's Boltzman Law: (T^4 Law)

$\frac{dQ}{dt}$ is directly proportional to A
to T^4

$$\frac{dQ}{dt} \propto AT^4 \Rightarrow \frac{dQ}{dt} = \sigma AT^4$$

(For perfect black body)

$$\boxed{\frac{dQ}{dt} = \epsilon \sigma AT^4}$$

ϵ is emitivity
(for Black body $\epsilon = 1$)

σ is stefan's Constant

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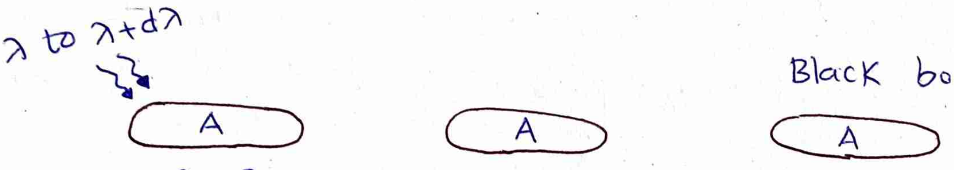
Kirchoff's Law:

In thermal eqbm

Good absorbers are also good emitter

Black body



a_1, e_1 a_2, e_2 $a = 1, e_\lambda$

$a_1 Q = E_1 At$ $a_2 Q = E_2 At$ $Q = E_3 At$

$E \propto e \rightarrow$ emisivity

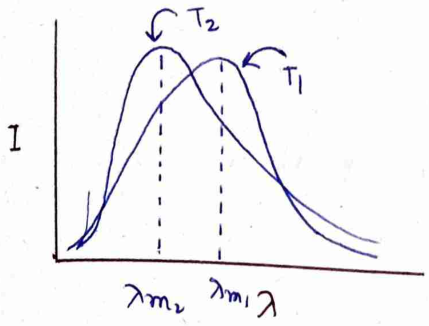
$$\boxed{\frac{a_1}{E_1} = \frac{a_2}{E_2} = \frac{1}{E_3}}$$

$$\frac{a_1}{\epsilon_1} = \frac{a_2}{\epsilon_2} = \frac{1}{\epsilon_b}$$

Emisivity $\propto$ Absorptivity ie $\boxed{\epsilon \propto a}$

Wein's displacement law:

Based on experiment, $\lambda_m T = \text{Constant}$



λ_m is the wavelength for which intensity is maximum

If Temperature increase λ_m shifted to left

Wein's distribution law:

Total energy $E = \int E_\lambda d\lambda$

E_λ is spectral energy density.

$$\boxed{E_\lambda d\lambda \propto \lambda^{-5} e^{-a\lambda T} d\lambda}$$

Rayleigh Jeans law:

Radiate energy is considered as standing wave

No of nodes $= \frac{8\pi}{\lambda^4} d\lambda$

Mode energy $= KT$

$$E_\lambda d\lambda = \frac{8\pi}{\lambda^4} KT d\lambda$$

Density of state:

$E = PC$

Here quantum concept is use like Heisenberg uncertainity Relation

$$E_\nu d\nu = \frac{d^3P d^3x}{h^3} = \frac{4\pi P^2 dP V}{h^3} = \frac{4\pi}{h^3}\left(\frac{h\nu}{c}\right)^2 \frac{h}{c} d\nu \; V$$

$$= \frac{4\pi}{h^3 c^3} h^3 \nu^2 d\nu \; V$$

$$E_\nu d\nu = \frac{8\pi\nu^2}{c^3} d\nu \; V$$ (EMW has 2 Polarization state)

$$\frac{E_\nu d\nu}{V} = \frac{8\pi\nu^2}{c^3} d\nu \; K_B T$$

or $$u_\nu d\nu = \left(\frac{8\pi\nu^2}{c^3} K_B T\right) d\nu$$

From Maxwell Boltzman distribution

$$N(E) = A e^{-E/K_B T}$$

$$\langle E \rangle = \frac{\sum E_i e^{-E_i/K_B T}}{\sum e^{-E_i/K_B T}}$$

or $$\langle E \rangle = \frac{h\nu e^{-\frac{h\nu}{KT}} + 2h\nu e^{\frac{-2h\nu}{KT}} + 3h\nu e^{\frac{-3h\nu}{KT}} + \cdots}{e^{-\frac{h\nu}{KT}} + e^{-2h\nu/KT} + e^{-3h\nu/KT}} = \frac{h\nu}{e^{h\nu/K_B T} - 1}$$

By Plank.

$$u_\nu d\nu = \left(\frac{8\pi\nu^2 d\nu}{c^3}\right) \frac{h\nu}{e^{h\nu/K_B T} - 1}$$

$$u_\lambda d\lambda = \frac{8\pi hc}{\lambda^5} \cdot \frac{1}{e^{hc/\lambda KT} - 1}$$

If $\lambda \to 0$, $\nu \to 0$ then $u_\lambda d\lambda \to 0$

(UV Catorphe)

- If $\frac{hc}{\lambda K_B T} << 1$ then $\lambda >> \frac{hc}{K_B T}$

$$e^{\frac{hc}{\lambda K_B T}} = 1 + \frac{hc}{\lambda K_B T}$$

$$u_\lambda d\lambda = \frac{8\pi hc}{\lambda^5} \times \frac{\lambda K_B T}{hc} = \frac{8\pi}{\lambda^4} K_B T d\lambda$$

$u_\lambda d\lambda = \frac{8\pi}{\lambda^4} K_B T d\lambda \Rightarrow$ Again Rayleigh Jeans law

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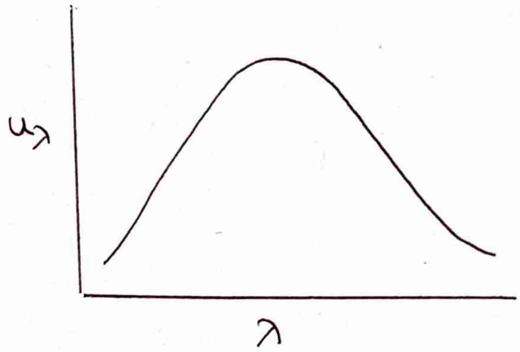
① Rayleigh Jeans law is invalid in lower wavelength or higher frequency.



② $\lambda >> \frac{hc}{K_B T}$ Here Rayleigh Jeans is follow

③ $\lambda << \frac{hc}{K_B T}$ then. $\frac{hc}{\lambda K_B T} >> 1$

$$u_\lambda d\lambda = \frac{8\pi hc}{\lambda^5} \times \frac{d\lambda}{e^{hc/\lambda KT}}$$

$$u_\lambda d\lambda = c\lambda^{-5} e^{-hc/\lambda KT} d\lambda \Rightarrow \text{Spectral Energy density.}$$

At lower wavelength wein's displacement law is Valid.

$$u = \int_0^\infty u_\lambda d\lambda = \int_0^\infty \frac{8\pi hc}{\lambda^5} \cdot \frac{1}{e^{hc/\lambda K_B} - 1} d\lambda$$

put $\frac{hc}{\lambda K_B T} = x \Rightarrow \lambda = \frac{hc}{x K_B T}$

$$\Rightarrow d\lambda = \frac{-hc}{K_B T} \cdot \frac{1}{x^2} dx$$

$$u = -\int_\infty^0 (8\pi hc) \frac{(hc)^{-5}}{x^{-5}(KT)^{-5}} \frac{1}{e^x - 1} \frac{hc}{x^2 K_B} dx$$

$$u = \frac{8\pi hc (KT)^5}{(hc)^5} \frac{hc}{KT} \int_0^\infty \frac{x^5}{x^2(e^x - 1)} dx$$

or $$u = \frac{8\pi(KT)^4}{(hc)^3} \int_0^\infty \frac{x^3}{(e^x - 1)} dx$$

$\Downarrow$

Reimann-zeta function $\Rightarrow \frac{\pi^4}{15}$

or $$u = \frac{8\pi (KT)^4}{(hc)^3} \cdot \frac{\pi^4}{15}$$

or $$u = \left[\frac{8\pi^5}{(hc)^3} \times \frac{K^4}{15}\right] T^4 \Rightarrow u = \sigma T^4$$

Here $\sigma = \frac{8\pi^5 K^4 \times 4}{(hc)^3 \times 15}$ = stefan's Constant

$$E = \frac{cu}{4} = \frac{c}{4} \times \frac{8}{15} \frac{\pi^5 K^4}{(hc)^3} \times T^4$$

$$E = \frac{2}{15} \frac{\pi^5 K^4}{(hc)^3} \times T^4$$

Here $\sigma = \frac{2}{15} \frac{\pi^5 K^4}{(hc)^3} = 5.67 \times 10^{-8}$

- $$u_\lambda = \frac{8\pi hc}{\lambda^5 (e^{hc/\lambda KT} - 1)}$$

$$\frac{du_\lambda}{d\lambda} = 8\pi hc \left[\frac{1}{\lambda^5} \cdot \frac{-e^{\frac{hc}{\lambda KT}}\left(-\frac{hc}{\lambda^2 KT}\right)}{(e^{hc/\lambda KT} - 1)^2} - \frac{5}{\lambda^6} \frac{1}{e^{hc/\lambda KT} - 1}\right]$$

For u_λ be maximum if $\frac{du_\lambda}{d\lambda} = 0$

$$\frac{1}{\lambda^7} \frac{e^{hc/\lambda KT}\, hc}{(e^{hc/\lambda KT} - 1)^2} \cdot \frac{1}{\lambda KT} = +\frac{5}{\lambda^6} \frac{1}{(e^{hc/\lambda KT} - 1)}$$

$$\frac{hc}{\lambda KT} \cdot e^{\frac{hc}{\lambda KT}} \frac{1}{e^{hc/\lambda KT} - 1} = 5$$

$$\frac{xe^x}{e^x - 1} = 5 \qquad \text{put } \frac{hc}{\lambda KT} = x$$

$$\frac{xe^x}{e^x-1} = 5$$

or $xe^x = 5e^x - 5$

or $e^x(x-5) = -5$

or $e^x = \frac{5}{5-x}$

or $x = \ln(5) - \ln(5-x)$

$y_1 = x$

$y_2 = \ln 5 - \ln(5-x)$

Solution will be

$(4.965, 4.965)$

$x = 4.965$

$$\frac{hc}{\lambda KT} = 4.965 \Rightarrow \lambda_{max} T = \frac{hc}{K_B \times 4.965}$$

$$\Rightarrow \lambda_{max} T = 2.98 \times 10^{-3}$$