Central Force Motion

Force depends on the distance.

$$\vec{F}(r) = \pm f(r)\hat{r}$$

Gravitational force $f(r) = \frac{Gm_1m_2}{r^2}$

Electrostatic force $f(r) = \frac{Kq_1q_2}{r^2}$

Spring force $f(x) = -Kx$

Now $\vec{\nabla}\times\vec{F}(r) = \pm\,\vec{\nabla}\times\left[\frac{f(r)}{r}\vec{r}\right]$

$$= \pm\left\{\frac{f(r)}{r}(\vec{\nabla}\times\vec{r}) + \vec{\nabla}\left(\frac{f(r)}{r}\right)\times\vec{r}\right\}$$

$$= \pm\left\{\frac{1}{r}\vec{\nabla}f(r)\times\vec{r} + \vec{\nabla}\left(\frac{1}{r}\right)f(r)\times\vec{r}\right\} = 0$$

As $\vec{\nabla}\times\vec{F} = 0$ (Irrotational)

$\oint \vec{F}\cdot d\vec{r} = 0$ (Conservative force field)

Torque, $\vec{\tau} = \vec{r}\times\vec{F} = \vec{r}\times f(r)\hat{r} = 0$

So, $\vec{\tau} = \frac{d\vec{L}}{dt} = 0 \Rightarrow \vec{L} = \text{constant} = mr^2\dot{\theta}$

① Effective Potential:

Acceleration in polar (r,θ) coordinate

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$$

$$\vec{F} = m\vec{a} = m(\ddot{r} - r\dot{\theta}^2)\hat{r} + m(r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta} = f(r)\hat{r}$$

So, $2m\dot{r}\dot{\theta} + mr\ddot{\theta} = 0$

$$\frac{d}{dt}(mr^2\dot{\theta}) = 0 \Rightarrow mr^2\dot{\theta} = \text{Constant}$$

So $L = mr^2\dot{\theta}$ = Angular momentum Conserved.

So basically we get. $f(r) = m(\ddot{r} - r\dot{\theta}^2)$

Total energy, $E = KE + PE$

$$= \frac{1}{2}mv^2 + V(r)$$

$$= \frac{1}{2}m(v_r^2 + v_\theta^2) + V(r)$$

$$E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + V(r)$$

$$= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\frac{L^2}{m^2r^4} + V(r)$$

$$= \frac{1}{2}m\dot{r}^2 + \underbrace{\frac{L^2}{2mr^2} + V(r)}$$

Totally depends on r so it is effective potential (V_{eff})

$$= \frac{1}{2}m\dot{r}^2 + V_{eff}$$

$$\boxed{V_{eff} = \frac{L^2}{2mr^2} + V(r)}$$

Effective potential

② <u>Stability Analysis</u>

For a given potential $V(r)$

(a) if $V(r) = 0$: points where curve cute the x-axis

(b) Get $r \to \pm\infty$ and $r = 0$

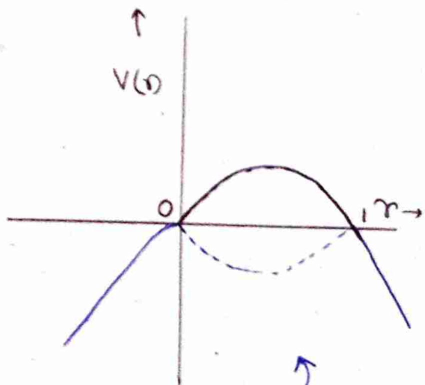
<u>Let $V(r) = r - r^2$</u>

$r \to \infty$. $V(\infty) = \infty - \infty^2 = -\infty$

$r \to -\infty$ $V(-\infty) = -\infty - \infty^2 = -\infty$

$V(r) = 0$ $r = 0, 1$

if $V(0.5) = \frac{1}{2} - \frac{1}{4} = +\frac{1}{4} > 0$



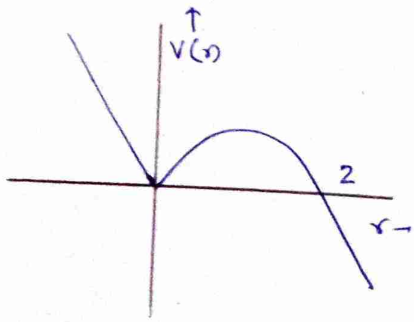
Rough estimated

<u>Let $V(r) = \frac{r^2}{2} - \frac{r^3}{6}$</u>

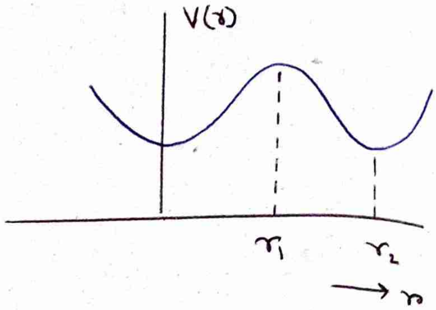
$r \to \infty$ $V(\infty) = -\infty$

$r \to -\infty$ $V(-\infty) + \infty$

$r = 0, 2$ at $V(r) = 0$



(ii) Maxima & Minima :



At local maximum:

unstable eqb^m point

$$\frac{\partial V}{\partial r} = 0 \quad \& \quad \frac{\partial^2 V}{\partial x^2} < 0$$

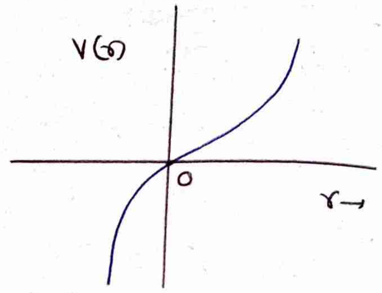
(Here $r = r_1$)

At local minima:

Stable eqb^m point

$$\frac{\partial V}{\partial r} = 0 \quad \& \quad \frac{\partial^2 V}{\partial x^2} > 0$$

(Here $r = r_2$)



(point o is Saddle point)

① For the given potential $V(x) = K(x^2 - x^4)$

(a) Find eqb^m points

(b) Max^m & Min^m

(c) Plot the graph

⇒ (a) $V(x) = K(x^2 - x^4)$

$$\frac{\partial V}{\partial x} = 0 \Rightarrow 2x - 4x^3 = 0 \Rightarrow x = 0, \pm \frac{1}{\sqrt{2}}$$

(b) $$\frac{\partial^2 V}{\partial x^2} = K(2 - 12x^2) \qquad \left.\frac{\partial^2 V}{\partial x^2}\right|_{x=0} = 2K > 0 \quad \text{(Minima)}$$

$$\left.\frac{\partial^2 V}{\partial x^2}\right|_{x=\pm\frac{1}{\sqrt{2}}} = K\left(2 - \frac{12}{2}\right) = -4K < 0 \quad \text{(Maxima)}$$

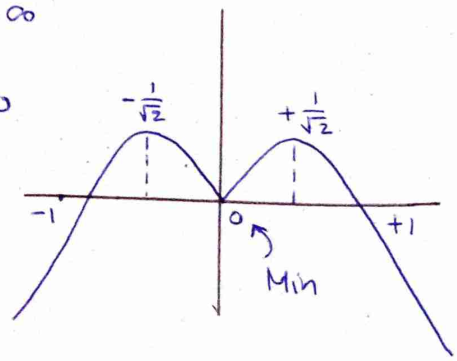
(c) $V(x) = K(x^2 - x^4) = 0 \Rightarrow x^2(1 - x^2) = 0 \Rightarrow x = 0, \pm 1$

$x \to \infty \quad V(x) = K(\infty^2 - \infty^4) = -\infty$

$x \to -\infty \quad V(x) = K(\infty^2 - \infty^4) = -\infty$

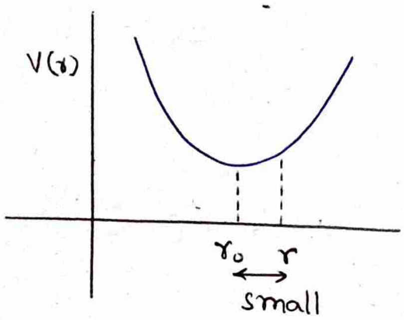
$x = -0.5 \quad V(-0.5) = K\left(\frac{1}{4} - \frac{1}{16}\right) > 0$

$x = 0.5 \quad V(0.5) > 0$



(III) Angular Frequency

Taylor series expansion about a point a is



$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n$$

Expansion of $V(r)$ about r_0 —

$$V(r) = V(r_0) + \cancel{\left.\frac{\partial V}{\partial r}\right|_{r=r_0}}^{\,0}(r-r_0) + \left.\frac{\partial^2 V}{\partial r^2}\right|_{r=r_0}\frac{(r-r_0)^2}{2!} + \cdots$$

$$\frac{\partial V(r)}{\partial r} = 0 + \left.\frac{\partial^2 V}{\partial r^2}\right|_{r=r_0}(r-r_0)$$

$$F = -\frac{\partial V(r)}{\partial r} = -\left.\frac{\partial^2 V}{\partial r^2}\right|_{r=r_0}(r-r_0) = -K\Delta r$$

So $K = \left.\frac{\partial^2 V}{\partial r^2}\right|_{r=r_0}$ So $\omega = \sqrt{\frac{K}{m}}$

Angular frequency $\omega = \sqrt{\dfrac{\left.\frac{\partial^2 V}{\partial r^2}\right|_{r=r_0}}{m}}$

> Now for the Previous question

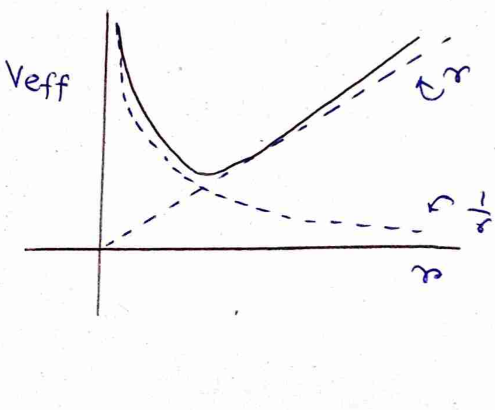
$V(r) = (r^2 - r^4)$ eqb^m point $r = 0, \pm\frac{1}{\sqrt{2}}$

$\left.\frac{\partial^2 V}{\partial r^2}\right|_{r=0} = 2$ (stable eqb^m) (unit mass)

$$\omega = \sqrt{\frac{\left.\partial^2 V/\partial r^2\right|_{r=r_0}}{m}} = \sqrt{2}$$

Plotting:

Potential, $V(r) = Kr$



$$V_{eff} = \frac{L^2}{2mr^2} + Kr$$

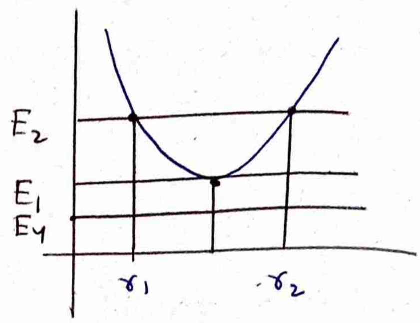
$r \to 0$ $V_{eff} \to \frac{L^2}{2mr^2}$

$r \to \infty$ $V_{eff} \to Kr$

For central force

$\vec{\nabla} \times \vec{F} = 0 \Rightarrow \vec{F}$ is conservative

$\vec{E}$ is Constant



E_1 given a single point
(Circular orbit)

E_2 give two turning point (r_1, r_2)
(Elliptical orbit)

$E = \frac{1}{2} m\dot{r}^2 + V_{eff}$

if $E = V_{eff}$ then $\dot{r} = 0$
(Turning point)

For the case E_4

$V_{eff} > E$

so $\frac{1}{2} m\dot{r}^2 < 0 \Rightarrow$ (Clasically forbiden region)

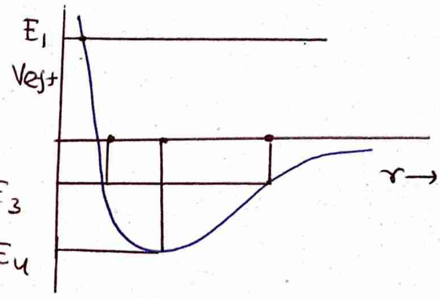
Gravitational Potential:

$V(r) = -\frac{K}{r}$ then $V_{eff}(r) = \frac{L^2}{2mr^2} - \frac{K}{r}$

$r \to \infty$ $\quad V_{eff} \to \frac{L^2}{2mr^2} - \frac{K}{r} \approx 0$

$V_{eff} = 0$

$\frac{L^2}{2mr^2} = \frac{K}{r} \Rightarrow r = \frac{L^2}{2mK}$



E_1 is either parabola or Hyperbola also E_0

E_3 is two turning point (Elliptical)

E_4 has one turning point (Circle)

> For the Circular orbit find Energy

$$V_{eff} = \frac{L^2}{2mr^2} - \frac{K}{r}$$

$$\frac{\partial V}{\partial r} = -\frac{L^2}{mr^3} + \frac{K}{r^2} = 0 \qquad \frac{L^2}{mr} = K \Rightarrow r = \frac{L^2}{mK}$$

$$E = \frac{L^2}{2m} \cdot \frac{m^2K^2}{L^4} - \frac{K}{L^2} mK = \frac{mK^2}{2L^2} - \frac{mK^2}{L^2} = -\frac{mK^2}{2L^2}$$

③ **Equation of orbit:**

For central force motion $f(r) = m(\ddot{r} - r\dot{\theta}^2)$ — ①

Angular momentum . $L = mr^2\dot{\theta} = mr^2 \frac{d\theta}{dt}$

So $\frac{d}{dt} = \frac{L}{mr^2} \frac{d}{d\theta}$

Now $\frac{d^2}{dt^2} = \frac{d}{dt}\left(\frac{d}{dt}\right) = \frac{L}{mr^2}\frac{d}{d\theta}\left(\frac{L}{mr^2}\frac{d}{d\theta}\right)$

$= \frac{L^2}{m^2r^2}\frac{d}{d\theta}\left(\frac{1}{r^2}\frac{d}{d\theta}\right)$

So $\frac{d^2r}{dt^2} = \frac{L^2}{m^2r^2}\frac{d}{d\theta}\left(\frac{1}{r^2}\frac{dr}{d\theta}\right) = -\frac{L^2}{m^2r^2}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)$ — ⑪

From eqn ① and ⑪ we have

$f(r) = m\frac{d^2r}{dt^2} - mr\dot{\theta}^2 = -\frac{L^2}{m^2r^2}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) - \frac{L^2}{mr^3}$

$f(r) = -\frac{L^2}{mr^2}\left[\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right]$ $(r = 1/u)$

$\frac{d^2u}{d\theta^2} + u = -\frac{mr^2}{L^2}f(r) \Rightarrow \boxed{\frac{d^2u}{d\theta^2} + u = -\frac{m}{L^2u^2}f\left(\frac{1}{u}\right)}$ ③

Eqn ③ is the Polar equation of orbit.

④ **For $r^n = a\cos n\theta$ find the dependency:**

$r^n = a\cos n\theta \Rightarrow u^n = \frac{1}{a}\sec n\theta$

$\Rightarrow n\ln u = \ln\left(\frac{1}{a}\right) + \ln(\sec n\theta)$

$\Rightarrow \frac{n}{u}\frac{du}{d\theta} = \frac{1}{\sec n\theta} n\sec n\theta\tan n\theta$

$\Rightarrow \frac{1}{u}\frac{du}{d\theta} = \tan n\theta \Rightarrow \frac{du}{d\theta} = u\tan n\theta$

$\Rightarrow \frac{d^2u}{d\theta^2} = \frac{du}{d\theta}\tan n\theta + n\sec^2 n\theta\, u$

$\Rightarrow \frac{d^2u}{d\theta^2} = nu\sec^2 n\theta + u\tan^2 n\theta$

Now from equation of orbit

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{L^2u^2} f\left(\frac{1}{u}\right)$$

or $\quad nu\sec^2 n\theta + u\tan^2 n\theta + u = -\frac{m}{L^2u^2} f\left(\frac{1}{u}\right)$

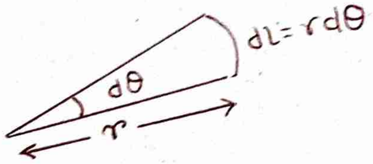
$\quad nu\sec^2 n\theta + u\sec^2 n\theta = -\frac{m}{L^2u^2} f(1/u)$

or $\quad u^3\sec^2 n\theta\,(n+1) = -\frac{m}{L^2} f\left(\frac{1}{u}\right)$

or $\quad f\left(\frac{1}{u}\right) \propto u^{3+2n} \Rightarrow f(r) \propto \frac{1}{r^{(2n+3)}}$

$$\boxed{f(r) \propto \frac{1}{r^{(2n+3)}}}$$ Dependency of force with distance r.

④ <u>Areal Velocity:</u>



Here dθ very small.

area $dA = \frac{1}{2} \times r \times r\,d\theta$

$dA = \frac{1}{2} r^2 d\theta$

$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} r^2 \dot{\theta} = \frac{L}{2m}$

As L, m are constant

$$\boxed{\frac{dA}{dt} = \frac{L}{2m} = \text{Constant}}$$ Kepler's second Law.

Area swift out per time (Areal Velocity) is constant

(Q) A particle of m moves under an attractive central force Kr^4 with angular momentum l. For circular orbit $r_0 = (l^2/Km)^{1/\alpha}$. Find α

$\Rightarrow \quad V(r) = -\int f(r)\,dr = -K\int r^4 dr = \frac{Kr^5}{5}$

$V_{eff} = \frac{L^2}{2mr^2} + \frac{Kr^5}{5} \Rightarrow \frac{\partial V_{eff}}{\partial r} = -\frac{L^2}{mr^3} + Kr^4 = 0$

$\Rightarrow r^7 = \frac{L^2}{mK} \Rightarrow r = \left(\frac{L^2}{mK}\right)^{1/7}$

So $\alpha = 7$

(b) Consider circular orbit in a central potential $V(r) = \frac{-K}{r^n}$ $(0<n<2)$. If the time period of a circular orbit of radius R is T_1 and that of radius 2R is T_2 then T_2/T_1 is

$\Rightarrow$ $$V_{eff} = \frac{L^2}{2mr^2} - \frac{K}{r^n}$$

$$\frac{\partial V_{eff}}{\partial r} = -\frac{L^2}{mr^3} + \frac{nK}{r^{n+1}} = 0 \Rightarrow \frac{L^2}{mr^3} = \frac{nK}{r^{n+1}}$$

$$\Rightarrow r^{n-2} = \frac{nKm}{L^2}$$

$$\frac{\partial^2 V_{eff}}{\partial r^2} = \frac{3L^2}{mr^4} - \frac{n(n+1)K}{r^{n+2}} > 0$$

$$\frac{3L^2}{mr^4} > \frac{n(n+1)K}{r^{n+2}} \Rightarrow r^{n-2} \cdot \frac{3L^2}{m} > n(n+1)K$$

$$\Rightarrow \frac{nKm}{L^2} \cdot \frac{3L^2}{m} > n(n+1)K \Rightarrow 3 > n+1 \Rightarrow n<2$$

$$r = r_0 = \left(\frac{nKm}{L^2}\right)^{\frac{1}{n-2}}$$

$$\frac{dA}{dt} = \frac{L}{2m} \Rightarrow T = \frac{2\pi m R^2}{L}$$

$$T \propto R^2 \cdot R^{\frac{n-2}{2}}$$

$$T \propto R^{\frac{n+2}{2}}$$

$$\frac{T_2}{T_1} = \left(\frac{2R}{R}\right)^{\frac{n+2}{2}}$$

$$\frac{T_2}{T_1} = 2^{(1+\frac{n}{2})}$$

(c) A particle of mass m moves along x axis under influence of $V(x) = -Kxe^{-ax}$. Period of small oscillation about eqlm position is

$\Rightarrow$ $$\frac{\partial V}{\partial x} = -K\left[e^{-ax} - axe^{-ax}\right] = 0 \Rightarrow x = \frac{1}{a}$$

$$\frac{\partial^2 V}{\partial x^2} = +aKe^{-ax} - a(e^{-ax} - axe^{-ax})$$

$$= aKe^{-1} - ae^{-1}K + a^2 \cdot \frac{1}{a} e^{-1} = e^{-1}aK = \frac{aK}{e}$$

$$T = 2\pi \sqrt{\frac{d^2V/dx^2}{m}} = 2\pi \sqrt{\frac{eM}{aK}}$$

(d) A particle moving under a central force describes a cardoid given by $r = \frac{a}{2}(e^{i\theta} + e^{-i\theta} + 2)$ The force is given by

$\Rightarrow$ $r = \frac{a}{2}(e^{i\theta} + e^{-i\theta} + 2) = \frac{a}{2}(2\cos\theta + 2) = 2a\cos^2\theta/_2$

$r^n = a\cos n\theta$

$r^{\frac{1}{2}} = \sqrt{2a}\cos\frac{\theta}{2}$

$f(r) \propto \frac{1}{r^{2n+3}}$

$n = \frac{1}{2}$

$\propto \frac{1}{r^4}$

(e) If L denotes the angular momentum of a particle in a circular orbit of radius r_0 in the potential $V(r)$ The orbit of the particle is $r = a(1+\cos\theta)$. L is proportional to $\left(\frac{1}{\sqrt{r_0}}\right)$ Ans

$\Rightarrow$

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{L^2u^2} f\left(\frac{1}{u}\right)$$

$r = a(1+\cos\theta)$

$r = 2a\cos^2\theta/_2$

$r^{1/2} = (2a)^{1/2}\cos\theta/_2$

$n = \frac{1}{2}$

so $f(r) \propto \frac{1}{r^4}$

$f(r) = \frac{K}{r^4}$

$V(r) = -\int f(r)dr = -\frac{K'}{r^3}$

$V_{eff} = \frac{L^2}{2mr^2} - \frac{K'}{r^3}$

$\frac{\partial V_{eff}}{\partial r} = -\frac{L^2}{mr^3} + \frac{3K'}{r^4} = 0$

$\frac{L^2}{mr^3} = \frac{3K'}{r^4} \Rightarrow r = \frac{3mK'}{L^2} \Rightarrow L \propto \frac{1}{\sqrt{r_0}}$

(f) A modified oscillator satisfies $\ddot{x} + 2\lambda\dot{x} + 25x = 0$

$$\lambda = \begin{cases} 0 & x < 0 \\ 4 & x > 0 \end{cases}$$

Time period of oscillation is

$\lambda = 4$

$\Rightarrow$ $m^2 + 2\lambda m + 25 = 0$

$m = \frac{-2\lambda \pm \sqrt{4\lambda^2 - 100}}{2} = \frac{-8 \pm \sqrt{64-100}}{2} = -4 \pm 3i$

$m^2 + 25 = 0 \quad (x<0)$

$\omega_2 = 5$

$x(t) = e^{-4t}(A\cos 3t + B\sin 3t)$

$\omega_1 = 3 \quad (x>0)$

$T = \frac{\pi}{\omega_1} + \frac{\pi}{\omega_2} = \left(\frac{\pi}{5} + \frac{\pi}{3}\right) s.$

Kepler's Problem:

In Rutherford Scattering

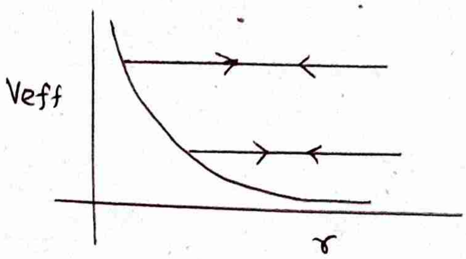
$$V_{eff} = \frac{L^2}{2mr^2} + \frac{K}{r}$$

(No minima exist)

Orbit will be either hyperbola or parabola

closed orbit will possible if $V_{eff}(r)$ has minima

$V_{eff}(r)$ is effective potential



Minima will not exist

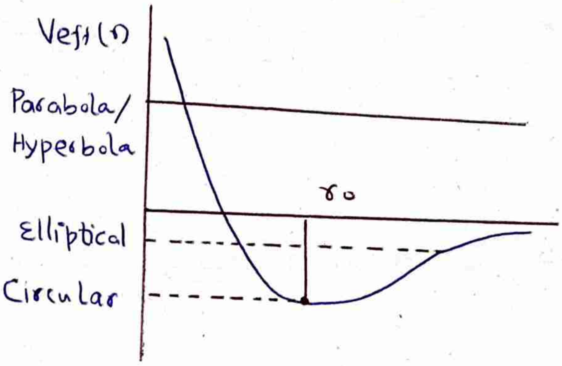
> In Kepler's problem $V(r) = -\frac{GMm}{r} = -\frac{K}{r}$

$$V_{eff} = \frac{L^2}{2mr^2} - \frac{K}{r}$$

$r \to \infty$ $V_{eff} \to 0$

$V_{eff} = 0$ $\frac{L^2}{2mr} = K$

$$r = \frac{L^2}{2mK}$$



$$\frac{\partial V_{eff}}{\partial r} = -\frac{L^2}{mr^3} + \frac{K}{r^2} = 0$$

$$\frac{L^2}{mr^3} = \frac{K}{r^2} \Rightarrow r_0 = \frac{L^2}{mK}$$

$r_0 = L^2/mK$

$$(V_{eff})_{min} = -\frac{mK^2}{2L^2}$$

$(V_{eff})_{min}$

$$= \frac{L^2}{2m}\left(\frac{mK}{L^2}\right)^2 - \frac{K}{L^2}\cdot mK$$

$$= \frac{mK^2}{2L^2} - \frac{mK^2}{L^2} = -\frac{mK^2}{2L^2}$$

⑥ Polar Equation of Ellipse

Total energy of the system

$L = \mu r^2 \dot{\theta}$

$$\frac{d\theta}{dt} = \frac{L}{\mu r^2}$$

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{L^2}{2\mu r^2} + V(r)$$

$$E = \frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2 + \frac{L^2}{2\mu r^2} + V(r)$$

or, $$\sqrt{\frac{2}{\mu}\left[E - \frac{L^2}{2\mu r^2} + V(r)\right]} = \frac{dr}{dt}$$

$$\text{Now}\quad \frac{dr}{d\theta} = \frac{dr/dt}{d\theta/dt}$$

$$u = \frac{1}{r}$$

$$du = -\frac{1}{r^2}dr$$

$$\frac{\sqrt{\frac{2}{\mu}\left[E - \frac{L^2}{2\mu r^2} - V(r)\right]}}{\left(L/2\mu r^2\right)} = \frac{dr}{d\theta}$$

$$\text{or}\quad d\theta = \frac{\frac{L}{\mu r^2}dr}{\sqrt{\frac{2}{\mu}\left(E - \frac{L^2}{2\mu r^2} - V(r)\right)}}$$

$$\text{or}\quad d\theta = \frac{\frac{L}{\mu}(-du)}{\sqrt{\frac{2}{\mu}\left(E - \frac{L^2u^2}{2\mu} - V(r)\right)}}$$

$$\text{or}\quad -d\theta = \frac{du}{\sqrt{\frac{2E}{\mu}\times\frac{\mu^2}{L^2} - \frac{2}{\mu}\cdot\frac{L^2u^2}{2\mu}\frac{\mu^2}{L^2} + \frac{2Ku}{\mu}\frac{\mu^2}{L^2}}}$$

$$\text{or}\quad -d\theta = \frac{du}{\sqrt{\frac{2E\mu}{L^2} - u^2 + \frac{2Ku\mu}{L^2}}}$$

$$\text{or}\quad -d\theta = \frac{du}{\sqrt{\left(\sqrt{\frac{2E\mu}{L^2} + \frac{K^2\mu^2}{L^4}}\right)^2 - \left(u - \frac{K\mu}{L^2}\right)^2}}$$

$$\text{or}\quad \theta + c = \cos^{-1}\left[\frac{u - \frac{K\mu}{L^2}}{\sqrt{\frac{2E\mu}{L^2} + \frac{K^2\mu^2}{L^2}}}\right]$$

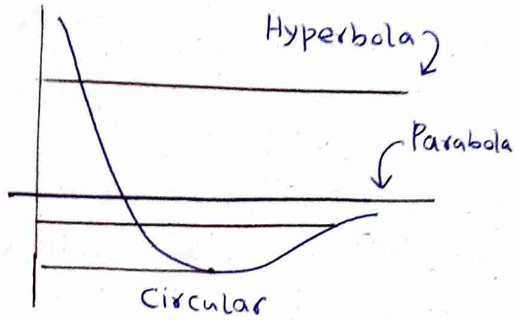
$$\text{or}\quad \cos(\theta + c) = \left(\frac{u - \frac{K\mu}{L^2}}{\sqrt{\frac{2E\mu}{L^2} + \frac{K^2\mu^2}{L^4}}}\right)$$

$$\therefore\quad \frac{1}{r} = \frac{K\mu}{L^2} + \sqrt{\frac{2E\mu}{L^2} + \frac{K^2\mu^2}{L^4}}\cos(\theta + c)$$

or $1+\sqrt{1+\frac{2EL^2}{\mu K^2}}\cos\theta = \frac{L^2/mK}{r} \Rightarrow l/r = 1+e\cos\theta$

$e = \sqrt{1+\frac{2EL^2}{\mu K^2}}$ $\qquad l = \frac{L^2}{mK}$ latus rectum

Now $e = \sqrt{1+\frac{2EL^2}{\mu K^2}}$



Parabola: $e=1, E=0$

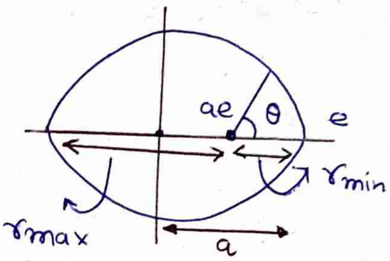
Hyperbola: $e>1, E>0$

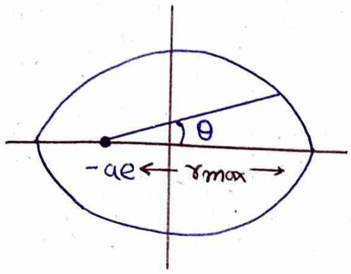
Ellipse: $e<1, E<0$

Circular: $e=0, E = -\frac{mK^2}{2L^2}$ (Verified).

> $\frac{l}{r} = 1 \pm e\cos\theta \Rightarrow r = \frac{l}{1+e\cos\theta}$

$r = \frac{l}{1-e\cos\theta}$





$r_{min} = a(1-e)$ $(\theta=0)$

$r_{max} = a(1+e)$ $(\theta=\pi)$

$r_{min} = a(1-e)$ $(\theta=\pi)$

$r_{max} = a(1+e)$ $(\theta=0)$

So basically we have, $r = \frac{L^2/\mu K}{1-\sqrt{1+\frac{2EL^2}{\mu K^2}}\cos\theta}$

$r_1 = \frac{L^2/\mu K}{1-\sqrt{1+\frac{2EL^2}{\mu K^2}}}$ $\qquad r_2 = \frac{L^2/\mu K}{1+\sqrt{1+\frac{2EL^2}{\mu K^2}}}$

or $1-\sqrt{1+\frac{2EL^2}{\mu K^2}} = \frac{L^2}{\mu K}\cdot\frac{1}{r_1}$ $\quad\cdot\quad 1+\sqrt{1+\frac{2EL^2}{\mu K^2}} = \frac{L^2}{\mu K}\cdot\frac{1}{r_2}$

By adding this two equation

$$2 = \frac{L^2}{\mu K}\left(\frac{r_1+r_2}{r_1 r_2}\right) \Rightarrow L = \sqrt{\frac{2\mu K r_1 r_2}{r_1+r_2}}$$

$$\Rightarrow \quad L = \sqrt{\frac{2r_1r_2}{r_1+r_2}\left(\frac{m_1m_2}{m_1+m_2}\right) G m_1 m_2}$$

Now, $r_1 = a(1-e)$ $\qquad e = \sqrt{1-\frac{b^2}{a^2}}$

$r_2 = a(1+e)$

$$r_1+r_2 = a(1-e)+a(1+e) \Rightarrow r_1+r_2 = 2a$$

$$r_1r_2 = a^2(1-e^2) \Rightarrow r_1r_2 = b^2$$

$$L = m_1m_2\sqrt{\frac{Gb^2}{a}\left(\frac{1}{m_1+m_2}\right)}$$

⑦ <u>Conservation of angular momentum:-</u>

$$\mathcal{L} = m_1 V_{max} r_{min} = m_2 V_{min} r_{max}$$

or $$\frac{V_{max}}{V_{min}} = \frac{r_{max}}{r_{min}}$$

or $$\frac{V_{max}}{V_{min}} = \frac{a(1+e)}{a(1-e)} \Rightarrow \boxed{\frac{V_{max}}{V_{min}} = \left(\frac{1+e}{1-e}\right)}$$

$$L = \sqrt{\frac{G(m_1m_2)^2}{m_1+m_2}\cdot\left(\frac{2r_1r_2}{r_1+r_2}\right)}$$

$$m_1 V_{max} r_1 = \sqrt{\frac{G(m_1m_2)^2}{m_1+m_2}\left(\frac{2r_1r_2}{r_1+r_2}\right)}$$

$$V_{max} = \sqrt{\frac{Gm_2^2}{m_1+m_2}\cdot\frac{1}{a}\left(\frac{1+e}{1-e}\right)} = \sqrt{\frac{Gm_2}{a}\left(\frac{1+e}{1-e}\right)}$$

$$V_{min} = \sqrt{\frac{Gm_2^2}{m_1+m_2}\,\frac{1}{a}\left(\frac{1-e}{1+e}\right)} = \sqrt{\frac{Gm_2}{a}\left(\frac{1-e}{1+e}\right)}$$

(1) The distance between the sun and earth is 1AU. Jupiter takes about 11.9 earth years to orbit around the sun. The distance between sun and Jupiter is

$\Rightarrow$ $T^2 \propto a^3$ $\qquad$ $a_1 = 1AU$

$T_2 = 11.9T_1$

or $\frac{T_1^2}{T_2^2} = \frac{a_1^3}{a_2^3}$ $\qquad$ $1AU = 150\times10^6 m$

or $\frac{1}{(11.9)^2} = \frac{1}{a_2^3} \Rightarrow a_2 = (11.9)^{2/3} = 5.4 AU$

(2) A particle of mass m moves under the action of a central force whose potential $V(r) = Kmr^3$. then for what angular momentum will the orbit be a circle of radius a, about the origin

$\Rightarrow$ $V_{eff} = \frac{L^2}{2mr^2} + Kmr^3$ $\qquad$ $\frac{\partial V_{eff}}{\partial r} = -\frac{L^2}{mr^3} + 3Kmr^2 = 0$

$3r^5m^2K = L \Rightarrow L = ma^2\sqrt{3Ka}$

(3) The maximum and minimum speeds of a statelite moving around a planet of mass 10^{30} Kg are 100 Km/s and 10 km/s respectively. The length of the semi major axis of its elliptical orbit is

$\Rightarrow$ $\frac{V_{max}}{V_{min}} = \frac{1+e}{1-e}$ $\qquad$ $V_{max} = \sqrt{\frac{Gm_2}{a}\left(\frac{1+e}{1-e}\right)}$

or $\frac{100}{10} = \frac{1+e}{1-e}$ $\qquad$ $100 = \sqrt{\frac{G\times10^{30}}{a}\times 10}$

or $10-10e = 1+e$ $\qquad$ $10000 = \frac{G\times10^{30}}{a}\times10$

or $9 = 11e$ $\qquad$ $10000 = \frac{6.67\times10^{-11}\times10^{31}}{a}$

or $e = 9/11$

$a = 6.67\times10^{16}$

$=$ 3.7 light second.

(4) A particle moving under gravitational force pericenter distance in parabolic orbit is r_p while the radius of the circular orbit with same angular momentum r_c. Value of r_c/r_p $\qquad$ Parabola $e=1$

$\Rightarrow$ $\frac{l}{r} = 1+e\cos\theta$ $\qquad$ Circle $r_c = l$ $\quad (e=0)$ $\qquad$ $r_p = l/2$

$\frac{r_c}{r_p} = 2$

(5) Eqn of orbit of a particle moving under central force is $r\theta = \beta$. The force acting on the particle

$\Rightarrow$ $r\theta = \beta$ $\qquad \frac{du}{d\theta} = \frac{1}{\beta}$ $\qquad \frac{d^2u}{d\theta^2} + u = -\frac{m}{L^2u^2} f(\frac{1}{u})$

$r = \frac{\beta}{\theta}$ $\qquad \frac{d^2u}{d\theta^2} = 0$ $\qquad -\frac{\theta}{\beta} \cdot \frac{L^2u^2}{m} = f(\frac{1}{u})$

$u = \frac{\theta}{\beta}$ $\qquad -\frac{L^2u}{m}$

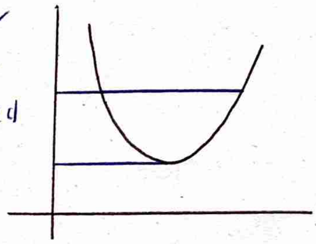
$\left(\frac{2}{\beta\theta^2} + \frac{1}{\beta\theta}\right)$ $\qquad f(r) \propto \frac{1}{r^3}$

(6) A particle of mass m moving in central force $f(r) = -Kr.\ (K>0)$ then which of following orbit possi..

$\Rightarrow$ $f = -\frac{\partial V}{\partial r} \Rightarrow V(r) = -\int f(r)\,dr = \frac{Kr^2}{2}$

$V_{eff} = \frac{\alpha^2}{2mr^2} + \frac{Kr^2}{2}$

only bounded
~~Par~~. Circle
Elliptical



(7) If $V(r) = Kr^2$ $(K>0)$ if r_1 and r_2 denote the radii of circular orbits when the particle possesses respective angular momentum L_0 and $2L_0$

$\Rightarrow$ $V_{eff} = \frac{L^2}{2\mu r^2} + Kr^2$ $\qquad \frac{L^2}{\mu r^3} = 2Kr$

$\frac{\partial V_{eff}}{\partial r} = -\frac{L^2}{\mu r^3} + 2Kr = 0$ $\qquad \frac{L^2}{2\mu K} = r^4$

$\frac{L_2}{L_1} = \left(\frac{r_2}{r_1}\right)^2 \Rightarrow \frac{2L_0}{L_0} = \left(\frac{r_2}{r_1}\right)^2$ $\qquad L \propto r^2$

or $r_2 = \sqrt{2}\, r_1$

(8) A particle of mass m and angular momentum is subjected to a central potential $V(r) = K\ln r$. Find Radius of circular orbit (Jest 2021)

$\Rightarrow$ $V_{eff} = \frac{\alpha^2}{2mr^2} + K\ln r$ $\qquad \frac{\partial^2 V}{eff} = \frac{3L^2}{mr^4} - \frac{K}{r^2}$

$\frac{\partial V}{eff} = \frac{-L^2}{mr^3} + \frac{K}{r} = 0$ $\qquad = \frac{3L^2}{m} \times \frac{m^2K^2}{L^4} - \frac{K\,mK}{L^2}$

$= 2\frac{mK^2}{L^2} > 0$

$\frac{L^2}{mr^3} = \frac{K}{r}$

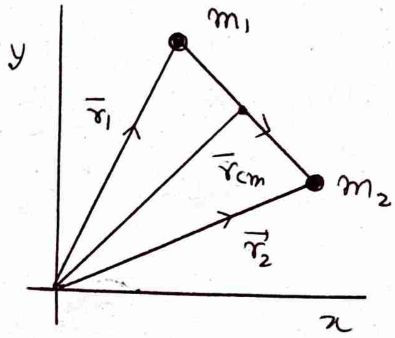
(stable orbit)

$\frac{L}{\sqrt{mK}} = R$

Centre of Mass

The point on an object at which the whole mass of the object is supposed to be Concentrated

Force on the system



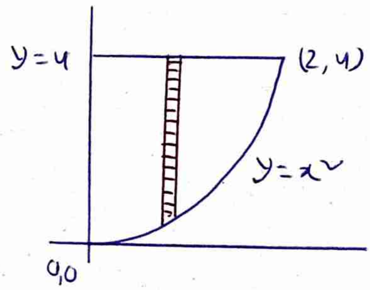
$$\vec{F} = m_1 \frac{d^2\bar{r}_1}{dt^2} + m_2 \frac{d^2\bar{r}_2}{dt^2}$$

$$= (m_1+m_2)\frac{d^2\bar{r}_{cm}}{dt^2}$$

or $$\frac{d^2\vec{R}_{cm}}{dt^2} = \frac{d^2}{dt^2}\left(\frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2}\right)$$ $$\vec{R}_{cm} = \frac{m_1\vec{r}_1+m_2\bar{r}_2}{m_1+m_2}$$

So $$\bar{R}_{cm} = \frac{\sum m_i\bar{r}_i}{\sum m_i}$$ $$R_{cm} = \frac{\int x\,dm}{\int dm}.$$

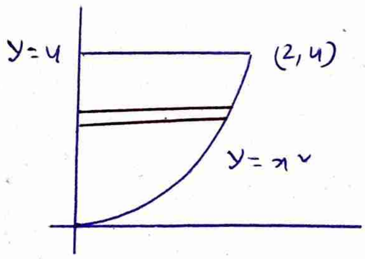
① If σ be the mass density then find the position of Centre of mass



$\Rightarrow$
$$X_{cm} = \frac{\int x\,dm}{\int dm} = \frac{\sigma\int x\,dx\,dy}{\sigma\int dx\,dy}$$

$$= \frac{\int_0^2 x\,dx\int_{x^2}^4 dy}{\int_0^2 dx\int_{x^2}^4 dy} = \frac{2x^2 - \frac{x^4}{4}\Big|_0^2}{4x - \frac{x^3}{3}\Big|_0^2}$$

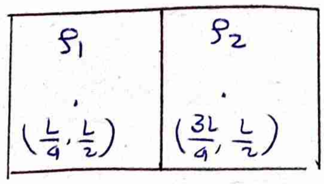
$$= \frac{\int_0^2 (4x-x^3)\,dx}{\int_0^2 (4-x^2)\,dx} = \frac{(8-4)}{(8-\frac{16}{3})} = 3/4$$



$$Y_{cm} = \frac{\int_0^2 dx\int_{x^2}^4 y\,dy}{\int_0^2 dx\int_{x^2}^4 dy} = \frac{12}{5}$$

$$(X_{cm}, Y_{cm}) = \left(\frac{3}{4}, \frac{12}{5}\right)$$

(2) There is a square plate of side length L Half of the square plate is made of a material of density ρ_1 and other half plate of density ρ_2. Locate the Centre of mass



$\Rightarrow$

$$M_1 = \rho_1 L \cdot \frac{L}{2} = \frac{\rho_1 L^2}{2}$$

$$M_2 = \rho_2 L \cdot \frac{L}{2} = \frac{\rho_2 L^2}{2}$$

$$X_{cm} = \frac{\left(\frac{\rho_1 L^2}{2} \times \frac{L}{4}\right) + \left(\frac{\rho_2 L^2}{2} \times \frac{3L}{4}\right)}{\frac{L^2}{2}(\rho_1 + \rho_2)} = \frac{(\rho_1 L + 3\rho_2 L)}{4(\rho_1 + \rho_2)}$$

$$Y_{cm} = \frac{L}{2} \qquad (X_{cm}, Y_{cm}) = \left[\frac{\rho_1 L + 3\rho_2 L}{4(\rho_1 + \rho_2)}, \frac{L}{2}\right]$$

(3) The linear mass density of a rod varies as $\lambda = \lambda_0 (1 + \frac{x}{L})$ The Centre of mass of the rod

$\Rightarrow$

$$X_{cm} = \frac{\int \lambda dx}{\int dx} = \frac{\lambda_0 \int x(1+\frac{x}{L})dx}{\lambda_0 \int (1+\frac{x}{L})dx}$$

$$= \frac{\int (x + \frac{x^2}{L})dx}{\int dx + \frac{1}{L} x dx} = \frac{\left.\frac{x^2}{2} + \frac{x^3}{3L}\right|_0^L}{L + \frac{1}{2L}L^2} = \frac{L^2 \times \frac{5}{6}}{L \times \frac{3}{2}} = \frac{5}{9}L$$

(4) A homogeneous semicircular plate of R = 3cm The distance of COM of the plate is

$\Rightarrow$

$$y_m = \frac{4R}{3\pi} = \frac{4 \times 3}{3 \times \pi} = 1.2\,m$$

(5) For a semicircular disc $\sigma \propto r^2$. Locate the COM

$\Rightarrow$

$$y_{cm} = \frac{\int y\,dm}{\int dm} = \frac{\int r \sin\theta\, K r^2\, r\,dr\,d\theta}{\int K r^2\, r\,dr\,d\theta}$$

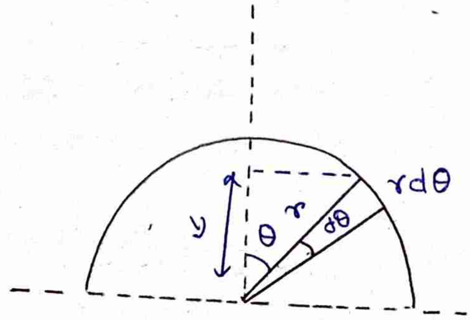
$$= \frac{\int r^4 dr \int \sin\theta\, d\theta}{\int r^3 dr \int d\theta} = \frac{\frac{R^5}{5}}{\frac{R^4}{4}} \times \frac{2}{\pi} = \frac{8R}{5\pi}$$

COM of a semi-circular ring:

Linear mass density

$$\lambda = \frac{M}{\pi r}$$

$$dm = \lambda r d\theta = \frac{M}{\pi} d\theta$$



$$y_{cm} = \frac{\int y\,dm}{\int dm} = \frac{\int r\cos\theta \frac{M}{\pi} d\theta}{\int \frac{M}{\pi} d\theta} = \frac{r\int \cos\theta\, d\theta}{\int d\theta}$$

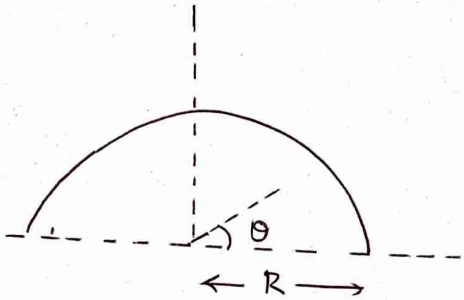
$$= \frac{r \sin\theta \Big|_{-\pi/2}^{\pi/2}}{\pi} = \frac{2r}{\pi}$$

$$y_{cm} = \frac{2R}{\pi}$$

COM of an uniform semicircular Disc.

Here also, $X_{cm} = 0$

$dm = \sigma dA = \sigma r dr d\theta$



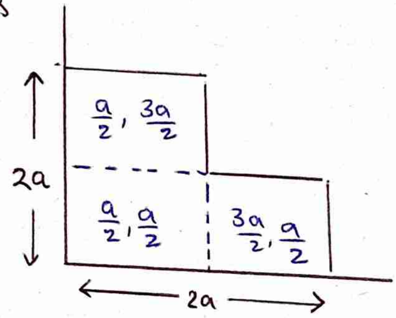
$$y_{cm} = \frac{\int y\,dm}{\int dm}$$

$$= \frac{\int r\sin\theta\, \sigma\, dr d\theta\, r}{\int \sigma r dr d\theta} = \frac{\int r^2 dr \int \sin\theta d\theta}{\int r dr \int d\theta} \qquad \begin{matrix} r: 0 \to R \\ \theta: 0 \to \pi. \end{matrix}$$

$$= \frac{4R}{3\pi}$$

(6) Find the Position of Centre of mass

⇒ Mass of each segment $M/3$



$$X_{cm} = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3}$$

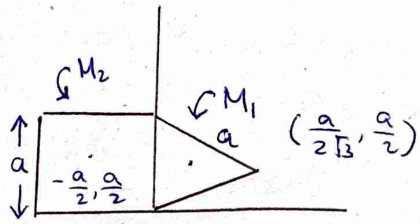
$$= \frac{\frac{M}{3}\left(\frac{a}{2} + \frac{a}{2} + \frac{3a}{2}\right)}{M} = \frac{5a}{6}$$

$$Y_{cm} = \frac{m_1y_1 + m_2y_2 + m_3y_3}{m_1 + m_2 + m_3} = \frac{\frac{M}{3}\left(\frac{3a}{2} + \frac{a}{2} + \frac{a}{2}\right)}{M} = \frac{5a}{6}$$

(7) A circular disc of radius r is melted and a square and an equilateral triangle of same side length are made. They are joined together. The distance of COM from origin is αr. Find α

$\Rightarrow$

σ be the mass density



$$\sigma(\pi R^2) = \sigma\left[a^2 + \frac{\sqrt{3}}{4}a^2\right]$$

$$\pi R^2 = a^2\left(1 + \frac{\sqrt{3}}{4}\right)$$

$$M_1 = \sigma\frac{\sqrt{3}}{4}a^2$$

$$a^2 = \frac{\pi R^2}{1.43} \Rightarrow a = 1.48R$$

$$M_2 = \sigma a^2$$

$$X_{cm} = \frac{\left(\sigma \cdot \frac{\sqrt{3}}{4}a^2 \times \frac{a}{2\sqrt{3}}\right) + \sigma a^2 \times \left(-\frac{a}{2}\right)}{a^2\sigma\left[\frac{\sqrt{3}}{4}+1\right]}$$

$$= \frac{a\left(\frac{1}{8}-\frac{1}{2}\right)}{\left(\frac{\sqrt{3}}{4}+1\right)} = \frac{-0.375a}{1.433} = -0.26a$$

$$Y_{cm} = \frac{\left(\sigma \cdot \frac{\sqrt{3}}{4}a^2 \times \frac{a}{2}\right) + \sigma a^2\left(\frac{a}{2}\right)}{a^2\sigma\left[\frac{\sqrt{3}}{4}+1\right]} = \frac{\left(\frac{\sqrt{3}}{8}+\frac{1}{2}\right)a}{\left(\frac{\sqrt{3}}{4}+1\right)} = \frac{a}{2}$$

$$r = \sqrt{X_{cm}^2 + Y_{cm}^2} = \sqrt{(0.26)^2 + (0.5)^2}\, a = 0.56a = 0.83R$$

(8) Density of a circular plate of radius R is $\sigma(r,\theta) = Kr(1-\sin\theta)$. If mass of plate is $\beta K R^{\alpha}$, then $\alpha+\beta$

$\Rightarrow$

$$dm = \sigma r\,dr\,d\theta$$

$$= \int Kr(1-\sin\theta)\,r\,dr\,d\theta$$

$$= K\int_0^R r^2 dr \int_0^{2\pi}(1-\sin\theta)\,d\theta$$

$$= \frac{KR^3}{3}\left.\theta + \cos\theta\right|_0^{2\pi} = \frac{KR^3}{3}\left[2\pi + 0\right]$$

$$= \frac{2}{3}(\pi + 0)KR^3 = \frac{2\pi}{3}KR^3$$

$$\beta = \frac{2\pi}{3} = 2.09$$

$$\alpha = 3$$

$$\alpha + \beta = 5.09$$

Gravitational Force

Acceleration due to gravity at a height h from the surface of earth

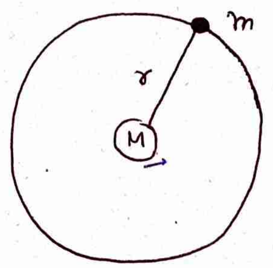
$$g' = g\left(1 - \frac{2h}{R}\right) \qquad R >> h$$

$$g' = \frac{GM}{(R+h)^2}$$

Acceleration at a depth d.

$$g' = g\left(1 - \frac{d}{R}\right) \quad \text{(Always valid)}$$

Orbitial Velocity



Here we have

$$\frac{mv^2}{r} = \frac{GMm}{r^2}$$

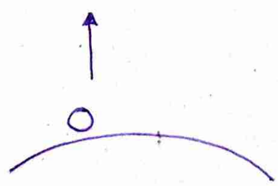
$$v = \sqrt{\frac{GM}{r}}$$

At height $v = \sqrt{\frac{GM}{R+h}}$

$$T = \frac{2\pi r}{v} = \frac{2\pi}{\sqrt{\frac{GM}{r}}} r \Rightarrow T^2 \propto r^3$$

(Kepler's 3rd law)

Escape velocity



$$E = \frac{1}{2}mv^2 - \frac{GMm}{R}$$

$$E_f = \frac{1}{2}mv_f^2 - \cancel{\frac{GMm}{R+h}} \rightarrow 0$$

Since object is not return back

so $v_f \geq 0$

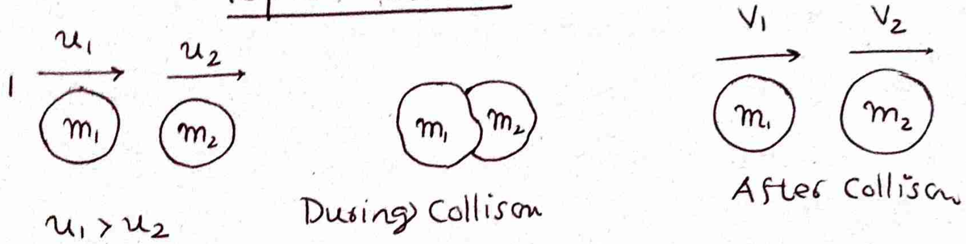
$E_f \geq 0$

$$\frac{1}{2}mV_{es}^2 - \frac{GMm}{R} \geq 0$$

$$V_{es} \geq \sqrt{\frac{2GM}{R}}$$

$$V_{es} \geq \sqrt{2gR}$$

Topic : Collison



From momentum Conservation

$$m_1 u_1 + m_2 u_2 = m_1 V_1 + m_2 V_2$$

$$m_1 (u_1 - V_1) = m_2 (V_2 - u_2) \quad —\ (i)$$

From KE Conservation

$$\frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2 = \frac{1}{2} m_1 V_1^2 + \frac{1}{2} m_2 V_2^2$$

or $\quad m_1 (u_1^2 - V_1^2) = m_2 (V_2^2 - u_2^2) \quad —\ (ii)$

(2) ÷ (1)

$$u_1 + V_1 = u_2 + V_2 \qquad \frac{V_2 - V_1}{u_1 - u_2} = e = 1$$

$$u_1 - u_2 = V_2 - V_1$$

$$u_1 + V_1 = u_2 + V_2$$

$$V_2 = u_1 + V_1 - u_2$$

$$m_1 u_1 + m_2 u_2 = m_1 V_1 + m_2 V_2$$

$$m_1 u_1 + m_2 u_2 = m_1 V_1 + m_2 (u_1 + V_1 - u_2)$$

$$m_1 u_1 + m_2 u_2 = m_1 V_1 + m_2 u_1 + m_2 V_1 - m_2 u_2$$

$$(m_1 - m_2) u_1 + 2 m_2 u_2 = V_1 (m_1 + m_2)$$

$$V_1 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) u_1 + \left(\frac{2 m_2}{m_1 + m_2}\right) u_2$$

Also $\quad u_1 + V_1 = u_2 + V_2 \Rightarrow V_1 = u_2 + V_2 - u_1$

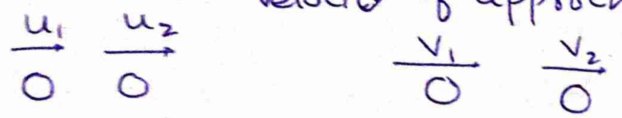
$$V_2 = \left(\frac{m_2 - m_1}{m_1 + m_2}\right) u_2 + \frac{2 m_1}{m_1 + m_2} u_1$$

If $\quad m_1 = m_2 \qquad V_1 = u_2$ and $V_2 = u_1$

(Velocity exchange)

Coefficient of Restitution

$$e = \frac{\text{Velocity of seperation}}{\text{Velocity of approch}} = \frac{V_2 - V_1}{u_1 - u_2}$$

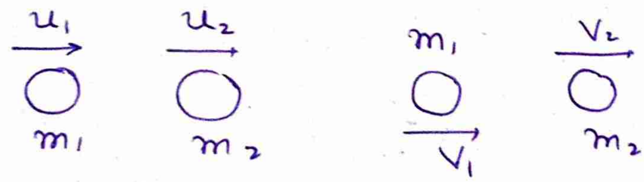


$u_1 > u_2$ $\qquad\qquad$ $V_2 > V_1$

$u_1 - u_2 = V_{sep}$ $\qquad\qquad$ $V_2 - V_1 = V_{sep}$

Collison
- Elastic Collison — $e = 1$ — KE, P = Conserved
- Inelastic Collison — $e = 0$, $V_2 = V_1$ — only P Conserved
- Partialy Elastic / Inelastic — $0 < e < 1$ — only P conserved

Partially Elastic / Partially Inelastic



Change in Kinetic energy $\Delta KE = KE_i - KE_f$

$$\Delta KE = \frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2 - \frac{1}{2} m_1 V_1^2 - \frac{1}{2} m_2 V_2^2$$

$$= \frac{1}{2(m_1+m_2)} (m_1+m_2) \left[m_1 u_1^2 + m_2 u_2^2 - m_1 V_1^2 - m_2 V_2^2 \right]$$

$$= \frac{1}{2(m_1+m_2)} \Big[(m_1 u_1)^2 + m_1 m_2 u_1^2 + m_1 m_2 u_2^2 + (m_2 u_2)^2 - \{ (m_1 V_1)^2 + m_1 m_2 V_1^2 + m_1 m_2 V_2^2 + (m_2 V_2)^2 \} \Big]$$

$$= \frac{1}{2(m_1+m_2)} \Big[(m_1 u_1)^2 + (m_2 u_2)^2 + 2(m_1 u_1)(m_2 u_2) - 2(m_1 u_1)(m_2 u_2) + m_1 m_2 u_1^2 + m_1 m_2 u_2^2 - \{ (m_1 V_1)^2 + (m_2 V_2)^2 + 2(m_1 V_1)(m_2 V_2) - 2(m_1 V_1)(m_2 V_2) + m_1 m_2 V_1^2 + m_1 m_2 V_2^2 \} \Big]$$

$$= \frac{1}{2(m_1+m_2)}\Big[(m_1u_1 \cancel{+} m_2u_2)^2 + m_1m_2(u_1^2+u_2^2-2u_1u_2) - (m_1v_1 \cancel{+} m_2v_2)^2 - m_1m_2(v_1^2+v_2^2-2v_1v_2)\Big]$$

$$= \frac{m_1m_2}{2(m_1+m_2)}\left[(u_1-u_2)^2 - (v_1-v_2)^2\right]$$

$v_2 - v_1 = e(u_1-u_2)$

$$= \frac{m_1m_2}{2(m_1+m_2)}\left[(u_1-u_2)^2 - e^2(u_1-u_2)^2\right]$$

$$= \frac{1}{2}\mu\,(u_1-u_2)^2(1-e^2)$$

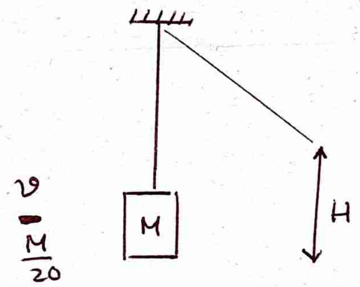
$\mu = \frac{m_1m_2}{m_1+m_2}$, Reduced mass

$$\boxed{\Delta KE = \frac{1}{2}\mu\,(u_1-u_2)^2(1-e^2)}$$

Elastic Collision $e = 1$, $\Delta KE = 0$

Inelastic $e = 0$ $\Delta KE = \frac{1}{2}\mu(u_1-u_2)^2$

① Attain height H will be



=) From momentum Conservation

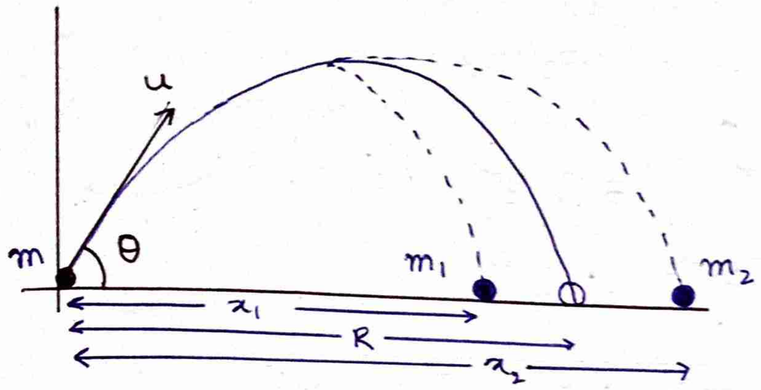
$$\frac{M}{20}v = \left(M + \frac{M}{20}\right)v_1$$

$$\frac{M}{20}v = \frac{21}{20}Mv_1$$

$$v_1 = \frac{v}{21}$$

$$\frac{1}{2}\left(M+\frac{M}{20}\right)\left(\frac{v}{21}\right)^2 = \left(M+\frac{M}{20}\right)gH$$

$$H = \frac{v^2}{2\times 441\times g}$$



The original path will be traced by the centre of mass

$$R = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

Data Mass = M, $\theta = 30^\circ$, $u = 100\,m/s$

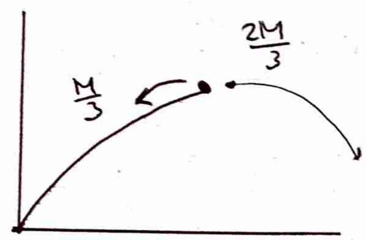
At Hmax M breaks into $\frac{M}{2}$ and $\frac{M}{2}$. one drop vertically then another will fall at what distance

$\Rightarrow$ Range $R = \frac{u^2 \sin 2\theta}{g} = \frac{(100)^2 \times \sin 60}{10} = 866\ m/s$

$R = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \Rightarrow \frac{\frac{M}{2} \times 433 + \frac{M}{2} x_2}{M} = 866$

$\frac{1}{2}(433 + x_2) = 866 \Rightarrow x_2 = 1299\ m$

Particle break into $\frac{M}{3}$ and $\frac{2M}{3}$ and $\frac{M}{3}$ back to origin then $\frac{2M}{3}$ will be



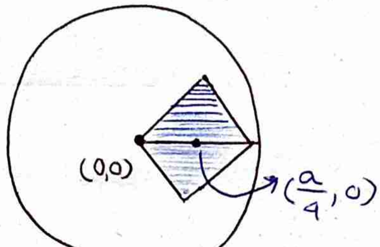
$\Rightarrow$

Range = 866

$$866 = \frac{\frac{M}{3} \times 0 + \frac{2M}{3} x_2}{M} \Rightarrow x_2 = 1299$$

(9) A square hole is punched out of a circular lamina. the diagonal of the Square being the radius of the circle. If a is the diameter of Circle. Find COM

⇒ If d is side of Square



$$\frac{a}{2} = d\sqrt{2} \Rightarrow d = \frac{a}{2\sqrt{2}}$$

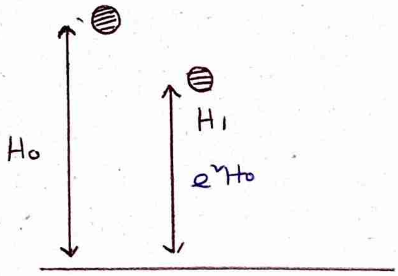
Mass density $\sigma = \frac{M}{\frac{\pi a^2}{4}} = \frac{4M}{\pi a^2}$

Mass of square : $\sigma \times d^2 = \frac{4M}{\pi a^2} \times \frac{a^2}{8} = \frac{M}{2\pi}$

$$X_{CM} = \frac{(M\times 0) + \left(-\frac{M}{2\pi}\times\frac{a}{4}\right)}{M - \frac{M}{2\pi}} = \frac{-\frac{Ma}{8\pi}}{M\left(1-\frac{1}{2\pi}\right)} = -\frac{Ma}{8\pi}\times\frac{2\pi}{M(2\pi-1)}$$

$$= -\frac{a}{4(2\pi-1)} \qquad y_{cm} = 0$$

<u>Collision of a falling body with Fixed Horizen:</u>



Let e is the Coefficient of Restitution

$u_0 = \sqrt{2gH_0}$

$$e = \frac{v_2 - v_1}{u_1 - u_2} = \frac{0 - v_1}{-\sqrt{2gH_0} - 0}$$

$v_1 = e\sqrt{2gH_0}$ — (1)

$$H_1 = \frac{v_1^2}{2g} = \frac{e^2\, 2g H_0}{2g}$$

$H_1 = e^2 H_0$

$$\boxed{H_n = e^{2n} H_0}$$

<u>Distance covered before come to rest</u>

Distance. $S = H_0 + 2H_1 + 2H_2 + \dots$

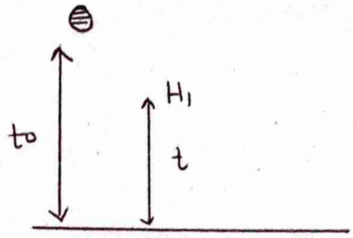
$= H_0 + 2H_0 e^2 + 2H_0 e^4 + \dots$

$= H_0\left[1 + 2e^2(1 + e^2 + e^4 + e^6 + \dots)\right]$

$= H_0\left[1 + 2e^2 \cdot \frac{1}{1-e^2}\right] = H_0\left(\frac{1+e^2}{1-e^2}\right)$

$$\boxed{S = H_0\left(\frac{1+e^2}{1-e^2}\right)}$$

Taken time before come to Rest



Taken time to Reach the height

$t_0 = \sqrt{\frac{2H_0}{g}}$

$0 = eV \cdot t - g\frac{t^2}{2} \Rightarrow t = 2e\sqrt{\frac{2H_0}{g}}$

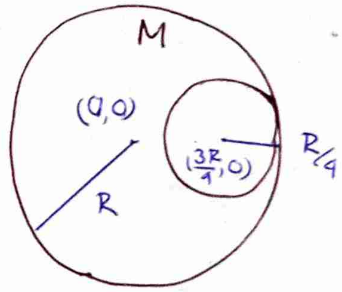
Total time $T = \sqrt{\frac{2H_0}{g}} + \sqrt{\frac{2H_0}{g}}\, 2e + \sqrt{\frac{2H_0}{g}}\, 2e^2 + \cdots$

$= \sqrt{\frac{2H_0}{g}}\,(1 + 2e + 2e^2 + \cdots \infty) = \sqrt{\frac{2H_0}{g}}\left[1 + \frac{2e}{1-e}\right]$

$$\boxed{T = \sqrt{\frac{2H_0}{g}}\left(\frac{1+e}{1-e}\right)}$$

① From a disc of mass m and radius R, a disc of Radius R/4 is cut and thrown find COM

⇒ Mass density $\sigma = \frac{M'}{\pi R^2}$



~~Remains~~

$(\pi R^2 - \pi \frac{R^2}{16}) \rightarrow \frac{M'}{\pi R^2} \times \frac{16\pi R^2}{16} = M$

$M' = \frac{16}{15} M$ $\quad M'' = \frac{M}{15}$

$R_{cm} = \dfrac{\frac{16}{15}M - \frac{M}{15} \times \frac{R}{4}}{\frac{16M}{15} - \frac{M}{15}} = -R/20$

② Two particles of mass m and 2m moving in opposite directions collide elastically with velocities V and 2V. then final velocity

⇒ $u_1 = V \quad m$

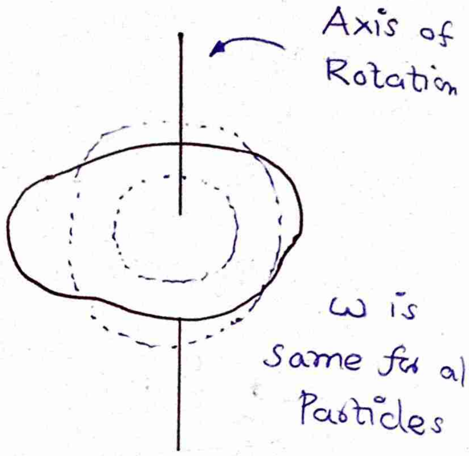
$u_2 = 2V \quad 2m$

$V_1 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) u_1 + \left(\frac{2m_2}{m_1 + m_2}\right) u_2$

$= \frac{-m}{3m} V + \frac{2 \cdot 2m \cdot (-2V)}{3m}$

$= -3V$

Rotational Motion

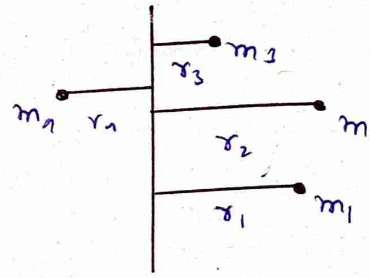


Total Kinetic Energy

$$KE = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \frac{1}{2}m_3v_3^2 + \frac{1}{2}m_4v_4^2 + \cdots$$

$$KE = \frac{1}{2}m_1(r_1\omega)^2 + \frac{1}{2}m_2(r_2\omega)^2 + \frac{1}{2}m_3(r_3\omega)^2 + \frac{1}{2}m_4(r_4\omega)^2$$

$$= \frac{1}{2}(m_1r_1^2 + m_2r_2^2 + \cdots)\omega^2$$



$$\boxed{KE = \frac{1}{2}I\omega^2}$$

So moment of Inertia $I = \sum m_i r_i^2$

Continous system $I = \int dm\, r_\perp^2$

$$I = m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \cdots + m_nr_n^2$$

$$= \frac{nm\,(r_1^2 + r_2^2 + \cdots + r_n^2)}{n} \qquad I = MK^2$$

$$= M\frac{(r_1^2 + r_2^2 + \cdots + r_n^2)}{n} = MK^2$$

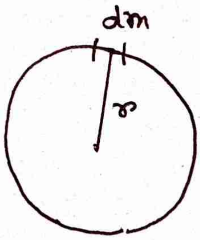
$$K = \sqrt{\frac{r_1^2 + r_2^2 + \cdots + r_n^2}{n}} = \text{RMS distance}$$

is Radius of Gyration

Ring

$dI = dm\,r^2$

$I = R^2\int dm \Rightarrow I = MR^2$



Disc:

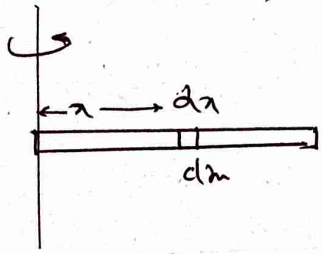
Elementary mass

$$dm = \sigma\, da = \left(\frac{M}{\pi R^2}\right) r\, dr\, d\theta$$

$$I = \int dm\, r^2$$

$$= \int \left(\frac{M}{\pi R^2}\right) r\, dr\, d\theta\, r^2 = \frac{M}{\pi R^2}\int_0^R r^3 dr \int_0^{2\pi} d\theta = \frac{1}{2}MR^2$$

Rectangle

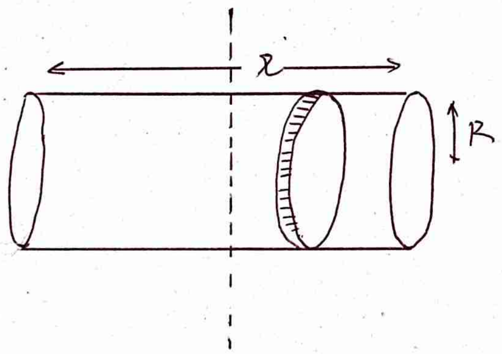


Elementary mass

$$dM = \frac{M}{L}\, dx \qquad I = \int dm\, x^2$$

$$I = \int \frac{M}{L} x^2 dx = \frac{1}{3}ML^2$$

About mid of the axis $I = \frac{1}{12}ML^2$

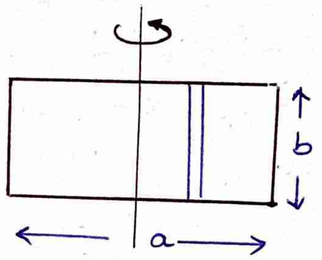


Mass density $\sigma = \dfrac{M}{2\pi R l}$

$$dA = 2\pi R\, dx$$

$$dM = \sigma\, dA = 2\pi R\, dx \times \frac{M}{2\pi R l}$$

$$= \frac{M}{l}\, dx$$



$$dm = (b\, dx)\,\sigma$$

$$I = \int x^2 dm = \sigma b \frac{a^3}{12}$$

$$= \frac{M}{ab} \times b \frac{a^3}{12}$$

$$= \frac{1}{12}Ma^2$$

$$dI = \frac{1}{2}dM R^2$$

$$dI = \frac{1}{2}\frac{M}{l}R^2 dx$$

From Parallel axis theorem $dI' = \frac{1}{2}\frac{M}{l}R^2 dx + \frac{M}{l}dx \cdot x^2$

$$I' = \int_{-l/2}^{l/2} dI' = \frac{1}{2}mr^2 + \frac{1}{12}ml^2$$

Simmilarly for Solid Cylinder

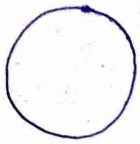
$$I = \frac{1}{4}MR^2 + \frac{1}{12}Ml^2$$

Time Period of Compound Pendulum $T = 2\pi\sqrt{\frac{I}{mgl}}$

l: distance b/w Pivoted Point & COM Point

A loop of radius R is Pivoted at a point on the Circumference. The Time Period

=)



$I = 2MR^2$

$l = R$

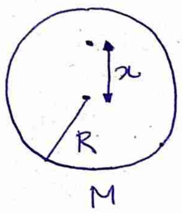
$T = 2\pi\sqrt{\frac{I}{mgl}} = 2\pi\sqrt{\frac{2R}{g}}$

$= 2\pi\sqrt{l/g}$

$l = 2R$ is equivalent length of the Pendulum

A Circular disc of radius R is suspended from a point on its surface such that it oscillates about an axis passing through point of suspension and perpendicular to its plane. For what distance a point of suspension from centre of disc time Period of oscillation is minimum

=)



Time Period

$T = 2\pi\sqrt{\frac{I}{mgl}}$

$T = 2\pi\sqrt{\frac{\frac{MR^2}{2} + Mx^2}{mxg}}$

$y = \frac{MR^2}{2mgx} + \frac{mx^2}{mgx}$

$y = \frac{R^2}{2gx} + \frac{x^2}{gx}$

$\frac{dy}{dx} = -\frac{R^2}{2gx^2} + \frac{1}{g}\left(\frac{2x^2 - x^2}{x}\right)$

$x = R/\sqrt{2}$

Angular Momentum

It is defined as $\vec{L} = \vec{r} \times \vec{p}$

$$\frac{d\vec{L}}{dt} = \vec{r} \times \frac{d\vec{P}}{dt} + \frac{d\vec{r}}{dt} \times \vec{P}$$

$\boxed{rF_{\perp} = Fr_{\perp}}$

$$\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} + m(\vec{v} \times \vec{v})$$

$\vec{r} \times \vec{F} = \vec{\tau}$

$$\frac{d\vec{L}}{dt} = \vec{\tau}$$

$\boxed{|\vec{\tau}| = r_{\perp} F = rF\sin\theta}$

$\vec{\tau}_{ext} = 0 \quad \vec{L} = \text{Constant}$

① A particle of mass m is bound by a linear Potential $u = Kr$ it moves in a circular orbit of radius r_0. If the Particle is slightly disturbed from its circular motion. Frequency of small osscillat

⇒ $V_{eff} = \frac{L^2}{2mr^2} + Kr$

$\frac{L^2}{mr^3} = K$

$\frac{\partial V_{eff}}{\partial r} = -\frac{L^2}{mr^3} + K = 0$

$r = \left(\frac{L^2}{mK}\right)^{1/3}$

$$\frac{\partial^2 V_{eff}}{\partial r^2} = \frac{3L^2}{mr^4} = \frac{3L^2}{m}\left(\frac{L^2}{mK}\right)^{4/3} = \frac{3L^2 \cdot L^{8/3}}{m \cdot m^{4/3} \cdot K^{4/3}}$$

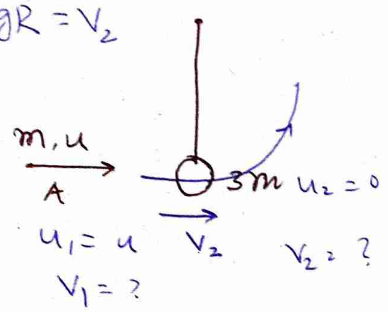
$$\frac{\partial^2 V_{eff}}{\partial r^2} \Big/ m = \frac{3L^{14/3}}{m^{10/3} K^{4/3}}$$

$$\omega = \sqrt{\frac{3K}{mr_0}}$$

② A Particle of mass 3m is hung from a pivol through a light string of length l. If a particle of mass m Collides headon elastically with it. the main value of u to shat B Completes a Circular revolution about o is equal B

⇒

$V_{min} = \sqrt{5gR} = V_2$



$$V_2 = \frac{2m_1u_1}{m_1+m_2} + \frac{m_2-m_1}{m_2+m_1} u_2$$

$$V_2 = \frac{2mu}{4m} = \frac{u}{2}$$

$$u = 2V_2 = 2\sqrt{5gR}$$

③ A particle of mass M moves along the x-axis under influence of $-Kx\exp(-\alpha x) = V(x)$ Period of small oscillation about eqbm Position

$\Rightarrow$ $V(x) = -Kxe^{-\alpha x}$

$$\frac{\partial V(x)}{\partial x} = -K\left[e^{-\alpha x} - \alpha e^{-\alpha x} \cdot x\right] = -Ke^{-\alpha x}(1-\alpha x)$$

$$1-\alpha x = 0$$

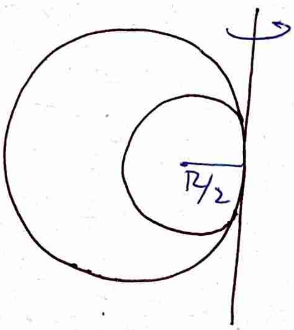
$$x = 1/\alpha$$

$$\frac{\partial^2 V(x)}{\partial x^2} = -K\left[-\alpha e^{-\alpha x} - \alpha\left(e^{-\alpha x} - \alpha x e^{-\alpha x}\right)\right]$$

$$= -K\left[-\alpha e^{-1} - \alpha\left(e^{-1} - e^{-\alpha}\right)\right] = \frac{K\alpha}{e}$$

frequency $\omega = \sqrt{\frac{\partial^2 V}{\partial x^2}/m} = \sqrt{\frac{K\alpha}{me}}$

④ If the object hass mass M. MOI about the given axis is.



Volume of the Sphere

$$\frac{4}{3}\pi R^3 - \frac{4}{3}\pi\left(\frac{R}{2}\right)^3$$

$$\frac{4}{3}\pi R^3\left(1-\frac{1}{8}\right) = \frac{28}{24}\pi R^3$$

$$8 \times \frac{28}{24}\pi R^3 = M$$

$$1 = \frac{M \times 24}{28 8 \pi R^3}$$

$$\frac{4}{3}\pi R^3 = \frac{M \times 24\ 8}{7\ 28\ \cancel{\pi} R^3} \times \frac{4}{\cancel{3}}\cancel{\pi}\cancel{R^3} = \frac{8}{7}M$$

cuting Portion $= \frac{1}{7}M$

Moment of inertia about tangential axis

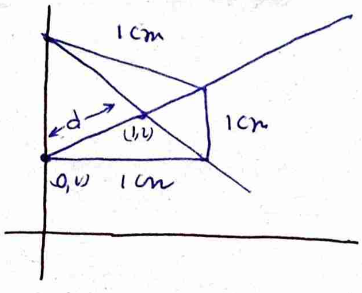
$$\frac{7}{5}\cdot\left(\frac{8M}{7}\right) \times R^2 = \frac{8M}{5}R^2$$

$$\frac{7}{5}\cdot\left(\frac{M}{7}\right) \times \left(\frac{R}{2}\right)^2 = \frac{1}{20}MR^2$$

$$I = \left(\frac{8}{5} - \frac{1}{20}\right)MR^2 = \frac{31}{20}MR^2$$

(5) A uniform plate of side 1 cm lie in xy plane. Diagonal of the plate are along the st. line $y = x+1$ and $y = -x+3$. If moment of inertia of the plate about z-axis be $62\ gm \cdot cm^2$. Mass of the square plate is —— gm

$$\frac{M}{6}a^2 + Md^2 = 62$$



$$M\left[\frac{a^2}{6} + d^2\right] = 62$$

$$M\left[\frac{1}{6} + 5\right] = 62$$

$$M \times \frac{31}{6} = 62 \Rightarrow M = 12\ gm$$

(6) A circular disc of radius R and mass M has non uniform surface mass density $\sigma = ar^3$. MOI about perpendicular central axis is

$$\Rightarrow I = \int r^2 dm = \int r^2 \sigma 2\pi r dr = 2\pi a \int_0^a r^6 dr = \frac{2\pi a R^7}{7}$$

$$M = \int dm = \int_0^R ar^3 2\pi r dr = \frac{2\pi a R^5}{5} \Rightarrow a = \frac{5M}{2\pi R^5}$$

$$I = \frac{2\pi R^7}{7} \times \frac{5M}{2\pi R^5} = \frac{5}{7}MR^2$$

(7) A modified oscillator satisfies

$$\ddot{x} + 2\lambda\dot{x} + 25x = 0$$

$$\lambda = \begin{cases} 0 & x < 0 \\ 4 & x > 0 \end{cases}$$

For different ω time period

Time period of oscillation is

$$\boxed{T = \frac{\pi}{\omega_1} + \frac{\pi}{\omega_2}}$$

$\Rightarrow$ for $x < 0$

$$\ddot{x} + 25x = 0$$

$$\frac{d^2x}{dt^2} + 5^2 x = 0$$

$$\omega = 5$$

for $x > 0$

$$\ddot{x} + 8\dot{x} + 25x = 0$$

Roots $m = -4 \pm 3i$

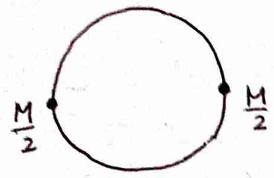
$$x(t) = e^{-4t}(A\cos 3t + B\sin 3t)$$

$$\omega = 3$$

$$T = \frac{\pi}{\omega_1} + \frac{\pi}{\omega_2} = \pi\left(\frac{1}{5} + \frac{1}{3}\right)$$

$$T = \frac{8\pi}{15}$$

06.07.2024

① 

Mass of the disc is M and the angular velocity ω. After putting the mass gentelly what will be angular velocity

$\Rightarrow$ Initial angular momentum $L_i = I\omega = \frac{1}{2}MR^2\omega$

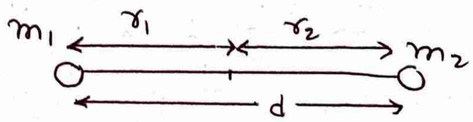
Final angular momentum $L_f = [\frac{1}{2}MR^2 + MR^2]\omega'$

$$L_f = \frac{3}{2}MR^2\omega'$$

So. $\frac{1}{2}MR^2\omega = \frac{3}{2}MR^2\omega' \Rightarrow \omega' = \omega/3$

② The MOI of a system of two particles having masses m_1 and m_2 seperated by distance d. about centre of mass is

$\Rightarrow$



From moment of mass $m_1r_1 = m_2r_2$ and $r_1 + r_2 = d$

$r_1 = \frac{m_2r_2}{m_1}$

$r_1 = d - r_2$

$r_1 = \frac{m_2 \cdot d}{m_1+m_2}$

$r_2 = \frac{m_1 d}{m_1+m_2}$

Moment of inertia about centre of mass

$$I = m_1r_1^2 + m_2r_2^2$$

$$= m_1\frac{m_2^2d^2}{(m_1+m_2)^2} + m_2\frac{m_1^2d^2}{(m_1+m_2)^2}$$

$$= \frac{m_1m_2d^2}{(m_1+m_2)^2}(m_1+m_2)$$

$$= \frac{m_1m_2}{m_1+m_2}d^2$$

power $P = \bar{F}\cdot\bar{V}$

$= \bar{\tau}\cdot\bar{\omega}$

For a Cylinder of mass M, L. moment of inertia $I = M\left(\frac{R^2}{4} + \frac{L^2}{12}\right)$

If Such a Cylinder is made for a given mass of material the ratio L/R for min Possible I is

$\Rightarrow$ Mass is Same

$$I = M\left(\frac{R^2}{4} + \frac{L^2}{12}\right) \qquad MR^2L\rho = C$$

$$R^2L = C$$

$$L = C/R^2$$

$$I = M\left(\frac{C}{4L} + \frac{L^2}{12}\right) \qquad \frac{dI}{dL} = 0$$

$$\frac{C}{4L^2} = \frac{2L}{12} \Rightarrow \frac{C}{L^2} = \frac{2}{3}L$$

$$\Rightarrow \frac{R^2}{L^2} = \frac{2}{3} \qquad \frac{L}{R} = \sqrt{\frac{3}{2}}$$

A particle of mass m is moving with a uniform Speed v in xy plane along a st line $y = x + k$. Angular momentum of the Particle about origin is

$$y = mx + c \qquad L = mvr\sin\theta$$

$$m = 1 = \tan\theta \qquad = mvr\sin 45 = \frac{mvr}{\sqrt{2}}$$

$$\theta = 45^\circ$$

A mass of 80 Kg is standing on the rim of a Circular Platform of mass 200 Kg. The Platform rotates about its axis at 12 rpm The man moves from rim to centre of Platform Angular velocity of The System is

$\Rightarrow$ $I_1 = \frac{MR^2}{2} + mR^2 = I_1 \qquad \omega_1 = 12\ \text{RPM}$

$$I_2 = \frac{MR^2}{2} \qquad \omega_2 = ?$$

$$I_1\omega_1 = I_2\omega_2$$

$$\left(\frac{M}{2} + m\right)\omega \cancel{R^2} = \frac{M}{2}\omega' \cancel{R^2}$$

$$180 \times 12 = 100\,\omega'$$

$$\omega' = 21.6\ \text{RPM}$$

03.07.2024

A sphere of mass M and radius R is moving on a rough fixed surface having coefficent of friction μ It will attain a min velocity after time.

For pure Rolling motion

$V = r\omega$

Sphere will have min linear velocity If it Rolling

For slipping + Rotation

$V_T \perp V$

$a = \mu g$

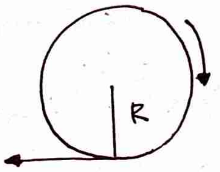
Velocity at any time t

$V(t) = \omega(t) R$

$V_0 - at$

$V_0 - \mu g t = (\omega_0 + \alpha t) R$

$V_0 - \mu g t = \left(\omega_0 + \frac{5}{2}\frac{\mu g}{R} t\right) R$



$V_0 - \omega_0 R = t\left(\frac{5}{2}\frac{\mu g}{R} R + \mu g\right)$

$V_0 - \omega_0 R = \frac{7}{2} \mu g t$

$t = \frac{2(V_0 - \omega_0 R)}{7 \mu g}$

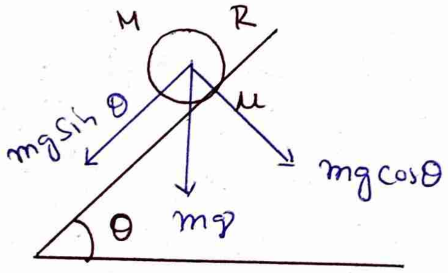
Torque $\tau = fR$

$I\alpha = \mu m g R$

$I = (\mu m g R / \alpha)$

$\frac{2}{5} M R^2 = \frac{\mu m g R}{\alpha}$

$\alpha = \frac{5}{2} \frac{\mu g}{R}$



The disc of mass Rolling. The frictional force acting on the disc is

$fr = I\alpha$

$m\alpha = mg \sin\theta - f$

$f = mg \sin\theta - m\alpha$

$= mg \sin\theta - mr\alpha$

$f = \frac{1}{3} mg \sin\theta$

04.06.2024

Moment of Inertia Tensor

Angular momentum $\vec{L} = m(\vec{r} \times \vec{v})$

$$\vec{L} = \sum m_i (\vec{r}_i \times \vec{v}_i)$$

$$= \sum m_i [\vec{r}_i \times (\vec{r}_i \times \vec{\omega})]$$

$$= \sum m_i \{\vec{r}_i (\vec{r}_i \cdot \vec{\omega}) - \vec{\omega}(\vec{r}_i \cdot \vec{r}_i)\}$$

so, $$\vec{L} = \sum m_i \{(x_i\hat{i} + y_i\hat{j} + z_i\hat{k})(\omega_x x_i + \omega_y y_i + \omega_z z_i) - (\omega_x\hat{i} + \omega_y\hat{j} + \omega_z\hat{k})(x_i^2 + y_i^2 + z_i^2)\}$$

or $$\vec{L} = \sum \hat{i} m_i [-\omega_x x_i^2 \mp x_i y_i \omega_y \mp x_i z_i \omega_z + \omega_x(x_i^2+y_i^2+z_i^2)]$$
$$+ \sum \hat{j} m_i [-x_i y_i \omega_x \mp \omega_y y_i^2 \mp y_i z_i \omega_z + \omega_y(x_i^2+y_i^2+z_i^2)]$$
$$+ \sum \hat{k} m_i [-x_i z_i \omega_x \mp \omega_y y_i z_i \mp \omega_z z_i^2 + \omega_z(x_i^2+y_i^2+z_i^2)]$$

$$I_{xx} = \sum m_i [-x_i^2 + (x_i^2+y_i^2+z_i^2)] = \sum m_i (r_i^2 - x_i^2)$$
$$= \sum m_i (y_i^2 + z_i^2)$$

$$I_{xy} = -\sum m_i x_i y_i = I_{yx}$$

$$I_{xz} = -\sum m_i x_i z_i = I_{zx}$$

$$I_{yy} = \sum m_i (x_i^2 + z_i^2)$$

$$I_{zz} = \sum m_i (x_i^2 + y_i^2)$$

$$\vec{L} = L_x\hat{i} + L_y\hat{j} + L_z\hat{k}$$

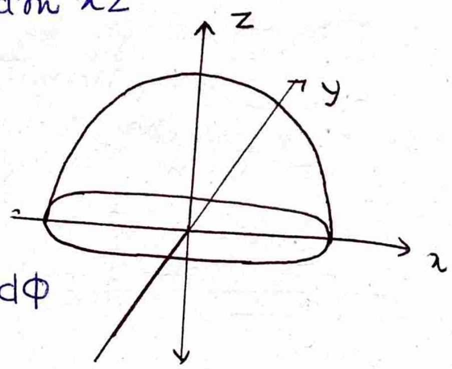
$$\begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix} = \begin{bmatrix} \sum m_i(r_i^2 - x_i^2) & -\sum m_i x_i y_i & -\sum m_i x_i z_i \\ -\sum m_i x_i y_i & \sum m_i (r_i^2 - y_i^2) & -\sum m_i y_i z_i \\ -\sum m_i x_i z_i & -\sum m_i y_i z_i & \sum m_i (r_i^2 - z_i^2) \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

$$\begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

$$I_{xx} = \sum m_i (y_i^2 + z_i^2) = \int dm (y^2+z^2)$$

$$I_{xy} = -\sum m_i x_i y_i = -\int dm\, xy$$

$$I_{xz} = -\sum m_i x_i z_i = -\int dm\, xz$$



Hemisphere

$$dm = \rho\, dV$$

$$dm = \frac{M}{\frac{2}{3}\pi R^3} r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$I_{xx} = \int dm (y^2+z^2)$$

$$I_{xx} = \frac{3M}{2\pi R^3} \int (r^2 \sin\theta\, dr\, d\theta\, d\phi)(r^2 \sin^2\theta \sin^2\phi + r^2 \cos^2$$

$$I_{xx} = \frac{3M}{2\pi R^3}\left[\int_0^R r^4 dr \int_0^{\pi/2} \sin^3\theta\, d\theta \int_0^{2\pi} \sin^2\phi\, d\phi + \int_0^R r^4 dr \int_0^{\pi/2} \cos^2\theta \sin\theta\, d\theta \int_0^{2\pi} d\phi \right] = \frac{2}{5} m r^2$$

$$I_{xy} = \int dm\, xy$$

$$= \frac{3M}{2\pi R^3} \int (r \sin\theta \cos\phi\; r \sin\theta \sin\phi)(r^2 \sin\theta . d\theta\, d\phi\, dr)$$

$$= \frac{3M}{2\pi R^3}\left[\int_0^R r^4 dr \int_0^{\pi/2} \sin^3\theta\, d\theta \int_0^{2\pi} \cos\phi\, d\phi\right] = 0$$

Similarly $I_{xz} = 0 = I_{zx}$ $\quad I_{yz} = 0 = I_{zy}$

$$I_{yy} = \int dm (x^2+z^2) = \frac{2}{5} MR^2$$

$$I_{zz} = \int dm (x^2+y^2) = \frac{2}{5} MR^2$$

(Perpendicular axis theorem is valid in 3D object)

$$I = \begin{bmatrix} \frac{2}{5}MR^2 & 0 & 0 \\ 0 & \frac{2}{5}MR^2 & 0 \\ 0 & 0 & \frac{2}{5}MR^2 \end{bmatrix}$$

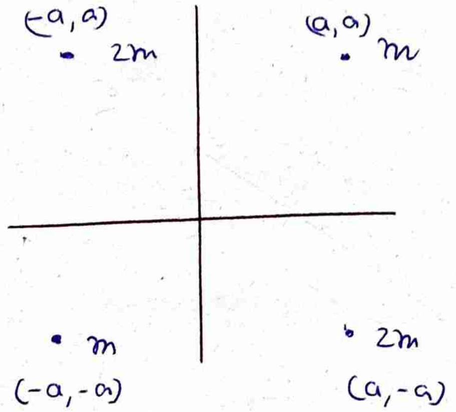
Moment of inertia tensor only diagonal term Present

⇓

Principal axis's

① Two masses m each, are placed at the Points (a,a) and $(-a,-a)$ and two masses $2m$ each are placed at the Points $(a,-a)$ and $(-a,a)$ The principal moment of inertia of the system be

=)



$I_{xx} = \sum m_i (z_i^2 + y_i^2)$

$= m(0+a^2) + 2m(a^2)$
$+ ma^2 + 2ma^2$

$= 6ma^2$

$I_{yy} = \sum m_i (x_i^2 + z_i^2)$

$= 6ma^2$ $\quad I_{zz} = 12ma^2$ $\quad I_{zx} = 0 = I_{zy}$

$I_{yx} = 2ma^2$ $\quad I_{xy} = 2ma^2$

$$\begin{bmatrix} 6ma^2 & 2ma^2 & 0 \\ 2ma^2 & 6ma^2 & 0 \\ 0 & 0 & 12ma^2 \end{bmatrix}$$

Moment of inertia about Principal axis

Eigen value $\quad 4ma^2, 8ma^2, 12ma^2$

To get Corresponding Eigen vector

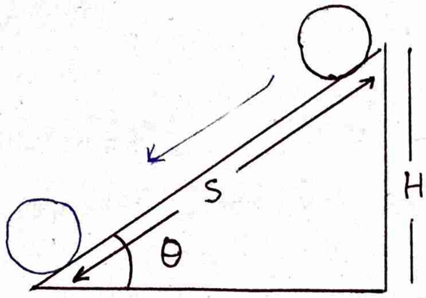
$$\begin{bmatrix} -2 & 2 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$[1 \; 1 \; 0]$ $\quad -2x_1 + 2x_2 = 0$

$x_1 = x_2$

$x\hat{i} + y\hat{j} + z\hat{k}$ $\quad x_3 = 0$

$= \hat{i} + \hat{j}$



$KE = \frac{1}{2} m v^2 \left(1 + \frac{K^2}{R^2}\right)$

$mgH = \frac{1}{2} m v^2 \left(1 + \frac{K^2}{R^2}\right)$

$\sin\theta = \frac{H}{S}$

$$\boxed{v = \sqrt{\frac{2gH}{(1 + K^2/R^2)}}}$$

If there is no rotation $v = \sqrt{2gH}$

$v^2 = u^2 + 2as$

$$\frac{2gH}{1 + \frac{K^2}{R^2}} = 2as$$

$$\boxed{a = \frac{g\sin\theta}{1 + K^2/R^2}}$$

If there no Rolling

$a = g\sin\theta$

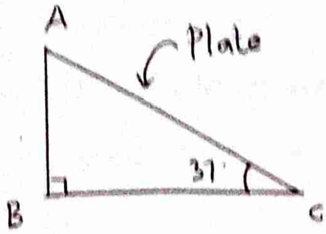
$$\frac{2gH}{1 + \frac{K^2}{R^2}} = 2a \times \frac{H}{\sin\theta}$$

$V = u + at$

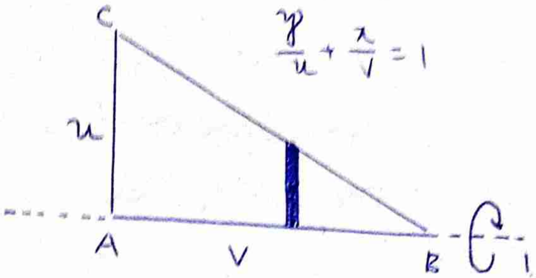
$$\sqrt{\frac{2gH}{1 + K^2/R^2}} = \frac{g\sin\theta}{1 + K^2/R^2} t$$

$$t = \left[\sqrt{\frac{2H}{g}\left(1 + \frac{K^2}{R^2}\right)}\right] \times \frac{1}{\sin\theta}$$

$$\boxed{t = \frac{1}{\sin\theta}\sqrt{\frac{2H}{g}\left(1 + \frac{K^2}{R^2}\right)}}$$

(b) 

Moment of inertia about AB, BC, AC are I_1, I_2 and I_3 then



→

$$I_1 = \int (dm)\, y^2$$

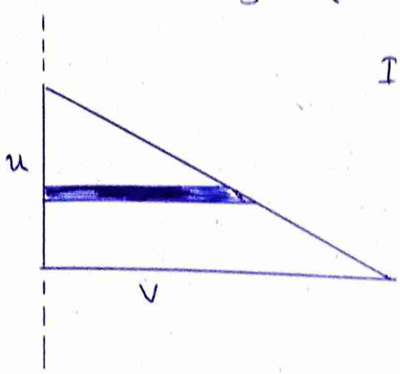
$$= \sigma \int y^2 dx\, dy$$

$$= \sigma \int_0^v dx \int_0^{u(1-\frac{x}{v})} y^2 dy = \sigma \int_0^v dx\, \frac{1}{3} u^3 \left(1-\frac{x}{v}\right)^3$$

$$= \frac{\sigma u^3}{3} \int_0^v \left(1 - \frac{x^3}{v^3} - 3\frac{x}{v} + \frac{3x^2}{v^2}\right) dx$$

$$= \frac{\sigma u^3}{3} \left. x - \frac{x^4}{4v^3} - \frac{3x^2}{2v} + \frac{3x^3}{3v^2} \right|_0^v$$

$$= \frac{\sigma u^3}{3} \left(v - \frac{v}{4} - \frac{3}{2}v + v\right) = \frac{\sigma}{12} u^3 v$$



$$I_2 = \sigma \int dx\, dy\, x^2$$

$$= \sigma \int_0^v x^2 dx \int_0^{u(1-\frac{x}{v})} dy$$

$$= u\sigma \int_0^v x^2\left(1-\frac{x}{v}\right) dx = u\sigma \int_0 \left(x^2 - \frac{x^3}{v}\right) dx$$

$$= u\sigma \left. \frac{x^3}{3} - \frac{x^4}{4v} \right|_0^v = \frac{u\sigma v^3}{12}$$

$I_1 = \frac{\sigma}{12} u^3 v$

$I_2 = \frac{\sigma}{12} u v^3$ $\quad \theta = 37 \quad \frac{u}{v} = \frac{3}{4} \quad I_2 > I_1$

(c) If the diameter of Earth is increased by 4% without changing the mass. Length of the day

→ $I_1\omega_1 = I_2\omega_2$ $\quad D_1 = 2R \quad D_2 = 2R \times \frac{104}{100}$

$R^2 \times \frac{2\pi}{24} = R^2 \times \left(\frac{104}{100}\right) \times \frac{2\pi}{T}$ $\quad R_1 = R \quad R_2 = R \times \frac{104}{100}$

$\frac{1}{24} = \frac{104}{100} \times \frac{1}{T}$ $\quad T = 24.96$ hours.

Fluid Mechanics 08.07.2024

Anything that can flow is fluid

Air, water, Mercury

- Statics
- fluid dynamics
- Surface Tension

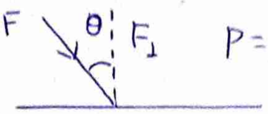
Ideal fluid

① Incompressible $\rho \neq \rho(P)$

② Non viscous (No tangential force act on it)

Exerted fluid by a fluid

Pressure $p = \frac{F_\perp}{A}$



$p = \frac{F\cos\theta}{A}$

(For isotrophic / Homogenous medium pressure is scalar quantity but otherwise it is tensor)

unit of Pressure N/m^2 (Pa)

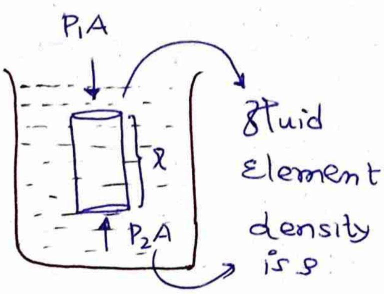
$1 \text{ atm} = 1.013 \times 10^5 \text{ Pa}$ $\quad$ $1 \text{ atm} = 76 \text{ cm Hg}$

$1 \text{ Bar} = 1 \times 10^5 \text{ Pa} = 1 \text{ atm}$ $\quad$ $1 \text{ Torr} = 1 \text{ mm of Hg}$

Pascal's Law

Ideal fluid Exert same Pressure in all direction

Gauge Pressure



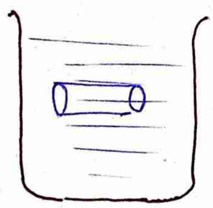
As fluid Element is in Eqbm $F_{net} = 0$

$(P_2A - P_1A) = mg$

$(P_2 - P_1)A = \rho A h g$

$P_2 - P_1 = \rho g h$ $\quad$ $\Delta P = \rho g h$ $\quad$ $\boxed{\Delta P = -\rho g \Delta z}$

(−) indicates with height ↑ Pressure decrease



Here

$P_2 - P_1 = \rho g h$

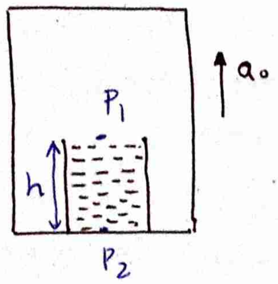
$P_2 - P_1 = 0$

$P_2 = P_1$

so for same height Pressure will same

⇓

Pascal's Law



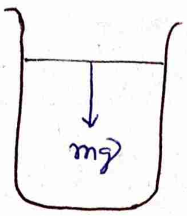
Here $P_2 = P_1 + \rho g_{eff} h$

$$P_2 = P_1 + \rho(g + a_0)h$$

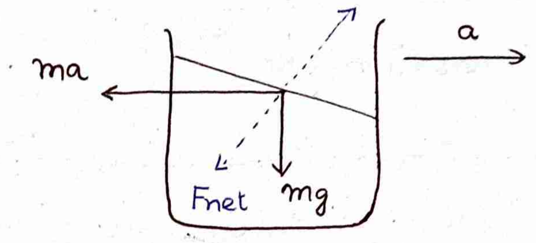
If lift goes downward

$$P_2 = P_1 + \rho(g - a_0)h$$

In any free surface of fluid resultant force is always perpendicular to its surface



(Stationary)



(Here also Fnet is $\perp$ to the surface)

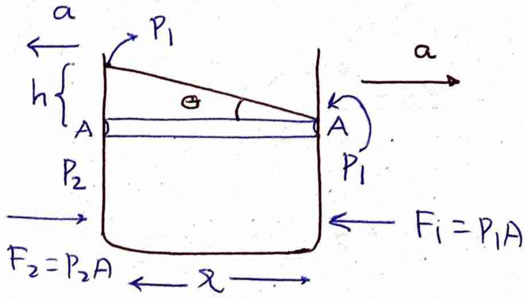
Horizontal Acceleration

So from Newton's 2^{nd} law of motion

$$P_2 A - P_1 A = ma$$

$$P_2 A - P_1 A = \rho x A a$$

$$P_2 - P_1 = \rho x a \quad \text{—(i)}$$



Also $P_2 - P_1 = \rho g h$ —(ii)

So $\rho g h = \rho x a$

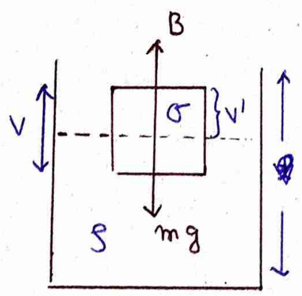
$$\frac{h}{x} = \frac{a}{g} = \tan\theta$$

$$\boxed{a = g\tan\theta}$$

09.07.2024

Density = $\dfrac{\text{Total mass}}{\text{Total volume}}$

Buoyancy

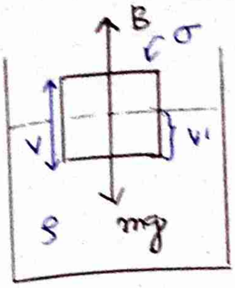


Submerged volume = $v - v'$

Buoyancy force

$$B = \rho(v - v')g$$

weight of displaced liquid

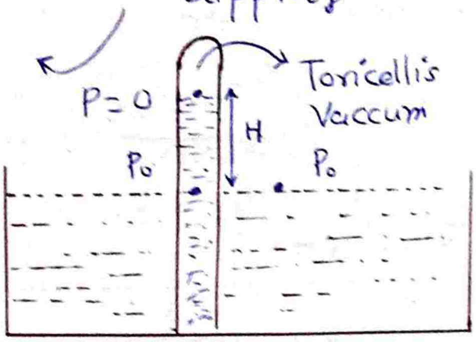


$B = mg$

$\rho v' g = \sigma V g$

$$\boxed{\frac{v'}{V} = \frac{\sigma}{\rho}}$$

Percentage of floating and dipping



<u>Barometer</u>

So $\quad P_0 = P + \rho g h$

$\Rightarrow \quad P_0 = \rho g h$

$\Rightarrow \quad P_0 = 13.6 \times 10^3 \times 9.8 \times 0.76 = 1.01 \times 10^5 \ N/m^2$

If we use water $\quad \rho = 1000 \ Kg/m^3$

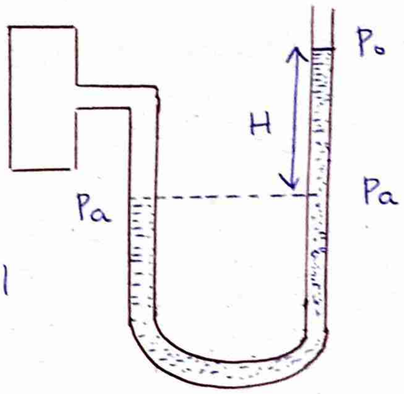
$P = \rho g H \qquad P = 1.013 \times 10^3 \ N/m^2$

$g = 10 \ m/s^2$

$10^5 = 10^3 \times 10 \times H$

$H = 10m \qquad$ So height be so big.

<u>Manometer</u>

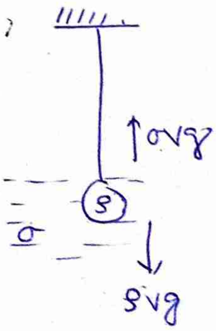


$P_a = P_0 + \rho g H$

If density is low. H will increase so we can measure more Precisly and Error will decrease.

1) Time Period of a Simple Pendulum is T. The relative density of the bob with respect to water is n. The bob is put into water and oscillate. The time period will be

$\Rightarrow$



$T = 2\pi \sqrt{\frac{l}{g_{eff}}}$

$T = 2\pi \sqrt{\frac{l}{g}}$

$r = \frac{\text{density of water}}{\text{density of material}}$

$T' = 2\pi \sqrt{\frac{l}{(\rho - \frac{\sigma}{\rho})g}}$

$T' = 2\pi \sqrt{\frac{l}{g(1 - \frac{1}{n})}}$

$g_{eff} = \frac{\rho V g - \sigma V g}{\rho V} = g\left(1 - \frac{\sigma}{\rho}\right)$

$= g\left(1 - \frac{1}{n}\right)$

$T' = T\sqrt{\frac{n}{n-1}}$

11.07.2024

Bernoulli's Theorem

① **Pressure Enenergy density**

work done $= PA \cdot \Delta x$

and $\dfrac{PA\Delta x}{V} = \dfrac{P(V)}{V} = P$ is Pressure energy density

② **Potential Energy density**

Potential Energy $= mgh$

density $\dfrac{mgh}{V} = \rho gh$

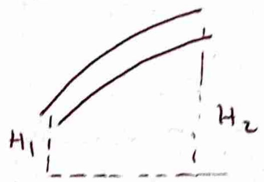
③ **Kinetic Energy density**

Kinetic energy, $KE = \frac{1}{2}mv^2$

density $= \dfrac{1}{2}\dfrac{mv^2}{V} = \frac{1}{2}\rho v^2$

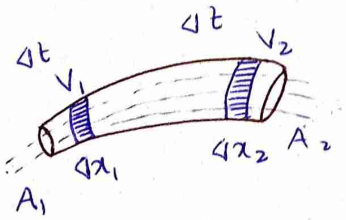
The energy density at a particular time is always constant. (Total energy density)

$$\boxed{U = P + \rho gh + \frac{1}{2}\rho v^2}$$



$$P_1 + \rho g h_1 + \frac{1}{2}\rho v_1^2 = P_2 + \rho g h_2 + \frac{1}{2}\rho v_2^2$$

Equation of Continuity



water incoming per unit time = water outgoing per unit time

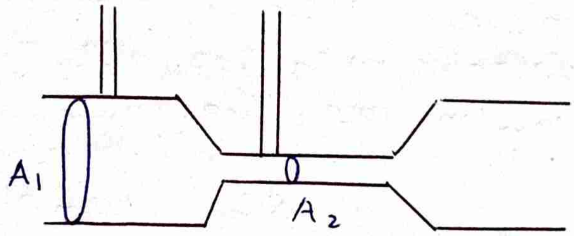
$$\left(\frac{\Delta m}{\Delta t}\right)_1 = \left(\frac{\Delta m}{\Delta t}\right)_2$$

$$\frac{\rho A_1 V_1 \Delta t}{\Delta t} = \frac{\rho A_2 V_2 \Delta t}{\Delta t}$$

$$\boxed{A_1V_1 = A_2V_2}$$

Venturimeter

From Bernoulis Equation we have



$$P_1 + \rho g H_1 + \frac{1}{2}\rho V_1^2 = P_1 + \rho g H_2 + \frac{1}{2}\rho V_2^2$$

At same height $H_1 = H_2$

$$P_1 + \frac{1}{2}\rho V_1^2 = P_2 + \frac{1}{2}\rho V_2^2$$

$$P_1 - P_2 = \frac{1}{2}\rho (v_2^2 - v_1^2)$$

$$\boxed{P_1 - P_2 = \frac{1}{2}\rho v_1^2 \left[\left(\frac{A_1}{A_2}\right)^2 - 1\right]}$$

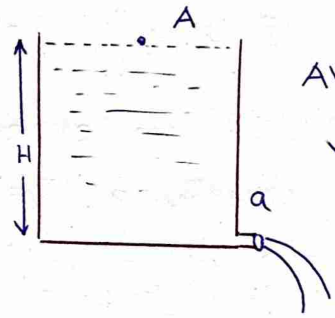
Now $A_1V_1 = A_2V_2$

$V_2 = \left(\frac{A_1}{A_2}\right)V_1$

$A_1 > A_2$

$V_2 > V_1$

Case If $A >> a$



$AV_1 = av_2$

$V_1 = \frac{a}{A} v_2$

$\frac{a}{A} << 1$

$V_1 = 0$

$$P_1 + \rho g H_1 + \frac{1}{2}\rho v_1^2 = P_2 + \rho g H_2 + \frac{1}{2}\rho v_2^2$$

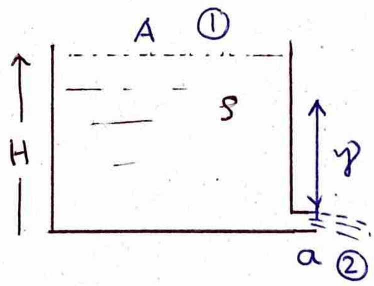
$(P_1 = P_2 =$ atmospheric pressure$)$

$$\rho g H + \frac{1}{2}\rho 0^2 = 0 + \frac{1}{2}\rho v^2$$

$$\boxed{v = \sqrt{2gH}} \quad \text{(Torricell's Law)}$$

12.07.2024

Required time for Empty



Eflux velocity $v_2 = \sqrt{2gy}$

$$aV_2 = AV_1$$

$$A\left(-\frac{dy}{dt}\right) = a\sqrt{2gy}$$

$$-\int_H^0 \frac{dy}{\sqrt{2gy}} = \frac{a}{A}\int_0^t dt$$

$$\boxed{-\int \frac{dy}{\sqrt{2gy}} = \frac{a}{A}\int dt}$$

Key points

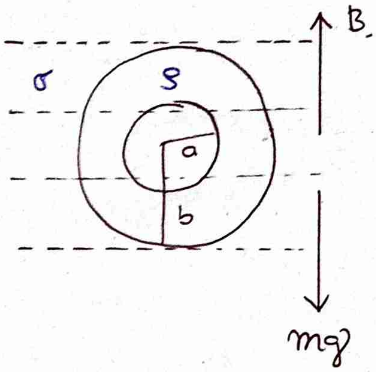
$$\frac{1}{\sqrt{2g}} \cdot \frac{y^{1/2}}{1/2}\Bigg|_0^H = \frac{a}{A} t$$

$$\sqrt{\frac{2}{g}}\sqrt{H} = \frac{a}{A} t$$

$$t = \frac{A}{a}\sqrt{\frac{2H}{g}}$$

① A Spherical aluminium ball of mass 1.26 Kg. Contains an empty spherical Cavity that is Concentric with the ball. The ball just barely floats in water. The radius of the Cavity is

⇒



$mg = B$

$\rho\left[\frac{4}{3}\pi(b^3 - a^3)\right]g = \sigma\frac{4}{3}\pi b^3 g$

$\rho(b^3 - a^3) = \sigma b^3$ — ①

$\frac{0.945}{1000} = b^3$

$1.26 = \rho\left[\frac{4}{3}\pi(b^3 - a^3)\right]$

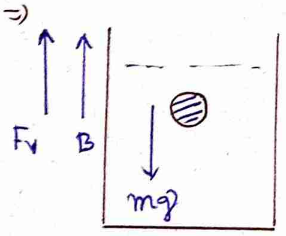
$b = 5.74$ cm

$1.26 = 2.7\times10^3 \times \frac{4}{3} \times 3.14(b^3 - a^3)$

$\rho(b^3 - a^3) = 0.945$

② In an experiment, a small steel ball falls through a liquid at a constant speed of 10 cm/s. If the steel ball is pulled upward with a force equal to twice its effective weight. How fast will it move upward

⇒



Terminal velocity

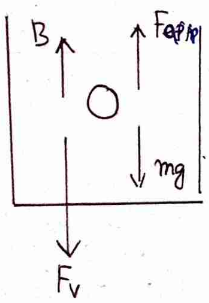
$V_T = 10$ cm/s

$F_v + B = mg$

Effective weight $= mg - B = F_{eff}$

For 2 time effective weight

$2F_{eff} - (mg - B) - 6\pi\eta r v$



$F_{app} = 2F_{eff}$

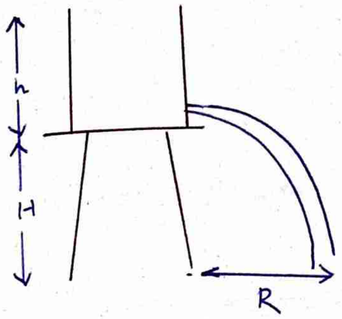
$F_{net} = F_{app} + B - F_v - mg$

$= 2(mg - B) + B - F_v - mg$

$F_{net} = mg - B - F_v$

$F_{net} = 0$

(3) A Cylinder Containing water stands on a table of height H. a small hole is punched in the side of Cylindrical hole. The stream of water strikes the ground at a horizontal distance R from table. The depth of water in the Cylinder is



$v = \sqrt{2gh}$

$t = \sqrt{2H/g}$

$R = vt$

$R = \sqrt{2gh \times \frac{2H}{g}}$

$R = 2\sqrt{Hh}$

$\frac{R^2}{4H} = h$

(4) Consider a fluid to be Compressible. If we assume that the density of air is proportional to pressure then find the Pressure Variation with depth

At ~~top~~. bottom

$\rho_0 = KP_0$

$K = \rho_0 / P_0$

$\rho \propto p$

$\rho = Kp$ — (i)

$\rho = \frac{\rho_0}{P_0} p$

$dP = -\rho g dy$

$dp = -\frac{\rho_0}{P_0} p g dy$

$\int_{P_0}^{P} \frac{dp}{p} = -\frac{\rho_0}{P_0} g \int_0^h dy$

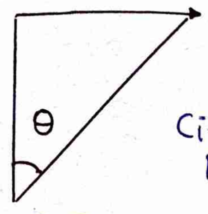
$\ln \frac{P}{P_0} = -\frac{\rho_0}{P_0} g h$

$P = P_0 e^{-\frac{\rho_0 g h}{P_0}}$

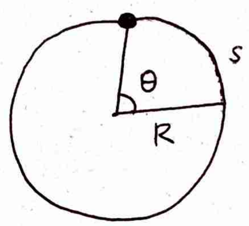
08.06.2024

Circular Motion:

If any object trace of a Circle then it is Circular motion



Also Circular Motion



Linear displacement

$S = R\theta$

Angular velocity :

$$\omega = \frac{d\theta}{dt} = \dot{\theta}$$

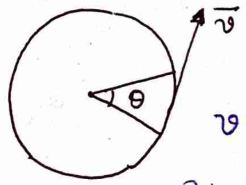
(ω is also axial Vector)

unit of θ is Radian

ω is Rad/sec

θ is axial Vector

(⊥ to the Plane)



$$v = \frac{\Delta s}{\Delta t}$$

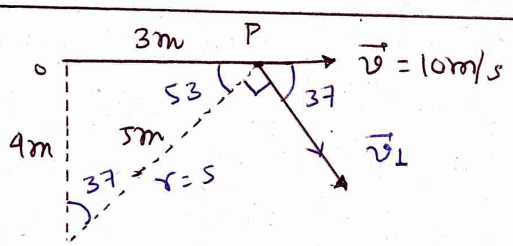
Linear Velocity

$$v = \frac{\Delta s}{\Delta t} = \frac{r\,d\theta}{dt}$$

$$v = r\omega \Rightarrow \boxed{\omega = \frac{v}{r}} \qquad \text{But } \omega = \frac{V_\perp}{r}$$

$V_\perp$ is the Component of velocity ⊥ to r.

In Circular motion $V \perp r$ always. so $\omega = \frac{V_\perp}{r} = \frac{V}{r}$



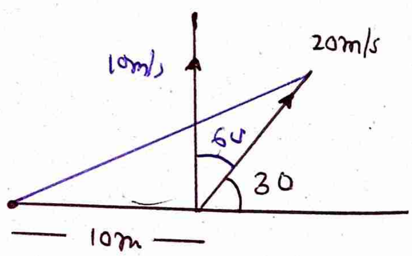
$$\vec{v}_\perp = v \cos 37 = 10 \times \frac{4}{5} = 8m/s$$

$$\omega = \frac{\vec{v}_\perp}{r} = \frac{8}{5} = 1.6 \text{ rad/sec}$$

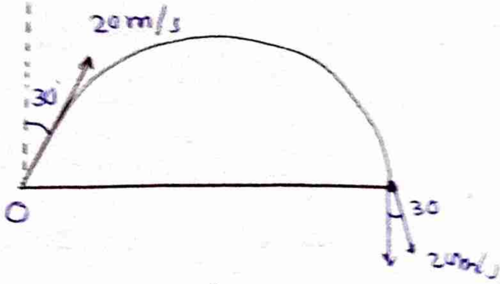
$V_\perp = 20 \cos 60$

$V_\perp = 10 \, m/s$

$$\omega = \frac{V_\perp}{R} = \frac{10}{10} = 1 \text{ rad/sec}$$



Angular velocity at the Point where it drop



Range $R = \frac{u^2 \sin 2\theta}{g}$

$= \frac{(20)^2 \times \sqrt{3}}{2 \times 10}$

$= 20\sqrt{3}\ m$

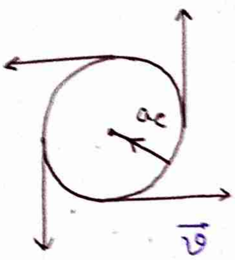
$V_\perp = V\cos 30$

$= 20 \times \frac{\sqrt{3}}{2} = 10\sqrt{3}\ m/s$

$\omega = \frac{V_\perp}{R} = \frac{10\sqrt{3}}{20\sqrt{3}} = \frac{1}{2}\ rad/s$

> **Centripetal Acceleration:**

uniform velocity



Centripetal Acceleration

$a_c = \frac{v^2}{R}$ Direction is always towards the Centre

It changes velocity direction but not magnitude

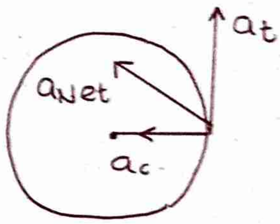
$F = ma_c = \frac{mv^2}{R}$ Always act

> **Tangential Acceleration**

Non uniform velocity

T. Acceleration $\frac{d|v|}{dt} = a_t$

It will act if velocity changes



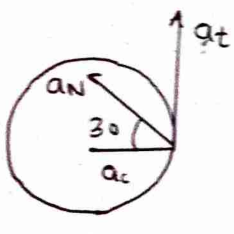
$a_{Net} = \sqrt{a_t^2 + a_c^2}$

If velocity constant $a_t = 0$ only a_c present

$\omega = 4t$, $R = 1m$ t at which a_{Net} makes an angle 30 with a_c. $a_c = ?$ $a_t = ?$ $a_{Net} = ?$

=> $\omega = \frac{V}{R}$ $\tan 30 = \frac{a_t}{a_c}$ $a_t = 4$

$a_t = a_c/\sqrt{3}$



$V = 4t$

$a_c = \frac{V^2}{R} = 16t^2$

$\frac{1}{\sqrt{3}} = \frac{4}{16t^2}$

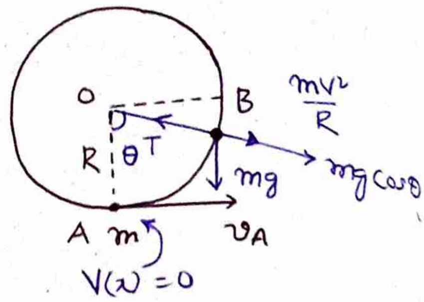
$t = \sqrt{\frac{\sqrt{3}}{4}}$ sec

Circular Motion in a Vertical Plane

From Equilibrium Condition

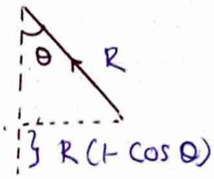
$$\frac{mv^2}{R} + mg\cos\theta = T$$

$$T - mg\cos\theta = \frac{mv^2}{R}$$



From 0 to 90 $T \not< 0$

So Slacking not Possible



$$KE_i + PE_i = KE_f + PE_f$$

$$\frac{1}{2}mV_A^2 + 0 = \frac{1}{2}mV_B^2 + mgR(1-\cos\theta)$$

(B is any Point)

$$V_B^2 = V_A^2 - 2gR(1-\cos\theta)$$

$$V_B = \sqrt{V_A^2 - 2gR(1-\cos\theta)}$$

$T < 0$

Slacking

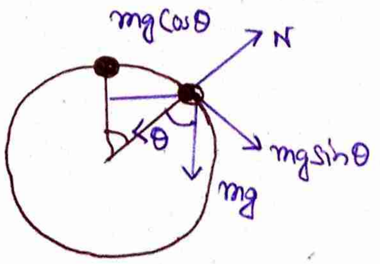
$$T = \frac{m}{R}\left[V_A^2 - 2gR(1-\cos\theta)\right] + mg\cos\theta$$

$$T = \frac{mV_A^2}{R} + 3mg\cos\theta - 2mg$$

Angular acceleration $\alpha = \frac{d\omega}{dt}$

$$\alpha = \frac{d}{dt}\left(\frac{V}{R}\right) = \frac{1}{R}\frac{dV}{dt} = \frac{1}{R}a_t$$

(a_t is tangential acceleration)



No fall of object

$$mg\cos\theta - N = \frac{mv^2}{R}$$

$$N = mg\cos\theta - \frac{mv^2}{R}$$

$$mgR = \frac{1}{2}mv^2 + mgR\cos\theta$$

$$v^2 = 2gR - 2gR\cos\theta$$

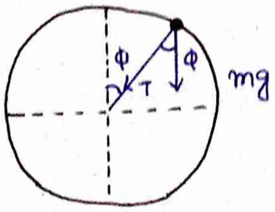
$$N = mg\cos\theta - 2gm + 2mg\cos\theta$$

$$N = 0$$

$$\cos\theta = 2/3$$

$$\theta = \cos^{-1}(2/3)$$

09.06.2024



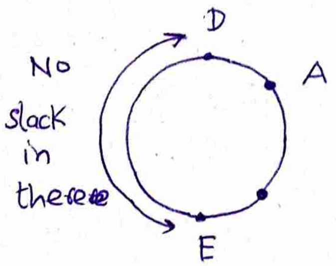
$$T + mg\cos\phi = \frac{mv^2}{R}$$

$$T = \frac{mv^2}{R} - mg\cos\theta$$

For Complete Cycle $T \geq 0$
(No slacking)

$$\frac{mv^2}{R} - mg\cos\theta \geq 0 \Rightarrow v \geq \sqrt{gR\cos\phi}$$

Critical Velocity



$T \geq 0$

Critical velocity for D

$$V \geq \sqrt{gR\cos\phi}$$

$$V \geq \sqrt{gR\cos 0} \Rightarrow V \geq \sqrt{gR}$$

V_E such that particle reaches at D.
By energy conservation

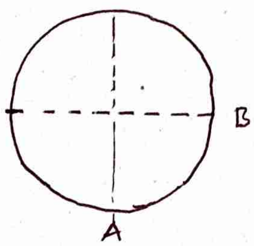
$$\frac{1}{2}mV_E^2 = \frac{1}{2}mV_D^2 + mg(2R)$$

or $\quad V_E^2 = V_D^2 + 4gR$

or $\quad V_E^2 \geq gR + 4gR$

or $\quad V_E \geq \sqrt{5gR}$

1



Minimum velocity at A so that it reached at Point B

$$T = \frac{mV_A^2}{R} - 2mg + 3mg\cos\theta$$

$T \geq 0$ without slacking $\theta = 90$

$$\frac{mV_A^2}{R} - 2mg \geq 0$$

$$V_A = \sqrt{2gR}$$

$\omega = \omega_0 + \alpha t$ $\omega \equiv \frac{2\pi}{60}$

$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$

$\omega^2 = \omega_0^2 + 2\alpha\theta$

10.06.2024

① A particle starts moving in a circular path of radius 2m with initial speed 2m/s at constant angular acceleration after 2 complete rotation. its speed became 8 m/s. Angular acceleration is

=) $R = 2m$ $u = 2m/s$ $\omega_0 = 2/2 = 1 \text{ rad/s}$

$\omega = 8/2 = 4 \text{ rad/s}$ $\omega = \omega_0 + \alpha t$

$\omega^2 = \omega_0^2 + 2\alpha\theta$ $\theta = 2 \times 2\pi = 4\pi$

$16 - 1 = 2 \times 4\pi \times \alpha$ =) $\alpha = 15/8\pi$

② A body rotates with an angular retardation equal to K/ω where ω is the angular velocity of the object. Find the time after which the body comes to rest if initial angular velocity is ω_0

=) Angular retardation $\alpha = -K/\omega$

$\alpha = \frac{d\omega}{dt}$ $\int_{\omega_0}^{0} \omega \, d\omega = -K \int_0^t K dt$

$-\frac{K}{\omega} = \frac{d\omega}{dt}$ $\frac{\omega_0^2}{2} = +Kt$ =) $t = \frac{\omega_0^2}{2K}$

$\omega \, d\omega = -K dt$

③ A particle moves in a circular path of radius 3m, where speed $2t^3$ m/s find a_c, a_t, a at $t=2$

=) $a_t = \frac{dv}{dt} = 6t^2$ $a_c = \frac{v^2}{R} = \frac{4t^6}{R} = \frac{4 \times 2^6}{3} = \frac{256}{3}$

$a_t = 6 \times 4 = 24$

$a = \sqrt{a_c^2 + a_t^2}$

$= \sqrt{576 + \left(\frac{256}{3}\right)^2} = 88.64$

Velocity and Acceleration in different Co-ordinate System

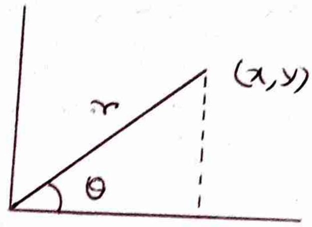
① **Cartesian System**

Position Vector, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\frac{d\vec{r}}{dt} = V_x\hat{i} + V_y\hat{j} + V_z\hat{k} = \vec{V}$$

$$\frac{d\vec{V}}{dt} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k} = \vec{a}$$

② **Polar System**



$$\vec{r} = x\hat{i} + y\hat{j}$$

$$\vec{r} = r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j}$$

$$\vec{r} = r(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}) = r\hat{r}$$

$$\hat{r} = (\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$$

$x = r\cos\theta$

$y = r\sin\theta$

$$\hat{r}\cdot\hat{\theta} = 0$$

$$\hat{\theta} = \frac{\frac{\partial\vec{r}}{\partial\theta}}{\left|\frac{\partial\vec{r}}{\partial\theta}\right|} = (-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})$$

$$\frac{d\hat{r}}{dt} = -\sin\theta\,\hat{i}\,(\dot{\theta}) + \cos\theta\,\dot{\theta}\,\hat{j} = \dot{\theta}\left[-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}\right] = \dot{\theta}\hat{\theta}$$

$$\frac{d\hat{\theta}}{dt} = -\cos\theta\,(\dot{\theta})\,\hat{i} - \sin\theta\,\dot{\theta}\,\hat{j} = -\dot{\theta}\,(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}) = -\dot{\theta}\hat{r}$$

$$\vec{r} = r\hat{r}$$

$$\frac{d\vec{r}}{dt} = r\frac{d\hat{r}}{dt} + \frac{dr}{dt}\hat{r} = r\dot{\theta}\hat{\theta} + \cancel{r\dot{\theta}}\,\dot{r}\hat{r}$$

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} \qquad |\vec{V}| = \sqrt{\dot{r}^2 + r^2\dot{\theta}^2}$$

$$\frac{d\vec{v}}{dt} = \ddot{r}\hat{r} + \dot{r}\frac{d\hat{r}}{dt} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\dot{\hat{\theta}}$$

$$\frac{d\vec{v}}{dt} = \ddot{r}\hat{r} + \dot{r}\dot{\theta}\hat{\theta} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} - r\dot{\theta}\dot{\theta}\hat{r}$$

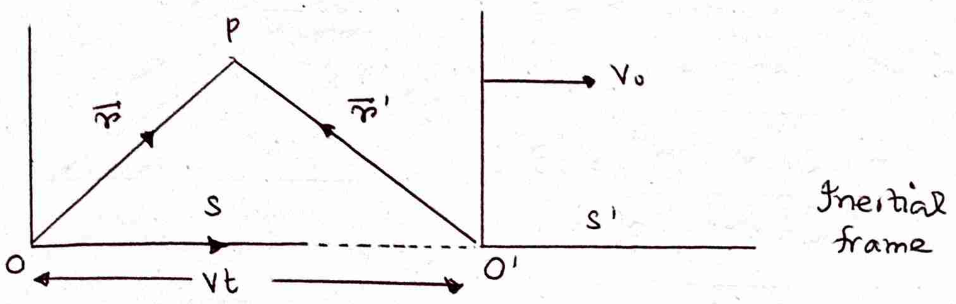
$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{r} = a_r\hat{r} + a_\theta\hat{\theta}$$

$a_r = \ddot{r} - r\dot{\theta}^2$ (Radial)

$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$ (Tangential)

Pseudo Force

Case ①



At $t=0$ O and O' Coincide

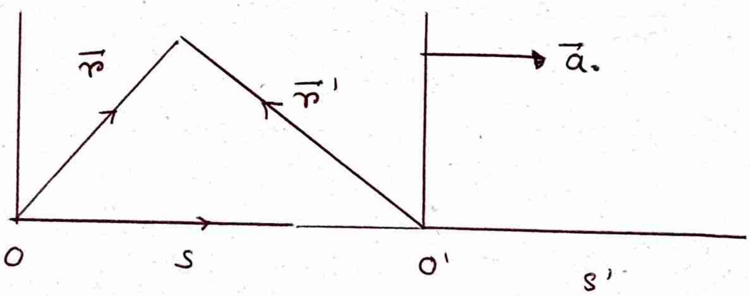
$\vec{r}' + \vec{v}_0 t = \vec{r} \Rightarrow \vec{r}' = \vec{r} - \vec{V}_0 t$

$\Rightarrow \frac{d\vec{r}'}{dt} = \frac{d\vec{r}}{dt} - \vec{v}_0 \Rightarrow \vec{v}' = \vec{v} - \vec{v}_0$

Again differentiation $\frac{d\vec{v}'}{dt} = \frac{d\vec{v}}{dt} - \frac{d\vec{v}_0}{dt}$

$\Rightarrow \vec{a}' = \vec{a} \Rightarrow m\vec{a}' = m\vec{a}$

Case ②



At $t=0$ they Coincide $\quad OO' = \frac{1}{2}\vec{a}_0 t^2$

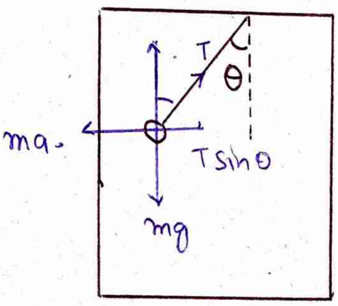
$\frac{1}{2}\vec{a}_0 t^2 + \vec{r}' = \vec{r} \Rightarrow \vec{r}' = \vec{r} - \frac{1}{2} a_0 t^2$

$\Rightarrow \frac{d\vec{r}'}{dt} = \frac{d\vec{r}}{dt} - a_0 t \Rightarrow \vec{v}' = \vec{v} - a_0 t$

On differentiating $\frac{d\vec{v}'}{dt} = \frac{d\vec{v}}{dt} - a_0$ $\qquad \vec{F}_0 = -m\vec{a}_0$

$\vec{a}' = \vec{a} - \vec{a}_0 \Rightarrow \vec{F}' = \vec{F} + \vec{F}_0$

$\vec{F}' = \vec{F} - m\vec{a}_0$ Pseudo force

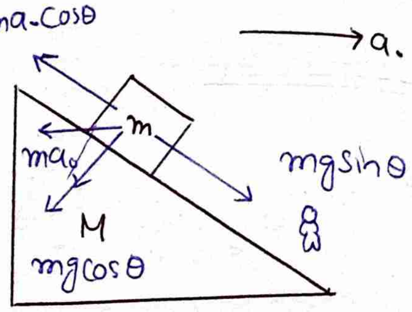


$T\sin\theta = ma_0$

$T\cos\theta = mg$

$\tan\theta = a_0/g$

① What should be the value of a_0 so that m doesn't slip



$\Rightarrow$

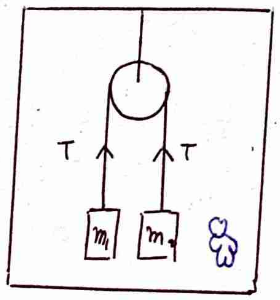
$$ma_0 \cos\theta = mg\sin\theta$$

$$a_0 = g\tan\theta$$

$$N = ma_0\sin\theta + mg\cos\theta$$

② Find the tension of the string if the system is lifting with an acceleration a_0

$\Rightarrow$



$m_1 > m_2$

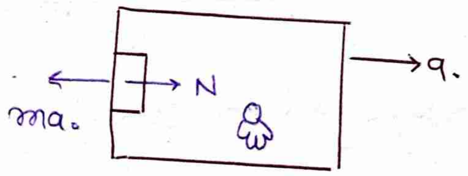
$$m_1 a_0 + m_1 g - T = m_1 a$$
$$T - m_2 g - m_2 a_0 = m_2 a$$

$$\frac{(m_1 - m_2)a_0 + (m_1 - m_2)g}{(m_1 + m_2)} = a$$

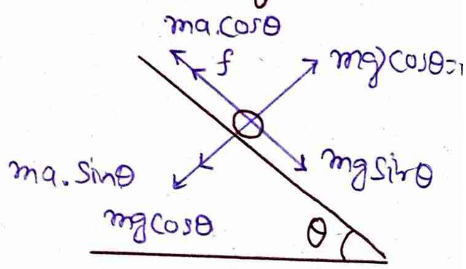
$$T = \frac{2m_1 m_2}{m_1 + m_2}(g + a_0)$$

③ Find the relation so that a_0, g, μ is eqsm

$$\mu N \geq mg$$
$$\mu ma_0 \geq mg$$
$$\mu a_0 \geq g$$



④ If the block doesn't slip then value of θ is



$$ma_0\cos\theta + \mu mg\cos\theta = mg\sin\theta$$
$$a_0\cos\theta + \mu g\cos\theta = g\sin\theta$$
$$\cos\theta\,(a_0 + \mu g) = g\sin\theta$$
$$\tan\theta =$$

$$f = \mu N \; \cancel{= mg\sin\theta}$$
$$= \mu(ma_0\sin\theta + mg\cos\theta)$$

$$f + ma_0\cos\theta = mg\sin\theta$$

24.06.2004

Torque

Torque $\tau \propto F$, $\tau \propto r$, $\tau \propto \sin\theta$

$\tau = rF\sin\theta$

$\vec{\tau} = \vec{r} \times \vec{F}$

(Torque is an axial Vector)

Translational Eqbm

$F_{net} = 0$ $\quad F_x, F_y, F_z = 0$

object at Rest or Constant motion

Rotational Eqbm

$\vec{\tau}_{net} = 0$ $\quad \tau_x = \tau_y = \tau_z = 0$

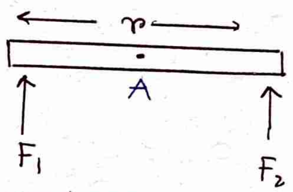
So $\alpha = 0$

$\omega =$ Constant

If both translational & Rotational Eqbm is present, torque about any point will be Zero

Torque about Point A,

$\tau = (F_1 \times \frac{r}{2}) + (F_2 \times \frac{r}{2})$



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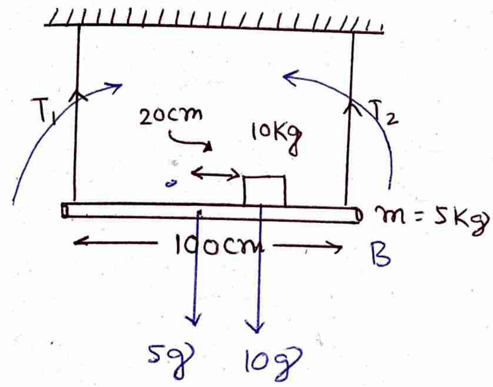
① 

System is in translational and Rotational Eqbm

$F_{net} = 0$

$T_1 + T_2 = (5 \times 10) + (10 \times 10)$

$T_1 + T_2 = 150$ N

For rotational Eqbm

torque about O: $\quad 0.5 \times T_2 - 0.5 \times T_1 - 10g \times 0.2 = 0$

$0.5\,(T_2 - T_1) = 20$

$T_2 - T_1 = 40$

$2T_2 = 190$

$T_2 = 95$ N $\quad T_1 = 55$N

Torque about B $\quad$ Again $T_1 + T_2 = 150$N

$-T_1 \times 1 + (10g \times 0.3) + (5g \times 0.5) = 0$

$T_1 = 30 + 25 = 55$ N $\quad T_2 = 95$ N

Angular Momentum

It is defined as $\vec{L} = \vec{r} \times \vec{p}$

$$\frac{d\vec{L}}{dt} = \vec{r} \times \frac{d\vec{P}}{dt} + \frac{d\vec{r}}{dt} \times \vec{P}$$

$\boxed{rF_{\perp} = Fr_{\perp}}$

$$\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} + m(\vec{v} \times \vec{v})$$

$$\frac{d\vec{L}}{dt} = \vec{\tau}$$

$\vec{r} \times \vec{F} = \vec{\tau}$

$\boxed{|\vec{\tau}| = r_{\perp}F = rF\sin\theta}$

$\vec{\tau}_{ext} = 0 \quad \vec{L} = \text{Constant}$

① A particle of mass m is bound by a linear Potential $u = Kr$ it moves in a circular orbit of radius r_0. If the Particle is slightly disturbed from its circular motion. Frequency of small osscillat

⇒ $V_{eff} = \frac{L^2}{2mr^2} + Kr$

$\frac{L^2}{mr^3} = K$

$r = \left(\frac{L^2}{mK}\right)^{1/3}$

$$\frac{\partial V_{eff}}{\partial r} = \frac{-L^2}{mr^3} + K = 0$$

$$\frac{\partial^2 V_{eff}}{\partial r^2} = \frac{3L^2}{mr^4} = \frac{3L^2}{m}\left(\frac{L^2}{mK}\right)^{4/3} = \frac{3L^2 \cdot L^{8/3}}{m \cdot m^{4/3} \cdot K^{4/3}}$$

$$\frac{\partial^2 V_{eff}}{\partial r^2} \Big/ m = \frac{3L^{14/3}}{m^{10/3}K^{4/3}}$$

$$\omega = \sqrt{\frac{3K}{mr_0}}$$

② A Particle of mass 3m is hung from a pivot 'O' through a light string of length l. If a particle of mass m Collides headon elastically with it. the main value of u to that B Completes a Circular revolution about O is equal to

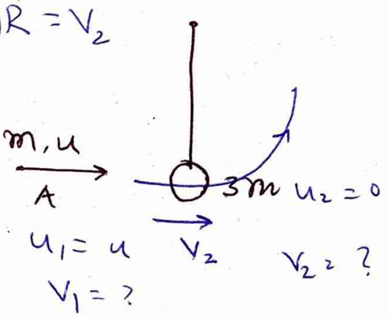
⇒

$V_{min} = \sqrt{5gR} = V_2$

$$V_2 = \frac{2m_1u_1}{m_1+m_2} + \frac{m_2-m_1}{m_2+m_1}u_2$$

$$V_2 = \frac{2mu}{4m} = \frac{u}{2}$$

$$u = 2V_2 = 2\sqrt{5gR}$$



③ A particle of mass M moves along the x-axis under influence of $-Kx\exp(-\alpha x) = V(x)$ Period of small oscillation about eqlm Position

$\Rightarrow$ $V(x) = -Kx e^{-\alpha x}$

$$\frac{\partial V(x)}{\partial x} = -K\left[e^{-\alpha x} - \alpha e^{-\alpha x} x\right] = -Ke^{-\alpha x}(1-\alpha x)$$

$$1 - \alpha x = 0$$

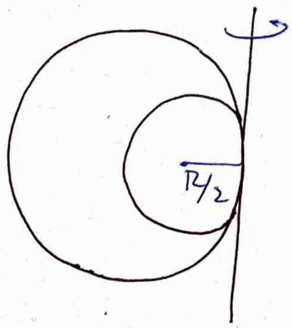
$$x = 1/\alpha$$

$$\frac{\partial^2 V(x)}{\partial x^2} = -K\left[-\alpha e^{-\alpha x} - \alpha\left(e^{-\alpha x} - \alpha x e^{-\alpha x}\right)\right]$$

$$= -K\left[-\alpha e^{-1} - \alpha\left(e^{-1} - e^{-\alpha}\right)\right] = \frac{K\alpha}{e}$$

frequency $\omega = \sqrt{\frac{\partial^2 V}{\partial x^2}/m} = \sqrt{\frac{K\alpha}{me}}$

④ If the object hass mass M. MOI about the given axis is.



Volume of the Sphere

$$\frac{4}{3}\pi R^3 - \frac{4}{3}\pi\left(\frac{R}{2}\right)^3$$

$$\frac{4}{3}\pi R^3\left(1 - \frac{1}{8}\right) = \frac{28}{24}\pi R^3$$

$$8 \times \frac{28}{24}\pi R^3 = M$$

$$1 = \frac{M \times 24}{28\,8\,\pi R^3}$$

$$\frac{4}{3}\pi R^3 = \frac{M \times 24\,8}{7\,28\,8\,\cancel{\pi}\,R^3} \times \frac{4}{\cancel{3}}\cancel{\pi}\cancel{R^3} = \frac{8}{7}M$$

cuting Portion $= \frac{1}{7}M$

Moment of inertia about tangential axis

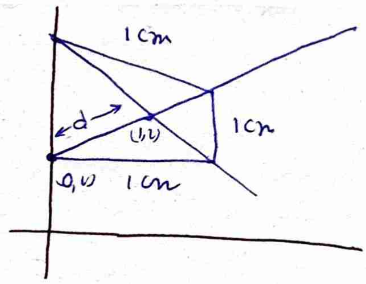
$$\frac{7}{5}\cdot\left(\frac{8M}{7}\right) \times R^2 = \frac{8M}{5}R^2$$

$$\frac{7}{5}\cdot\left(\frac{M}{7}\right) \times \left(\frac{R}{2}\right)^2 = \frac{1}{20}MR^2$$

$$I = \left(\frac{8}{5} - \frac{1}{20}\right)MR^2 = \frac{31}{20}MR^2$$

(5) A uniform plate of side 1 cm lie in xy plane. Diagonal of the plate are along the st. line $y = x+1$ and $y = -x+3$. If moment of inertia of the plate about z-axis be $62\ gm \cdot cm^2$. Mass of the square plate is — gm

$$\frac{M}{6}a^2 + Md^2 = 62$$



$$M\left[\frac{a^2}{6} + d^2\right] = 62$$

$$M\left[\frac{1}{6} + 5\right] = 62$$

$$M \times \frac{31}{6} = 62 \Rightarrow M = 12\ gm$$

(6) A circular disc of radius R and mass M has non uniform surface mass density $\sigma = ar^3$. MOI about perpendicular central axis is

$\Rightarrow$ $I = \int r^2 dm = \int r^2 \sigma 2\pi r dr = 2\pi a \int_0^a r^6 dr = \frac{2\pi a R^7}{7}$

$M = \int dm = \int_0^R ar^3\, 2\pi r dr = \frac{2\pi a R^5}{5} \Rightarrow a = \frac{5M}{2\pi R^5}$

$$I = \frac{2\pi R^7}{7} \times \frac{5M}{2\pi R^5} = \frac{5}{7}MR^2$$

(7) A modified oscillator satisfies

$$\ddot{x} + 2\lambda\dot{x} + 25x = 0$$

$$\lambda = \begin{cases} 0 & x < 0 \\ 4 & x > 0 \end{cases}$$

For different ω time period

Time period of oscillation is

$$\boxed{T = \frac{\pi}{\omega_1} + \frac{\pi}{\omega_2}}$$

$\Rightarrow$ for $x < 0$

$\ddot{x} + 25x = 0$

$\frac{d^2x}{dt^2} + 5^2 x = 0$

$\omega = 5$

for $x > 0$

$\ddot{x} + 8\dot{x} + 25x = 0$

Roots $m = -4 \pm 3i$

$x(t) = e^{-4t}(A\cos 3t + B\sin 3t)$

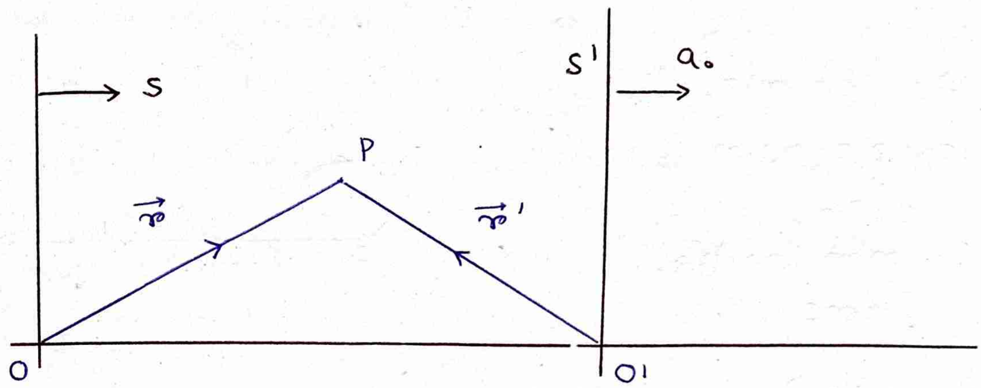
$\omega = 3$

$$T = \frac{\pi}{\omega_1} + \frac{\pi}{\omega_2} = \pi\left(\frac{1}{5} + \frac{1}{3}\right)$$

$$T = \frac{8\pi}{15}$$

Different Types of Frame

27.06.24



Non inertial frame

From traingle OO'P

$$\overline{OO'} + \vec{r}\,' = \vec{r}$$

or, $\vec{r}\,' = \vec{r} - \overrightarrow{OO'}$

or, $\vec{r}\,' = \vec{r} - \vec{v}_0 t$

or $\dfrac{d\vec{r}\,'}{dt} = \dfrac{d\vec{r}}{dt} - \vec{v}_0$

or $\vec{v}\,' = \vec{v} - \vec{v}_0$

$\vec{v}\,' \Rightarrow$ velocity of P wrt S'

$\vec{v} \Rightarrow$ velocity of P wrt S

$\vec{v}_0 \Rightarrow$ velocity of S' frame

$$\frac{d\vec{v}\,'}{dt} = \frac{d\vec{v}}{dt} - \frac{d\vec{v}_0}{dt}$$

$$\vec{a}\,' = \vec{a} - \vec{a}_0$$

$$m\vec{a}\,' = m\vec{a} - m\vec{a}_0$$

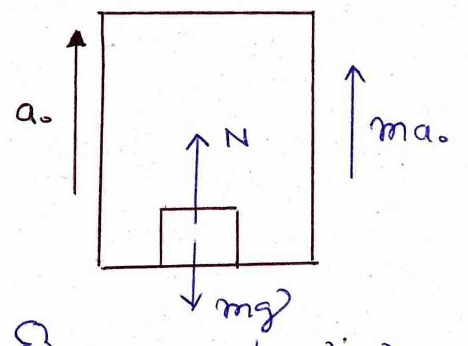
$$\boxed{\vec{F}' = \vec{F} - m\vec{a}_0}$$

$\vec{F}'$ accelerating frame

$\vec{F}$ is accelerating frame

① Measure wrt S' frame

② Pseudo force is opposite to the direction

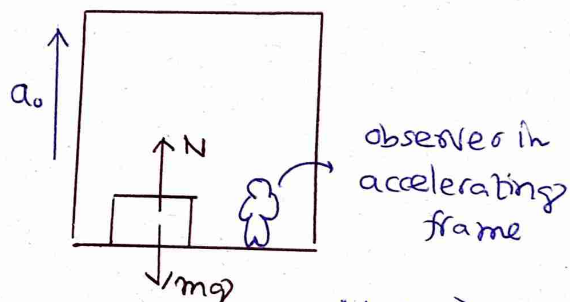


Non acelerating frame

$N - mg = ma_0$

$N = m(g + a_0)$

Pseudo force not required as Non accelerating frame

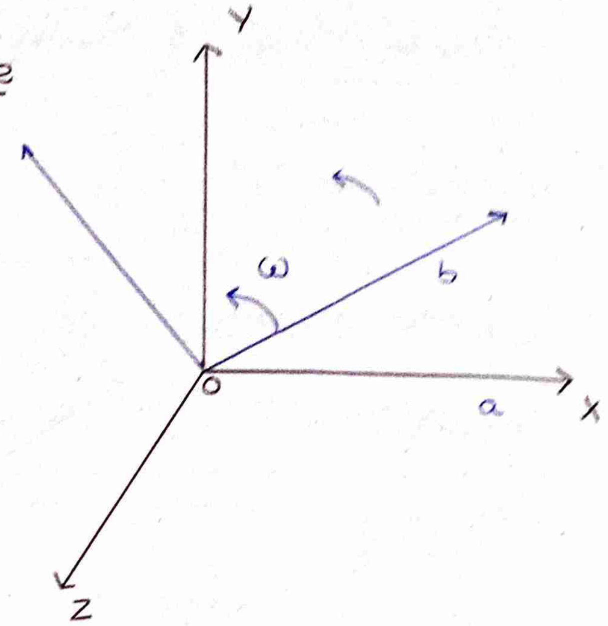


$N = mg$

But for accelerating frame due to pseudo force ma_0 should added

$N = mg + ma_0$

Rotating Frame



As centripetal acceleration is Present it is a non inertial frame

Rotating : b

Stationary : a

$$\boxed{\vec{F_b} = \vec{F_a} - m\,\vec{r} \times \frac{d\vec{\omega}}{dt} - 2m(\vec{\omega} \times V_a) - m\vec{\omega} \times (\vec{\omega} \times \vec{r})}$$

$\vec{F_b}$ ⇒ Fource measuring in accelerating frame

$\vec{F_a}$ ⇒ Force on non accelerating frame

$m\vec{r} \times \frac{d\vec{\omega}}{dt}$ ⇒ Azimuthal force

If ω is constant then $F_{AZM} = 0$

If measure in centre $r = 0$ $F_{AZM} = 0$

$F_c = -2m(\vec{\omega} \times V_a)$ is Coriolis force

$\vec{F}_{ce} = -m\,\vec{\omega} \times (\vec{\omega} \times \vec{r})$ is centrifugal force

For earth $\omega = \frac{2\pi}{T} = 7.2 \times 10^{-5}$ Rad/sec

ω is constant so

Azimuthal force is no acting

Stationary body at earth only centrifugal force will act as $V_a = 0$ so here sonly Coriolis force is considerable

$\vec{\omega} = \omega[\cos\lambda\,\hat{j} + \sin\lambda\,\hat{k}]$

$\vec{\omega} = \sqrt{\omega^2\cos^2\lambda + \omega^2\sin^2\lambda} = \omega$

Northern Hemisphere

λ is Positive

$\vec{\omega} = \omega[\cos\lambda\,\hat{j} + \sin\lambda\,\hat{k}]$

Southern Hemisphere

λ is negative

$\vec{\omega} = \omega[\cos\lambda\,\hat{j} - \sin\lambda\,\hat{k}]$

28.06.24

At Equator $\lambda = 0$, $\vec{\omega} = \omega\hat{j}$

Pole $\lambda = \frac{\pi}{2}$ $\vec{\omega} = \omega\hat{k}$

$t = \sqrt{\frac{2H}{g}}$

$\vec{v} = gt(-\hat{k})$

$\vec{F}_{cor} = -2m[\vec{\omega} \times \vec{v}]$

$= -2m[\omega\hat{j} * gt(-\hat{k})]$

$\vec{F}_{cor} = +2m\omega gt\,\hat{i}$

$\vec{F}_{cor} = 2m\omega gt\,\hat{i}$

Towards East

$m\frac{d^2x}{dt^2} = 2m\omega gt$

$m\frac{dx}{dt} = m\omega gt^2 \Rightarrow \frac{dx}{dt} = \omega gt^2 + c$

At $t = 0$ $\vec{v} = 0 \Rightarrow c = 0$

$x = \frac{1}{3}\omega gt^3 + c$ $\qquad \frac{dx}{dt} = \omega gt^2$

At $t = 0$ $x = 0$ $c = 0$

$\boxed{x = \frac{1}{3}\omega gt^3}$

$\boxed{x = \frac{1}{3}\omega g\left(\frac{2H}{g}\right)^{3/2}}$ $\qquad t = \sqrt{2H/g}$

Here x is the shift due to Coriolis Force.

① If we drop a object from $H = 100\,m$

$x = \frac{1}{3} \times 7.3 \times 10^{-5} \times 10 \times \left(\frac{2 \times 100}{10}\right)^{3/2} = 2.1\,cm$

① Consider a man running from north to South with a Constant Velocity in Southern hemisphere The Coriolis force on Particle

$\Rightarrow$ $\vec{v} = -vj$ $\quad \vec{\omega} = \omega[\cos\lambda j - \sin\lambda \hat{k}]$

$\vec{F} = -2m(\vec{\omega} \times \vec{v})$ $\quad$ Along East.

$= -2m(-i) = 2mi$

② A bullet is fired along vertically upward direction with a velocity in upward direction with velocity 500 m/s at 60° latitude in northern Hemisphere. Mass of bullet 100 gm Magnitude of Coriolis acceleration is

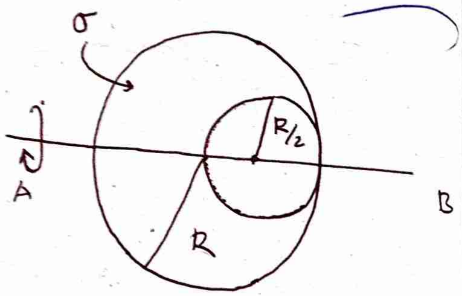
$\Rightarrow$ $\vec{v} = 500\hat{k}$ $\quad \vec{\omega} = \omega\left[\frac{1}{2}\hat{j} + \frac{\sqrt{3}}{2}\hat{k}\right]$

$\frac{\vec{F}}{m} = -2(\vec{\omega} \times \vec{v}) = \frac{-2}{0.1}\left(\frac{\omega}{2} \times 500\right)$

$= 7.2 \times 10^{-5} \times 10 \times 500$

$= 36 \times 10^{-2} = 3.6 \text{ cm/s}^2$

③ Surface mass density σ a disc of $R/2$ of radius Cut. Then moment of inertia about AB is $\frac{15}{16K} \sigma \pi R^4$. then K is



$\Rightarrow$ Mass of disc. $M = \sigma \pi R^2$

then MOI $\quad I = \frac{1}{4}MR^2 = \frac{1}{4}\sigma\pi R^4$

$M' = \sigma\pi\left(\frac{R}{2}\right)^2 = \frac{1}{4}\sigma\pi R^2$

$I' = \frac{1}{4}M'\left(\frac{R}{2}\right)^2 = \frac{1}{4} \cdot \frac{1}{4}\sigma\pi R^2 \times \frac{R^2}{4} = \frac{1}{64}\sigma\pi R^4$

$I = \frac{\sigma\pi R^4}{4}\left(1 - \frac{1}{16}\right) = \frac{15}{16}\frac{\sigma\pi R^4}{4}$

K is 4.

Classical Mechanics

› Lagrangian Mechanics

Lagrangian of a System, $L = T - V$

L = Kinetic energy $(\dot{x}, \dot{\theta}, \dot{\phi})$

V = Potential energy (x, θ, ϕ)

Generalized Co-ordinate

Lagrangian has no Physical Significance.

In 2D Kinetic energy $T = \frac{1}{2}m(V_x^2 + V_y^2)$

$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$

Polar System:

$x = r\cos\theta$ $\qquad$ $y = r\sin\theta$

$\dot{x} = \dot{r}\cos\theta - r\sin\theta\,\dot{\theta}$ $\qquad$ $\dot{y} = \dot{r}\sin\theta + r\cos\theta\,\dot{\theta}$

$$T = \frac{1}{2}m\left[(\dot{r}\cos\theta - r\dot{\theta}\sin\theta)^2 + (\dot{r}\sin\theta + r\cos\theta\,\dot{\theta})^2\right]$$

$$= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$$

≡ Cylindrical System

Here, $x = \rho\cos\theta$, $y = \rho\sin\theta$, $z = z$

$$T = \frac{1}{2}m[\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2]$$

≡ Spherical System

$x = r\sin\theta\cos\phi$

or $\dot{x} = \dot{r}\sin\theta\cos\phi + r\cos\theta\,\dot{\theta}\cos\phi - r\sin\theta\sin\phi\,\dot{\phi}$

$y = r\sin\theta\sin\phi$

or $\dot{y} = \dot{r}\sin\theta\sin\phi + r\cos\theta\,\dot{\theta}\sin\phi + r\sin\theta\cos\phi\,\dot{\phi}$

$z = r\cos\theta$

or $\dot{z} = \dot{r}\cos\theta - r\sin\theta\,\dot{\theta}$

$$T = \frac{1}{2}m[\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2]$$

Equation of Motion

To get the equation of motion from Lagrangian we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \qquad L = L(q, \dot{q}, t)$$

① **Free Particle:**

Kinetic energy. $T = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2$

potential energy $V(x) = 0$

Lagrangian of the System $L = \frac{1}{2}m\dot{x}^2$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

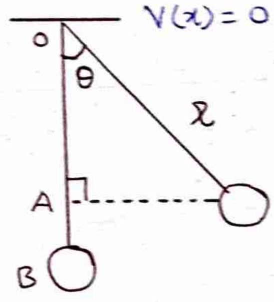
Now $\frac{\partial L}{\partial x} = 0$

$\frac{\partial L}{\partial \dot{x}} = m\dot{x}$

or. $\frac{d}{dt}(m\dot{x}) = 0$

or, $m\ddot{x} = 0$

② **Simple Pendulum**

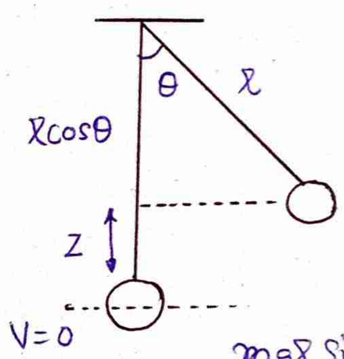


Kinetic energy of the System $T = \frac{1}{2}I\omega^2 = \frac{1}{2}ml^2\dot{\theta}^2$

Potential Energy

$V(x) = -mgl\cos\theta$

Lagrangian of the System. $L = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta$



Here $z = l - l\cos\theta = l(1-\cos\theta)$

Potential energy.

$V = mgl(1-\cos\theta)$

$L = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1-\cos\theta)$

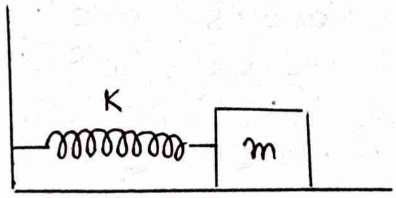
$mgl\sin\theta + ml^2\ddot{\theta} = 0$

$\Rightarrow \ddot{\theta} + g/l \sin\theta = 0$

$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$

$\frac{\partial L}{\partial \theta} = -mgl\sin\theta$

③ Spring mass system



Kinetic energy

$T = \frac{1}{2} m \dot{x}^2$

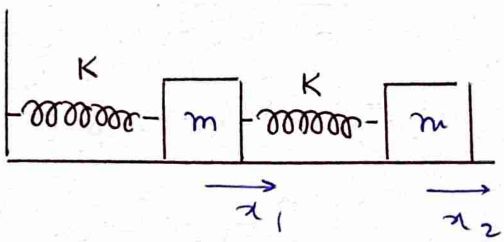
Potential energy $V = \frac{1}{2} K x^2$

Lagrangian of the system. $L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} K x^2$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x}$ $\quad \frac{\partial L}{\partial x} - Kx$

so $\quad m\ddot{x} + Kx = 0 \Rightarrow \ddot{x} + \frac{K}{m} x = 0$

④ Two mass system



Kinetic energy of the system

$T = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2$

$V = \frac{1}{2} K x_1^2 + \frac{1}{2} K (x_1 - x_2)^2 + \frac{1}{2} K x_2^2$

Lagrangian of the system is

$L = \frac{1}{2} m (\dot{x}_1^2 + \dot{x}_2^2) - \frac{1}{2} K x_1^2 - \frac{1}{2} K (x_1 - x_2)^2 - \frac{1}{2} K x_2^2$

$\frac{\partial L}{\partial x_1} = -Kx_1 - K(x_1 - x_2) = -2Kx_1 + Kx_2$

$\frac{\partial L}{\partial \dot{x}_1} = m\dot{x}_1$ $\quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0$

$m\ddot{x}_1 + K(2x_1 - x_2) = 0$

$\ddot{x}_1 + (2x_1 - x_2)\frac{K}{m} = 0$

Now $\frac{\partial L}{\partial x_2} = + K(x_1 - 2x_2)$ $\quad \frac{\partial L}{\partial \dot{x}_2} = m\dot{x}_2$

$\ddot{x}_2 + \frac{(2x_2 - x_1)}{m} K = 0$

A particle of mass m is constrained to move in a vertical plane along a trajectory given by $x = A\cos\theta$ and $y = A\sin\theta$. The Lagrangian of the system

$\Rightarrow$

$x = A\cos\theta$ $\qquad$ $y = A\sin\theta$

so $\dot{x} = -A\sin\theta\,\dot{\theta}$ $\qquad$ or $\dot{y} = A\cos\theta\,\dot{\theta}$

$\dot{x}^2 + \dot{y}^2 = A^2\dot{\theta}^2$ $\qquad$ Potential. $V = mgy = mgA\sin\theta$

Kinetic energy. $T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) = \frac{1}{2}mA^2\dot{\theta}^2$

Lagrangian of the system

$$L = T - V = \frac{1}{2}mA^2\dot{\theta}^2 - mgA\sin\theta$$

<u>Equation of motion</u>

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{\partial L}{\partial \dot{\theta}} = mA^2\dot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -mgA\cos\theta$$

$$A^2\ddot{\theta} + gA\cos\theta = 0$$

$$\frac{d^2\theta}{dt^2} + \frac{g}{A}\cos\theta = 0$$

Equation of motion

≡ <u>Cyclic Coordinate:</u>

Lagrangian of a system $L = L(q, \dot{q}, t)$

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + mgr$$

If $\dot{q}_i$ is present but q_i is absent then q_i is cyclic coordinate.

> Generalized momentum, $p_i = \frac{\partial L}{\partial \dot{q}_i}$

$$L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2$$

$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$ and $p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y}$

> For a cyclic coordinate it's corresponding momentum will be conserved.

Here p_θ is conserved.

> P_θ Conserved means force is zero so Corresponding Potential is zero. Free Particle w.r.t θ. Potential energy term doesn't Contain any term of θ.

Take Lagrangian of the system

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + mgr$$

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r} \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = m\ddot{r}$$

$$\frac{\partial L}{\partial r} = mg + mr\dot{\theta}^2 \qquad \text{So, } m\ddot{r} - mg - mr\dot{\theta}^2 = 0$$

$$\ddot{r} = g + r\dot{\theta}^2$$

Now, $\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$ $\quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta}$

$$\frac{\partial L}{\partial \theta} = 0$$

So $mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} = 0$

or $r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0$

a $a_\theta = 0 \Rightarrow \vec{F}_\theta = 0$

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

$$\vec{a} = \underbrace{(\ddot{r} - r\dot{\theta}^2)}_{a_r}\hat{r} + \underbrace{(r\ddot{\theta} + 2\dot{r}\dot{\theta})}_{a_\theta}\hat{\theta}$$

P_θ is Constant Here.

① The Lagrangian of a particle of mass m moving in a central Potential $V(r)$ is

$$L = \frac{1}{2}m(\dot{r}^2 + r\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2) - V(r)$$

$\Rightarrow$ $r, \dot{r}$ both are Present

$\theta, \dot{\theta}$ both are Present

$\dot{\phi}$ is present but ϕ is absent so ϕ is Cyclic

② Lagrangian $L = \dot{q}^2 - q\dot{q}$ then

$\Rightarrow$ $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$

$$\frac{\partial L}{\partial \dot{q}} = 2\dot{q} - q$$

$$\frac{\partial L}{\partial q} = -\dot{q}$$

$2\ddot{q} - \dot{q} + \dot{q} = 0 \Rightarrow \ddot{q} = 0$

(Free Particle)

③ The Lagrangian $L = \frac{1}{2}m(\dot{x}-\lambda x)^2$. If it starts at the origin $t=0$. equation of motion

$\Rightarrow$ $L = \frac{1}{2}m(\dot{x}-\lambda x)^2$ $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m(\ddot{x}-\lambda\dot{x})$

$\frac{\partial L}{\partial \dot{x}} = m(\dot{x}-\lambda \bar{x})$

$\frac{\partial L}{\partial x} = m(-\lambda)(\dot{x}-\lambda x)$

$= -\lambda m(\dot{x}-\lambda x)$

Eqn of motion

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$

The displacement

$x = Ae^{\lambda t} + Be^{-\lambda t}$

$m(\ddot{x}-\lambda\dot{x}+\lambda\dot{x}) - \lambda^2 m x = 0$

or $\ddot{x} - \lambda^2 x = 0 \Rightarrow \alpha = \lambda, -\lambda$

Now at $t=0$, $x=0$ $0 = A+B \Rightarrow B = -A$

$x = Ae^{\lambda t} - Ae^{-\lambda t} = A(e^{\lambda t} - e^{-\lambda t}) = c\sinh \lambda t$

④ Lagrangian of the system $L = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) - \frac{\lambda}{4}(x+y)^4$

The intial Conditions are given by $y(0)=0$, $x(0)=42$

$\dot{x}(0) = \dot{y}(0) = 0$. what is the value of $x(t)-y(t)$ at $t = 25$ seconds

$\Rightarrow$ $\frac{\partial L}{\partial \dot{x}} = m\dot{x}$ $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x}$ $\frac{\partial L}{\partial x} = -\lambda(x+y)^3$

$\frac{\partial L}{\partial \dot{y}} = m\dot{y}$ $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = m\ddot{y}$ $\frac{\partial L}{\partial y} = -\lambda(x+y)^3$

Eqn of motions $m\ddot{y} + \lambda(x+y)^3 = 0$

$m\ddot{x} + \lambda(x+y)^3 = 0$

So $\ddot{x} = \ddot{y} \Rightarrow \dot{x} + c_1 = \dot{y} + c_2 \Rightarrow c_1 = c_2 = 0$

$\Rightarrow \dot{x} = \dot{y}$

$\Rightarrow x = y + c$

$\Rightarrow 42 = 0 + c \Rightarrow c = 42$

So $x - y = 42$ (At any time)

(5) Lagrangian of a particle $L = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) - \frac{1}{2}m\omega^2(x^2+y^2)$.
At $t=0$ $x=0$, $y=y_0$, $\dot{x}=V_0=\dot{y}$ Angular momentum of the Particle at time t is

$\Rightarrow$ Lagrangian of the System is given by

$$L = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) - \frac{1}{2}m\omega^2(x^2+y^2)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \Rightarrow m\ddot{x} + m\omega^2 x = 0$$

$$\Rightarrow \ddot{x} + \omega^2 x = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0 \Rightarrow m\ddot{y} + m\omega^2 y = 0$$

$$\Rightarrow \ddot{y} + \omega^2 y = 0$$

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

$x = C_1 \cos t + C_2 \sin t$

$\dot{x} = -C_1 \sin x + C_2 \cos x$

At $t=0$, $x=0$

$0 = C_1$

$x = C_2 \sin t$

$\dot{x} = C_2 \cos t$

$V_0 = C_2$

So $x = V_0 \sin t$

$$\frac{d^2y}{dt^2} + \omega^2 y = 0$$

$y = C_1 \cos t + C_2 \sin t$

$y_0 = C_1 \Rightarrow y = y_0 \cos t + C_2 \sin t$

$\dot{y} = -y_0 \sin t + C_2 \cos t$

$V_0 = C_2$

$y = y_0 \cos t + V_0 \sin t$

17.05.2024

① The dynamics of a particle is governed by the Lagrangian $L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}Kx^2 - Kx\dot{x}t$ describes

$\Rightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$

$\frac{\partial L}{\partial \dot{x}} = m\dot{x} - Kxt$

$\frac{\partial L}{\partial x} = -Kx - K\dot{x}t$

$\frac{d}{dt}(m\dot{x} - Kxt) + Kx + K\dot{x}t = 0$

or $\quad m\ddot{x} - (K\dot{x}t + Kx) + Kx + K\dot{x}t = 0$

or $\quad m\ddot{x} = 0 \Rightarrow m\frac{d^2x}{dt^2} = 0$

This is free Particle

② In Cartesian Coordinate $L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - \frac{K}{2}(\dot{x}y - \dot{y}x)$.
The Lagrangian in polar Coordinate is

$\Rightarrow \quad x = r\cos\theta \qquad y = r\sin\theta$

$\dot{x} = \dot{r}\cos\theta - r\sin\theta\,\dot{\theta} \qquad \dot{y} = \dot{r}\sin\theta + r\cos\theta\,\dot{\theta}$

$\dot{x}y - \dot{y}x = (\dot{r}\cos\theta - r\sin\theta\,\dot{\theta})(r\sin\theta) - (\dot{r}\sin\theta + r\cos\theta\,\dot{\theta})(r\cos\theta)$

$= r\dot{r}\sin\theta\cos\theta - r^2\sin^2\theta\,\dot{\theta} - r\dot{r}\sin\theta\cos\theta - r^2\cos^2\theta\,\dot{\theta}$

$= -r^2\dot{\theta}(\cos^2\theta + \sin^2\theta) = -r^2\dot{\theta}$

$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2}Kr^2\dot{\theta}$

③ Lagrangian of a System $L = \dot{q}_K q_K - \sqrt{1 - \dot{q}_K^2}$. Equation of motion of the System is

$\Rightarrow \quad L = \dot{q}_K q_K - \sqrt{1 - \dot{q}_K^2}$

$\frac{\partial L}{\partial \dot{q}_K} = q_K + \frac{1}{2}\frac{2\dot{q}_k}{\sqrt{1 - \dot{q}_K^2}} = q_K + \frac{\dot{q}_k}{\sqrt{1 + \dot{q}_k^2}}$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \dot{q}_k + \frac{\sqrt{1 - \dot{q}_K^2}\;\ddot{q}_K}{\sqrt{1 - \dot{q}_K^2}} = \dot{q}_k + \frac{(1 - \dot{q}_k^2)\ddot{q}_k + \ddot{q}_k\dot{q}^2}{(1 - \dot{q}_k^2)^{3/2}}$

Again $\frac{\partial L}{\partial q} = \dot{q}$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q_k} = 0$

$\frac{d^2 q_k}{dt^2} = 0$

(4) The Lagrangian $L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2x^2$. If at $t=0$, $x=A$, $\dot{x}=0$ then at what time particle reaches $x = A/2$ for the first time.

$\Rightarrow$

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2x^2 \qquad \frac{\partial L}{\partial x} = -m\omega^2 x$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x}$$

Equation of motion $m\ddot{x} + m\omega^2 x = 0$

$$\Rightarrow \ddot{x} + \omega^2 x = 0$$

The Solution of the equation will be

$$x = A'\sin\omega t + B'\cos\omega t$$

$t=0$, $x=A$. $A = B'$ so $x = A'\sin\omega t + A\cos\omega t$

$$\dot{x} = \omega(A'\cos\omega t - A\sin\omega t)$$

$$0 = \omega A' \Rightarrow A' = 0$$

So $x = A\cos\omega t$

$$\frac{A}{2} = A\cos\omega t$$

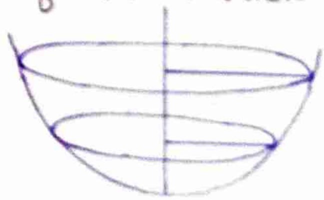
$$\frac{1}{2} = \cos\omega t$$

$\omega t = \pi/3$

$t = \pi/3\omega$

After time $\pi/3\omega$

(5) A particle of mass m moves inside a bowl. If the surface of the bowl is given by the equation $z = \frac{1}{2}a(x^2+y^2)$. The Lagrangian of the Particle is



$\Rightarrow$

$$z = \frac{1}{2}a(x^2+y^2) \qquad x = r\cos\theta$$

$$z = \frac{1}{2}ar^2 \qquad y = r\sin\theta$$

$$\dot{z} = ar\dot{r} \qquad \dot{x}^2+\dot{y}^2 = \dot{r}^2 + r^2\dot{\theta}^2$$

$$L = \frac{1}{2}m\left[(\dot{x}^2+\dot{y}^2+\dot{z}^2) - mgz\right]$$

$$= \frac{1}{2}m\left[\dot{r}^2 + r^2\dot{\theta}^2 + a^2r^2\dot{r}^2 - mg\times\frac{1}{2}ar^2\right]$$

The Lagrangian of a particle of mass $L = \frac{1}{2}m\dot{x}^2 - bx$.
The Coordinate of the Particle $x(t)$ is

$\Rightarrow$ Equation of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \qquad \frac{\partial L}{\partial \dot{x}} = m\dot{x} \qquad \frac{\partial L}{\partial x} = -b$$

$$m\ddot{x} + b = 0 \qquad x = -\frac{b}{2m}t^2 + c_1 t + c_2$$

$$\ddot{x} = -b/m$$

$$\dot{x} = -\frac{b}{m}t + c_1$$

c_1 and c_2 are arbitary constant

Convert: Lagrangian to Hamiltonian

Lagrangian of a system $L = L(q, \dot{q}, t)$

Hamiltonian of system $H = H(P, \bar{q}, t)$

$$H = \sum P\dot{q} - L \qquad P_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

$$L = \frac{1}{2}m\dot{x}^2 - Kx^2 \qquad \text{so } \dot{x} = P_x/m$$

$$H = P.\dot{x} - L$$

$$H = \sum P\dot{q} - L$$

or $H = P_x \cdot \frac{P_x}{m} - \left(\frac{1}{2}m\dot{x}^2 - Kx^2\right)$

or $H = \frac{P_x^2}{m} - \frac{1}{2}m.\frac{P_x^2}{m^2} + Kx^2$

a $H = \frac{P_x^2}{2m} + Kx^2$

For a Conservative system Hamiltonian = Energy

Equation of motion from Hamiltonian

$$H = H(P, q, t) \qquad H = \frac{p^2}{2m} + Kx^2$$

$$\left.\begin{aligned} \dot{p} &= -\frac{\partial H}{\partial q} \\ \dot{q} &= \frac{\partial H}{\partial p} \end{aligned}\right\} \text{Hamilton's Eqn of motion}$$

$$\dot{p} = -\frac{\partial H}{\partial x} = -2Kx$$

$$\dot{p} = -2Kx$$

$$F = -2Kx$$

Same outcome by Lagrangian and Hamiltonian

$A = A(p, q, t)$

$$\frac{dA}{dt} = \frac{\partial A}{\partial p}\frac{\partial p}{\partial t} + \frac{\partial A}{\partial q}\frac{\partial q}{\partial t} + \frac{\partial A}{\partial t}$$

or $$\frac{dA}{\partial t} = \frac{\partial A}{\partial p}\left(-\frac{\partial H}{\partial q}\right) + \frac{\partial A}{\partial q}\cdot\frac{\partial H}{\partial p} + \frac{\partial A}{\partial t}$$

or $$\frac{dA}{dt} = -\frac{\partial A}{\partial p}\frac{\partial H}{\partial q} + \frac{\partial A}{\partial q}\cdot\frac{\partial H}{\partial p} + \frac{\partial A}{\partial t}$$

$$\frac{dA}{dt} = \frac{\partial A}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial H}{\partial q} + \frac{\partial A}{\partial t}$$

or $$\frac{dA}{dt} = \{A, H\} + \frac{\partial A}{\partial t}$$ (Poisson Bracket)

so $$\{A, H\} = \frac{\partial A}{\partial q}\cdot\frac{\partial H}{\partial p} - \frac{\partial A}{\partial p}\cdot\frac{\partial H}{\partial q}$$

$$\{A.B\} = \frac{\partial A}{\partial q}\cdot\frac{\partial B}{\partial p} - \frac{\partial B}{\partial q}\cdot\frac{\partial A}{\partial p}$$

> $$\{x, P_x\} = \frac{\partial x}{\partial x}\cdot\frac{\partial P_x}{\partial P_x} - \frac{\partial x}{\partial P_x}\cdot\frac{\partial P_x}{\partial x} = 1 \times 1 = 1$$

If $\{A, B\} = 1$ A, B are Cannonical Variables

> $$\{A, BC\} = \frac{\partial A}{\partial x}\cdot\frac{\partial (BC)}{\partial P_x} - \frac{\partial (BC)}{\partial x}\cdot\frac{\partial A}{\partial P_x}$$

$$\{A, BC\} = \frac{\partial A}{\partial x}\left(B\frac{\partial C}{\partial P_x} + C\frac{\partial B}{\partial P_x}\right) - \left[\frac{\partial A}{\partial P_x}\left(B\frac{\partial C}{\partial x} + C\frac{\partial B}{\partial x}\right)\right]$$

$$\{A, BC\} = B\left[\frac{\partial A}{\partial x}\cdot\frac{\partial C}{\partial P_x} - \frac{\partial C}{\partial x}\cdot\frac{\partial A}{\partial P_x}\right] + C\left[\frac{\partial A}{\partial x}\cdot\frac{\partial B}{\partial P_x} - \frac{\partial B}{\partial x}\cdot\frac{\partial A}{\partial P_x}\right]$$

$$= B\{A, C\} + C\{A, B\}$$

$$\{A, BC\} = B\{A, C\} + C\{A, B\}$$

$$\{A,\{B,C\}\} + \{B,\{C,A\}\} + \{C,\{A,B\}\}$$

$\Rightarrow$ For the First Part

$$\{A,\{B,C\}\}$$

$$= \frac{\partial A}{\partial x}\cdot\frac{\partial}{\partial p_x}\{B,C\} - \frac{\partial A}{\partial p_x}\frac{\partial}{\partial x}\{B,C\}$$

$$= \frac{\partial A}{\partial x}\frac{\partial}{\partial p_x}\left(\frac{\partial B}{\partial x}\cdot\frac{\partial C}{\partial p_x} - \frac{\partial B}{\partial p_x}\frac{\partial C}{\partial x}\right) - \frac{\partial A}{\partial p_x}\frac{\partial}{\partial x}\left(\frac{\partial B}{\partial x}\cdot\frac{\partial C}{\partial p_x} - \frac{\partial B}{\partial p_x}\cdot\frac{\partial C}{\partial x}\right)$$

$$= \frac{\partial A}{\partial x}\cdot\frac{\partial B}{\partial x}\cdot\frac{\partial^2 C}{\partial p_x^2} + \frac{\partial A}{\partial x}\cdot\frac{\partial^2 B}{\partial p_x\partial x}\frac{\partial C}{\partial p_x}$$
$$- \frac{\partial A}{\partial x}\frac{\partial C}{\partial x}\frac{\partial^2 B}{\partial^2 p_x^2} - \frac{\partial A}{\partial x}\frac{\partial B}{\partial p_x}\cdot\frac{\partial^2 C}{\partial p_x\partial x}$$
$$- \frac{\partial A}{\partial p_x}\frac{\partial^2 B}{\partial x^2}\frac{\partial^2 C}{\partial x\partial p_x} - \frac{\partial A}{\partial p_x}\frac{\partial^2 B}{\partial x\partial p_x}\frac{\partial^2 C}{\partial x^2}$$
$$+ \frac{\partial A}{\partial p_x}\frac{\partial^2 B}{\partial x^2}\frac{\partial C}{\partial p_x} - \frac{\partial A}{\partial p_x}\frac{\partial B}{\partial p_x}\cdot\frac{\partial^2 C}{\partial x^2}$$

Second Part

$$\{B,\{C,A\}\}$$

$$\frac{\partial B}{\partial x}\cdot\frac{\partial C}{\partial x}\frac{\partial^2 A}{\partial p_x^2} + \frac{\partial B}{\partial x}\cdot\frac{\partial^2 C}{\partial p_x\partial x}\frac{\partial A}{\partial p_x}$$
$$- \frac{\partial B}{\partial x}\frac{\partial B}{\partial x}\frac{\partial^2 C}{\partial p_x^2} - \frac{\partial B}{\partial x}\frac{\partial C}{\partial p_x}\frac{\partial^2 A}{\partial p_x\partial x}$$
$$- \frac{\partial B}{\partial p_x}\frac{\partial^2 C}{\partial x^2}\frac{\partial^2 A}{\partial x\partial p_x} - \frac{\partial B}{\partial p_x}\cdot\frac{\partial^2 C}{\partial p_x\partial x}\frac{\partial^2 A}{\partial x^2}$$
$$+ \frac{\partial B}{\partial p_x}\frac{\partial^2 C}{\partial x^2}\frac{\partial A}{\partial p_x} - \frac{\partial B}{\partial p_x}\frac{\partial C}{\partial p_x}\cdot\frac{\partial^2 A}{\partial x^2}$$

Third Part $\{C,\{A,B\}\}$

$$= \frac{\partial C}{\partial x}\cdot\frac{\partial A}{\partial x}\cdot\frac{\partial^2 B}{\partial p_x^2} + \frac{\partial C}{\partial x}\cdot\frac{\partial^2 A}{\partial p_x\partial x}\cdot\frac{\partial B}{\partial p_x} - \frac{\partial C}{\partial x}\frac{\partial^2 A}{\partial p_x^2}\frac{\partial B}{\partial x}$$
$$- \frac{\partial C}{\partial x}\cdot\frac{\partial A}{\partial p_x}\cdot\frac{\partial^2 B}{\partial p_x\partial x} - \frac{\partial C}{\partial p_x}\frac{\partial^2 A}{\partial x^2}\frac{\partial^2 B}{\partial x\partial p_x} - \frac{\partial C}{\partial p_x}\frac{\partial^2 A}{\partial x\partial p_x}\frac{\partial^2 B}{\partial x^2}$$
$$+ \frac{\partial C}{\partial p_x}\frac{\partial^2 A}{\partial x^2}\frac{\partial B}{\partial p_x} - \frac{\partial C}{\partial p_x}\frac{\partial A}{\partial p_x}\frac{\partial^2 B}{\partial x^2}$$

By adding them we get

$$\{A,\{B,C\}\} + \{B,\{C,A\}\} + \{C,\{A,B\}\} = 0$$

18.05.2024

> Lagrangian of a system, $L = L(q,\dot{q},t)$

> Equation of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

> Hamiltonian $H = H(q,p,t)$

$$H = \sum p\dot{q} - L$$

> Poisson Bracket $\{A,B\} = \sum\left(\frac{\partial A}{\partial q}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial q}\right)$

> $\{A,BC\} = B\{A,C\} + C\{A,B\}$

> $\{AB,C\} = A\{B,C\} + B\{A,C\}$

> $\{A,\{B,C\}\} + \{B,\{C,A\}\} + \{C,\{A,B\}\} = 0$ Jacobi Identity

> $\{A,B\} = -\{B,A\}$ > $\{x,P_x\} = 1$

> $\{x,P_y\} = \left(\frac{\partial x}{\partial x}\frac{\partial P_y}{\partial x} - \frac{\partial x}{\partial P_x}\frac{\partial P_y}{\partial x}\right) + \left(\frac{\partial x}{\partial y}\frac{\partial P_y}{\partial P_y} - \frac{\partial x}{\partial P_y}\frac{\partial P_y}{\partial y}\right) = 0$

> $\{x,y\} = 0$ > $\{P_x,P_y\} = 0$ > $\{y,P_y\} = 1$ > $\{z,P_z\} = 1$

> $\{x, x^2+2y+z\} = 0$ doesn't contain momentum

> $\{x, P_x^2 + P_y + P_z\} = \{x,P_x^2\} + \{x,\cancel{P_y^2}\}^{0} + \{x,\cancel{P_z^2}\}^{0}$

(From Property) $= \frac{\partial x}{\partial x}\frac{\partial}{\partial P_x}(P_x^2) - \frac{\partial x}{\partial P_x}\frac{\partial P_x^2}{\partial x}$

$= 2P_x - 0 = 2P_x$

> $\{x_i,P_j\} = \delta_{ij}$ > $\{P_i,L_j\} = \epsilon_{ijk}P_k$

> $\{x_i,L_j\} = \epsilon_{ijk}x_k$ > $\{L_i,L_j\} = \epsilon_{ijk}L_k$

Angular momentum. $\vec{L} = \vec{r} \times \vec{P}$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\vec{P} = P_x\hat{i} + P_y\hat{j} + P_z\hat{k}$

Now $\vec{r} \times \vec{P}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ P_x & P_y & P_z \end{vmatrix}$$

So $\vec{r} \times \vec{P}$

$= (yP_z - zP_y)\hat{i} + (zP_x - xP_z)\hat{j} + \hat{k}(xP_y - yP_x)$

$= L_x\hat{i} + L_y\hat{j} + L_z\hat{k}$

$L_x = yP_z - zP_y$

$L_y = zP_x - xP_z$

$L_z = xP_y - yP_x$

Now

$\{x, L_x\}$

$= \dfrac{\partial x}{\partial x}\dfrac{\partial L_x}{\partial P_x} - \dfrac{\partial x}{\partial P_x} \times \dfrac{\partial L_y}{\partial x}$

$= \dfrac{\partial}{\partial P_x}(yP_z - zP_y) = 0$

So $\{x, L_x\} = 0$

$\{x, L_y\}$

and $\{x, L_y\} = \dfrac{\partial x}{\partial x}\dfrac{\partial L_y}{\partial P_x} - \dfrac{\partial x}{\partial P_x}\dfrac{\partial L_y}{\partial x} = z$

and $\{x, L_z\} = \{x, xP_y - yP_x\} = \{x, xP_y\} - \{x, yP_x\}$

$= x\{x, P_y\} + P_y\{x, x\} - y\{x, P_x\} - P_x\{x, y\} = -y$

> $\{y, L_z\} = \{y, xP_y - yP_x\} = \{y, xP_y\} - \{y, yP_x\}$

$= x\{x, P_y\} + P_y\{y, x\} - y\{y, P_x\} - P_x\{y, y\} = x$

> $\{P_x, L_z\} = \{P_x, xP_y - yP_x\} = \{P_x, xP_y\} - \{P_x, yP_x\}$

$= P_y\{P_x, x\} + x\{P_x, P_y\} - y\{P_x, P_x\} - P_x\{P_x, y\}$

$= -P_y\{x, P_x\} = -P_y$

$\{x, P_x\} = 1$

$\{x, P_y\} = 0$

$\{x, P_z\} = 0$

$\{y, P_x\} = 0$

$\{y, P_y\} = 0$

$\{x_i, P_j\} = \delta_{ij}$

$\delta_{ij} = 1, \quad i = j$

$= 0, \quad i \neq j$

$x_1 = x$

$x_2 = y$

$x_3 = z$

$P_1 = P_x$

$P_2 = P_y$

$P_3 = P_z$

$\{x, L_x\} = 0$

$\{x, L_y\} = z$

$\{x, L_z\} = -y$

$\{y, L_x\} = -z$

$\{y, L_y\} = 0$

$\{y, L_z\} = x$

$\{z, L_x\} = y$

$\{z, L_y\} = -x$

$\{z, L_z\} = 0$

$\{x_i, L_j\} = \epsilon_{ijk} x_k$

$\epsilon_{ijk} = 1$ Cyclic
$= -1$ Non Cyclic
$= 0$ Repeating
} Levisivita Tensor

> $\{x, L_x\} = \epsilon_{11k} x_k = 0$ (Repeating)

> $\{z, L_x\} = \{x_3, L_1\} = \epsilon_{31k} x_k = \epsilon_{312} x_2 = 1 \cdot y = y$

> $\{y, L_z\} = \{x_2, L_3\} = \epsilon_{23k} x_k = \epsilon_{231} x_1 = 1 \cdot x = x$

$\{P_x, L_x\} = 0$

$\{P_x, L_y\} = P_z$

$\{P_x, L_z\} = -P_y$

$\{P_y, L_x\} = -P_z$

$\{P_y, L_y\} = 0$

$\{P_y, L_z\} = P_x$

$\{P_i, L_j\} = \epsilon_{ijk} P_k$

> $\{P_y, L_y\} = \epsilon_{22k} P_k = 0$

> $\{P_x, L_z\} = \epsilon_{13k} P_k = -P_y$

> $$\{L_x, L_y\} = \frac{\partial L_x}{\partial x}\frac{\partial L_y}{\partial P_x} - \frac{\partial L_x}{\partial P_x}\frac{\partial L_y}{\partial x}$$

$$= \{yP_z - zP_y, zP_x - xP_z\}$$

$$= \{yP_z - zP_y, zP_x\} - \{yP_z - zP_y, xP_z\}$$

$$= \{yP_z, zP_x\} - \{zP_y, \overset{0}{\cancel{zP_x}}\} - \{yP_z, \overset{0}{\cancel{xP_z}}\} + \{zP_y, xP_z\}$$

$$= \{yP_z, zP_x\} + \{zP_y, xP_z\}$$

$$= y\{P_z, zP_x\} + P_z\{y, \overset{0}{\cancel{zP_x}}\} + z\{\overset{0}{\cancel{P_y}}, xP_z\} + P_y\{z, xP_z\}$$

$$= y\{P_z, zP_x\} + P_y\{z, xP_z\}$$

$$= y[P_x\{P_z, z\}] + P_y x\{z, P_z\}$$

$$= -yP_x + xP_y = xP_y - yP_x = L_z$$

$> \{L_x\, L_y\} = L_z$ $\quad > \{L_y, L_z\} = L_x$ $\quad \{L_i, L_j\} = \epsilon_{ijk} L_k$

$> \{L_x, L_z\} = -L_y$ $\quad > \{L_z, L_x\} = L_y$

① The Lagrangian of a system $L = \frac{1}{2} q\dot{q}^2$, then the solution of Lagrange's equation yields ($q=0$ at $t=0$)

$\Rightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \qquad \frac{\partial L}{\partial \dot{q}} = q\dot{q} \qquad \frac{\partial L}{\partial q} = \frac{1}{2}\dot{q}^2$

$\frac{d}{dt}(q\dot{q}) - \frac{1}{2}\dot{q}^2 = 0 \qquad q\ddot{q} + \frac{1}{2}\dot{q}^2 = 0$

$q\ddot{q} + \dot{q}^2 - \frac{1}{2}\dot{q}^2 = 0$

$\ddot{q} = \frac{d\dot{q}}{dt}\cdot\frac{dq}{dq} = \frac{d\dot{q}}{dt}\cdot\frac{dq}{dt} = \dot{q}\frac{d\dot{q}}{dq}$

Now $q\ddot{q} + \frac{1}{2}\dot{q}^2 = 0 \qquad \int \frac{d\dot{q}}{\dot{q}} = -\frac{1}{2}\int \frac{dq}{q}$

$q\dot{q}\frac{d\dot{q}}{dq} + \frac{1}{2}\dot{q}^2 = 0 \qquad \log \dot{q} = -\frac{1}{2}\log q + \log c$

$q\frac{d\dot{q}}{dq} = -\frac{\dot{q}}{2}. \qquad \dot{q} = cq^{-1/2}$

$\int q^{+1/2} dq = c\int dt \qquad \frac{dq}{dt} = cq^{-1/2}$

$q^{3/2} = ct \Rightarrow q^{3/2} \propto t \Rightarrow q \propto t^{2/3}$

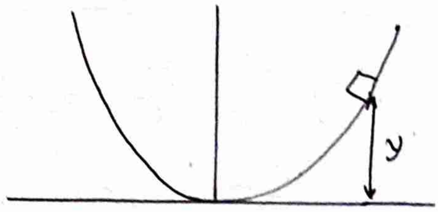
② Lagrangian of a particle $L = \frac{1}{2}m\dot{r}^2 + r^2\dot{\theta}^2 - V(r)$. which of the following is correct

$\Rightarrow$ θ is cyclic coordinate so P_θ constant

$P_\theta = \frac{\partial L}{\partial \dot{\theta}} = 2r^2\dot{\theta} = \text{Constant} \Rightarrow r^2 \propto \frac{1}{\dot{\theta}} \Rightarrow \dot{\theta} \propto \frac{1}{r^2}$

③ A particle of mass m slides under the gravity without friction along Parabolic path $y = ax^2$. The Lagrangian for the Particle is

$\Rightarrow$ $KE = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$



$y = ax^2$

$\dot{y} = 2ax\dot{x}$

$\dot{y}^2 = 4a^2x^2\dot{x}^2$.

$KE = \frac{1}{2}m(\dot{x}^2 + 4a^2x^2\dot{x})$

$PE = mgy = mgax^2$

$$L = \frac{1}{2}m(1 + 4a^2x^2)\dot{x}^2 - mgax^2$$

$$\frac{\partial L}{\partial \dot{x}} = m(1+4a^2x^2)\dot{x} = m\dot{x} + 4a^2x^2\dot{x}$$

$$\frac{\partial L}{\partial x} = 8a^2x\dot{x} - 2mgax$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = m\ddot{x} + 4a^2(x^2\ddot{x} + 2x\dot{x}^2).$$
$$- 8a^2x\dot{x} + 2mgax$$
$$= m\ddot{x} + 2mgax - 4a^2x^2\ddot{x}$$

④ Given $L = \frac{\dot{x}^2}{2x} - V(x)$ Hamiltonian is

$\Rightarrow$ $H = H(q, p t) = p\dot{q} - L = p\dot{q} - \frac{x^2}{2x} + V(x)$

$P = \frac{\partial L}{\partial \dot{x}} = \frac{\dot{x}}{x}$

$$= \frac{\dot{x}^2}{x} - \frac{\dot{x}^2}{2x} + V(x)$$

$$= \frac{\dot{x}^2}{2x} = \frac{1}{2}xp^2 + V(x)$$

⑤ $L = e^{\gamma t}[\dot{x}^2 - x^2]$ Then Hamiltonian

$\Rightarrow$ $H = p\dot{q} - L = p\dot{x} - e^{\gamma t}(\dot{x}^2 - x^2)$

$P_x = \frac{\partial L}{\partial \dot{x}} = 2\dot{x}e^{\gamma t}$

$$= \frac{P_x^2}{2}e^{-\gamma t} - e^{\gamma t}\frac{P_x^2}{4}e^{-2t} + x^2e^{\gamma t}$$

$\dot{x} = \frac{P_x}{2e^{\gamma t}}$

$$= \frac{P_x^2}{4}e^{-\gamma t} + x^2e^{\gamma t}$$

Hamiltonian of a system $H = \frac{xp^2}{2m} + \frac{1}{2}Kx^2$

Euler lagrange Equation of motion

$\Rightarrow$

$$H = \frac{xp^2}{2m} + \frac{1}{2}Kx^2$$

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{xp}{m} \Rightarrow p = \frac{m\dot{x}}{x}$$

Lagrangian $L = p\dot{q} - H$

$$= \frac{m\dot{x}}{x}.\dot{x} - \frac{xp^2}{2m} - \frac{1}{2}Kx^2$$

$$= \frac{m\dot{x}^2}{x} - \frac{x}{2m}\frac{m^2\dot{x}^2}{x^2} - \frac{1}{2}Kx^2$$

$$= \frac{m\dot{x}^2}{x} - \frac{m\dot{x}^2}{2x} - \frac{1}{2}Kx^2 = \frac{m\dot{x}^2}{2x} - \frac{1}{2}Kx^2$$

Equation of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\frac{d}{dt}\left(\frac{2m\dot{x}}{2x} - 0\right) - \left(-\frac{m\dot{x}^2}{2x^2} - Kx\right) = 0$$

$$m\ddot{x} + Kx^2 = 0$$

Hamiltonian of a System $H = \sqrt{p^2+m^2} + V(x)$. The Lagrangian will be

$$\Rightarrow \quad \dot{x} = \frac{\partial H}{\partial p} = \frac{p}{\sqrt{p^2+m^2}}$$

$$p^2(\dot{x}^2-1) = -m^2\dot{x}^2$$

$$p = \frac{m\dot{x}}{\sqrt{1-\dot{x}^2}}$$

Lagrangian

$$L = p\dot{x} - H$$

$$L = \frac{m\dot{x}^2}{\sqrt{1-\dot{x}^2}} - \sqrt{p^2+m^2} - V(x)$$

$$L = \frac{m\dot{x}^2}{\sqrt{1-\dot{x}^2}} - \sqrt{\frac{m^2\dot{x}^2}{1-\dot{x}^2}+m^2} - V(x)$$

$$L = -m\sqrt{1-\dot{x}^2} - V(x)$$

$H = \frac{1}{2m}(P_x^2 + P_y^2) - \alpha x$ if at $t=0, x=0, y=0, \dot{y}=\beta$ the path of the Particle on xy plane is.

$$\Rightarrow \quad \dot{x} = \frac{\partial H}{\partial P_x} = \frac{P_x}{m} \quad \text{and} \quad \dot{y} = \frac{\partial H}{\partial P_y} = \frac{P_y}{m}$$

Lagrangian of the System

$$L = \frac{1}{2m}(m^2\dot{x}^2 + m^2\dot{y}^2) - \alpha x = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - \alpha x$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\Rightarrow \quad m\ddot{x} + \alpha = 0$$

$$\Rightarrow \quad m\ddot{x} = -\alpha$$

$$\Rightarrow \quad \ddot{x} = -\frac{\alpha}{m}$$

$$\Rightarrow \quad \dot{x} = -\frac{\alpha}{m}t + c$$

$$\Rightarrow \quad x = -\frac{\alpha}{2m}t^2 + ct$$

$$x = -\frac{\alpha}{2m}t^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0$$

$$\ddot{y} = 0$$

$$\dot{y} = c$$

$$y = ct \Rightarrow t = \frac{y}{c}$$

$$x = -\frac{\alpha}{2m}\frac{y^2}{c^2} \Rightarrow y^2 \alpha x$$

(Parabola)

The Hamiltonian of a classical 1-D oscillator is $H = \frac{1}{2}(P^2 + x^2)$. The total time derivative of $P + 2\sqrt{x}$ is

$$\Rightarrow \quad \frac{dA}{dt} = \{A, H\} + \frac{\partial A}{\partial t}$$

$A = P + 2\sqrt{x}$

$\frac{\partial A}{\partial t} = 0$

$$= \left(\frac{\partial A}{\partial x}\cdot\frac{\partial H}{\partial p} - \frac{\partial A}{\partial P}\frac{\partial H}{\partial x}\right) + 0$$

$$= \left(\frac{1}{\sqrt{2}}P - x\right) = (P/\sqrt{2} - x)$$

⑩ The Value of Poisson bracket $[\vec{a}\cdot\vec{L}, \vec{b}\cdot\vec{r}]$ is

$$\Rightarrow \quad \{a_i L_i, b_j r_j\} = a_i b_j \{L_i, r_j\}$$

$$= -a_i b_j \{r_j L_i\} = -a_i b_j \epsilon_{jik} r_k = \epsilon_{ijk} a_i b_j r_k$$

19.05.2024

① The conditions that need to be satisfied by the real constants a, b, c, d to that $Q = apq^c$, $P = bq^d$ is canonical is

⇒ For canonical variable $\{Q, P\} = 1$

$$\frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} = 1$$

or $acpq^{c-1} \cdot 0 - aq^c \cdot dbq^{d-1} = 1$

or, $-adb\, q^{c+d-1} = 1$

$c+d-1 = 0$ $\qquad\qquad abd = -1$

$c+d = 1$

② If the following transformation is canonical

$Q = q \tan\alpha p$, $\qquad P = 2 \ln(\sin\beta p)$

The values of α and β are respectively

⇒ $\frac{\partial Q}{\partial q} = \tan\alpha p$ $\qquad \frac{\partial P}{\partial q} = \frac{2\beta}{\sin\beta p} \cos\beta p \times 0 = 0$

$\frac{\partial Q}{\partial p} = q\alpha \sec^2\alpha p$ $\qquad \frac{\partial P}{\partial p} = 2\beta \cot\beta p$

$2\beta \tan\alpha p \cdot \cot\beta p = 1$

$\beta \tan\alpha p \cot\beta p = \frac{1}{2}$ $\qquad \alpha = \beta = \frac{1}{2}$

③ The value of $\{A, \{B, C\}\} - \{\{A, B\}, C\}$ is given by

we have $\{A, \{B, C\}\} + \{B, \{C, A\}\} + \{C, \{A, B\}\} = 0$

$\{A, \{B, C\}\} - \{\{A, B\}, C\} = -\{B, \{C, A\}\}$

$\{A, \{B, C\}\} - \{\{A, B\}, C\} = \{\{C, A\}, B\}$

④ If L_z be the z component of angular momentum then $\{L_z, x P_y\}$ is equal to

$\Rightarrow$
$$\{L_z, x P_y\} = x\{L_z, P_y\} + P_y\{L_z, x\}$$
$$= -x\, \epsilon_{321} P_1 + (-P_y)\, \epsilon_{312}\, x_2$$
$$= y P_y - x P_x .$$

---

⑤ $Q = (2q)^a \cos^b p$, $P = (2q)^a \sin^b p$. If the transformation is cannonical then

$\Rightarrow$
$$\frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p} = 1$$

or $2^a a q^{a-1} \cdot b \cos^{b-1} p \sin p$

or
$$2^a a q^{a-1} \cos^b p\, (2q)^a \sin^{b-1} p \cos p$$
$$+ 2^a a q^{a-1} \sin^b q\, (2q)^a b \cos^{b-1} p \sin p = 1$$

or
$$2^{2a} ab\, q^{2a-1} \cos^{b+1} p \sin^{b-1} p + 2^{2a} ab \sin^{b+1} p \cos^{b-1} p = 1$$
$$\cdot\, q^{2a-1}$$

$q^{2a-1} = q^0 \Rightarrow a = \frac{1}{2}$ $\quad 2ab = 1 \Rightarrow b = 2/2 = 1$

---

⑥ $\{L_x, y L_z\}$ is equal to

$\Rightarrow$
$$\{L_x, y L_z\} = y\{L_x, L_z\} + L_z\{L_x, y\}$$
$$= y\, \epsilon_{132} L_2 + L_z\, \epsilon_{213}\, x_3$$
$$= -y L_y + z L_z$$

---

⑦ $P = \alpha(p^2+q^2)$ $\quad Q = \beta \tan^{-1}\left(\frac{\alpha}{p}\right)$ If P, Q are Canonical then α, β are

$\Rightarrow$ $\quad$ Question is made wrong

$$\frac{\partial P}{\partial p} = 2\alpha p \qquad \frac{\partial Q}{\partial q} = 0$$

$$\frac{\partial P}{\partial q} = 2\alpha q \qquad \frac{\partial Q}{\partial p} = \frac{\beta}{1+\frac{\alpha^2}{p^2}}\left(-\frac{\alpha}{p^2}\right) = \frac{-\alpha\beta}{p^2+\alpha^2}$$

⑧ $P = \alpha(p^2+q^2)$ and $Q = \beta \tan^{-1}\frac{q}{p}$ If P and Q are Canonical then α and β are

$\Rightarrow$ $P = \alpha(p^2+q^2)$ $\qquad Q = \beta \tan^{-1}\frac{q}{p}$

$\frac{\partial P}{\partial q} = 2\alpha q$ $\qquad \frac{\partial Q}{\partial p} = \frac{\beta}{1+\frac{q^2}{p^2}}\left(-\frac{q}{p^2}\right) = \frac{-q\beta}{p^2+q^2}$

$\frac{\partial P}{\partial p} = 2\alpha p$ $\qquad \frac{\partial Q}{\partial q} = \frac{\beta}{1+\frac{q^2}{p^2}}\frac{1}{p} = \frac{p\beta}{p^2+q^2}$

If P, Q are Canonical then

$\left\{\frac{\partial P}{\partial x}\cdot\frac{\partial Q}{\partial p} - \frac{\partial P}{\partial p}\frac{\partial Q}{\partial x}\right\} = -1$ $\qquad \alpha = \beta = \frac{1}{2}$

$-2\alpha q \times \frac{q\beta}{p^2+q^2} - 2\alpha p \times \frac{p\beta}{p^2+q^2} = -1$

$\frac{4\alpha\beta(q^2+p^2)}{p^2+q^2} = 1 \Rightarrow 4\alpha\beta = 1 \Rightarrow \alpha\beta = \frac{1}{4}$

⑨ $\{L_i, \vec{r}\cdot\vec{p}\}$ is equal to

$\Rightarrow$ $\{L_i, r_j p_j\} = r_j\{L_i, p_j\} + p_j\{L_i, r_j\}$

$= -r_j\{p_j, L_i\} - p_j\{r_j, L_i\}$

$= -r_j\,\epsilon_{jik}\,p_k - p_j\,\epsilon_{jik}\,r_k$

$= -\epsilon_{jik}\,r_j p_k - \epsilon_{jik}\,p_j r_k$

$= \epsilon_{ijk}\,r_j p_k + \epsilon_{ijk}\,p_j r_k = (\vec{r}\times\vec{p}) + (\vec{p}\times\vec{r}) = 0$

⑩ $L = \alpha\dot{q}^2 + \beta q^2$. If p_q denotes the Canonical momentum then

$\Rightarrow$ $p_q = \frac{\partial L}{\partial \dot{q}} = 2\alpha\dot{q}$ $\qquad$ q is not cyclic so $p_q = 2\alpha\dot{q}$ is not Conserved quantity

MCQs on Poisson's Bracket 25.05.2024

① The Poisson Bracket of $\alpha p^2 + 2\beta pq$ with the Hamiltonian $H = ap^2 + bq^2$ is

$\Rightarrow$ $\{\alpha p^2 + 2\beta pq, ap^2 + bq^2\}$

$$= \frac{\partial}{\partial q}(2\beta pq)\frac{\partial}{\partial p}(\alpha p^2) - \frac{\partial}{\partial p}(ap^2)\frac{\partial}{\partial q}(bq^2) = 4\alpha\beta p^2 - 4abpq$$

---

② The Hamiltonian of a Simple Pendulum consist of mass m attached to a massless string of length l is $H = \frac{P_\theta^2}{2ml^2} + mgl(1-\cos\theta)$ then $\frac{dL}{dt}$

$\Rightarrow$ $L = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1-\cos\theta)$

$L = \sum P_\theta \dot{\theta} - H$ $\qquad P_\theta = ml^2\dot{\theta}$

$\dot{\theta} = \frac{P_\theta}{ml^2}$

so $L = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1-\cos\theta)$

$$\{L, H\} = \left(\frac{\partial L}{\partial \theta}\frac{\partial H}{\partial P_\theta} - \frac{\partial L}{\partial P_\theta}\frac{\partial H}{\partial \theta}\right) = -\frac{2g}{l}P_\theta \sin\theta$$

and $\frac{\partial L}{\partial t} = 0$ so $\frac{dL}{dt} = \{L, H\} + \frac{\partial L}{\partial t} = -\frac{2g}{l}P_\theta\sin\theta$

---

③ If $L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{qB}{2}(x\dot{y} - y\dot{x})$ then $\{\dot{x}, \dot{y}\}$ is equal to

$\Rightarrow$ $\frac{\partial L}{\partial \dot{x}} = P_x = m\dot{x} - \frac{qB}{2}y \Rightarrow \dot{x} = \frac{P_x}{m} + \frac{qBy}{2m}$

$\frac{\partial L}{\partial \dot{y}} = P_y = m\dot{y} + \frac{qBx}{2} \Rightarrow \dot{y} = \frac{P_y}{m} - \frac{qBx}{2m}$

$$\{\dot{x}, \dot{y}\} = \frac{\partial \dot{x}}{\partial x}\frac{\partial \dot{y}}{\partial p} - \frac{\partial \dot{x}}{\partial p}\frac{\partial \dot{y}}{\partial x} = \frac{qB}{2m^2} - \left(-\frac{qB}{2m^2}\right) = \frac{qB}{m^2}$$

Normal Modes

$$\hat{T} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$$

$T_{11} \to$ Coeff of $\dot{x}_1^2$

$T_{22} \to$ Coeff of $\dot{x}_2^2$

$T_{12} = T_{21} \to \frac{1}{2}$ [Coeff of $\dot{x}_1 \dot{x}_2$]

$$\hat{V} = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}$$

$V_{11} \to$ Coeff of x_1^2

$V_{22} \to$ Coeff of x_2^2

$V_{12} = V_{21} \to \frac{1}{2}$ (Coeff of $x_1 x_2$)

No to find ω $\quad |V - \omega^2 T| = 0$

① $L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}(\omega_1^2 x^2 + \omega_2^2 y^2)$ Frequency of Small oscillation of the System

$\Rightarrow$ $T = \frac{1}{2}(\dot{x}^2 + \dot{y}^2)$ $\qquad V = \frac{1}{2}(\omega_1^2 x^2 + \omega_2^2 y^2)$

$$\hat{T} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \qquad \hat{V} = \begin{bmatrix} \frac{\omega_1^2}{2} & 0 \\ 0 & \frac{\omega_2^2}{2} \end{bmatrix}$$

$$|\hat{V} - \omega^2 T| = \begin{vmatrix} \frac{\omega_1^2}{2} - \frac{\omega^2}{2} & 0 \\ 0 & \frac{\omega_2^2}{2} - \frac{\omega^2}{2} \end{vmatrix} = 0$$

$$\left(\frac{\omega_1^2}{2} - \frac{\omega^2}{2}\right)\left(\frac{\omega_2^2}{2} - \frac{\omega^2}{2}\right) = 0$$

or $(\omega_1^2 - \omega^2)(\omega_2^2 - \omega^2) = 0$

$\omega_1^2 - \omega^2 = 0 \qquad \omega_2^2 - \omega^2 = 0$

$\omega_1 = \omega$

$\omega_2 = \omega$

These are two normal modes

② $T(\dot{x}\dot{y}\dot{z}) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

$V(x,y,z) = \frac{1}{2}m\omega^2(3x^2 + 3y^2 + 2z^2 + 2xy)$.

The oscillation frequency of 3 normal modes

$$V(x,y,z) = \frac{1}{2}m\omega^2(3x^2+3y^2+2z^2+2xy)$$

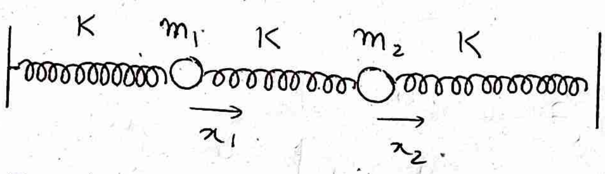
$$T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)$$

$$\dot{T} = \begin{bmatrix} \frac{m}{2} & 0 & 0 \\ 0 & \frac{m}{2} & 0 \\ 0 & 0 & \frac{m}{2} \end{bmatrix} \qquad \dot{V} = \begin{bmatrix} \frac{3}{2}m\omega^2 & \frac{1}{2}m\omega^2 & 0 \\ \frac{1}{2}m\omega^2 & \frac{3}{2}m\omega^2 & 0 \\ 0 & 0 & m\omega^2 \end{bmatrix}$$

$$|\dot{V} - \omega^2 T| = \begin{vmatrix} \frac{3}{2}m\omega^2 - \frac{1}{2}m\omega_0^2 & \frac{1}{2}m\omega_0^2 & 0 \\ \frac{1}{2}m\omega^2 & \frac{3}{2}m\omega^2 - \frac{1}{2}m\omega_0^2 & 0 \\ 0 & 0 & m\omega^2 - \frac{m\omega^2}{2} \end{vmatrix}$$

$$= \begin{vmatrix} m\omega^2 & \frac{1}{2}m\omega^2 & 0 \\ \frac{1}{2}m\omega^2 & m\omega^2 & 0 \\ 0 & 0 & \frac{m\omega^2}{2} \end{vmatrix} = m\omega^2[\quad]$$

(2)



$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 \qquad (\text{take } m_1 = m_2 = m)$$

$$= \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2$$

$$V = \frac{1}{2}Kx_1^2 + \frac{1}{2}K_2^2x_2^2 + \frac{1}{2}K(x_1-x_2)^2$$

$$L = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 - \frac{1}{2}Kx_1^2 - \frac{1}{2}K_2^2x_2^2 - \frac{1}{2}K(x_1-x_2)^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0 \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = 0$$

$$\frac{d}{dt}(m\dot{x}_1) - (-Kx_1 - K(x_1-x_2)) \qquad m\ddot{x}_2 + K[2x_2 - x_1] = 0$$

$$m\ddot{x}_1 + Kx_1 + Kx_1 - Kx_2 = 0$$

$$m\ddot{x}_1 + (2x_1 - x_2)K = 0$$

$$\ddot{x}_1 + (2x_1 - x_2)\frac{K}{m} = 0$$

By adding them

$$m(\ddot{x}_1 + \ddot{x}_2) + K(x_1 + x_2) = 0$$

$$x_1 + x_2 = S_1$$

$$(\ddot{x}_1 + \ddot{x}_2) + \frac{K}{m}(x_1 + x_2) = 0$$

$$\omega_1 = \sqrt{\frac{K}{m}}$$

$$\ddot{S}_1 + \frac{K}{m} S_1 = 0 \Rightarrow \ddot{S}_1 + \omega_1^2 S_1 = 0$$

Now $(\ddot{x}_1 - \ddot{x}_2) + \frac{3K}{m}(x_1 - x_2) = 0$

$$x_1 - x_2 = S_2$$

$$\ddot{S}_2 + \frac{3K}{m} S_2 = 0$$

$$\omega_2 = \sqrt{\frac{3K}{m}}$$

$$\Rightarrow \ddot{S}_2 + \omega_2^2 S_2 = 0$$

Again $T = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2$

$$V = \frac{1}{2} K x_1^2 + \frac{1}{2} K x_2^2 + \frac{1}{2}(K x_1^2 + K x_2^2 - 2K x_1 x_2)$$

$$= K x_1^2 + K x_2^2 - K x_1 x_2$$

$$T = \begin{bmatrix} \frac{m}{2} & 0 \\ 0 & \frac{m}{2} \end{bmatrix} \qquad V = \begin{bmatrix} K & -K/2 \\ -K/2 & K \end{bmatrix}$$

$$|V - \omega^2 T| = \begin{vmatrix} K - \frac{\omega^2 m}{2} & -\frac{K}{2} \\ -\frac{K}{2} & K - \frac{\omega^2 m}{2} \end{vmatrix} = 0$$

$$\left(K - \frac{\omega^2 m}{2}\right)^2 - \frac{K^2}{4} = 0$$

$$K^2 - \omega^2 m K + \frac{\omega^4 m^2}{4} - \frac{K^2}{4} = 0$$

$$\frac{3K^2}{4} - \omega^2 m K + \frac{\omega^4 m^2}{4} = 0$$

$$\omega^2 = \frac{\frac{4K}{m} \pm \sqrt{\frac{16K^2}{m^2} - \frac{12K^2}{m^2}}}{2}$$

$$\omega^4 \frac{m^2}{4} - \omega^2 m K + \frac{3K^2}{4} = 0$$

$$\frac{\frac{4K}{m} \pm \frac{2K}{m}}{2}$$

$$\omega^4 m^2 - 4\omega^2 m K + 3K^2 = 0$$

$$\omega^4 - \left(\frac{4K}{m}\right)\omega^2 + \frac{3K^2}{m^2} = 0$$

$$\omega_1 = \sqrt{K/m}$$

$$\omega_2 = \sqrt{3K/m}$$

③ Consider two coupled harmonic oscillators of mass m in each. The Hamiltonian

$$\hat{H} = \frac{\hat{p}_1^2}{2m} + \frac{\hat{p}_2^2}{2m} + \frac{1}{2} m\omega^2 \left[x_1^2 + x_2^2 + (x_1 - x_2)^2 \right]$$

The eigen values of $\hat{H}$ is (Jest 2018)

$\Rightarrow$

$$T = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2$$

$$V = \frac{1}{2} m\omega^2 x_1^2 + \frac{1}{2} m\omega^2 x_2^2 + \frac{1}{2} m\omega^2 (x_1^2 + x_2^2 - 2x_1x_2)$$

$$= \frac{1}{2} m\omega^2 x_1^2 + \frac{1}{2} m\omega^2 x_2^2 + \frac{1}{2} m\omega^2 x_1^2 + \frac{1}{2} m\omega^2 x_2^2 - x_1 x_2 m\omega^2$$

$$= m\omega^2 x_1^2 + m\omega^2 x_2^2 - x_1 x_2 m\omega^2$$

$$T = \begin{bmatrix} \frac{m}{2} & 0 \\ 0 & \frac{m}{2} \end{bmatrix} \qquad V = \begin{bmatrix} m\omega^2 & -\frac{m\omega^2}{2} \\ -\frac{m\omega^2}{2} & m\omega^2 \end{bmatrix}$$

$$|V - \omega_0^2 T| = \begin{vmatrix} m\omega^2 - \frac{\omega_0^2 m}{2} & -\frac{m\omega^2}{2} \\ -\frac{m\omega^2}{2} & m\omega^2 - \frac{\omega_0^2 m}{2} \end{vmatrix}$$

$$\left(m\omega^2 - \frac{\omega_0^2 m}{2}\right)^2 - \frac{m^2\omega^4}{4} = 0$$

$$m^2\omega^4 - \omega^2\omega_0^2 m^2 + \frac{\omega_0^4 m^2}{4} - \frac{m^2\omega^4}{4} = 0$$

$$\frac{\omega_0^4 m^2}{4} + \frac{3m^2\omega^4}{4} - \omega^2\omega_0^2 m^2 = 0$$

$$\omega_0^4 + 3\omega^4 - 4\omega^2\omega_0^2 = 0.$$

$$\omega_0^4 - (4\omega^2)\omega_0^2 + 3\omega^4 = 0$$

$$\omega_0^2 = \frac{4\omega^2 \pm \sqrt{16\omega^4 - 12\omega^4}}{2} = \frac{4\omega^2 \pm 2\omega^2}{2} = 2\omega^2 \pm \omega^2$$

$\omega_0 = \sqrt{3}\,\omega$ and $\omega_0 = \omega$

① A particle of mass m is thrown upward with velocity v and there is retarding air resistance proportional to the square of the velocity with proportionality constant K. If the particle attains a maximum height after time t and g. What is the velocity (Jest 2013)

⇒ Equation of motion $m\frac{dv}{dt} = mg + Kv^2$

$$\Rightarrow \frac{dv}{dt} = g + \frac{K}{m}v^2 \Rightarrow \frac{dv}{g+\frac{K}{m}v^2} = dt \Rightarrow \frac{m}{K}\cdot\frac{dv}{v^2+\frac{gm}{K}} = dt$$

$$\Rightarrow \frac{m}{K}\cdot\frac{1}{\sqrt{gm/K}}\tan^{-1}\left(\frac{v}{\sqrt{\frac{gm}{K}}}\right) = t \Rightarrow V = \sqrt{\frac{mg}{K}}\tan\left(\sqrt{\frac{Kg}{m}}\,t\right)$$

② A light beam is propagating through a block of glass with index of refraction n. If the glass is moving at constant velocity v in the same direction as the beam, the velocity of the light in the glass block as measured by an observe in labratory is (Jest 2013)

⇒
$$u = \frac{V+\frac{c}{n}}{1+\frac{V.c}{c^2n}} = \left(V+\frac{c}{n}\right)\left(1+\frac{V}{cn}\right)^{-1} = \left(V+\frac{c}{n}\right)\left(1-\frac{V}{cn}+\frac{V^2}{c^2n^2}\right)$$

$$= V - \frac{V^2}{cn} + \frac{V^3}{c^2n^2} + \frac{c}{n} - \frac{V}{n^2} + \frac{cV^2}{cn^3} = \frac{c}{n} + V\left(1-\frac{1}{n^2}\right)$$

③ A dynamical system with two generalized coordinates q_1 and q_2 has Lagrangian $L = \dot{q}_1^2 + \dot{q}_2^2$. The Hamiltonian is (Jest 2014)

⇒ Hamiltonian. $H = \sum p\dot{q} - L$

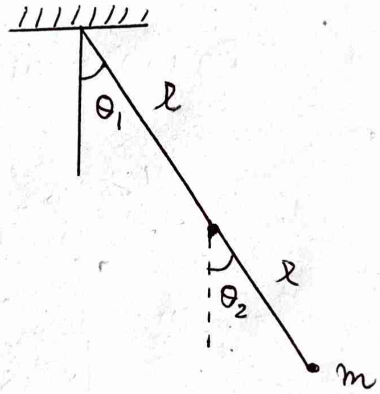
$$P_1 = \frac{\partial L}{\partial \dot{q}_1} = 2\dot{q}_1$$

$$= (P_1\dot{q}_1 + P_2\dot{q}_2) - \frac{P_1^2}{4} - \frac{P_2^2}{4}$$

$$P_2 = \frac{\partial L}{\partial \dot{q}_2} = 2\dot{q}_2$$

$$H = \frac{P_1^2}{2} + \frac{P_2^2}{2} - \frac{P_1^2}{4} - \frac{P_2^2}{4} = \frac{(P_1^2+P_2^2)}{4}$$

(4) A double pendulum consists of two equal masses m suspended by two strings of length l. What is the Lagrangian of this system for oscillations in a plane? Assume the angles θ_1 and θ_2 made by the two strings are small [$\cos\theta = 1 - \theta^2/2$)



$\Rightarrow$ $x_1 = l\sin\theta_1$, $y_1 = l\cos\theta_1$

$x_2 = x_1 + l\sin\theta_2 = l\sin\theta_1 + l\sin\theta_2$

$y_2 = y_1 + l\cos\theta_2 = l\cos\theta_1 + l\cos\theta_2$

$\dot{x}_2 = l\cos\theta_1\dot{\theta}_1 + l\cos\theta_2\dot{\theta}_2$

$\dot{y}_2 = -l\sin\theta_1\dot{\theta}_1 - l\sin\theta_2\theta_2$

$$\dot{x}_2^2 + \dot{y}_2^2 = l^2\cos^2\theta_1\dot{\theta}_1^2 + l^2\cos^2\theta_2\dot{\theta}_2^2 + 2l^2\cos\theta_1\cos\theta_2\dot{\theta}_1\dot{\theta}_2$$
$$+ l^2\sin^2\theta_1\dot{\theta}_1^2 + l^2\sin^2\theta_2\dot{\theta}_2^2 + 2l^2\sin\theta_1\sin\theta_2\dot{\theta}_1\dot{\theta}_2$$
$$= l^2\dot{\theta}_1^2 + l^2\dot{\theta}_2^2 + 2l^2\cos(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2$$

also $\dot{x}_1^2 + \dot{y}_1^2 = l^2\dot{\theta}_1^2$

$$T = \frac{1}{2}m(\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2)$$
$$= \frac{1}{2}m(l^2\dot{\theta}_1^2 + l^2\dot{\theta}_1^2 + l^2\dot{\theta}_2^2 + 2l^2\cos(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2)$$

$$V = mgy_1 + mgy_2 = -(2mgl\cos\theta_1 + mgl\cos\theta_2)$$

Lagrangian $\mathcal{L} = T - V$

$$\mathcal{L} = \frac{1}{2}m(l^2\dot{\theta}_1^2 + l^2\dot{\theta}_1^2 + l^2\dot{\theta}_2^2 + 2l^2\cos(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2)$$
$$+ 2mgl\cos\theta_1 + mgl\cos\theta_2$$

$$\mathcal{L} = ml^2\left(\dot{\theta}_1^2 + \frac{1}{2}\dot{\theta}_2^2 + \dot{\theta}_1\dot{\theta}_2 + \frac{g}{l}\left(1 - \frac{\theta_1^2}{2}\right) + \frac{g}{2l}\left(1 - \frac{\theta_2^2}{2}\right)\right]$$

$$\mathcal{L} = ml^2\left(\dot{\theta}_1^2 + \frac{1}{2}\dot{\theta}_2^2 + \dot{\theta}_1\dot{\theta}_2 - \frac{\omega_0^2\dot{\theta}_1^2}{2} - \frac{1}{4}\omega_0\dot{\theta}_2^2\right)$$

⑤ If the diameter of the earth is increased 4% without changing the mass. then the length of the day is ____ hours (Jam 2019)

$\Rightarrow$ From angular momentum conservation

$$I_1\omega_1 = I_2\omega_2 \Rightarrow R_1^2 \frac{2\pi}{T_1} = R_2^2 \frac{2\pi}{T_2}$$

$$\Rightarrow T_2 = \left[\frac{R_2}{R_1}\right]^2 T_1 \qquad R_2 = R_1 \times 1.04$$

$$\Rightarrow T_2 = (1.04)^2 \times 24 \qquad \frac{R_2}{R_1} = (1.04)$$

$$\Rightarrow T_2 = 25.96 \text{ hr}$$

⑥ Consider a classical particle subjected to an attractive inverse - square force field. The total energy of the particle is E and eccentricity ϵ. The particle will follow parabolic (Jam 2019)

$\Rightarrow$ Eccentricity. $\varepsilon = \sqrt{1 + \frac{2EL^2}{mk^2}}$

For parabolic path $\varepsilon = 1$. $E = 0$

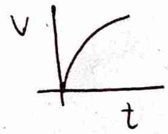
⑦ Consider an object moving with a velocity $\vec{v}$ in a frame which rotates with a constant angular Velocity $\vec{\omega}$. The Coriolis force experend by object (Jam 2019)

$\Rightarrow$ $F_c = -2m(\vec{\omega} \times \vec{v})$

perpendicular both $\vec{v}$ and $\vec{\omega}$

⑧ A ball of mass m is falling under gravity through a viscous medium in which drag force proportional to instanteneous velocity (Jam 2019)

$\Rightarrow$ $F = KV \Rightarrow ma = KV \Rightarrow m\frac{dV}{dt} = KV \Rightarrow V \alpha t$



⑨ The mass per unit length of a rod length 2m Varies as $\rho = 3x$ Kg/m. Moment of interia is (2019)

$\Rightarrow$ $I = \int x^2 dm = \int x^2 \rho dx = \int x^2 \cdot 3x \cdot dx = 3\frac{x^4}{4}\Big|_0^2 = 12$

(10) A particle of mass m and angular momentum L moves in space where potential is $U(r) = Kr^2$ $(r>0)$. If particle moves in circular orbit then radius of the orbit is (Jam 2021)

$\Rightarrow$ Effective potential

$$V_{eff} = \frac{L^2}{2mr^2} + Kr^2 \qquad \frac{L^2}{mr^3} = 2Kr$$

$$\frac{\partial V_{eff}}{\partial r} = -\frac{L^2}{mr^3} + 2Kr = 0 \qquad r = \left(\frac{L^2}{2mK}\right)^{1/4}$$

(11) Consider a particle of mass m moving in a plane with a constant radial speed $\dot{r}$ and constant angular speed $\dot{\theta}$. The acceleration (Jam 2021)

$\Rightarrow$ $$\bar{a}_r = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$$

$$\ddot{r} = 0 \qquad \ddot{\theta} = 0 \qquad = -r\dot{\theta}^2\hat{r} + 2\dot{r}\dot{\theta}\hat{\theta}$$

(12) A planet of mass m moves in an elliptical orbit. It's maximum and minimum distance from the Sun R and r. Assume $M \gg m$. The angular momentum of the plannet (Jam 2022)

$\Rightarrow$ $$\frac{l}{r} = 1 \pm \epsilon\cos\theta \Rightarrow r_1 = \frac{l}{1+\epsilon\cos\theta} = \frac{L^2/\mu K}{1+\sqrt{1+\frac{2EL^2}{mK^2}}\cos\theta}$$

$$1+\sqrt{1+\frac{2EL^2}{mK^2}} = \frac{L^2}{\mu K}\cdot\frac{1}{r_1} \qquad 1-\sqrt{1+\frac{2EL^2}{mK^2}} = \frac{L^2}{\mu K}\cdot\frac{1}{r_2}$$

$$2 = \frac{L^2}{\mu K}\left(\frac{1}{R}+\frac{1}{r}\right) \qquad \mu = \frac{mM}{m+M}$$

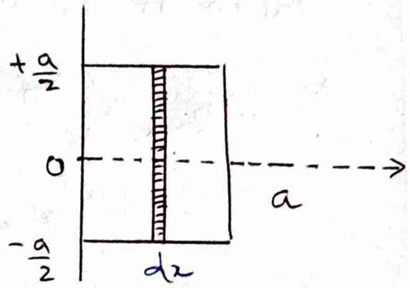
$$2 = \frac{L^2}{\mu K}\left(\frac{R+r}{Rr}\right) \qquad L^2 = \frac{2mM}{m+M}\,\frac{GMmRr}{R+r}$$

$$L^2 = \frac{2\mu KRr}{R+r} \qquad L^2 = \frac{G(Mm)^2}{m+M}\,\frac{2Rr}{R+r}$$

$$L = m\sqrt{\frac{GMRr}{2(R+r)}} \qquad L = Mm\sqrt{\frac{2GRr}{(R+r)}\,\frac{1}{m+M}}$$

(13) A square laminar sheet with side a mass M. The mass per unit area is $\sigma(x) = \sigma_0 (1 - \frac{x}{a})$ Moment of inertia of sheet about Y axis (Jam 2022)

$\Rightarrow$ Mass of the sheet



$$M = \int \sigma \, dx dy$$

$$= \int_0^a \sigma_0 (1 - \frac{x}{a}) dx \int_0^a dy$$

$$= \sigma_0 a \left[x - \frac{x^2}{2a} \right]_0^a = \frac{\sigma_0 a^2}{2}$$

$$I = \int x^2 dm = \int x^2 \sigma \, dx dy = \int x^2 \sigma_0 (1 - \frac{x}{a}) dx \int dy$$

$$= \sigma_0 a \int_0^a (x^2 - \frac{x^3}{a}) dx = \sigma_0 a \left. \frac{x^3}{3} - \frac{x^4}{4a} \right|_0^a$$

$$= \sigma_0 a^4 \times \frac{1}{12} = \frac{2M}{a^2} \times \frac{a^4}{12} = \frac{1}{6} Ma^2$$

(14) Let (r, θ) denote the polar coordinates of a particle moving in plane. If $\hat{r}$ and $\hat{\theta}$ represent unit vectors then (Jam 2022)

$\Rightarrow$ $x = r \cos\theta$ $\quad \vec{R} = r\cos\theta \, \hat{i} + r \sin\theta \, \hat{j}$

$y = r \sin\theta$

$$\frac{\partial \vec{R}}{\partial r} = \cos\theta \, \hat{i} + \sin\theta \, \hat{j}$$

$$\hat{r} = \frac{\frac{\partial \vec{R}}{\partial r}}{\left| \frac{\partial \vec{R}}{\partial r} \right|} = (\cos\theta \, \hat{i} + \sin\theta \, \hat{j})$$

$$\hat{\theta} = \frac{\partial \vec{R}/\partial \theta}{|\partial \vec{R}/\partial \theta|} = \frac{-r \sin\theta \, \hat{i} + r \cos\theta \, \hat{j}}{r} = (-\sin\theta \, \hat{i} + \cos\theta \, \hat{j})$$

$$\frac{d\hat{r}}{d\theta} = (-\sin\theta \, i + \cos\theta \, j) = \hat{\theta}$$

$$\frac{d\hat{\theta}}{d\theta} = -(\cos\theta \, i + \sin\theta \, j) = -\hat{r}$$

(13) Three mass $m_1=1, m_2=2$ and $m_3=3$ are located on X-axis such that COM is at $x=1$. Another mass $m_4=4$ is placed at x_0 and new COM is $x=3$. The Value of x_0 is (Jam 2022).

$\Rightarrow$

$$R_{cm} = \frac{m_1x_1+m_2x_2+m_3x_3}{m_1+m_2+m_3}$$

$$1 = \frac{x_1+2x_2+3x_3}{6} \Rightarrow x_1+2x_2+3x_3 = 6$$

$$3 = \frac{m_1x_1+m_2x_2+m_3x_3+m_4x_0}{m_1+m_2+m_3+m_4}$$

$$= \frac{x_1+2x_2+3x_3+4x_0}{10} = \frac{6+4x_0}{10} = 3 \Rightarrow x_0 = 6$$

(14) A satelite is revolving around earth in a closed orbit. The height of the satelite above Earth's surface at perigee and apogee are 2500 Km and 4500 Km. Consider the radius of earth 6500 Km. The eccentricty of Satelight orbit is

$\Rightarrow$

$$r_{max} = a(1+e) = 4500+6500 = 11000\text{ Km}$$

$$r_{min} = a(1-e) = 2500+6500 = 9000\text{ Km}$$

$$\frac{1+e}{1-e} = \frac{11}{9} \Rightarrow 11-11e = 9+9e \Rightarrow 2 = 20e \Rightarrow e = 0.1$$

(15) A planet is in a highlly eccentric orbit about a star. The distance of its closest approach is 300 times smaller than its farthest distance from star. V_c/V_f is (Jam 2021)

$\Rightarrow$ Conservation of angular momentum

$$V_c r_c = V_f r_f$$

$$d_c = \frac{d_f}{300}$$

$$\frac{V_c}{V_f} = \frac{d_f}{d_c} = 300$$

(16) The moment of inertia of a solid sphere (R, M) about the axis which is at a distance $R/2$ from the centre is (Jam 2021)

$\Rightarrow$ $I = I_{CM} + Md^2$

$$= \frac{2}{5}MR^2 + M\frac{R^2}{4} = MR^2\left(\frac{2}{5}+\frac{1}{4}\right) = \frac{13}{20}MR^2$$

(17) A particle initially at the origin in an inertial frame S, has constant velocity $v\hat{i}$. Frame S' is rotating about the Z-axis with angular velocity ω (ACW). The coordinate axes of S' coincide with those of S at $t=0$. The velocity of particle (V_x', V_y') in S' frame at $t = \pi/2\omega$ (Jam 2021)

$\Rightarrow$ From transformation matrix

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$x' = x\cos\theta + y\sin\theta = x\cos\omega t + y\sin\omega t$$

$$y' = -x\sin\theta + y\cos\theta = -x\sin\omega t + y\cos\omega t$$

$v_x' = (v_x\cos\omega t + v_y\sin\omega t) +$ $\quad t = \pi/2\omega \quad v_x' = -v\pi/2$

(18) A current density for a flow is given by

$$\vec{J}(x,y,z,t) = \frac{8et}{(1+x^2+y^2+z^2)}\hat{x}$$ At time $t=0$ the mass density $\rho(x,y,z,0) = 1$ Find $\rho(1,1,1,1)$

$\Rightarrow$

$$\vec{\nabla}\cdot\vec{J} = 8et\frac{d}{dx}(1+x^2+y^2+z^2)^{-1} = \frac{8et\,(-1)\cdot 2x}{(1+x^2+y^2+z^2)^2}$$

$$= \frac{-16xet}{(1+x^2+y^2+z^2)^2} = -\frac{\partial\rho}{\partial t}$$

$$\rho = \frac{16xet}{(1+x^2+y^2+z^2)^2} + c \Rightarrow 1 = \frac{16x}{(1+x^2+y^2+z^2)^2} + c$$

$$\rho = \frac{16xet}{(1+x^2+y^2+z^2)^2} + 1 - \frac{16x}{(1+x^2+y^2+z^2)^2} = \frac{16x}{(1+x^2+y^2+z^2)^2}(e^t - 1) + 1$$

$\rho(1,1,1,1) = 2.718$

(19) A thin circular disc lying in the xy-plane has a surface mass density σ, given by

$$\sigma(r) = \begin{cases} \sigma_0\left(1-\frac{r^2}{R^2}\right) & \text{if } r \le R \\ 0 & \text{if } r > R. \end{cases}$$ (Jam 2021)

Its moment of inertia about the z axis, passing through its centre is

$\Rightarrow$ Total mass, $m = \int dm = \int \sigma_0\left(1-\frac{r^2}{R^2}\right) r\,dr\,d\theta$

$$= 2\pi\sigma_0 \int\left(r - \frac{r^3}{R^2}\right)dr$$

$$= 2\pi\sigma_0\left(\frac{R^2}{2} - \frac{R^2}{4}\right) = 2\pi\sigma_0 \cdot \frac{R^2}{4} = \frac{\pi\sigma_0 R^2}{2}$$

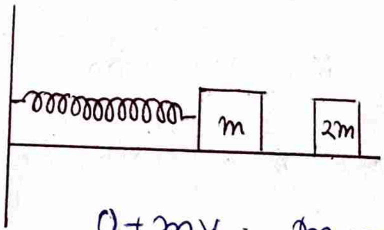
$$I = \int r^2 dm = \int \sigma_0\left(1-\frac{r^2}{R^2}\right) r^2 \cdot r\,dr\,d\theta = 2\sigma_0\pi\int\left(r^3 - \frac{r^5}{R^2}\right)dr$$

$$= 2\sigma_0\pi\left[\frac{r^4}{4} - \frac{r^6}{6R^2}\right]_0^R$$

$$= 2\sigma_0\pi R^4 \frac{1}{12} = \frac{\sigma_0\pi R^4}{6}$$

(20) A mass m is connected to a massless spring of spring constant K which is fixed to a wall. Another mass $2m$ having kinetic energy E collides collinearly with the mass m completely inelastically. (Jam 2021)

$\Rightarrow$



$$E = \frac{1}{2}(2m)V_1^2$$

$$V_1^2 = \frac{E}{m}$$

$$0 + m_1V_1 = (m_1+m_2)V_2$$

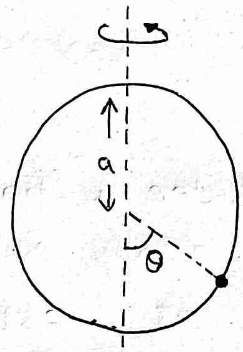
$$V_2 = \frac{2}{3}V_1$$

$$\frac{1}{2}Kx^2 = \frac{1}{2}(m_1+m_2)V_2^2$$

$$x^2 = \frac{4}{3}\frac{E}{K}$$

$$x = \sqrt{\frac{4}{3}\frac{E}{K}}$$

(4) A hoop of radius a rotates with constant angular velocity ω about the vertical axis as shown. A bead of mass m can side on the hoop without friction. If $g < \omega^2 a$. at what angle θ apart from θ and π is bead stationary ($\dot{\theta} = \ddot{\theta} = 0$) (Jest 2016)



⇒ Kinetic energy

$T = \frac{1}{2} ma^2 (\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2)$

$V = -mga \cos\theta$

Lagrangian of the system

$\mathcal{L} = \frac{1}{2} ma^2 (\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2) + mga\cos\theta$

$\frac{\partial L}{\partial \dot{\theta}} = ma^2 \dot{\theta}$ $\frac{\partial L}{\partial \theta} = -mga\sin\theta$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \Rightarrow ma^2\ddot{\theta} + mga\sin\theta = ma^2(\sin\theta\cos\theta\dot{\phi}^2)$

$\Rightarrow ma^2\ddot{\theta} = mga\sin\theta + ma^2(\sin\theta\cos\theta\dot{\phi}^2)$

when bead is stationary

$\frac{d\theta}{dt} = \frac{d^2\theta}{dt^2} = 0 \Rightarrow -ma^2(\sin\theta\cos\theta\,\dot{\phi}^2) + mga\sin\theta = 0$

$\cos\theta = g/\omega^2 a$

(5) Consider the motion of a particle in two dimensions given by the Lagrangian

$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - \frac{\lambda}{4}(x+y)^2$

where $\lambda > 0$ $y(0) = 0$, $x(0) = 42$ and

$\dot{x}(0) = \dot{y}(0) = 0$ Find $x(t) - y(t)$ at $t = 25$

⇒ $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x}$ $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0$

$m\ddot{x} + \frac{\lambda}{2}x + \frac{\lambda}{2}y = 0$ $m\ddot{y} + \frac{\lambda}{2}y + \frac{\lambda}{2}x = 0$

Subsracting $\quad m(\ddot{x}-\ddot{y}) = 0 \Rightarrow \ddot{x} = \ddot{y}$

$\dot{x} - \dot{y} = c_1 \qquad \dot{x}(0) = \dot{y}(0)$. and $c_1 = 0$

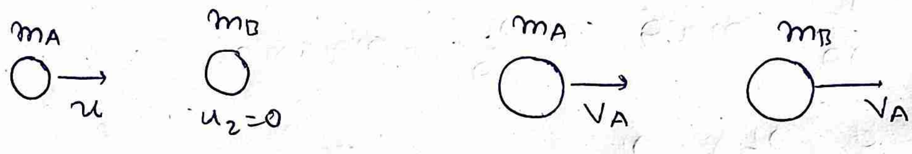
$\dot{x} - \dot{y} = 0 \Rightarrow x - y = c_2$

$x(0) = 42. \quad y(0) = 0 \Rightarrow c_2 = 42$

$x - y = 42$

⑥ Consider a point particle A of mass m_A Colliding elastically with another point particle B of mass m_B at rest. where $m_B = \gamma m_A$. After collision the ratio of the kinetic energy of Particle B of the initial kinetic energy of Particle A

$\Rightarrow$



$\vec{P_1} = \vec{P_2}$

$m_A u + 0 = m_B V_B + m_A u_A$

$u = V_A + \gamma V_B$

$KE_1 = KE_2$

$\frac{1}{2} m_A u^2 + 0 = \frac{1}{2} m_A V_A^2 + \frac{1}{2} m_B V_B^2$

$V_B = \frac{2\gamma u}{\gamma + \gamma^2} \Rightarrow \frac{V_B}{u} = \frac{2}{\gamma + 1}$

$$\frac{KE_B}{KE_A} = \frac{\frac{1}{2} m_B V_B^2}{\frac{1}{2} m_A u^2}$$

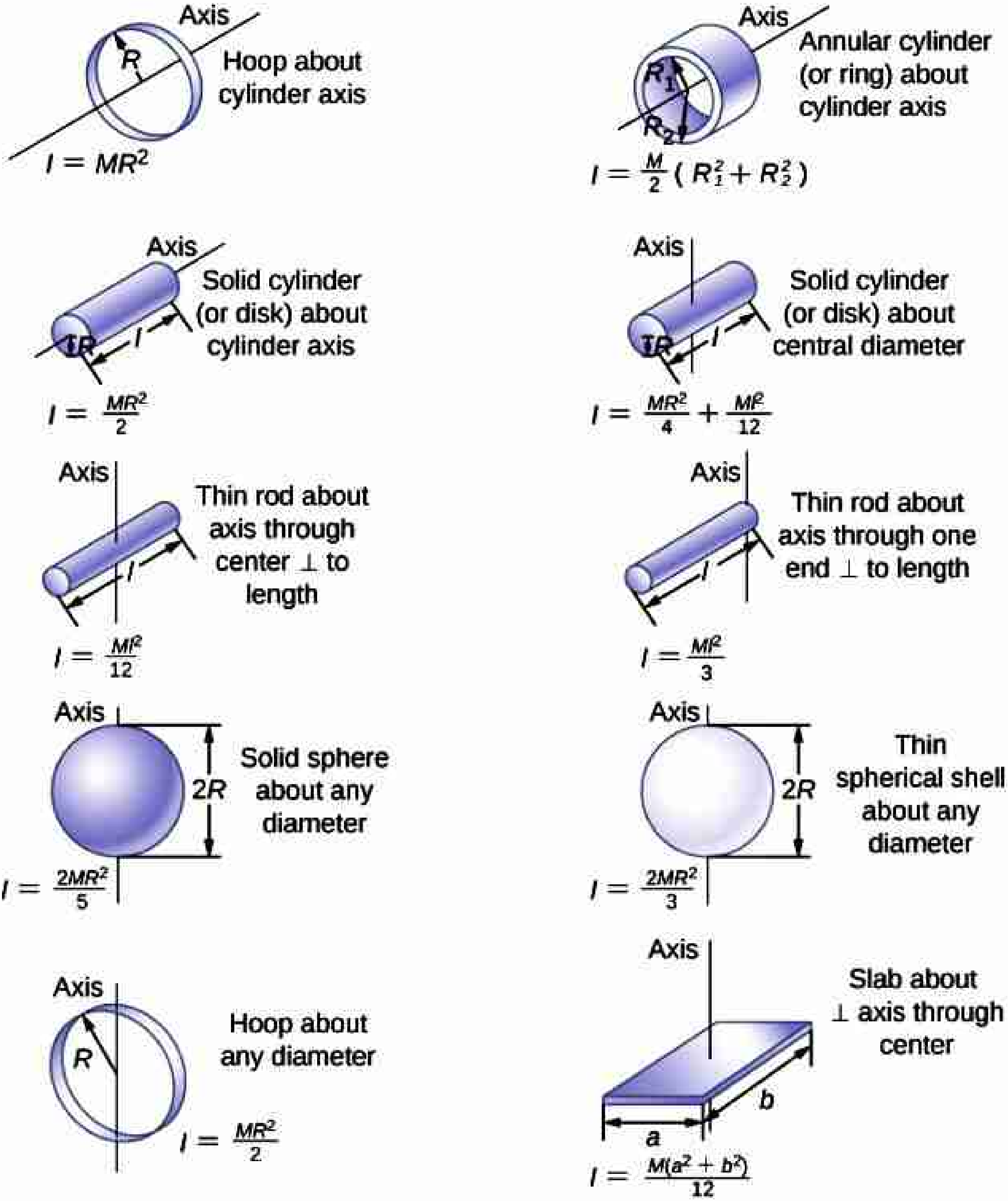
$$= \gamma \times \left(\frac{2}{\gamma+1}\right)^2 = \frac{4\gamma}{\gamma^2 + 2\gamma + 1} = \frac{4}{\gamma + 2 + 1/\gamma}$$

⑦ A bullet with initial speed V_0 is fired at a long of wood. The resistive force by wood on the bullet is given by ηv^{α}. what is the time taken to stop the bullet inside the wood log.

$\Rightarrow$

$$m \frac{dv}{dt} = -\eta v^{\alpha} \qquad t = \frac{m}{\eta} \frac{V_0^{1-\alpha}}{1-\alpha}$$

$$\int_0^t dt = -\frac{m}{\eta} \int_{v_0}^{0} \frac{dv}{v^{\alpha}}$$


Axis
R
Hoop about cylinder axis
$I = MR^2$
Axis
R_1
R_2
Annular cylinder (or ring) about cylinder axis
$I = \frac{M}{2}(R_1^2 + R_2^2)$
Axis
R
l
Solid cylinder (or disk) about cylinder axis
$I = \frac{MR^2}{2}$
Axis
R
l
Solid cylinder (or disk) about central diameter
$I = \frac{MR^2}{4} + \frac{Ml^2}{12}$
Axis
l
Thin rod about axis through center ⊥ to length
$I = \frac{Ml^2}{12}$
Axis
l
Thin rod about axis through one end ⊥ to length
$I = \frac{Ml^2}{3}$
Axis
2R
Solid sphere about any diameter
$I = \frac{2MR^2}{5}$
Axis
2R
Thin spherical shell about any diameter
$I = \frac{2MR^2}{3}$
Axis
R
Hoop about any diameter
$I = \frac{MR^2}{2}$
Axis
a
b
Slab about ⊥ axis through center
$I = \frac{M(a^2 + b^2)}{12}$