CartPole: dynamics and control

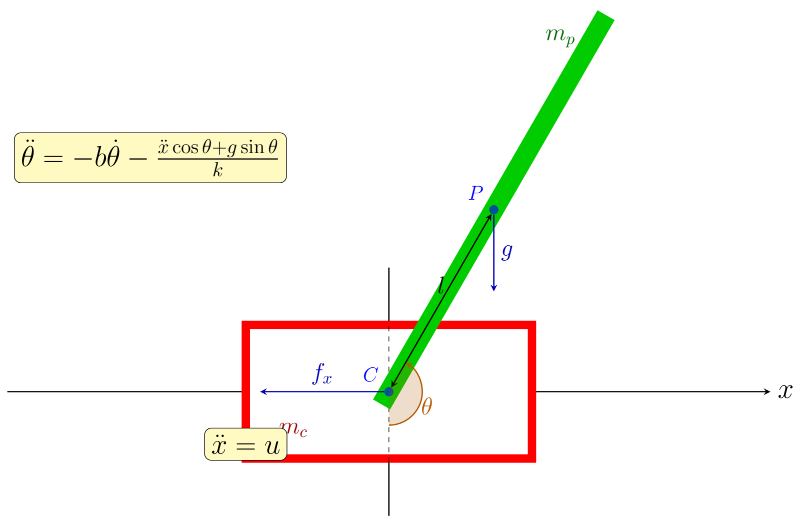
Robotics Group

1 Dynamics

CartPole is a classic control problem. It involves cart moving along a horizontal line with a freely hanging pendulum (metal pole) attached to it. Device is controlled with cart acceleration with stepper motor, the goal is to bring up the pendulum to unstable equilibrium position and maintain it there.

1.1 Linear CartPole

A cart with mass m_c moves along the x-axis, so its center C has coordinates $(x, 0)^T$. A pole with mass m_p is attached to the cart with a hinge at point C, and rotates around with viscous friction. The pole's center of mass is at point P, moment of inertia is I_p. The angle of rotation is denoted as θ, measured counterclockwise from the axis $-y$. A force f_x is applied to the cart, and the force of gravity g acts on the pole.



$$C = \begin{pmatrix} x \\ 0 \end{pmatrix} \qquad \dot{C} = \begin{pmatrix} \dot{x} \\ 0 \end{pmatrix} \tag{1}$$

$$P = \begin{pmatrix} x + l\sin\theta \\ -l\cos\theta \end{pmatrix} \qquad \dot{P} = \begin{pmatrix} \dot{x} + l\dot{\theta}\cos\theta \\ l\dot{\theta}\sin\theta \end{pmatrix} \tag{2}$$

The kinetic energy of the cart is

$$T_c = \frac{1}{2}m_c \left\|\dot{C}\right\|_2^2 = \frac{1}{2}m_c\dot{x}^2. \tag{3}$$

Kinetic energy of the pendulum is the sum of translational and rotational energies

$$\begin{aligned}
T_p &= T_p^t + T_p^r \\
&= \frac{1}{2} m_p \left\| \dot{P} \right\|_2^2 + \frac{1}{2} I_p \dot{\theta}^2 \\
&= \frac{1}{2} m_p (\dot{x} + l\dot{\theta}\cos\theta)^2 + \frac{1}{2} m_p (l\dot{\theta}\sin\theta)^2 + \frac{1}{2} I_p \dot{\theta}^2 \\
&= \frac{1}{2} m_p \dot{x}^2 + m_p \dot{x} l \dot{\theta} \cos\theta + \frac{1}{2}\dot{\theta}^2 \left(m_p l^2 + I_p\right). \qquad (4)
\end{aligned}$$

As a result energy of the whole system is following

$$\begin{aligned}
T &= T_c + T_p \\
&= \frac{1}{2} m_c \dot{x}^2 + \frac{1}{2} m_p \dot{x}^2 + m_p \dot{x} l \dot{\theta} \cos\theta + \frac{1}{2}\dot{\theta}^2 \left(m_p l^2 + I_p\right) \\
&= \frac{1}{2} \dot{x}^2 (m_c + m_p) + m_p \dot{x} l \dot{\theta} \cos\theta + \frac{1}{2}\dot{\theta}^2 \left(m_p l^2 + I_p\right); \qquad (5)
\end{aligned}$$

$$U = \underbrace{U_c}_{0} + U_p = -m_p g l \cos\theta. \qquad (6)$$

To find the dynamics of the system, let's use the Euler-Lagrange differential equation, where $L = T - U$, Q is the generalized force, and $q = (x, \theta)^T$. In our case, we have to deal with two forces: f_x and friction $f_\theta(\theta, \dot{\theta}) = -\mu\dot{\theta}$, as we assume it is viscous.

$$\begin{aligned}
Q &= \frac{d}{dt}\frac{dL}{d\dot{q}} - \frac{dL}{dq} \\
Q &= \begin{pmatrix} f_x \\ f_\theta \end{pmatrix}.
\end{aligned}$$

As a result we get the following equations of motion

$$m_p \ddot{x} l \cos\theta + \ddot{\theta}\left(m_p l^2 + I_p\right) + m_p g l \sin\theta = f_\theta(\theta, \dot{\theta}) \qquad (7)$$

$$\ddot{x}(m_c + m_p) + m_p l \ddot{\theta} \cos\theta - m_p l \dot{\theta}^2 \sin\theta = f_x. \qquad (8)$$

A more detailed calculation is below

$$L = \frac{1}{2}\dot{x}^2(m_c + m_p) + m_p \dot{x} l \dot{\theta}\cos\theta + \frac{1}{2}\dot{\theta}^2\left(m_p l^2 + I_p\right) + m_p g l \cos\theta. \qquad (9)$$

$$
\begin{aligned}
\frac{dL}{d\theta} &= -m_p\dot{x}l\dot{\theta}\sin\theta - m_pgl\sin\theta && (10)\\
\frac{dL}{d\dot{\theta}} &= m_p\dot{x}l\cos\theta + \left(m_pl^2 + I_p\right)\dot{\theta} && (11)\\
\frac{d}{dt}\frac{dL}{d\dot{\theta}} &= m_p\ddot{x}l\cos\theta - m_p\dot{x}l\dot{\theta}\sin\theta + \left(m_pl^2 + I_p\right)\ddot{\theta} && (12)\\
\frac{d}{dt}\frac{dL}{d\dot{\theta}} - \frac{dL}{d\theta} &= m_p\ddot{x}l\cos\theta - m_p\dot{x}l\dot{\theta}\sin\theta + \left(m_pl^2 + I_p\right)\ddot{\theta} + m_p\dot{x}l\dot{\theta}\sin\theta + m_pgl\sin\theta\\
&= m_p\ddot{x}l\cos\theta + \left(m_pl^2 + I_p\right)\ddot{\theta} + m_pgl\sin\theta && (13)
\end{aligned}
$$

$$
\begin{aligned}
\frac{dL}{dx} &= 0 && (14)\\
\frac{dL}{d\dot{x}} &= (m_c + m_p)\dot{x} + m_pl\dot{\theta}\cos\theta && (15)\\
\frac{d}{dt}\frac{dL}{d\dot{x}} &= (m_c + m_p)\ddot{x} + m_pl\ddot{\theta}\cos\theta - m_pl\dot{\theta}^2\sin\theta && (16)\\
\frac{d}{dt}\frac{dL}{d\dot{x}} - \frac{dL}{dx} &= (m_c + m_p)\ddot{x} + m_pl\ddot{\theta}\cos\theta - m_pl\dot{\theta}^2\sin\theta. && (17)\\
& && (18)
\end{aligned}
$$

1.2 Acceleration control

Let's make the assumption that the motor can generate any force necessary for the cart to reach acceleration in $[-a, a]$ on a fixed cart velocity range. This fact allows us to consider the cart acceleration as a control input and significantly simplify the equations of motion

$$
\begin{aligned}
\left(m_pl^2 + I_p\right)\ddot{\theta} &= f_\theta(\theta, \dot{\theta}) - m_p\ddot{x}l\cos\theta - m_pgl\sin\theta && (19)\\
\ddot{x} &= u, \quad u \in [-a, a]. && (20)
\end{aligned}
$$

But for practice it's more convenient to use another form

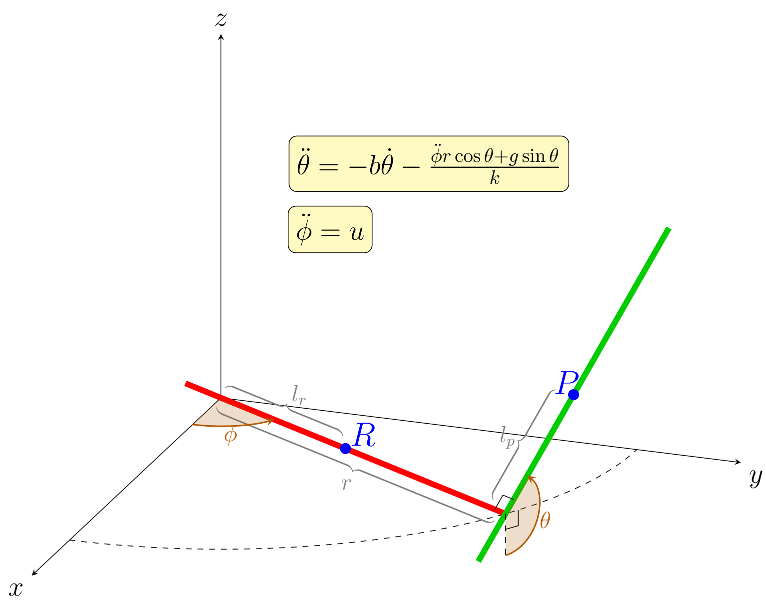
$$
\begin{aligned}
\left(m_pl^2 + I_p\right)\ddot{\theta} &= -\mu\dot{\theta} - m_p\ddot{x}l\cos\theta - m_pgl\sin\theta\\
\ddot{\theta} &= -\frac{\mu}{(m_pl^2 + I_p)}\dot{\theta} - \frac{\ddot{x}\cos\theta - g\sin\theta}{l + \frac{I_p}{m_pl}}.
\end{aligned}
$$

Since all parameters do not change over time, we can greatly simplify the motion equations

$$
\begin{aligned}
\ddot{\theta} &= -b\dot{\theta} - \frac{\ddot{x}\cos\theta - g\sin\theta}{k} \\
\ddot{x} &= u.
\end{aligned}
\tag{21}
$$

1.3 Radial CartPole

Linear CartPole is classic kinematic scheme, but it has some disadvantages, e.g. it requires a lot of space and you need periodically make homing (return cart to the initial position at the end of the episode). Radial CartPole is a modification, which allows to avoid these problems, keeping the same motion equations. In this case cart moves along the circle with radius r and the pole is attached to that cart.



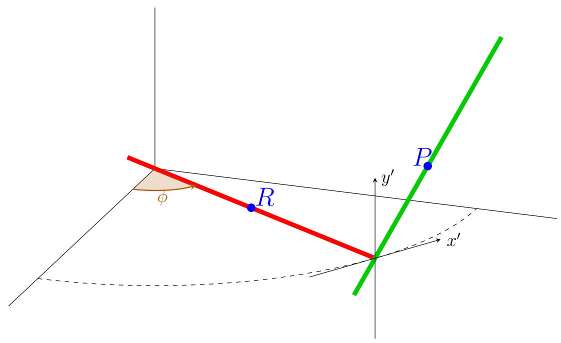
Pole's mass m_p, moment of inertian I_p and distance from hinge to center of mass l_p are known again. But instead of position x there is angle ϕ (in some sense $x = r\phi$) and control input is radial acceleration $\ddot{\phi}$. Also, as shown in previous section, we can consider mass of cart is zero.

Actually after some calculations we can get the same motion equations as for classic version, but linear variables are replaced by angular ones

$$\left(m_p l^2 + I_p\right)\ddot{\theta} = f_\theta(\theta, \dot{\theta}) - m_p r\ddot{\phi} l\cos\theta - m_p g l\sin\theta \tag{22}$$

$$\ddot{\phi} = u, \quad u \in [-u_{min}, u_{max}]. \tag{23}$$

Note that $T_p^r = \frac{1}{2} I_p \dot{\theta}^2$ and $T_p^t = \frac{1}{2} m_p \left\| \dot{P} \right\|_2^2$. To calculate $\left\| \dot{P} \right\|_2^2$, consider the coordinates system plane, which is attached to the cart and tangent to the circle (pole fully lies in that plane).



$$\left\| \dot{P} \right\|_2^2 = \left(\underbrace{\dot{\phi} r + \dot{\theta} l_p \cos\theta}_{\dot{x}'} \right)^2 + \left(\underbrace{\dot{\theta} l_p \sin\theta}_{\dot{y}'} \right)^2 = \dot{\phi}^2 r^2 + 2\dot{\phi}\dot{\theta} r l_p \cos\theta + \dot{\theta}^2 l_p^2 \tag{24}$$

It remains to do the trivial calculations

$$T_p = \frac{1}{2} I_p \dot{\theta}^2 + \frac{1}{2} m_p \left(\dot{\phi}^2 r^2 + 2\dot{\phi}\dot{\theta} r l_p \cos\theta + \dot{\theta}^2 l_p^2 \right) \tag{25}$$

$$U_p = -m_p g l_p \cos\theta \tag{26}$$

$$L = T_p - U_p = \frac{1}{2} I_p \dot{\theta}^2 + \frac{1}{2} m_p \left(\dot{\phi}^2 r^2 + 2\dot{\phi}\dot{\theta} r l_p \cos\theta + \dot{\theta}^2 l_p^2 \right) + m_p g l_p \cos\theta \tag{27}$$

$$\begin{aligned}
\frac{dL}{d\theta} &= -m_p l_p (r\dot{\phi}\dot{\theta} \sin\theta + g \sin\theta) \\
\frac{dL}{d\dot{\theta}} &= I_p \dot{\theta} + m_p l_p (\dot{\phi} r \cos\theta + l_p \dot{\theta}) \\
\frac{d}{dt}\frac{dL}{d\dot{\theta}} &= I_p \ddot{\theta} + m_p l_p (r\ddot{\phi} \cos\theta + l_p \ddot{\theta} - r\dot{\phi}\dot{\theta} \sin\theta) \\
\frac{d}{dt}\frac{dL}{d\dot{\theta}} - \frac{dL}{d\theta} &= \ddot{\theta}(I_p + m_p l_p^2) + m_p l_p (r\ddot{\phi} \cos\theta + g \sin\theta) = 0 \qquad (28)
\end{aligned}$$