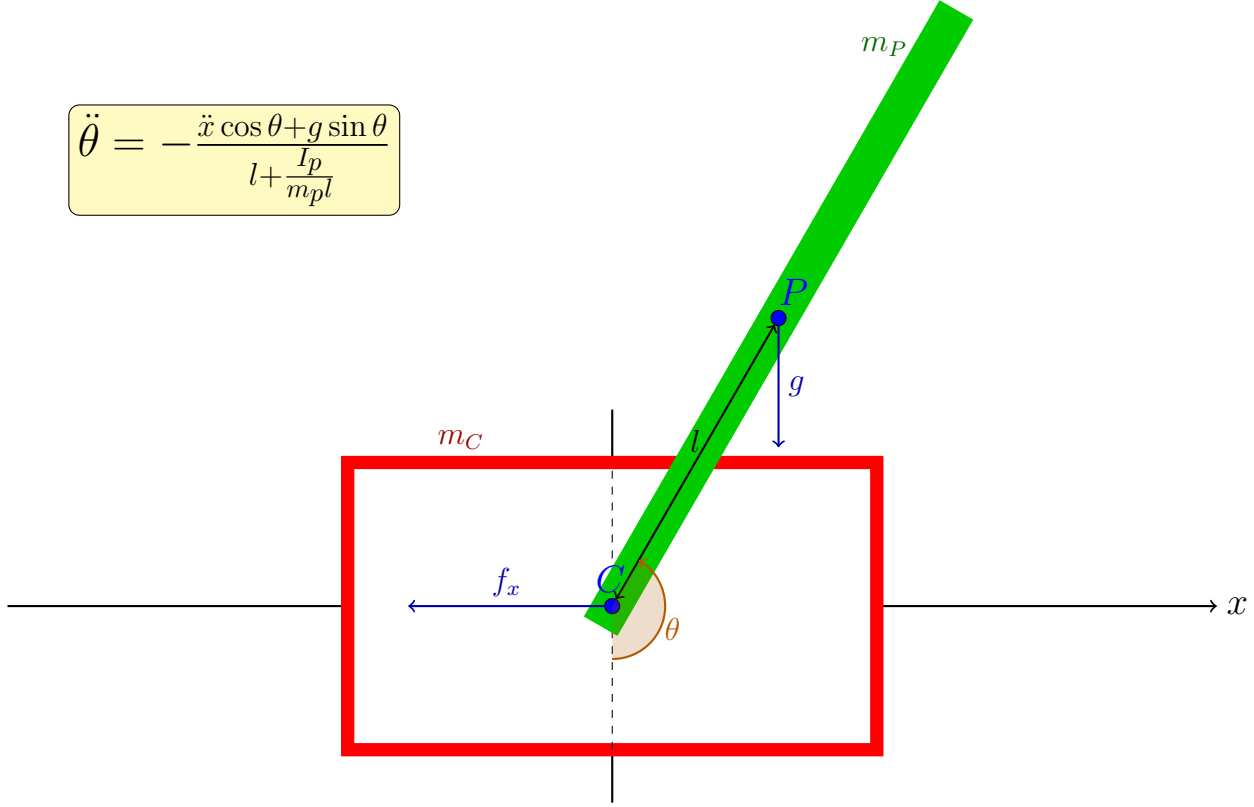


1 Classic Cart Pole

1.1 Dynamic



Coordinates of points and their derivatives

$$\begin{aligned} C &= \begin{pmatrix} x \\ 0 \end{pmatrix}; & \dot{C} &= \begin{pmatrix} \dot{x} \\ 0 \end{pmatrix}; \\ P &= \begin{pmatrix} x + l \sin \theta \\ -l \cos \theta \end{pmatrix}; & \dot{P} &= \begin{pmatrix} \dot{x} + l \dot{\theta} \cos \theta \\ l \dot{\theta} \sin \theta \end{pmatrix}; \end{aligned}$$

Then the kinetic energy of the carriage is

$$T_C = \frac{1}{2} m_c \|\dot{C}\|_2^2 = \frac{1}{2} m_c \dot{x}^2$$

Kinetic energy of the pendulum

$$\begin{aligned} T_p &= T_p^t + T_p^r = \\ &= \frac{1}{2} m_p \|\dot{P}\|_2^2 + \frac{1}{2} I_p \dot{\theta}^2 = \\ &= \frac{1}{2} m_p (\dot{x} + l \dot{\theta} \cos \theta)^2 + \frac{1}{2} m_p (l \dot{\theta} \sin \theta)^2 + \frac{1}{2} I_p \dot{\theta}^2 = \\ &= \frac{1}{2} m_p \dot{x}^2 + m_p \dot{x} l \dot{\theta} \cos \theta + \frac{1}{2} \dot{\theta}^2 (m_p l^2 + I_p) \end{aligned}$$

Then the kinetic and potential energy of the whole system

$$\begin{aligned} T &= T_c + T_p = \\ &= \frac{1}{2} m_c \dot{x}^2 + \frac{1}{2} m_p \dot{x}^2 + m_p \dot{x} l \dot{\theta} \cos \theta + \frac{1}{2} \dot{\theta}^2 (m_p l^2 + I_p) \\ &= \frac{1}{2} \dot{x}^2 (m_c + m_p) + m_p \dot{x} l \dot{\theta} \cos \theta + \frac{1}{2} \dot{\theta}^2 (m_p l^2 + I_p) \end{aligned}$$

$$U = \underbrace{U_c}_0 + U_p = -m_p g l \cos \theta$$

To calculate the dynamics of the system we will use the Euler-Lagrange differential equation

$$\frac{d}{dt} \frac{dL}{dq} - \frac{dL}{dq} = Q$$

where $L = T - U$, Q is the generalized force and $q = \begin{pmatrix} x \\ \theta \end{pmatrix}$. From this we derive the equation of motion

$$\begin{cases} m_p \ddot{x} l \cos \theta + \ddot{\theta} (m_p l^2 + I_p) + m_p g l \sin \theta = 0 \\ \ddot{x} (m_c + m_p) + m_p l \ddot{\theta} \cos \theta - m_p l \dot{\theta}^2 \sin \theta = f_x \end{cases}$$

A more detailed calculation is below

$$L = \frac{1}{2} \dot{x}^2 (m_c + m_p) + m_p \dot{x} l \dot{\theta} \cos \theta + \frac{1}{2} \dot{\theta}^2 (m_p l^2 + I_p) + m_p g l \cos \theta$$

$$\frac{dL}{d\theta} = -m_p \dot{x} l \dot{\theta} \sin \theta - m_p g l \sin \theta$$

$$\frac{dL}{d\dot{\theta}} = m_p \dot{x} l \cos \theta + \dot{\theta} (m_p l^2 + I_p)$$

$$\frac{d}{dt} \frac{dL}{d\dot{\theta}} = m_p \ddot{x} l \cos \theta - m_p \dot{x} l \dot{\theta} \sin \theta + \ddot{\theta} (m_p l^2 + I_p)$$

$$\begin{aligned} \frac{d}{dt} \frac{dL}{d\dot{\theta}} - \frac{dL}{d\theta} &= m_p \ddot{x} l \cos \theta - m_p \dot{x} l \dot{\theta} \sin \theta + \ddot{\theta} (m_p l^2 + I_p) + m_p \dot{x} l \dot{\theta} \sin \theta + m_p g l \sin \theta = \\ &= m_p \ddot{x} l \cos \theta + \ddot{\theta} (m_p l^2 + I_p) + m_p g l \sin \theta \end{aligned}$$

$$\frac{dL}{dx} = 0$$

$$\frac{dL}{d\dot{x}} = \dot{x} (m_c + m_p) + m_p l \dot{\theta} \cos \theta$$

$$\frac{d}{dt} \frac{dL}{d\dot{x}} = \ddot{x} (m_c + m_p) + m_p l \ddot{\theta} \cos \theta - m_p l \dot{\theta}^2 \sin \theta$$

$$\frac{d}{dt} \frac{dL}{d\dot{x}} - \frac{dL}{dx} = \ddot{x} (m_c + m_p) + m_p l \ddot{\theta} \cos \theta - m_p l \dot{\theta}^2 \sin \theta$$

1.2 Controlling acceleration

Convert the second equation in the resulting system

$$\begin{cases} m_p \ddot{x} l \cos \theta + \ddot{\theta} (m_p l^2 + I_p) + m_p g l \sin \theta = 0 \\ \ddot{x} = \frac{f_x - m_p l \ddot{\theta} \cos \theta + m_p l \dot{\theta}^2 \sin \theta}{m_c + m_p} \end{cases}$$

Consider that we can obtain any such f_x at any time that \ddot{x} can take any value from $[-a, a]$. Then the second equality does not make sense further, instead we can consider the first one

$$\begin{aligned} m_p \ddot{x} l \cos \theta + \ddot{\theta} (m_p l^2 + I_p) + m_p g l \sin \theta &= 0 \\ \ddot{\theta} &= -\frac{m_p \ddot{x} l \cos \theta + m_p g l \sin \theta}{m_p l^2 + I_p} = -\frac{\ddot{x} \cos \theta + g \sin \theta}{l + \frac{I_p}{m_p l}} \end{aligned}$$

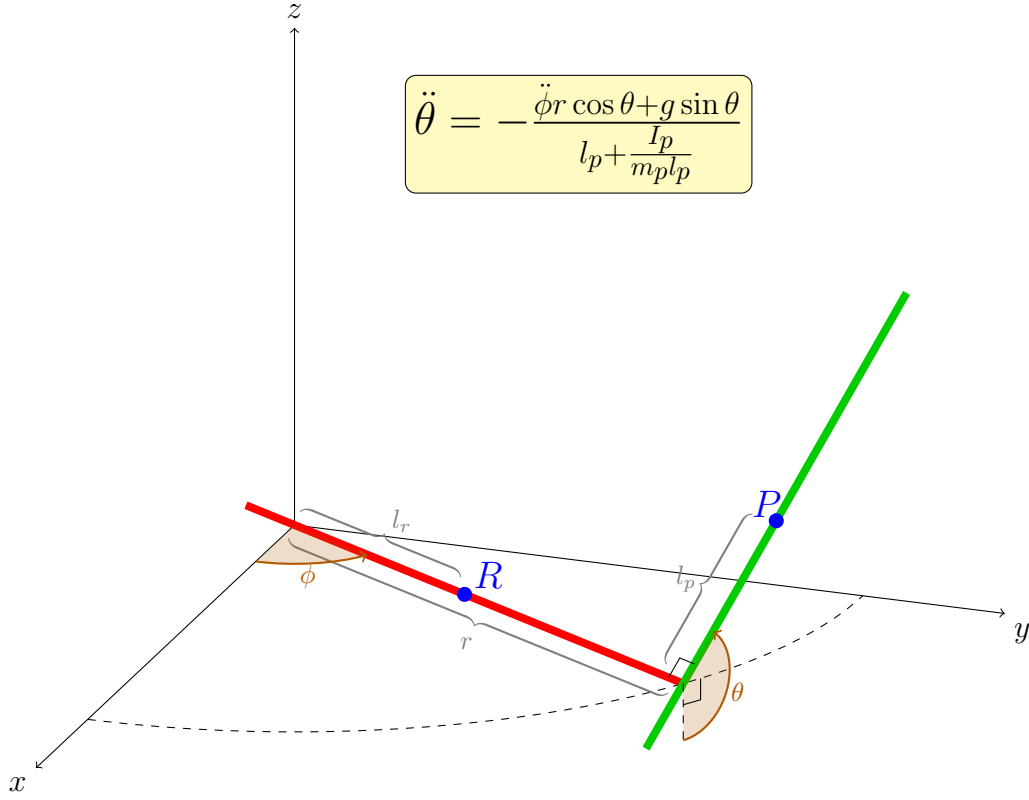
It turns out that the equation of motion is described by this equation, where \ddot{x} is given by any in the segment $[-a, a]$.

1.3 Friction

The friction affecting \ddot{x} is not considered, since it is given by any number from the segment $[-a, a]$. One can introduce friction on $\ddot{\theta}$ of the general form $f(\theta, \dot{\theta})$ (for example, linear from angular velocity $f(\theta, \dot{\theta}) = \mu \dot{\theta}$, where μ is a constant), then the equation of motion rewrites as

$$\ddot{\theta} = -\frac{\ddot{x} \cos \theta + g \sin \theta}{l + \frac{I_p}{m_p l}} + f(\theta, \dot{\theta})$$

2 Radial Cart Pole



In this problem, too, we consider that we control $\ddot{\phi}$ instead of f_ϕ at once, which means that the Euler-Lagrange can be derived only by the coordinate θ

$$\frac{d}{dt} \frac{dL}{d\dot{\theta}} - \frac{dL}{d\theta} = 0$$

After calculating all the components we obtain the equation

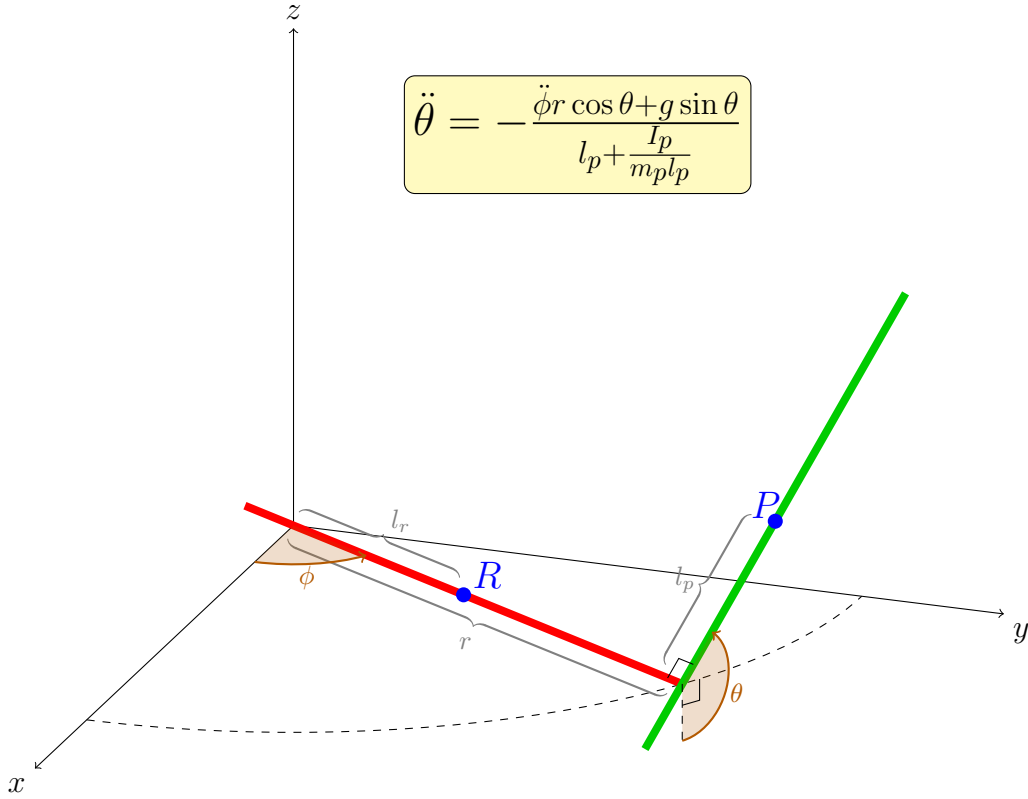
$$\ddot{\theta} = -\frac{\ddot{\phi} r \cos \theta + g \sin \theta}{l_p + \frac{I_p}{m_p l_p}}$$

A more detailed calculation

$$T_r = \frac{1}{2} \dot{\phi}^2 (I_r + m_r l_r^2)$$

$$T_p = T_p^t + T_p^c$$

Note that $T_p^c = \frac{1}{2} I_p \dot{\theta}^2$ and $T_p^t = \frac{1}{2} m_p \|\dot{P}\|_2^2$. To calculate $\|\dot{P}\|_2^2$, consider the system with respect to the coordinate system in the figure below



velocity $\|\dot{P}\|_2^2$ is decomposed by the coordinates y and z , we get that

$$\|\dot{P}\|_2^2 = \left(\underbrace{\dot{\phi} r + \dot{\theta} l_p \cos \theta}_{\dot{y}} \right)^2 + \left(\underbrace{\dot{\theta} l_p \sin \theta}_{\dot{z}} \right)^2 = \dot{\phi}^2 r^2 + 2\dot{\phi}\dot{\theta} r l_p \cos \theta + \dot{\theta}^2 l_p^2$$

It remains to do the trivial calculations

$$T_p = \frac{1}{2} I_p \dot{\theta}^2 + \frac{1}{2} m_p \left(\dot{\phi}^2 r^2 + 2\dot{\phi}\dot{\theta} r l_p \cos \theta + \dot{\theta}^2 l_p^2 \right)$$

$$T = T_r + T_p = \frac{1}{2} \dot{\phi}^2 (I_r + m_r l_r^2) + \frac{1}{2} I_p \dot{\theta}^2 + \frac{1}{2} m_p \left(\dot{\phi}^2 r^2 + 2\dot{\phi}\dot{\theta} r l_p \cos \theta + \dot{\theta}^2 l_p^2 \right)$$

$$U = \left(\underbrace{U_r + U_p}_0 \right) = -m_p g l_p \cos \theta$$

$$L = T - U = \frac{1}{2} \dot{\phi}^2 (I_r + m_r l_r^2) + \frac{1}{2} I_p \dot{\theta}^2 + \frac{1}{2} m_p \left(\dot{\phi}^2 r^2 + 2\dot{\phi}\dot{\theta} r l_p \cos \theta + \dot{\theta}^2 l_p^2 \right) + m_p g l_p \cos \theta$$

$$\frac{dL}{d\theta} = -m_p l_p (r \dot{\phi} \dot{\theta} \sin \theta + g \sin \theta)$$

$$\frac{dL}{d\dot{\theta}} = I_p \dot{\theta} + m_p l_p (\dot{\phi} r \cos \theta + l_p \dot{\theta})$$

$$\frac{d}{dt} \frac{dL}{d\dot{\theta}} = I_p \ddot{\theta} + m_p l_p (r \ddot{\phi} \cos \theta + l_p \ddot{\theta} - r \dot{\phi} \dot{\theta} \sin \theta)$$

$$\frac{d}{dt} \frac{dL}{d\dot{\theta}} - \frac{dL}{d\theta} = I_p \ddot{\theta} + m_p l_p (r \ddot{\phi} \cos \theta + l_p \ddot{\theta} - r \dot{\phi} \dot{\theta} \sin \theta) + m_p l_p (r \dot{\phi} \dot{\theta} \sin \theta + g \sin \theta) =$$

$$= I_p \ddot{\theta} + m_p l_p (r \ddot{\phi} \cos \theta + l_p \ddot{\theta}) + m_p l_p g \sin \theta =$$

$$= \ddot{\theta} (I_p + m_p l_p^2) + m_p l_p (r \ddot{\phi} \cos \theta + g \sin \theta) = 0$$