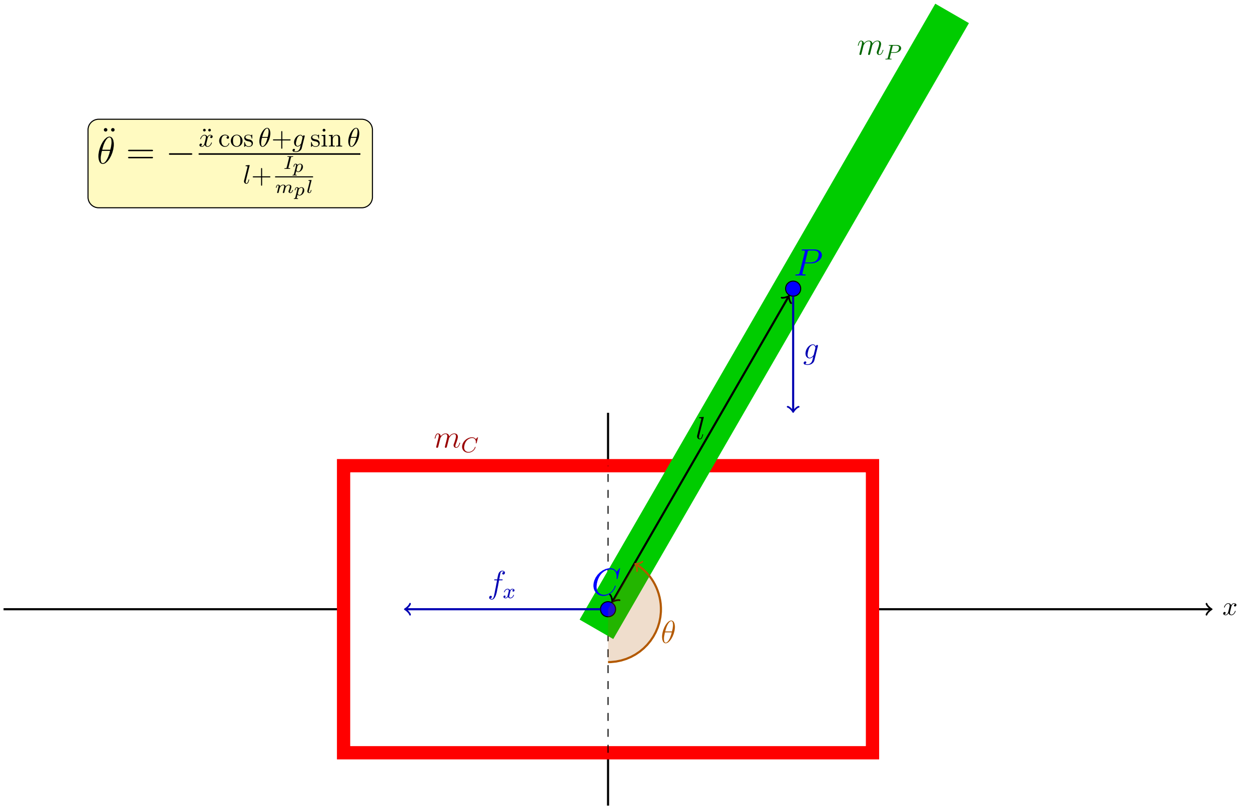
1 Classic Cart Pole

1.1 Dynamic



Coordinates of points and their derivatives

$$C = \begin{pmatrix} x \\ 0 \end{pmatrix}; \qquad \dot{C} = \begin{pmatrix} \dot{x} \\ 0 \end{pmatrix};$$

$$P = \begin{pmatrix} x + l\sin\theta \\ -l\cos\theta \end{pmatrix}; \qquad \dot{P} = \begin{pmatrix} \dot{x} + l\dot{\theta}\cos\theta \\ l\dot{\theta}\sin\theta \end{pmatrix};$$

Then the kinetic energy of the carriage is

$$T_C = \frac{1}{2}m_c \left\|\dot{C}\right\|_2^2 = \frac{1}{2}m_c\dot{x}^2$$

Kinetic energy of the pendulum

$$\begin{aligned} T_p &= T_p^t + T_p^r = \\ &= \frac{1}{2}m_p \left\|\dot{P}\right\|_2^2 + \frac{1}{2}I_p\dot{\theta}^2 = \\ &= \frac{1}{2}m_p(\dot{x} + l\dot{\theta}\cos\theta)^2 + \frac{1}{2}m_p(l\dot{\theta}\sin\theta)^2 + \frac{1}{2}I_p\dot{\theta}^2 = \\ &= \frac{1}{2}m_p\dot{x}^2 + m_p\dot{x}l\dot{\theta}\cos\theta + \frac{1}{2}\dot{\theta}^2\left(m_pl^2 + I_p\right) \end{aligned}$$

Then the kinetic and potential energy of the whole system

$$\begin{aligned} T &= T_c + T_p = \\ &= \frac{1}{2}m_c\dot{x}^2 + \frac{1}{2}m_p\dot{x}^2 + m_p\dot{x}l\dot{\theta}\cos\theta + \frac{1}{2}\dot{\theta}^2\left(m_pl^2 + I_p\right) \\ &= \frac{1}{2}\dot{x}^2(m_c + m_p) + m_p\dot{x}l\dot{\theta}\cos\theta + \frac{1}{2}\dot{\theta}^2\left(m_pl^2 + I_p\right) \end{aligned}$$

$$U = \underbrace{U_c}_{0} + U_p = -m_pgl\cos\theta$$

To calculate the dynamics of the system we will use the Euler-Lagrange differential equation

$$\frac{d}{dt}\frac{dL}{d\dot{q}} - \frac{dL}{dq} = Q$$

where $L = T - U$, Q is the generalized force and $q = \begin{pmatrix} x \\ \theta \end{pmatrix}$. From this we derive the equation of motion

$$\begin{cases} m_p \ddot{x} l \cos\theta + \ddot{\theta}\left(m_p l^2 + I_p\right) + m_p g l \sin\theta = 0 \\ \ddot{x}(m_c + m_p) + m_p l \ddot{\theta}\cos\theta - m_p l \dot{\theta}^2 \sin\theta = f_x \end{cases}$$

A more detailed calculation is below

$$L = \frac{1}{2}\dot{x}^2(m_c + m_p) + m_p \dot{x} l \dot{\theta}\cos\theta + \frac{1}{2}\dot{\theta}^2\left(m_p l^2 + I_p\right) + m_p g l \cos\theta$$

$$\begin{aligned}
\frac{dL}{d\theta} &= -m_p \dot{x} l \dot{\theta} \sin\theta - m_p g l \sin\theta \\
\frac{dL}{d\dot{\theta}} &= m_p \dot{x} l \cos\theta + \dot{\theta}\left(m_p l^2 + I_p\right) \\
\frac{d}{dt}\frac{dL}{d\dot{\theta}} &= m_p \ddot{x} l \cos\theta - m_p \dot{x} l \dot{\theta} \sin\theta + \ddot{\theta}\left(m_p l^2 + I_p\right) \\
\frac{d}{dt}\frac{dL}{d\dot{\theta}} - \frac{dL}{d\theta} &= m_p \ddot{x} l \cos\theta - m_p \dot{x} l \dot{\theta} \sin\theta + \ddot{\theta}\left(m_p l^2 + I_p\right) + m_p \dot{x} l \dot{\theta} \sin\theta + m_p g l \sin\theta = \\
&= m_p \ddot{x} l \cos\theta + \ddot{\theta}\left(m_p l^2 + I_p\right) + m_p g l \sin\theta
\end{aligned}$$

$$\begin{aligned}
\frac{dL}{dx} &= 0 \\
\frac{dL}{d\dot{x}} &= \dot{x}(m_c + m_p) + m_p l \dot{\theta}\cos\theta \\
\frac{d}{dt}\frac{dL}{d\dot{x}} &= \ddot{x}(m_c + m_p) + m_p l \ddot{\theta}\cos\theta - m_p l \dot{\theta}^2 \sin\theta \\
\frac{d}{dt}\frac{dL}{d\dot{x}} - \frac{dL}{dx} &= \ddot{x}(m_c + m_p) + m_p l \ddot{\theta}\cos\theta - m_p l \dot{\theta}^2 \sin\theta
\end{aligned}$$

1.2 Controlling acceleration

Convert the second equation in the resulting system

$$\begin{cases} m_p \ddot{x} l \cos\theta + \ddot{\theta}\left(m_p l^2 + I_p\right) + m_p g l \sin\theta = 0 \\ \ddot{x} = \frac{f_x - m_p l \ddot{\theta}\cos\theta + m_p l \dot{\theta}^2 \sin\theta}{m_c + m_p} \end{cases}$$

Consider that we can obtain any such f_x at any time that $\ddot{x}$ can take any value from $[-a, a]$. Then the second equality does not make sense further, instead we can consider the first one

$$m_p \ddot{x} l \cos\theta + \ddot{\theta}\left(m_p l^2 + I_p\right) + m_p g l \sin\theta = 0$$

$$\ddot{\theta} = -\frac{m_p \ddot{x} l \cos\theta + m_p g l \sin\theta}{m_p l^2 + I_p} = -\frac{\ddot{x}\cos\theta + g\sin\theta}{l + \frac{I_p}{m_p l}}$$

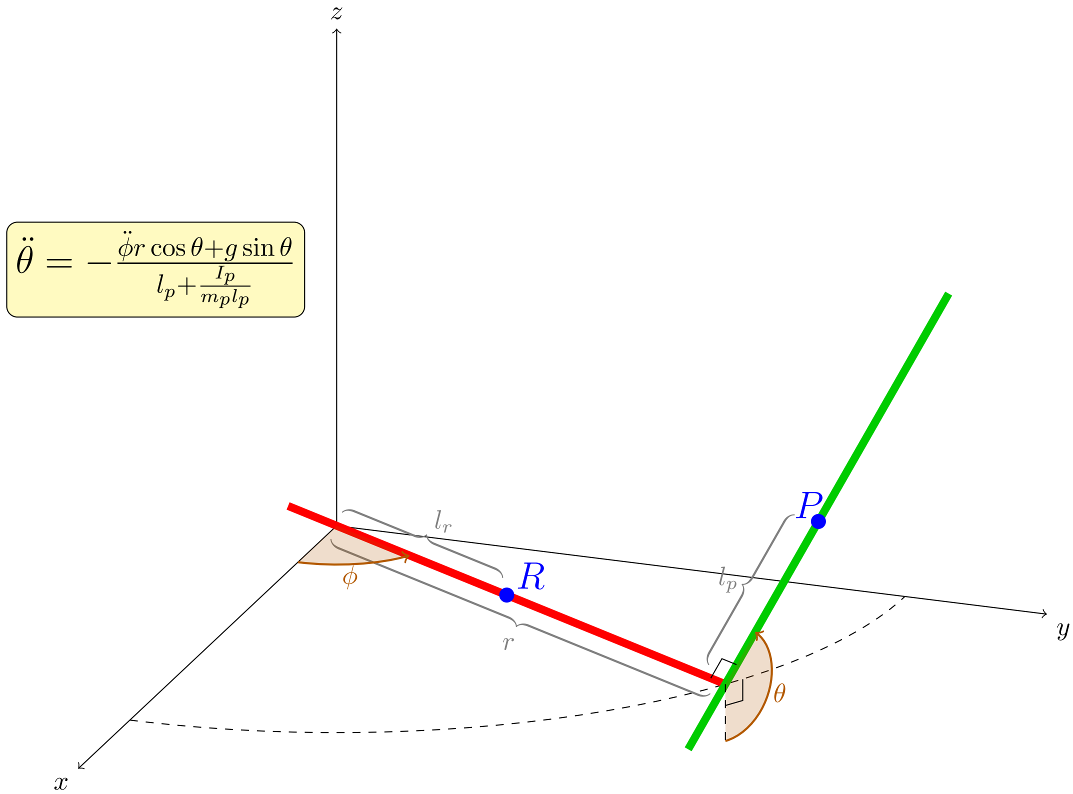
It turns out that the equation of motion is described by this equation, where $\ddot{x}$ is given by any in the segment $[-a, a]$.

1.3 Friction

The friction affecting $\ddot{x}$ is not considered, since it is given by any number from the segment $[-a, a]$. One can introduce friction on $\ddot{\theta}$ of the general form $f(\theta, \dot{\theta})$ (for example, linear from angular velocity $f(\theta, \dot{\theta}) = \mu\dot{\theta}$, where μ is a constant), then the equation of motion rewrites as

$$\ddot{\theta} = -\frac{\ddot{x}\cos\theta + g\sin\theta}{l + \frac{I_p}{m_p l}} + f(\theta, \dot{\theta})$$

2 Radial Cart Pole



In this problem, too, we consider that we control $\ddot{\phi}$ instead of f_ϕ at once, which means that the Euler-Lagrange can be derived only by the coordinate θ

$$\frac{d}{dt}\frac{dL}{d\dot{\theta}} - \frac{dL}{d\theta} = 0$$

After calculating all the components we obtain the equation

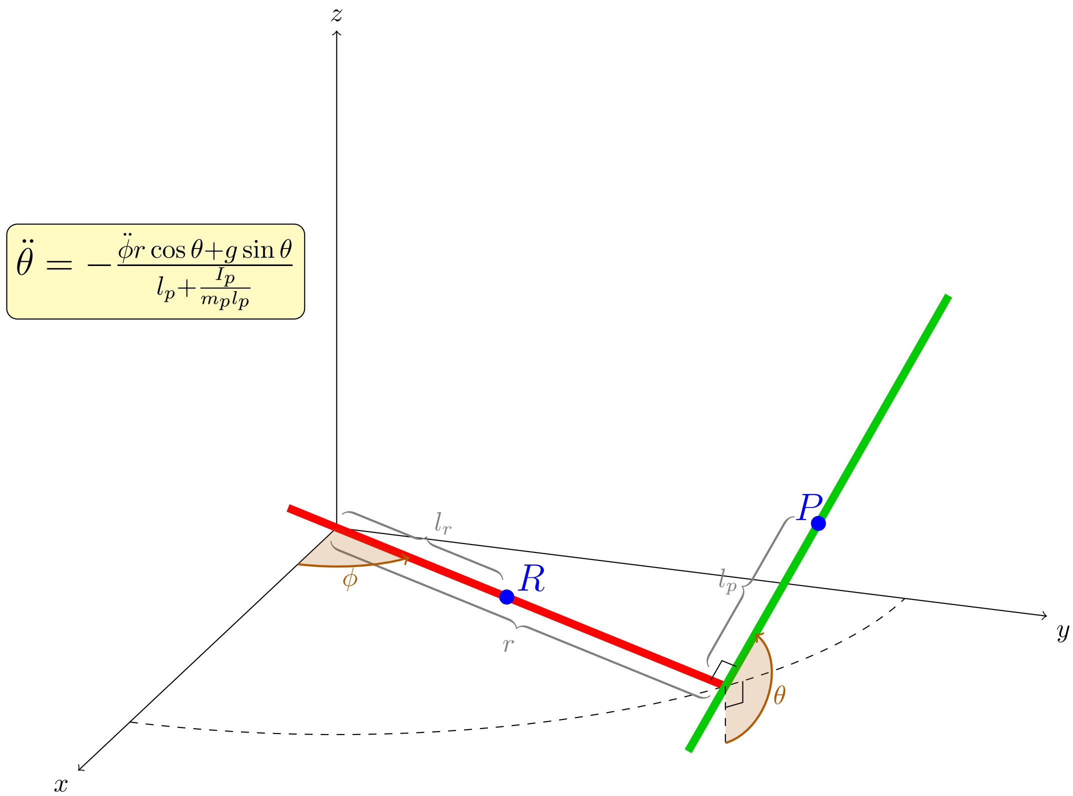
$$\ddot{\theta} = -\frac{\ddot{\phi} r \cos\theta + g \sin\theta}{l_p + \frac{I_p}{m_p l_p}}$$

A more detailed calculation

$$T_r = \frac{1}{2}\dot{\phi}^2(I_r + m_r l_r^2)$$

$$T_p = T_p^t + T_p^c$$

Note that $T_p^c = \frac{1}{2} I_p \dot{\theta}^2$ and $T_p^t = \frac{1}{2} m_p \left\|\dot{P}\right\|_2^2$. To calculate $\left\|\dot{P}\right\|_2^2$, consider the system with respect to the coordinate system in the figure below



velocity $\left\|\dot{P}\right\|_2^2$ is decomposed by the coordinates y and z, we get that

$$\left\|\dot{P}\right\|_2^2 = \left(\underbrace{\dot{\phi} r + \dot{\theta} l_p \cos\theta}_{\dot{y}}\right)^2 + \left(\underbrace{\dot{\theta} l_p \sin\theta}_{\dot{z}}\right)^2 = \dot{\phi}^2 r^2 + 2\dot{\phi}\dot{\theta} r l_p \cos\theta + \dot{\theta}^2 l_p^2$$

It remains to do the trivial calculations

$$T_p = \frac{1}{2} I_p \dot{\theta}^2 + \frac{1}{2} m_p \left(\dot{\phi}^2 r^2 + 2\dot{\phi}\dot{\theta} r l_p \cos\theta + \dot{\theta}^2 l_p^2\right)$$

$$T = T_r + T_p = \frac{1}{2}\dot{\phi}^2 (I_r + m_r l_r^2) + \frac{1}{2} I_p \dot{\theta}^2 + \frac{1}{2} m_p \left(\dot{\phi}^2 r^2 + 2\dot{\phi}\dot{\theta} r l_p \cos\theta + \dot{\theta}^2 l_p^2\right)$$

$$U = \left(\underbrace{U_r}_{0} + U_p\right) = -m_p g l_p \cos\theta$$

$$L = T - U = \frac{1}{2}\dot{\phi}^2 (I_r + m_r l_r^2) + \frac{1}{2} I_p \dot{\theta}^2 + \frac{1}{2} m_p \left(\dot{\phi}^2 r^2 + 2\dot{\phi}\dot{\theta} r l_p \cos\theta + \dot{\theta}^2 l_p^2\right) + m_p g l_p \cos\theta$$

$$\begin{aligned}
\frac{dL}{d\theta} &= -m_p l_p (r\dot{\phi}\dot{\theta}\sin\theta + g\sin\theta) \\
\frac{dL}{d\dot{\theta}} &= I_p\dot{\theta} + m_p l_p (\dot{\phi} r\cos\theta + l_p\dot{\theta}) \\
\frac{d}{dt}\frac{dL}{d\dot{\theta}} &= I_p\ddot{\theta} + m_p l_p (r\ddot{\phi}\cos\theta + l_p\ddot{\theta} - r\dot{\phi}\dot{\theta}\sin\theta) \\
\frac{d}{dt}\frac{dL}{d\dot{\theta}} - \frac{dL}{d\theta} &= I_p\ddot{\theta} + m_p l_p (r\ddot{\phi}\cos\theta + l_p\ddot{\theta} - r\dot{\phi}\dot{\theta}\sin\theta) + m_p l_p (r\dot{\phi}\dot{\theta}\sin\theta + g\sin\theta) = \\
&= I_p\ddot{\theta} + m_p l_p (r\ddot{\phi}\cos\theta + l_p\ddot{\theta}) + m_p l_p g\sin\theta = \\
&= \ddot{\theta}(I_p + m_p l_p^2) + m_p l_p (r\ddot{\phi}\cos\theta + g\sin\theta) = 0
\end{aligned}$$