NOTES ON GENERALIZED SKEW DERIVATIONS

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DEFINITION:: A ring R is said to be a **Prime ring** if and only if the zero ideal in R is a Prime ideal in R.

Similarly, we define the semiprime ring by

DEFINITION:: A ring R is said to be a **Semiprime ring** if and only if the zero ideal in R is a semiprime ideal of R.

The following theorems give equivalent conditions for a ring to be prime or semiprime. These are well known results and thus their proof is omitted.

THEOREM-1:: The following conditions are equivalent:

(i) R is a prime ring.
(ii) If A,B are ideals of R such that AB=0, then A=0 or B=0.
(iii) If a,b $\in$ R such that aRb=0 then a=0 or b=0.

THEOREM-2:: The following conditions are equivalent:

(i) R is a semiprime ring.
(ii) If A is an ideal of R such that $A^2=0$, then A=0.
(iii) If a $\in$ R such that aRa=0 then a=0.

EXAMPLES OF PRIME RING ::

(i) Integral domain is a Prime ring.
(ii) A simple ring R with $R^2 \neq 0$, is a Prime ring.
(iii) Let T be a ring with unity, then complete matrix ring T_n is prime iff T is a Prime ring.

DEFINITION:: A map f : R→R is said to be **additive** if $f(x+y)=f(x)+f(y)$ holds for all $x,y \in R$.

DEFINITION:: An additive mapping $d : R \to R$ is called a **derivation**, if $d(xy) = d(x)y + xd(y)$ holds for all $x,y \in R$.

One motivating example is the usual derivation d on the polynomial ring $R = F[x]$ given by

$$d(\sum_{i=0}^{t} a_i x^i) = \sum_{i=1}^{t} i\, a_i x^{i-1} = \sum_{i=0}^{t} i a_i x^{i-1}$$

However, there is another fundamental class of derivation. The mapping $d_a : R \to R$ defined by $d_a(x) = [a,x]$ for all $x \in R$ is a derivation, called the inner derivation. If $d_a = 0$ then $[a,R] = 0$ $i.e.,\ a \in Z(R)$.

We have the following properties.

(1) $d([x,y]) = [d(x),y] + [x,d(y)]$

(2) $d(Z) \subseteq Z$, because for $z \in Z$ and $r \in R, 0 = d([z,r]) = [d(z),r] + [z,d)] = [d(z),r]$ implies $d(z) \in Z$ for all $z \in Z$.

DEFINITION:: Let $n \geq 1$ be a fixed integer. A ring R is said to be n-torsion free, if for some $x \in R, nx = 0$ implies $x = 0$.

DEFINITION:: An additive mapping $F : R \rightarrow R$ is called a **generalized derivation**, if there exists a derivation $D : R \rightarrow R$ such that $F(xy) = F(x)y + xD(y)$ holds for all $x, y \in R$.

Examples::

(i) Every derivation is a generalized derivation (when $F = d$).

(ii) Every left multiplier map (i.e. $F(xy) = F(x)y$ for all $x, y \in R$) is a generalized derivation (when $d = 0$). Thus generalized derivation map generalizes the concept of derivation as well as left multiplier map.

(iii) For any derivation d of R, the map $F(x) = ax + d(x)$ for all $x \in R$ is an example of generalized derivation.

(iv) For fixed $a, b \in R$, the mapp $F(x) = ax + xb$ for all $x \in R$ is a generalized derivation, because $F(x) = ax + xb = (a + b)x - [b, x]$. This kind of generalized derivations are called as inner generalized derivation of R.

DEFINITION:: Let (R,+,∗) be a ring with multiplicative identity 1. A function α : R→R is a **Ring Automorphism** if:

(i) $\alpha(a + b) = \alpha(a) + \alpha(b)$

(ii) $\alpha(a * b) = \alpha(a) * \alpha(b)$

(iii) $\alpha(1) = 1$

(iv) α is bijective.

In others words, a ring automorphism of a ring (R,+,∗) is a bijective ring homomorphism from the ring onto itself.

THEOREM:: Let $(R, +, *)$ be an ring with multiplicative identity e. A function $\alpha : R \rightarrow R$ is a ring automorphism, then $\alpha(e) = e$.

Proof. Let a be any element in R.

$\therefore \alpha(e) * a = \alpha(e) * \alpha(b)$ [$\because a$ be any element in R and α is an automorphism from $R \rightarrow R$, $\exists\, b \in R$ such that $\alpha(b) = a$.]

$= \alpha(e * b)$ [$\because \alpha$ is homomorphism]

$= \alpha(b)$

$= a$

Similarly, $a * \alpha(e) = a$

$\therefore \alpha(e)$ is identity.

$\therefore \alpha(e) = e$.

<u>DEFINITION</u>:: Let α be an automorphism of a ring R. An additive mapping $D : R \to R$ is called an **α-derivation or skew derivation** on R if $D(xy) = D(x)y + \alpha(x)D(y)$ for all pairs $x, y \in R$. In this case α is called an associated automorphism of D.

<u>DEFINITION</u>:: An additive mapping $F : R \to R$ is said to be a **generalized skew derivation** of R if there exists a skew derivation D of R with associated automorphism α such that $F(xy) = F(x)y + \alpha(x)D(y)$ for all $x, y \in R$.

<u>DEFINITION</u>:: Let α be an automorphism of a ring R. An additive mapping $D : R \to R$ is called a **Jordan skew derivation** on R if $D(x^2) = D(x)x + \alpha(x)D(x)$ for all pairs $x \in R$. In this case α is called an associated automorphism of D.

<u>DEFINITION</u>:: An additive mapping $F : R \to R$ is said to be a **Jordan generalized skew derivation** of R if there exists a Jordan skew derivation D of R with associated automorphism α such that $F(x^2) = F(x)x + \alpha(x)D(x)$ for all $x \in R$.

Remark : From above definitions it is clear that, every skew derivation is Jordan skew derivation and every generalized skew derivation is Jordan generalized skew derivation. But the converse is not true in general. Converse is true under certain conditions.

Liu and Shiue proved in [2] the following:

Let R be a 2-torsion free semiprime ring and let θ, φ be automorphisms of R. If δ : R → R is a Jordan (θ, φ)-derivation, then δ is a (θ, φ)-derivation.

Note that *(θ, φ)-derivation* means an additive mapping $D : R \to R$ satisfying $D(xy) = D(x)\theta(y) + \phi(x)D(y)$ for all $x, y \in R$.

Remark-1: Therefore, as a particular case of above result of Liu and Shiue [2], we can say that

"If R is a 2-torsion free semiprime ring, then every Jordan skew derivation of R is a skew derivation."

Under this project work, our target is to generalize the concept of Jordan skew derivation by considering the situation

$$D(x^{n+1}) = D(x)x^n + \alpha(x)D(x)x^{n-1} + (\alpha(x))^2 D(x)x^{n-2} + \cdots + (\alpha(x))^n D(x)$$

where D is an additive mapping and α is an automorphism of R. It is very clear that for n=1, D becomes a Jordan skew derivation.

Similarly, we generalize the concept of Jordan generalized skew derivation by considering the situation

$$F(x^{n+1}) = F(x)x^n + \alpha(x)D(x)x^{n-1} + (\alpha(x))^2 D(x)x^{n-2} + \cdots + (\alpha(x))^n D(x)$$

where F, D are additive mappings and α is an automorphism of R. It is very clear that for n=1, D becomes a Jordan generalized skew derivation.

In this project work, we investigate these mappings and prove under some conditions when these mappings to be Jordan skew derivation or Jordan generalized skew derivation.

MAIN RESULTS

THEOREM-1. *Let $n \geq 1$ be a fixed integer and let R be a $(n+1)\,!$ -torsion free any ring with identity element. If $D : R \to R$ be an additive mapping such that*

$$D(x^{n+1}) = D(x)x^n + \alpha(x)D(x)x^{n-1} + \left(\alpha(x)\right)^2 D(x)x^{n-2} + \cdots + \left(\alpha(x)\right)^n D(x)$$

holds for all $x \in R$, then D is Jordan Skew-Derivation.

Proof. We have the relation

$$D(x^{n+1}) = D(x)x^n + \alpha(x)D(x)x^{n-1} + \left(\alpha(x)\right)^2 D(x)x^{n-2} + \cdots + \left(\alpha(x)\right)^n D(x)$$
$$= D(x)x^n + \sum_{i=1}^{n} (\alpha(x))^i \, D(x)x^{n-i} \cdots\cdots (1)$$

for all $x \in R$. Let e be the identity element of R, then replacing x by e in (1), we get

$D(e) = D(e) + nD(e)$ which implies $nD(e) = 0$.

Since R is n-torsion free, we may conclude that $D(e) = 0$.

Now replacing x by $x + ke$ in (1), where k is any positive integer, we get

$$D((x+ke)^{n+1}) = D(x+ke)(x+ke)^n + \sum_{i=1}^{n} \left(\alpha(x+ke)\right)^i D(x+ke)(x+ke)^{n-i}$$

for all $x \in R$. Expanding the power values of $(x + ke)$ and using the fact $D(e) = 0$, we have

$$D[\, x^{n+1} + \binom{n+1}{1} kx^n + \binom{n+1}{2} k^2x^{n-1} + \cdots + \binom{n+1}{n-1} k^{n-1}x^2 + \binom{n+1}{n} k^n x + k^{n+1}e]$$
$$= D(x)\,\{x^n + \cdots + \binom{n}{n-2} k^{n-2}x^2 + \binom{n}{n-1} k^{n-1}x + k^n e\,\}$$
$$+ \sum_{i=1}^{n} \{\left(\alpha(x)\right)^i + \cdots + \binom{i}{i-2} k^{i-2}\left(\alpha(x)\right)^2 + \binom{i}{i-1} k^{i-1}\alpha(x)$$
$$+ k^i e\}\, D(x)\{x^{n-i} + \cdots + \binom{n-i}{n-i-2} k^{n-i-2}x^2 + \binom{n-i}{n-i-1} k^{n-i-1}x$$
$$+ k^{n-i}e\} \qquad \cdots\cdots (2)$$

for all $x \in R$. Using relation (1), this can be written as

$$kf_1(x,e) + k^2 f_2(x,e) + \cdots + k^n f_n(x,e) = 0 \quad \cdots\cdots (3)$$

for all $x \in R$. Now, replacing k by 1,2,3, ... n in turn and considering the resulting system of n homogeneous equations, we see that the coefficient matrix of the system is a Van-der-Monde matrix.

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 2 & 2^2 & 2^3 & \cdots & 2^n \\ \vdots & \vdots & \vdots & \dots & \vdots \\ n & n^2 & n^3 & \cdots & n^n \end{pmatrix}$$

Since the determinant of the matrix is equal to a product of positive integers, each of which is less than n, and since R is $(n+1)!$ -torsion free, it follows immediately that $f_i(x,e) = 0$ for all $x \in R$, $i = 1,2,\dots n$. Now, $f_{n-1}(x,e) = 0$ gives that

$$\binom{n+1}{n-1}D(x^2) = \binom{n}{n-1}D(x)x + \sum_{i=1}^{n}\{\binom{i}{i-1}\alpha(x)D(x) + \binom{n-i}{n-i-1}D(x)x\}$$

This implies

$$\frac{n(n+1)}{2}D(x^2) = nD(x)x + \sum_{i=1}^{n}\{i\alpha(x)D(x) + (n-i)D(x)x\}$$

Multiplying both sides by *2*, it reduces to

$\Rightarrow n(n+1)D(x^2) = 2nD(x)x + n(n+1)\alpha(x)D(x) + n(n-1)D(x)x$

Since R is n torsion free, this gives

$$(n+1)D(x^2) = 2D(x)x + (n+1)\alpha(x)D(x) + (n-1)D(x)x$$

$$\Rightarrow (n+1)D(x^2) = (n+1)\alpha(x)D(x) + (n+1)D(x)x$$

Again, since R is (n+1) torsion free, we have

$$D(x^2) = \alpha(x)D(x) + D(x)x$$

for all $x \in R$ which means ***D* is a Jordan Skew Derivation** in R.

Thus the proof is complete.

THEOREM-2. Let $n \geq 1$ be a fixed integer and let R be a $(n+1)!$-torsion free any ring with identity element. If $F : R \to R$ and $D : R \to R$ are two additive mapping such that

$$F(x^{n+1}) = F(x)x^n + \alpha(x)D(x)x^{n-1} + (\alpha(x))^2D(x)x^{n-2} + \cdots + (\alpha(x))^nD(x)$$

holds for all $x \in R$, then D is a Jordan Skew Derivation and F is a Jordan generalized Skew Derivation.

Proof. We have the relation

$$F(x^{n+1}) = F(x)x^n + \alpha(x)D(x)x^{n-1} + (\alpha(x))^2D(x)x^{n-2} + \cdots + (\alpha(x))^nD(x)$$
$$= F(x)x^n + \sum_{i=1}^{n}(\alpha(x))^iD(x)x^{n-i} \cdots\cdots \quad (1)$$

for all $x \in R$. Let e be the identity element of R, then replacing x by e in (1), we get

$F(e) = F(e) + nD(e)$ which implies $nD(e) = 0$.

Since R is n-torsion free, we may conclude that $D(e) = 0$.

Now replacing x by $x + ke$ in (1), where k is any positive integer, we get

$$F((x+ke)^{n+1}) = F(x+ke)(x+ke)^n + \sum_{i=1}^{n}(\alpha(x+ke))^i D(x+ke)(x+ke)^{n-i}$$

for all x ∈ R. Expanding the power values of $(x + ke)$ and using the fact $D(e) = 0$, we have

$$F(x^{n+1} + \binom{n+1}{1}kx^n + \binom{n+1}{2}k^2x^{n-1} + \cdots + \binom{n+1}{n-1}k^{n-1}x^2 + \binom{n+1}{n}k^nx + k^{n+1}e)$$

$$= (F(x)+kF(e))\{x^n + \cdots + \binom{n}{n-2}k^{n-2}x^2 + \binom{n}{n-1}k^{n-1}x + k^ne\}$$

$$+\sum_{i=1}^{n}\{(\alpha(x))^i + \cdots + \binom{i}{i-2}k^{i-2}(\alpha(x))^2 + \binom{i}{i-1}k^{i-1}\alpha(x) + k^ie\}\ D(x)$$

$$\{x^{n-i} + \cdots + \binom{n-i}{n-i-2}k^{n-i-2}x^2 + \binom{n-i}{n-i-1}k^{n-i-1}x + k^{n-i}e\} \cdots\cdots (2)$$

for all x ∈ R. Using relation (1), this can be written as

$$kf_1(x,e) + k^2f_2(x,e) + \cdots + k^nf_n(x,e) = 0 \qquad \cdots\cdots (3)$$

for all x ∈ R. Now, replacing k by 1,2,3 ... n in turn and considering the resulting system of n homogeneous equations, we that the coefficient matrix of this system is van-der-Monde matrix.

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 2 & 2^2 & 2^3 & \dots & 2^n \\ \vdots & \vdots & \vdots & \dots & \vdots \\ n & n^2 & n^3 & \dots & n^n \end{pmatrix}$$

Since the determinant of the matrix is equal to a product of positive integers, each of which is less than n, and since R is $(n+1)!$-torsion free, it follows immediately that $f_i(x,e) = 0$ for all $x \in R, i = 1,2,\dots,n$. Now, $f_n(x,e) = 0$ gives that

$$\binom{n+1}{n}F(x) = F(x) + \binom{n}{n-1}F(e)x + nD(x)$$

$$\Rightarrow (n+1)F(x) = F(x) + nF(e)x + nD(x)$$

$$\Rightarrow nF(x) = nF(e)x + nD(x)$$

Since R is n-torsion free

$\Rightarrow F(x) = F(e)x + D(x)$ for all x ∈ R

Now, $f_{n-1} = 0$ gives

$$\binom{n+1}{n-1}F(x^2) = \binom{n}{n-1}F(x)x + \binom{n}{n-2}F(e)x^2 + \sum_{i=1}^{n}\{\binom{i}{i-1}\alpha(x)D(x) + \binom{n-i}{n-i-1}D(x)x\}$$

$$\Rightarrow \frac{n(n+1)}{2}F(x^2) = nF(x)x + \frac{n(n-1)}{2}F(e)x^2 + \sum_{i=1}^{n}\{(i\alpha(x)D(x) + (n-i)D(x)x\}$$

Multiplying both sides by 2, it reduces to

$$\Rightarrow n(n+1)F(x^2) = 2nF(x)x + n(n-1)F(e)x^2 + n(n+1)\alpha(x)D(x) + n(n-1)D(x)x$$

Since R is n-torsion free, this gives

$$(n+1)F(x^2) = 2F(x)x + (n-1)F(e)x^2 + (n+1)\alpha(x)D(x) + (n-1)D(x)x$$

Using $F(x) = F(e)x + D(x)$, this implies

$$\Rightarrow (n+1)\{F(e)x^2 + D(x^2)\} = 2\{F(e)x^2 + D(x)x\} + (n-1)F(e)x^2 + (n+1)\alpha(x)D(x) + (n-1)D(x)x$$

$\Rightarrow (n+1)D(x^2) = (n+1)\alpha(x)D(x) + (n+1)D(x)x$

Again since R is (n+1) -torsion free, we have

$\Rightarrow D(x^2) = \alpha(x)D(x) + D(x)x$, for all $x \in R$

which means **D is a Jordan Skew Derivation** in R.

Therefore from $F(x) = F(e)x + D(x)$,

we obtain $F(x^2) = F(e)x^2 + \alpha(x)D(x) + D(x)x$

$$= F(x)x + \alpha(x)D(x),$$

for all $x \in R$, implying **F is a Jordan generalized Skew Derivation** in R.

Thus the proof is complete.

Using Remark-1, following Corollaries are straightforward.

Corollary-3. *Let $n \geq 1$ be a fixed integer and let R be a $(n+1)$! -torsion free semiprime ring with identity element. If $D : R \to R$ be an additive mapping such that*

$$D(x^{n+1}) = D(x)x^n + \alpha(x)D(x)x^{n-1} + \big(\alpha(x)\big)^2 D(x)x^{n-2} + \cdots + \big(\alpha(x)\big)^n D(x)$$

holds for all $x \in R$, then D is Skew-Derivation.

Corollary-4. Let $n \geq 1$ be a fixed integer and let R be a $(n+1)$!-torsion free semiprime ring with identity element. If $F : R \to R$ and $D : R \to R$ are two additive mapping such that

$$F(x^{n+1}) = F(x)x^n + \alpha(x)D(x)x^{n-1} + (\alpha(x))^2D(x)x^{n-2} + \cdots + (\alpha(x))^nD(x)$$

holds for all $x \in R$, then D is a Skew Derivation and F is a generalized Skew Derivation.

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