MATH 20410

PSet 5

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Problem 9.17.

Solution. a) The range of f is certainly all of $\mathbb{R}^2 n(0,0)$. It can be no more than $\mathbb{R}^2$, so suffices to show that it is at least $\mathbb{R}^2 n(0,0)$. Any ratio between $f_1 \& f_2$ can be achieved, since it is the range of tan. Since e^x goes to infinity, the size of the points with any ratio is not limited. The only trouble is that $e^x > 0$, $\forall x$. Since $\frac{0}{0}$ is not a ratio, and the scalar can never be 0, (0.0) is far out.

b) In particular, the Jacobian Matrix of the function is $\begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix}$
e^x is never 0. Computing determinant gives $\det(D(f(x))) = ((e^x \cos y)(e^x \cos y) - (-e^x \sin y)(e^x \sin y)) = e^{2x}$ by pythagoreon identity. So done.

c)But look, in general, $g_2(b) = a_2 = \tan^{-1}(\frac{f_2(a)}{f_1(a)}) = \tan^{-1} \frac{b_2}{b_1}$. Pick the obvious value (since certaily the functuion is not one to one, due to periodicity of sin/cos), or indeed chose any of them: just the same one for the whole neighborhood. Solving for $g_1(b)$ is $g_1(b) = a_1 = \ln(\frac{f_1(a)}{\cos(g_2(b))}) = \ln(\frac{b_1}{\cos(g_2(b))})$ (or flipped, but with sin).
Evaluating the Jacobian Matrix at a

$$\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

I want to differentiate g using the explicit writing out of it which I did above. I can't though.

d) Parallel to the x axis, and constant y, they look like linear functions, through $(0,0)$. Parallel to y axis, constant y, they look like circles of various diameters.

□

Problem 1.

Solution. a) Certainly *det* is a polynomial function, it is therefore known to be continuous wrt to each entry in the matrix. Now, we know that a function is open iff if open sets map to open images and vise versa. Now the set of invertible matrices described is s.t $\mathbf{GL} = \mathbf{det}^{-1}(\mathbb{R}\backslash\{0\})$. The inside fo the bracket is clearly open, and hence by continuity, the set which it maps to (by lin alg definition, invertible matrices) is open. Done.

b) 2 thing: First, if $||\mathbf{A}|| = c$, then, since for any vector $\vec{v}$ $||\mathbf{A}^{-1}\mathbf{A}v|| = ||v||$, it means that $||\mathbf{A}^{-1}|| \geq \frac{1}{c}$. Second, if a matrix is invertible, it means that there does exist a $c > 0$ $s.t||\mathbf{A}v|| \geq c||v||$. Combining these, in particular $\frac{1}{||\mathbf{A}^{-1}}|| = c$ suffices. Now, I'm note sure why I wrote that previous part, but I spent a while on it, and I'm not deleting it.

Look at this though. Continuity is the same as trying to show that close enough $\mathbf{A}, \mathbf{B}$ lead to close enough $\mathbf{A}^{-1}$, $\mathbf{B}^{-1}$. Now, pick an x n-D unit vector. Proving that when the transformations $\mathbf{A}, \mathbf{B}$ get closer, then, regardless of that vector, $(\mathbf{A}^{-1} - \mathbf{B}^{-1})x$ goes to 0 is literally (don't quote me) the same as continuity. $\mathbf{A}^{-1}(\mathbf{B} - \mathbf{A})x \leq ||\mathbf{A}^{-1}||\mathbf{B} - \mathbf{A}||$. Now $||\mathbf{A}^{-1}||$ is fixed, and we are ourselves bringing the right part to 0, so done. □

Problem 9.18.

Solution. a) Pick any value for v. Then $v = 2xy \Rightarrow y = \frac{v}{2x} \Rightarrow u = x^2 - \frac{v^2}{4x^2} \iff x^4 - ux^2 - \frac{v^2}{4} = 0$. By treating this as a hidden quadratic it is observed that this has a root iff $x^4 + 2x^2y^2 + y^4 = x^4 - 2x^2y^2 + y^4 + 4x^2y^2 = u^2 + v^2 \geq 0$, which is of course known to be as such, for any u and any v. Subsituting that into the first equation gives the value of y for the x. So the range is all of $\mathbb{R}^2$

b) In particular the Jacobian Matrix is

$$\begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix}$$

Jacobian is $\mathbf{det}(dF(x,y)) = 4x^2 + 4y^2$ The square of a real number results in non-negative numbers, so Jacobian is 0 iff $x\&y = 0$. We know that roots of 4th order polynomials come in pairs of two, hence by the same manner that a single root is guaranteed, two are also (see above). So not one-one.

c) Solving the previous hidden quadratic explicitly:

$$x^2 = \frac{u \pm \sqrt{u^2 + v^2}}{2} \implies x = \sqrt{\frac{u \pm \sqrt{u^2 + v^2}}{2}}$$

To receive y, first use the u part of the jingle-jangle, which like spurrs, doth guide mine ramble. Since

$$x^2 = \frac{u \pm \sqrt{u^2 + v^2}}{2}, \ \& \ y^2 = x^2 - u = \frac{-u \pm \sqrt{u^2 + v2}}{2}.$$

So,

$$x = \pm\sqrt{\frac{u + \sqrt{u^2 + v2}}{2}}; \ y = \pm\sqrt{\frac{-u + \sqrt{u^2 + v2}}{2}}$$

To ensure positivity, since we are taking square roots, $\pm$ is forgotten about, replaced by its more handsome brother. Also, from the defintion of v, and it follows that we pick a combination of the postives and the negative roots s.t the number of roots taken negative is even if v not negative, and odd if it is.
The given values imply that $|x| = |y|$, so $x = y = \pm\sqrt{\frac{\pi}{3}}$ In the one case, the Jacobian Matrix is:

$$\begin{pmatrix} \left(\frac{1}{2}\frac{u+\sqrt{u^2+v^2}}{2}\right)^{\frac{-1}{2}}\left(1+\frac{2u}{2\sqrt{(u^2+v^2)}}\right) & \left(\frac{1}{2}\frac{u+\sqrt{u^2+v^2}}{2}\right)^{\frac{-1}{2}}\frac{2v}{2\sqrt{(u^2+v^2)}} \\ \left(\frac{1}{2}\frac{-u+\sqrt{u^2+v^2}}{2}\right)^{\frac{-1}{2}}\left(-1+\frac{2u}{2\sqrt{(u^2+v^2)}}\right) & \left(\frac{1}{2}\frac{-u+\sqrt{u^2+v^2}}{2}\right)^{\frac{-1}{2}}\frac{2v}{2\sqrt{(u^2+v^2)}} \end{pmatrix}$$

Either that, or (-1) times that (since there must even amount of negative signs between x and y, so 0 or 2, so 1 or -1 scalar)

d) They look like graphs of $\frac{1}{x}$ (y constant) or $\frac{1}{y}$ (x constant) stretched in v direction by $\frac{1}{2}$ and shifted depending on where exactly the thing is parallel. □

Problem 9.19.

Solution. It is visually clear that $R1 = R2 + R3$, exlcuding the $R(u)$ component. I.e, taking $R1 \to R1 - R2 - R3$:

$$\begin{aligned} 3x+ \quad y- \quad 3z + u^2 &= 0 \\ x- \quad y+ \quad 2z + u &= 0 \\ 2x+ \quad 2y- \quad 3z + 2u &= 0 \end{aligned}$$

$$\begin{aligned} u^2- \quad 3u &= 0 \\ x- \quad y+ \quad 2z + u &= 0 \\ 2x+ \quad 2y- \quad 3z + 2u &= 0 \end{aligned}$$

So we can clearly solve for which u lead to a valid system, (indeed, $u = 0\ or u = 3$), and then we are solving for the remaining 3 variables in the remaining 3 equations. But, conversely, the x, y, z parts of $R2$ & $R3$ together specify exactly $R1$, hence the only thing we can get (this has been done many times in the linear algebra class; if 1 equation 'agrees' with 2 others, it is irrelevant). So, WHEN the system is consistent, it means that exactly $R1 = R2 + R3$. So we have really a system of 2 3 variable equations, so from lin alg they can indeed all be solved in terms of any of those variables. The solution then will be 2 families in each of the 3 cases (where we are solving for given $x\ or y\ or z$), one for each possible solution of u. General u on the other hand does not have a solution, so there. □