MATH 20250

PSet 4

February 4, 2025

Problem 9.17.

Solution. a) The range of f is certainly all of $\mathbb{R}^2 n(0,0)$. It can be no more than $\mathbb{R}^2$, so suffices to show that it is at least $\mathbb{R}^2 n(0,0)$. Any ratio between $f_1 \& f_2$ can be achieved, since it is the range of tan. Since e^x goes to infinity, the size of the points with any ratio is not limited. The only trouble is that $e^x > 0$, $\forall x$. Since $\frac{0}{0}$ is not a ratio, and the scalar can never be 0, (0.0) is far out.

b) In particular, the Jacobian Matrix of the function is $\begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix}$
e^x is never 0. Computing determinant gives $det(D(f(x))) = ((e^x \cos y)(e^x \cos y) - (-e^x \sin y)(e^x \sin y)) = e^{2x}$ by pythagoreon identity. So done.

c)But look, in general, $g_2(b) = a_2 = \tan^{-1}(\frac{f_2(a)}{f_1(a)}) = \tan^{-1}\frac{b_2}{b_1}$. Pick the obvious value (since certaily the functuion is not one to one, due to periodicity of sin/cos), or indeed chose any of them: just the same one for the whole neighborhood. Solving for $g_1(b)$ is $g_1(b) = a_1 = \ln(\frac{f_1(a)}{\cos(g_2(b))}) = \ln(\frac{b_1}{\cos(g_2(b))})$ (or flipped, but with sin).
Evaluating the Jacobian Matrix at a:

$$\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

I want to differentiate g using the explicit writing out of it which I did above. I can't though.

d) Parallel to the x axis, and constant y, they look like linear functions, through $(0,0)$. Parallel to y axis, constant y, they look like circles of various diameters.

□

Problem 9.18.

Solution. a) Pick any value for v. Then $v = 2xy \Rightarrow y = \frac{v}{2x} \Rightarrow u = x^2 - \frac{v^2}{4x^2} \iff x^4 - ux^2 - \frac{v^2}{4} = 0$. By treating this as a hidden quadratic it is observed that this has a root iff $x^4 + 2x^2y^2 + y^4 = x^4 - 2x^2y^2 + y^4 +$

$4x^2y^2 = u^2 + v^2 \geq 0$, which is of course known to be as such, for any u and any v. Subsituting that into the first equation gives the value of y for the x. So the range is all of $\mathbb{R}^2$

b) In particular the Jacobian Matrix is

$$\begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix}$$

Jacobian is $\mathbf{det}(dF(x,y)) = 4x^2 + 4y^2$ The square of a real number results in non-negative numbers, so Jacobian is 0 iff $x \& y = 0$. We know that roots of 4th order polynomials come in pairs of two, hence by the same manner that a single root is guaranteed, two are also (see above). So not one-one. c) Solving the previous hidden quadratic explicitly:

$$x^2 = \frac{u \pm \sqrt{u^2 + v^2}}{2} \implies x = \sqrt{\frac{u \pm \sqrt{u^2 + v^2}}{2}}$$

To receive y, first use the u part of the jingle-jangle, which like spurrs, doth guide mine ramble. Since

$$x^2 = \frac{u \pm \sqrt{u^2 + v^2}}{2}, \ \& \ y^2 = x^2 - u = \frac{-u \pm \sqrt{u^2 + v2}}{2}.$$

So,

$$x = \pm\sqrt{\frac{u + \sqrt{u^2 + v2}}{2}}; \ y = \pm\sqrt{\frac{-u + \sqrt{u^2 + v2}}{2}}$$

To ensure positivity, since we are taking square roots, $\pm$ is forgotten about, replaced by its more handsome brother. Also, from the defintion of v, and it follows that we pick a combination of the postives and the negative roots s.t the number of roots taken negative is even if v not negative, and odd if it is.

The given values imply that $|x| = |y|$, so $x = y = \pm\sqrt{\frac{\pi}{3}}$ In the one case, the Jacobian Matrix is:

$$\begin{pmatrix} \left(\frac{1}{2}\frac{u+\sqrt{u^2+v^2}}{2}\right)^{\frac{-1}{2}} \left((1 + \frac{2u}{2\sqrt{u^2}+v^2})\right) & \left(\frac{1}{2}\frac{u+\sqrt{u^2+v^2}}{2}\right)^{\frac{-1}{2}} \left((\frac{2v}{2\sqrt{u^2}+v^2})\right) \\ \left(\frac{1}{2}\frac{-u+\sqrt{u^2+v^2}}{2}\right)^{\frac{-1}{2}} \left((-1 + \frac{2u}{2\sqrt{u^2}+v^2})\right) & \left(\frac{1}{2}\frac{-u+\sqrt{u^2+v^2}}{2}\right)^{\frac{-1}{2}} \left((\frac{2v}{2\sqrt{u^2}+v^2})\right) \end{pmatrix}$$

d) They look like graphs of $\frac{1}{x}$ (y constant) or $\frac{1}{y}$ (x constant) stretched in v direction by $\frac{1}{2}$ and shifted depending on where exactly the thing is parallel. $\square$