MATH 20410 PSet 6

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February 13, 2025

Problem 9.23.

Solution. Begin by a simple computation

$$f(0,1,-1) = (0)^2 1 + e^0 + (-1) = 0 + 1 - 1 = 0.5$$

The 1st partial derivative wrt to x is

$$D_1 f(x, y_1, y_2) = 2xy_1 + e^x + 0; \text{evaluated at the desired point being: } = 0 + 1 = 1 \neq 0$$

Clearly, this represents an invertible transformation, hence, we have found a point

$$(x_0, y_0) \text{ for which the derivative is invertible, and the value is } 0$$

The function, say, g which is defined, by $g(y_0)$ the correspoding value of x for which is (x, y_0) is a root of f is C^1, in a neighborhood of y_0.
By the same theorem from which we obtained the previous result, the implicit value theorem, we know that each derivative of this g is given by

$$D_1 g(y_1, y_2) = -D_{1,x}(g(y), y)^{-1} D_{1,y}(g(y), y)$$

But this y we are given, we know something about it:

$$g(y) = 0; \text{hence} - D_{1,x}(g(y), y)^{-1} D_{1,y}(g(y), y) = -D_{1,x}(g(y), 0)^{-1} D_{1,y}(g(y), 0) = (-h, k)^T$$

□

Problem 9.24.

Solution. I suppose begin by parameterising, and call the first output a and the second b. Hence, though the - in front of y^2 may irritate:

$$a^2 + 4b^2 = \frac{x^4 - 2x^2y^2 + y^4 + 4x^2y^2}{(x^2+y^2)^2} = \frac{x^4 + 2x^2y^2 + y^4}{(x^2+y^2)^2} = \frac{(x^2+y^2)^2}{(x^2+y^2)^2} = 1$$

Returning to the original expression, to $a^2 + 4b^2 = 1$, and notice that any value of a or b is allowed. Furthermore, pick the fixed variable to be b wlog, and observe that

$$a = \frac{-4 \pm \sqrt{16+4}}{2}$$

So that range of that is the range of the function. To compute the rank of the derivative, first compute each partial (and observe they are continuous, and therefore rank makes sense as a notion):

$$D_1 f_1(x,y) = \frac{(2x)(x^2+y^2) - (x^2)(2x)}{(x^2+y^2)^2} = \frac{2xy^2}{(x^2+y^2)^2}$$

$$D_2 f_1(x,y) = \frac{(2y)(x^2+y^2) - (y^2)(2y)}{(x^2+y^2)^2} = \frac{2yx^2}{(x^2+y^2)^2}$$

$$D_1 f_2(x,y) = \frac{(y)(x^2+y^2) - (xy)(2x)}{(x^2+y^2)^2} = \frac{y^3 - x^2 y}{(x^2+y^2)^2}$$

$$D_2 f_2(x,y) = \text{ same as before, x,y flipped, by symmetry } = \frac{x^3 - y^2 x}{(x^2+y^2)^2}$$

Hence, the derivative is the following matrix:

$$\begin{matrix} \frac{2xy^2}{(x^2+y^2)^2} & \frac{2yx^2}{(x^2+y^2)^2} \\ \frac{y^3-x^2y}{(x^2+y^2)^2} & \frac{x^3-y^2x}{(x^2+y^2)^2} \end{matrix}$$

Clear by the consistent addition of powers of $x \& y$ across the diagnonals, the determinant is identically 0. (I really don't want to write out the whole thing in latex). So undetermined rank. To be more specific, observe that the determinant is only undefined if $x = y = 0$, which we know never to be the case. To observe if the rank is ever 0 (clearly, they are not identically zero, as the left and right columns can be made distinct, try $x = 0$ $y = 1$) we can ignore the constant scalar factor $\frac{1}{(x^2+y^2)^2}$. Hence, looking to see if there are are any values (x, y) s.t

$$\begin{pmatrix} 2xy^2 \\ y^3 - x^2 y \end{pmatrix} = \lambda \begin{pmatrix} 2yx^2 \\ x^3 - y^2 x \end{pmatrix}$$

Or otherwise:

$$2xy^2 = \lambda 2y^2 x$$

$$y^3 - x^2 y = \lambda(x^3 - y^2 x)$$

Now, this means taht $x = \lambda y$, which added to the second equation means:

$$y^3 - \lambda^2 y^3 = \lambda(\lambda^3 y^3 - \lambda y^3)$$

This surely means that $x = y$, for any such combination. So there, the rank is 1 one otherwise, and 0 in that case.

$\square$

Problem 9.27.

Solution. (a) Continuity means:

$$\forall \epsilon > 0 \exists \delta \text{ s.t } |p-q| < \delta \implies ||f(p) - f(q)|| < \epsilon$$

Beginning from the second part,

$$\frac{p_1 p_2 (p_1^2 - p_2^2)}{p_1^2 + p_1^2} = \frac{p_1 p_2 (p_1^2 - p_2^2)}{p_1^2 + p_2^2} - \frac{q_1 q_2 (q_1^2 - q_2^2)}{q_1^2 + q_2^2}$$

And clearly since the sum of a 2 squares is greater than the subtraction of them, provided neither is 0, then we get by algebraic manipulation:

$$\text{above} < p_1 p_2 - q_1 q_2$$

We do not even need to discuss why there exists a δ for which the above is less by ϵ, and where hence the desired inequality is less than ϵ.

In particular,

$$D_1 f(x,y) = \frac{(y(x^2 - y^2) - xy(2x))(x^2 + y^2) - xy(x^2 - y^2))(x^2 + y^2)}{(x^2 + y^2)^2}$$

In particular, and ignoring y as it is a constant, we have a rational function, and those are always continuous. We know that $D_2 f$ is identical, except multiplied by (-1) and flipped wrt $x\&y$, hence for the same reason continuous.

(b) Rational functions, so clearly the functions have derivatives. The derivatives are also rational, so they are also continuous; except of course when the denominator, which is $(x^2 + y^2)^n$, is 0, which is if and only if $x = y = 0$. $\square$

Problem 9.29.

Solution. The definition of successive partial derivatives is their successive application:

$$D_{e_i} \dots D_{e_j}.$$

Now by the inductive step, we know that whichever derivative is after any derivative upto k, can be interchanged. To take

$$D_{e_1} \dots D_{e_k}$$

To any other permutation of the derivatives, define the following algorithm: index the positions from 1 to k in the obvious way. Then, if theD_{e_1} partial belongs in the i_1^{th} index, do the shuffles as defined by theorem 9.41 $(i_1 - 1)$ times. Then, repeat this for the nth derivatives, performing $i_n -$ current index shuffles, where a negative number indicates left-ward shuffles. This defines the algorithm, which, by successive applications of 9.41, gets any permutation σ_k

$\square$

Problem .

Solution. ∎