triangulate

Consider replace symbol of refraction:

- replace v_1 with r to represent unit direction vector (FROM)
- replace v_2 with r' to represent unit direction vector (TO)

Then the formula of r' becomes:

$$r' = n\left[(\frac{n_1}{n_2})[-n^T r] - \sqrt{1-(\frac{n_1}{n_2})^2(1-[r^T n]^2)}\right] + r(\frac{n_1}{n_2})$$

symbol table

SMYBOL	MEANING
r'_1	unit direction vector of ray 1 in water
r'_2	unit direction vector of ray 2 in water
I_1	intersection point of interface plane and ray 1
I_2	intersection point of interface plane and ray 2
M_1	closest point of ray 1 to ray 2
M_2	closest point of ray 2 to ray 1
M	mid point of M_1 and M_2
k_1	scalar factor from I_1 to M_1
k_2	scalar factor from I_2 to M_2

Our goal is to express M with what we know r'_1, r'_2, I_1, I_2

analysis

Defination of M_1, M_2,
the closest point pair $M_1 - M_2$ must be perpendicular to r'_1, r'_2:

$$
\begin{aligned}
{r'}_1^T(M_1 - M_2) = 0 \\
{r'}_2^T(M_1 - M_2) = 0
\end{aligned}
$$

Definition of k_1, k_2:

$$
\begin{aligned}
k_1 r'_1 \equiv M_1 - I_1 \\
k_2 r'_2 \equiv M_2 - I_2
\end{aligned}
$$

Replace M_1, M_2 with unknow k_1, k_2
and what we know r'_1, r'_2, I_1, I_2,
To solve k_1, k_2 firstly

$$
\begin{aligned}
{r'}_1^T\left([I_1 - I_2] + k_1 r'_1 - k_2 r'_2\right) = 0 \\
{r'}_2^T\left([I_1 - I_2] + k_1 r'_1 - k_2 r'_2\right) = 0
\end{aligned}
$$

It is equivalent to

$$
\begin{aligned}
[{r'}_1^T r'_1]k_1 - [{r'}_1^T r'_2]k_2 = -{r'}_1^T[I_1 - I_2] \\
-[{r'}_2^T r'_1]k_1 + [{r'}_2^T r'_2]k_2 = {r'}_2^T[I_1 - I_2]
\end{aligned}
$$

In matrix form

$$
\begin{pmatrix} [{r'}_1^T r'_1] & -[{r'}_1^T r'_2] \\ -[{r'}_1^T r'_2] & [{r'}_2^T r'_2] \end{pmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{pmatrix} -{r'}_1^T[I_1 - I_2] \\ {r'}_2^T[I_1 - I_2] \end{pmatrix}
$$

With Cramer's rule

$$
\begin{aligned}
k_1 &= \frac{\begin{vmatrix} -{r'}_1^T[I_1 - I_2] & -[{r'}_1^T r'_2] \\ {r'}_2^T[I_1 - I_2] & [{r'}_2^T r'_2] \end{vmatrix}}{\begin{vmatrix} [{r'}_1^T r'_1] & -[{r'}_1^T r'_2] \\ -[{r'}_1^T r'_2] & [{r'}_2^T r'_2] \end{vmatrix}} = \frac{[-{r'}_2^T r'_2 {r'}_1^T + {r'}_2^T r'_1 {r'}_2^T][I_1 - I_2]}{1 - [{r'}_1^T r'_2]^2} \\
&= {r'}_2^T \left[\frac{1}{1 - [{r'}_1^T r'_2]^2} \left(r'_1 {r'}_2^T - r'_2 {r'}_1^T \right) [I_1 - I_2] \right] \\
k_2 &= \frac{\begin{vmatrix} [{r'}_1^T r'_1] & -{r'}_1^T[I_1 - I_2] \\ -[{r'}_1^T r'_2] & {r'}_2^T[I_1 - I_2] \end{vmatrix}}{\begin{vmatrix} [{r'}_1^T r'_1] & -[{r'}_1^T r'_2] \\ -[{r'}_1^T r'_2] & [{r'}_2^T r'_2] \end{vmatrix}} = \frac{[{r'}_1^T r'_1 {r'}_2^T - {r'}_1^T r'_2 {r'}_1^T][I_1 - I_2]}{1 - [{r'}_1^T r'_2]^2} \\
&= {r'}_1^T \left[\frac{1}{1 - [{r'}_1^T r'_2]^2} \left(r'_1 {r'}_2^T - r'_2 {r'}_1^T \right) [I_1 - I_2] \right]
\end{aligned}
$$

expression of M

Definition of M is

$$
\begin{aligned}
M \equiv \frac{M_1 + M_2}{2} &= \frac{I_1 + I_2}{2} + \frac{k_1 r'_1 + k_2 r'_2}{2} \\
&= \frac{I_1 + I_2}{2} + \frac{1}{2} \left(r'_1 {r'}_2^T + r'_2 {r'}_1^T \right) \left[\frac{1}{1 - [{r'}_1^T r'_2]^2} \left(r'_1 {r'}_2^T - r'_2 {r'}_1^T \right) [I_1 - I_2] \right] \\
&= \frac{I_1 + I_2}{2} + \frac{1}{2} \frac{1}{1 - [{r'}_1^T r'_2]^2} \left[\left(r'_1 {r'}_2^T + r'_2 {r'}_1^T \right) \left(r'_1 {r'}_2^T - r'_2 {r'}_1^T \right) \right] [I_1 - I_2] \\
&= \frac{I_1 + I_2}{2} + \frac{1}{2} \frac{1}{1 - [{r'}_1^T r'_2]^2} \left[r'_1 [{r'}_2^T r'_1] {r'}_2^T - r'_2 [{r'}_1^T r'_2] {r'}_1^T \right] [I_1 - I_2] \\
&= \frac{I_1 + I_2}{2} + \frac{1}{2} \frac{[{r'}_1^T r'_2]}{1 - [{r'}_1^T r'_2]^2} \left(r'_1 {r'}_2^T - r'_2 {r'}_1^T \right) [I_1 - I_2]
\end{aligned}
$$

Consider the cross product, and its cross product matrix form

$$a \times b = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

$$\begin{aligned}(a \times b)\times &= \begin{bmatrix} 0 & -[a_1 b_2 - a_2 b_1] & a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 & 0 & -[a_2 b_3 - a_3 b_2] \\ -[a_3 b_1 - a_1 b_3] & a_2 b_3 - a_3 b_2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & a_2 b_1 - a_1 b_2 & a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 & 0 & a_3 b_2 - a_2 b_3 \\ a_1 b_3 - a_3 b_1 & a_2 b_3 - a_3 b_2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} a_1 b_1 & a_2 b_1 & a_3 b_1 \\ a_1 b_2 & a_2 b_2 & a_3 b_2 \\ a_1 b_3 & a_2 b_3 & a_3 b_3 \end{bmatrix} - \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix} \\ &= ba^T - ab^T\end{aligned}$$

Similarly, we have

$$\left(r'_1 {r'}_2^T - r'_2 {r'}_1^T\right) = -(r'_1 \times r'_2) \times$$

$$M = \frac{I_1 + I_2}{2} - \frac{1}{2} \frac{[{r'}_1^T r'_2]}{1 - [{r'}_1^T r'_2]^2} (r'_1 \times r'_2) \times [I_1 - I_2]$$