Z transform

Sampling

sampling interval T

$$f_T(t) \equiv \sum_{k=0}^{\infty} f(kT)\delta(t-kT)$$

Laplace Transform

Let's' try to derive Z transform from Laplace transform

$$\begin{aligned}
F_T(s) &\equiv \int_{0_-}^{\infty} f_T(t)e^{-st}dt = \sum_{k=0}^{\infty} f(kT)\int_{0_-}^{\infty} \delta(t-kT)e^{-st}dt \\
&= \sum_{k=0}^{\infty} f(kT)[e^{-Ts}]^k \\
f_T^*(t) &\equiv \frac{1}{2\pi j}\int_{\beta-j\infty}^{\beta+j\infty} F_T(s)e^{st}ds = \sum_{k=0}^{\infty} f(kT)\frac{1}{2\pi j}\int_{\beta-j\infty}^{\beta+j\infty} [e^{(t-kT)}]^s ds \\
&= \sum_{k=0}^{\infty} e^{(t-kT)\beta} f(kT)\{\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-jkT\omega}e^{j\omega t}d\omega\} \\
&= \sum_{k=0}^{\infty} e^{(t-kT)\beta} f(kT)\delta(t-kT) \\
&= \sum_{k=0}^{\infty} f(kT)\delta(t-kT)
\end{aligned}$$

think about $\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-jkT\omega}e^{j\omega t}d\omega$, introduce the parameter a, then

$$
\begin{aligned}
\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-jkT\omega}e^{j\omega t}d\omega &= \lim_{a\to 0}\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{j(t-kT)\omega}e^{-a\omega^2}\,d\omega \\
&= \lim_{a\to 0} e^{-\frac{(t-kT)^2}{4a}}\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-a[\omega-j\frac{(t-kT)}{2a}]^2}\,d\omega \\
&= \lim_{a\to 0} e^{-\frac{(t-kT)^2}{4a}}\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-a\omega^2}\,d\omega \\
&= \lim_{a\to 0} e^{-\frac{(t-kT)^2}{4a}}\frac{1}{2\pi}\pi^{\frac{1}{2}}a^{-\frac{1}{2}} \\
&= \lim_{a\to 0}\frac{\pi^{-\frac{1}{2}}a^{-\frac{1}{2}}}{2}e^{-\frac{(t-kT)^2}{4a}} \qquad [\frac{\pi^{-\frac{1}{2}}a^{-\frac{1}{2}}}{2}\int_{-\infty}^{+\infty} e^{-\frac{(t-kT)^2}{4a}}\,dt = \frac{\pi^{-\frac{1}{2}}a^{-\frac{1}{2}}}{2}\pi^{\frac{1}{2}}2a^{\frac{1}{2}} = 1] \\
&= \delta(t-kT)
\end{aligned}
$$

To sum up, we can choose β wisely, to make sure the convergence of $F_T(s)=\sum_{k=0}^{\infty} f(kT)[e^{-Ts}]^k$, thus

$$
f_T(t) = f_T^*(t) = \sum_{k=0}^{\infty} f(kT)\delta(t-kT)
$$
$$
F_T(s) = \sum_{k=0}^{\infty} f(kT)[e^{-Ts}]^k
$$

Keep the signal

$\hat{f}_T(t) = f(kT)$ keep the signal for $t \in [kT,(k+1)T)$

then, we have

$$
\begin{aligned}
\hat{f}_T(t) &\equiv \sum_{k=0}^{\infty}[f(kT)-f((k-1)T)]u(t-kT)\\
\hat{F}_T(s) &\equiv \sum_{k=0}^{\infty}[f(kT)-f((k-1)T)]\int_{0_-}^{\infty} u(t-kT)e^{-st}dt\\
&= \sum_{k=0}^{\infty}[f(kT)-f((k-1)T)]\int_{kT}^{\infty} e^{-st}dt\\
&= \sum_{k=0}^{\infty}[f(kT)-f((k-1)T)]\frac{[e^{-Ts}]^k}{s}\\
&=\Big\{\sum_{k=0}^{\infty} f(kT)[e^{-Ts}]^k\Big\}\frac{[1-e^{-Ts}]}{s}\\
&= F_T(s)\frac{[1-e^{-Ts}]}{s}\\
\hat{f}^*_T(t) &\equiv \frac{1}{2\pi j}\int_{\beta-j\infty}^{\beta+j\infty} \hat{F}(s)e^{st}ds\\
&= \sum_{k=0}^{\infty}[f(kT)-f((k-1)T)]\frac{1}{2\pi j}\int_{\beta-j\infty}^{\beta+j\infty} \frac{[e^{(t-kT)}]^s}{s}ds\\
&= \sum_{k=0}^{\infty} e^{(t-kT)\beta}[f(kT)-f((k-1)T)]\{\frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{e^{-jkT\omega}e^{j\omega t}}{\beta+j\omega}d\omega\}\\
&= \sum_{k=0}^{\infty} e^{(t-kT)\beta}[f(kT)-f((k-1)T)]\{e^{-(t-kT)\beta}u(t-kT)\}\\
&= \sum_{k=0}^{\infty}[f(kT)-f((k-1)T)]u(t-kT)\\
[u(t)|_{t=0} &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{1}{\beta+j\omega}d\omega = \frac{1}{2\pi j}\lim_{A\to\infty}\ln(\frac{A-j\beta}{-A-j\beta}) = \frac{1}{2}]
\end{aligned}
$$

On the other hand, let's set $t \to 0$, then

$$
\begin{aligned}
f(t) &= \lim_{T\to 0}\hat{f}_T(t)\\
F(s) &= \lim_{T\to 0}\hat{F}_T(s) = \lim_{T\to 0}F_T(s)\lim_{T\to 0}\frac{[1-e^{-Ts}]}{s}\\
&= T\lim_{T\to 0}F_T(s)\\
&= T\lim_{T\to 0}\Big\{\sum_{k=0}^{\infty} f(kT)[e^{-Ts}]^k\Big\}\\
&\equiv \int_{0_-}^{\infty} f(t)e^{-st}dt \quad [T=dt, kT=t]
\end{aligned}
$$

Z transform and Fourier transform

Think about the flip side, the inverse transform of Z transform and Fourier transform

$$
\begin{aligned}
x(n) &= \frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz \quad [C \text{ contains poles of } X(z)] \\
X(z) &= \sum_{n=0}^{\infty} x(n)z^{-n}
\end{aligned}
$$

Here replace $z = e^{sT}$, we have

$$
\begin{aligned}
f(nT) = x(n) &= \frac{1}{2\pi j}\oint_C X(z)e^{s(n-1)T}de^{sT} \\
&= \frac{1}{2\pi j}\int_{\beta-\frac{\pi}{T}j}^{\beta+\frac{\pi}{T}j} TX(z)e^{snT}ds \\
X(z) &= \sum_{n=0}^{\infty} x(n)z^{-n} \\
&= \sum_{n=0}^{\infty} f(nT)e^{-snT}
\end{aligned}
$$

Think about the following function

$$
\begin{aligned}
f_z(t)|_{t=nT} &= f(nT) \\
F_z(s) &= TX(z)[u(\mathrm{Im}(s) + \frac{\pi}{T}) - u(\mathrm{Im}(s) - \frac{\pi}{T})] \\
&= [\sum_{n=0}^{\infty} f(nT)e^{-snT}] \cdot T[u(\mathrm{Im}(s) + \frac{\pi}{T}) - u(\mathrm{Im}(s) - \frac{\pi}{T})] \\
&= F_T(s) \cdot F_u(s)
\end{aligned}
$$

If we know $f_T(t) = \mathbf{L^{-1}}[F_T(s)], f_u(t) = \mathbf{L^{-1}}[F_u(s)]\backslash$
Then we have $f_z(t) = f_T(t) * f_u(t)$

$$
\begin{aligned}
f_T(t) &= \sum_{n=0}^{\infty} f(nT)\delta(t-nT) \\
f_u(t) &= \frac{1}{2\pi j}\int_{\beta-\infty}^{\beta+\infty} T[u(\mathrm{Im}(s)+\frac{\pi}{T})-u(\mathrm{Im}(s)-\frac{\pi}{T})]e^{st}ds \\
&= \frac{1}{2\pi}e^{\beta t}T\int_{-\frac{\pi}{T}}^{+\frac{\pi}{T}} e^{j\omega t}d\omega \\
&= e^{\beta t}\frac{\sin(\frac{\pi}{T}t)}{(\frac{\pi}{T}t)} \quad [t\in(-\infty,+\infty)] \\
f_z(t) &= f_T(t)*f_u(t) = \sum_{n=0}^{\infty} f(nT)e^{\beta(t-nT)}\frac{\sin(\frac{\pi}{T}(t-nT))}{(\frac{\pi}{T}(t-nT))} \quad [t\in(-\infty,+\infty)] \\
&= e^{\beta t}\sum_{n=0}^{\infty} f(nT)e^{-\beta nT}\frac{\sin(\frac{\pi}{T}(t-nT))}{(\frac{\pi}{T}(t-nT))} \quad [t\in(-\infty,+\infty)] \\
F_z(s) &= \mathbf{L}(f_z(t)) = \mathbf{F}(f_z(t)e^{-\beta t}) \\
&= \mathbf{F}[\sum_{n=0}^{\infty} f(nT)e^{-\beta nT}\frac{\sin(\frac{\pi}{T}(t-nT))}{(\frac{\pi}{T}(t-nT))}] \\
&= \sum_{n=0}^{\infty} f(nT)e^{-\beta nT}\mathbf{F}[\frac{\sin(\frac{\pi}{T}(t-nT))}{(\frac{\pi}{T}(t-nT))}] \\
&= [\sum_{n=0}^{\infty} f(nT)e^{-(\beta+j\omega)nT}]\int_{-\infty}^{+\infty}\frac{\sin(\frac{\pi}{T}(t-nT))}{(\frac{\pi}{T}(t-nT))}e^{-j\omega(t-nT)}d(t-nT) \\
&= [\sum_{n=0}^{\infty} f(nT)e^{-(\beta+j\omega)nT}]\int_{-\infty}^{+\infty}\frac{\sin(\frac{\pi}{T}x)}{(\frac{\pi}{T}x)}e^{-j\omega x}dx \\
&= [\sum_{n=0}^{\infty} f(nT)e^{-(\beta+j\omega)nT}]\int_{-\infty}^{+\infty}\frac{\sin(\frac{\pi}{T}x)}{(\frac{\pi}{T}x)}e^{j\omega x}dx \\
&= [\sum_{n=0}^{\infty} f(nT)e^{-(\beta+j\omega)nT}]\frac{T}{2\pi j}\int_{-\infty}^{+\infty}\frac{e^{j\frac{\pi}{T}x}-e^{-j\frac{\pi}{T}x}}{x}e^{j\omega x}dx \\
&= [\sum_{n=0}^{\infty} f(nT)e^{-(\beta+j\omega)nT}]\cdot T[u(\omega+\frac{\pi}{T})-u(\omega-\frac{\pi}{T})] \\
&= [\sum_{n=0}^{\infty} f(nT)e^{-snT}]\cdot T[u(\mathrm{Im}(s)+\frac{\pi}{T})-u(\mathrm{Im}(s)-\frac{\pi}{T})] \\
&[\int_{-\infty}^{+\infty}\frac{e^{j(\omega+a)x}-e^{j(\omega-a)x}}{x}dx = 2\pi j[u(\omega+a)-u(\omega-a)] \quad ,a=\frac{\pi}{T}]
\end{aligned}
$$

The reason is

$$
\begin{aligned}
\int_{-\infty}^{+\infty} \frac{e^{j(\omega+a)x} - e^{j(\omega-a)x}}{x} dx &= \lim_{\epsilon\to0+}[\int_{+\epsilon}^{+\infty} + \int_{-\epsilon}^{+\epsilon} + \int_{-\infty}^{-\epsilon}] \\
&= \lim_{\epsilon\to0+}[\int_{+\epsilon}^{+\infty} + \int_{-\infty}^{-\epsilon}] + \lim_{\epsilon\to0+}\int_{-\epsilon}^{+\epsilon}[j2a + o(1)]dx \\
&= \lim_{\epsilon\to0+}[\int_{+\epsilon}^{+\infty} + \int_{-\infty}^{-\epsilon}] + \lim_{\epsilon\to0+} j4a\epsilon \\
&= \lim_{\epsilon\to0+} j2\int_{+\epsilon}^{+\infty} \frac{\sin((\omega+a)x) - \sin((\omega+a)x)}{x} dx + 0 \\
&= j\pi[\mathrm{sgn}(\omega+a) - \mathrm{sgn}(\omega-a)] \\
&= 2\pi j[u(\omega+a) - u(\omega-a)]
\end{aligned}
$$

we will find poles for $F_z(s)$, here

$$
s_k = \beta_k + j\omega_k + jm\frac{2\pi}{T} \quad m \in Z
$$

In the other way

$$
\begin{aligned}
f_z^*(t) &= \frac{1}{2\pi j}\int_{\beta-j\infty}^{\beta+j\infty} F_z(s)e^{st}ds \\
&= \frac{1}{2\pi j}\int_{\beta-j\infty}^{\beta+j\infty} [\sum_{n=0}^{\infty} f(nT)e^{-snT}] \cdot T[u(\mathrm{Im}(s) + \frac{\pi}{T}) - u(\mathrm{Im}(s) - \frac{\pi}{T})]e^{st}ds \\
&= \sum_{n=0}^{\infty} f(nT)[\frac{T}{2\pi j}\int_{\beta-j\frac{\pi}{T}}^{\beta+j\frac{\pi}{T}} e^{s(t-nT)}ds] \\
&= \sum_{n=0}^{\infty} f(nT)[\frac{T}{2\pi j}e^{\beta(t-nT)}2j\frac{\sin(\frac{\pi}{T}(t-nT))}{(t-nT)}] \\
&= \sum_{n=0}^{\infty} f(nT)e^{\beta(t-nT)}\frac{\sin(\frac{\pi}{T}(t-nT))}{(\frac{\pi}{T}(t-nT))} \\
f_z^*(nT) &= f(nT)
\end{aligned}
$$

Z transform and Laplace transform

define $x(n), X(z)$, with $e^{sT} = z, \mathrm{Im}(s) \in (-\frac{\pi}{T}, +\frac{\pi}{T}), z \in C$, here we have

$$
\begin{aligned}
x(n) &\equiv f_z(t)|_{t=nT} = f(nT) \\
X(z) &\equiv \frac{F_z(s)}{T} = [\sum_{n=0}^{\infty} f(nT)e^{-snT}] = [\sum_{n=0}^{\infty} f(nT)z^{-n}]
\end{aligned}
$$

So, inverse Z transformation is derived from inverse Laplace transform

$$
\begin{aligned}
x(n) \equiv f_z(t)|_{t=nT} &= \mathbf{L^{-1}}[F_z(s)]|_{t=nT} \\
&= [\frac{1}{2\pi j}\int_{\beta-j\infty}^{\beta+j\infty} F_z(s)e^{st}ds]|_{t=nT} \\
&= [\frac{1}{2\pi j}\int_{\beta-j\frac{\pi}{T}}^{\beta+j\frac{\pi}{T}} F_z(s)e^{st}ds]|_{t=nT} \\
&= \frac{1}{2\pi j}\oint_{|z|=e^{\beta}} TX(z)z^n d\frac{\ln(z)}{T} \\
&= \frac{1}{2\pi j}\oint_{|z|=e^{\beta}} X(z)z^{n-1}dz
\end{aligned}
$$

Z transform and DTFT

when $n \in (\infty, +\infty), \beta = 0$

$$
\begin{aligned}
f_{DTFT}(t) &= \sum_{n=-\infty}^{\infty} f(nT)\frac{\sin(\frac{\pi}{T}(t-nT))}{(\frac{\pi}{T}(t-nT))} \\
F_{DTFT}(j\omega) &= F_z(s)|_{\mathrm{Re}(s)=0} \\
&= [\sum_{n=0}^{\infty} f(nT)e^{-snT}]\cdot T[u(\mathrm{Im}(s)+\frac{\pi}{T}) - u(\mathrm{Im}(s)-\frac{\pi}{T})]|_{\mathrm{Re}(s)=0} \\
&= [\sum_{n=0}^{\infty} f(nT)e^{-j\omega Tn}]\cdot T[u(\omega+\frac{\pi}{T}) - u(\omega-\frac{\pi}{T})] \\
&= [\sum_{n=0}^{\infty} f(nT)e^{-j\Omega n}]\cdot T[u(\Omega+\pi) - u(\Omega-\pi)] \quad [\Omega \equiv \omega T]
\end{aligned}
$$

Here we have $e^{sT} = e^{(\beta+j\omega)T} = e^{B}e^{j\Omega}$

$$
\begin{aligned}
x(n) &\equiv f_{DTFT}(t)|_{t=nT} = f(nT) \\
X(e^{j\Omega}) &\equiv \frac{F_{DTFT}(j\omega)}{T} \\
&= [\sum_{n=0}^{\infty} f(nT)e^{-j\omega nT}] = [\sum_{n=0}^{\infty} x(n)e^{-j\Omega n}] \\
x(n) &= [\frac{1}{2\pi j}\int_{-j\infty}^{+j\infty} F_{DTFT}(j\omega)e^{j\omega t}dj\omega]|_{t=nT} \\
&= \frac{1}{2\pi}\int_{-\pi/T}^{+\pi/T} F_{DTFT}(j\omega)e^{j\omega Tn}d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{+\pi} TX(e^{j\Omega})e^{j\Omega n}d\frac{\Omega}{T} \\
&= \frac{1}{2\pi}\int_{-\pi}^{+\pi} X(e^{j\Omega})e^{j\Omega n}d\Omega
\end{aligned}
$$