under-water model

The unit vector on the laser plane v_m

$$egin{aligned} v_m &= c_1 \pi + c_2 n \ \pi^T v_m &= \pi^T (c_1 \pi + c_2 n) = c_1 + c_2 (\pi^T n) = 0 \ v_m^T v_m &= (c_1^2 + c_2^2) + 2 c_1 c_2 (\pi^T n) = 1 \end{aligned}$$

Then we can have that

$$egin{aligned} c_1 &= -c_2(\pi^T n) \ [1 + (\pi^T n)^2] c_2^2 - 2(\pi^T n)^2 c_2^2 &= 1 \end{aligned}$$

Then we solve

$$egin{align} c_2 &= rac{-1}{\sqrt{1-(\pi^T n)^2}} \ c_1 &= rac{(\pi^T n)}{\sqrt{1-(\pi^T n)^2}} \ v_m &= \pi [rac{(\pi^T n)}{\sqrt{1-(\pi^T n)^2}}] + n [rac{-1}{\sqrt{1-(\pi^T n)^2}}] \ \end{array}$$

expression of r, r'

Consider replace symbol of refraction:

- replace v_1 with r to represent unit direction vector (FROM)
- replace v_2 with r' to represent unit direction vector (TO)

Here r_{θ} is the unit vector after rotating v_m around rotation axis π for angle θ :

$$egin{aligned} r_{ heta} &= R_{ heta} v_m \ &= [\cos heta I + \sin heta(\pi imes) + (1 - \cos heta)\pi\pi^T] v_m \ &= [\cos heta I + \sin heta(\pi imes) + (1 - \cos heta)\pi\pi^T] \Big(\pi [rac{(\pi^T n)}{\sqrt{1 - (\pi^T n)^2}}] + n [rac{-1}{\sqrt{1 - (\pi^T n)^2}}]\Big) \ &= \cos heta \Big(\pi [rac{(\pi^T n)}{\sqrt{1 - (\pi^T n)^2}}] + n [rac{-1}{\sqrt{1 - (\pi^T n)^2}}]\Big) \ &+ \sin heta(\pi imes n) [rac{-1}{\sqrt{1 - (\pi^T n)^2}}] \end{aligned}$$

After refraction, the formula of r'_{θ} is:

$$r_{ heta}' = n \left[(rac{n_1}{n_2}) [-n^T r_{ heta}] - \sqrt{1 - (rac{n_1}{n_2})^2 (1 - [r_{ heta}^T n]^2)}
ight] + r_{ heta} (rac{n_1}{n_2})$$

Here, the inner product, is a function of angle θ :

$$[-n^T r_{ heta}] = \cos heta \sqrt{1-(\pi^T n)^2}$$

Thus

$$egin{aligned} r_{ heta}' &= n \Bigg[-(rac{n_1}{n_2}) [\cos heta rac{(\pi^T n)^2}{\sqrt{1-(\pi^T n)^2}}] - \sqrt{1-(rac{n_1}{n_2})^2 \Big(1-\cos^2 heta [1-(\pi^T n)^2]\Big)} \Bigg] \ &+ \sin heta (\pi imes n) (rac{n_1}{n_2}) [rac{-1}{\sqrt{1-(\pi^T n)^2}}] \ &+ \cos heta \, \pi (rac{n_1}{n_2}) [rac{(\pi^T n)}{\sqrt{1-(\pi^T n)^2}}] \end{aligned}$$

find intersection I

set the interface plane

$$n^T I = h$$

Here it must on the ray, all start from laser point *P*:

$$I_{ heta} = P + k_{ heta} r_{ heta}$$

Solve k_{θ} , I_{θ} respectively

$$n^T P - k_ heta \cos heta \sqrt{1 - (\pi^T n)^2} = h$$
 $k_ heta = rac{n^T P - h}{\cos heta \sqrt{1 - (\pi^T n)^2}} \ I_ heta = P + [n^T P - h] igg[\pi [rac{(\pi^T n)}{1 - (\pi^T n)^2}] + n [rac{-1}{1 - (\pi^T n)^2}] igg] + [n^T P - h] an heta(\pi imes n) [rac{-1}{1 - (\pi^T n)^2}]$