Design Via State Space

Symbols

$\dot{X}(t) = AX(t) + BU(t)$
$Y(t) = CX(t) + DU(t)$

$\dot{X}(t)$ derivative of state vector

$X(t)$ state vector | nx1

$Y(t)$ output vector | px1

$U(t)$ input/control vector | mx1

A system matrix | nxn

B input matrix | nxm

C output matrix | pxn

D feedforward matrix | pxm

set $X(t)|_{t=0} = 0$, do Laplace Transform
$sX(s) = AX(s) + BU(s)$
$Y(s) = CX(s) + DU(s)$

so,
$X(s) = (sI - A)^{-1}BU(s) = \frac{\mathrm{adj}(sI-A)B}{\det(sI-A)}U(s)$
$Y(s) = [\frac{Cadj(sI-A)B}{det(sI-A)} + D]U(s)$

thus, transfer function $G(s)$
$G(s) \equiv \frac{Y(s)}{U(s)} = \frac{Cadj(sI-A)B}{det(sI-A)} + D$

set $X(t)|_{t=0} = X(0)$

$$sX(s) - X(0) = AX(s) + BU(s)$$
$$Y(s) = CX(s) + DU(s)$$

then

$$Y(s) = C(sI - A)^{-1}X(0) + C(sI - A)^{-1}BU(s) + DU(s)$$

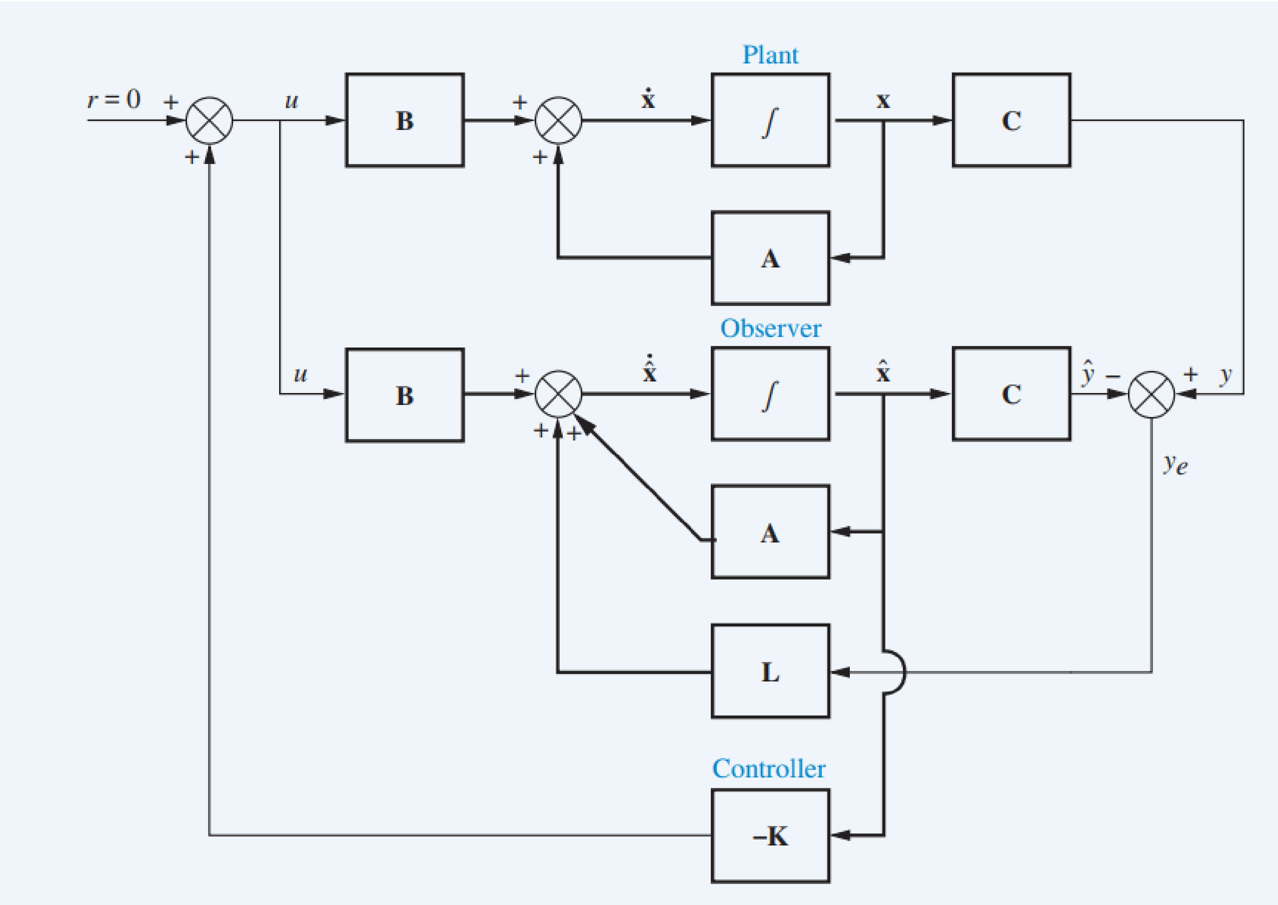
Here set $\Phi(t) \equiv L[(sI - A)^{-1}] = e^{At} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}$

In another way

$$(sI - A)^{-1} = \begin{cases} -A^{-1}(I - sA^{-1})^{-1} = -A^{-1}[\sum_{k=0}^{\infty} s^k A^{-k}] & |s| < \lambda_{min} \\ s^{-1}(I - \frac{A}{s})^{-1} = \frac{1}{s}[\sum_{k=0}^{\infty} s^{-k} A^k] & |s| > \lambda_{max} \end{cases}$$

Here $\frac{1}{s^{k+1}} = L^{-1}[\frac{t^k}{k!}]$

So we can get $(sI - A)^{-1}|_{s=0} = -A^{-1}, \lim_{s\to\infty} s(sI - A)^{-1} = I$



Controllability

The basic equation set: (when D always = 0)

$$\dot{X} = AX + BU$$
$$Y = CX + D$$

Introduce the Controller K always 1xN, \
where U always 1x1

$$U = r - KX$$

So, we obtain

$$\dot{X} = AX + B(r - KU) = (A - BK)X + Br$$

If we could manipulate the poles of $|sI - (A - BK)|$
That means

$$Y = C\mathbb{L}^{-1}[(sI - (A - BK))^{-1}] * \mathbb{L}^{-1}[BR(s)]$$

Transformation

Here
$Z = PX$
We have

$$\dot{Z} = AZ + BU$$
$$Y = CZ$$
$$U = r - KZ$$

That means we have

$$\dot{X} = P^{-1}APX + P^{-1}BU$$
$$Y = CPX$$
$$U = r - KPX$$

Compare with

$$\dot{X} = A_x X + B_x U$$
$$Y = C_x X$$
$$U = r - K_x X$$

So we have

$$
\begin{aligned}
A_x &= P^{-1}AP \\
B_x &= P^{-1}B \\
C_x &= CP \\
K_x &= KP
\end{aligned}
$$

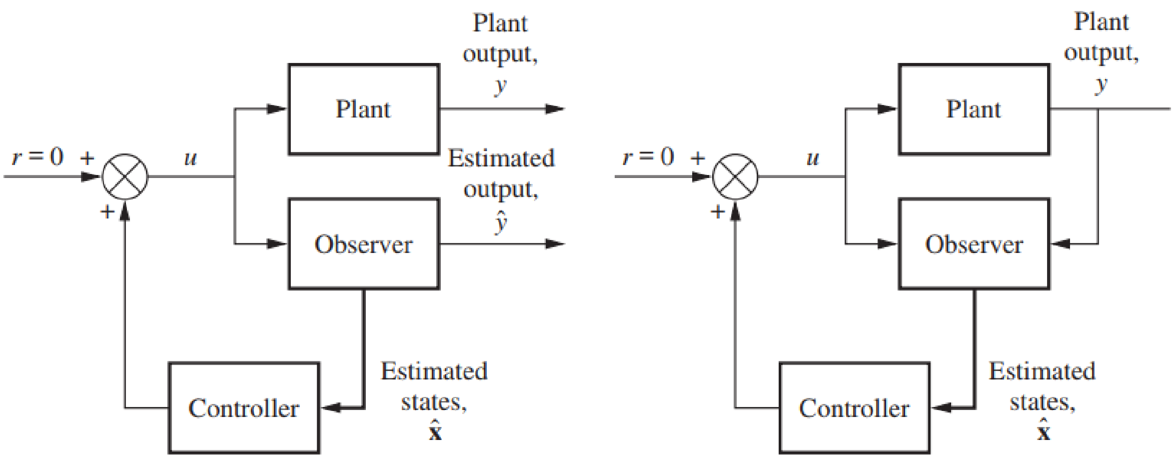
Then

$$C_{Mx} = [B_x \ A_x B_x \cdots A_x^{N-1} B_x] = P^{-1} C_{Mz}$$

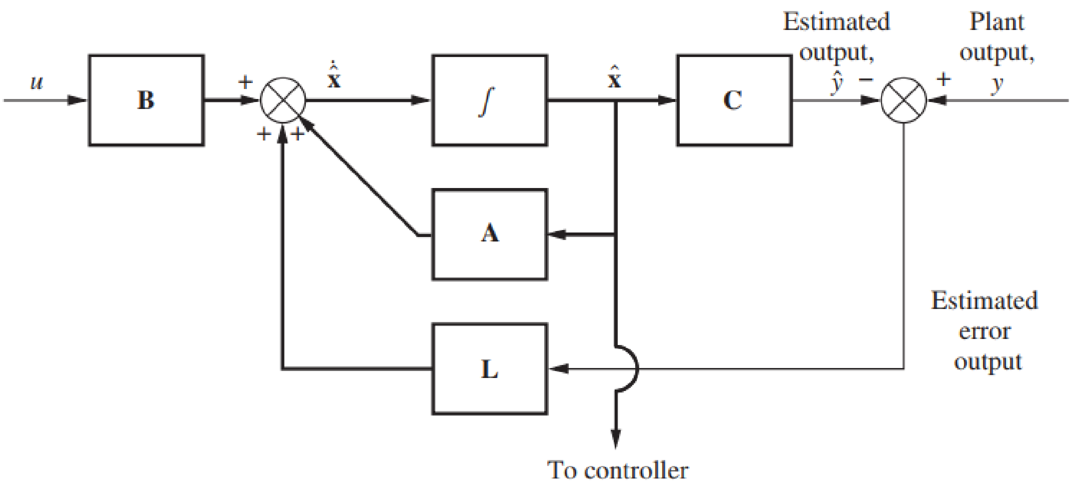
where X is observer canonical form\
Z is another form (like phase variable form, cascade form)

$$P = C_{Mz} C_{Mx}^{-1} \ K_z = K_x P^{-1}$$

Observability



(a) (b)



(c)

$$
\begin{aligned}
\dot{\hat{X}} &= A\hat{X} + BU + L(Y - \hat{Y}) \\
\hat{Y} &= C\hat{X}
\end{aligned}
$$

so with

$$\begin{aligned}\dot{X} &= AX + BU \\ Y &= CX\end{aligned}$$

then obtain

$$\begin{aligned}\dot{X} - \dot{\hat{X}} &= A(X - \hat{X}) - LC(X - \hat{X}) \\ &= (A - LC)(X - \hat{X})\end{aligned}$$

define $e_X \equiv (X - \hat{X})$, we have

$$\dot{e}_X = (A - LC)e_X$$

If all poles of (A-LC) in the left plane

$$\lim_{t \to \infty} e_X = (X - \hat{X}) = 0$$

Then we could use $\hat{X}$ to estimate X\
regardless the influence of initial value $\hat{X}(0)$ and $X(0)$

Transformation

$Z = PX$
where X is observer canonical form\
Z is another form (like phase variable form, cascade form)

$$(\dot{Z} - \dot{\hat{Z}}) = (A - LC)(Z - \hat{Z})$$

Then we have

$$(\dot{X} - \dot{\hat{X}}) = P^{-1}(A - LC)P(X - \hat{X})$$

So we have:

$$\begin{aligned}A_x &= P^{-1}AP \\ B_x &= P^{-1}B \\ C_x &= CP \\ L_x &= P^{-1}L\end{aligned}$$

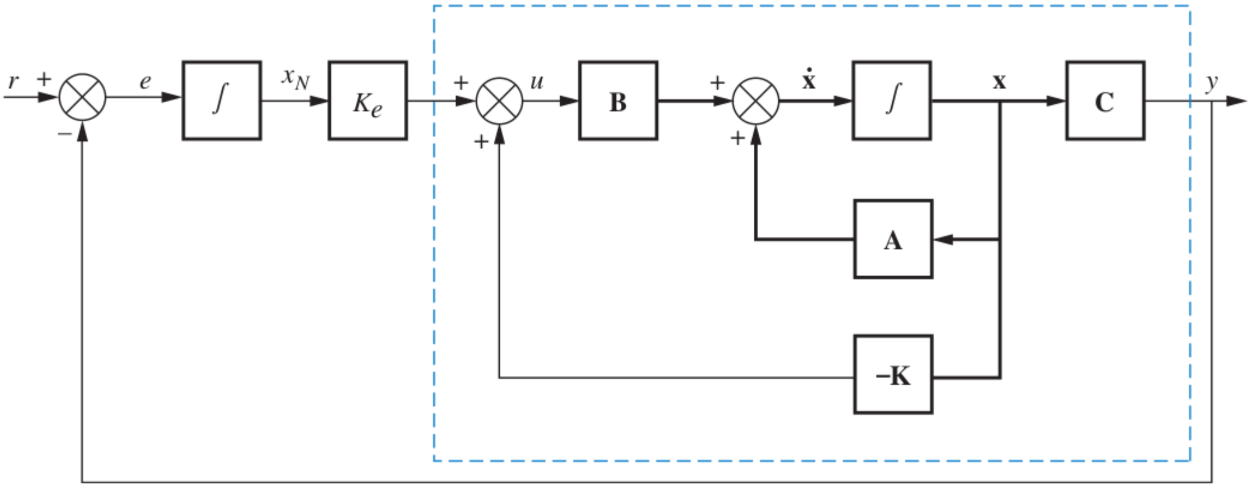
Now calculate O_{Mx}

$$O_{Mx} = \begin{bmatrix} C_x \\ C_x A_x \\ \vdots \\ C_x A_x^{N-1} \end{bmatrix} = O_{Mz} P$$

So, in conclusion:

$$P = O_{Mz}^{-1} O_{Mx} \ L_z = P L_x$$

Integral Control with 0 Steady-State Error



$$U = V - KX$$
$$\frac{(R-Y)}{s} K_e = X_N K_e = V \equiv \frac{Y}{T(s)}$$

So

$$\frac{Y}{R} = \frac{K_e \frac{T(s)}{s}}{1 + K_e \frac{T(s)}{s}}$$

Then

$$\begin{aligned} e_{ss} &= \lim_{s \to 0+} sR(s)(1 - \frac{Y(s)}{R(s)}) \\ &= \lim_{s \to 0+} \frac{1}{1 + K_e \frac{T(s)}{s}} \\ &= \lim_{s \to 0+} \frac{s}{s + K_e T(s)} = 0 \end{aligned}$$

Because

$$\dot{x}_N = R - Y = R - CX = [-C\ 0]\begin{bmatrix} X \\ x_N \end{bmatrix} + R$$

Thus

$$\begin{bmatrix} \dot{X} \\ \dot{x}_N \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix}\begin{bmatrix} X \\ x_N \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} U + \begin{bmatrix} 0 \\ 1 \end{bmatrix} R$$

Because

$$U = K_e x_N - KX = [-K\ K_e]\begin{bmatrix} X \\ x_N \end{bmatrix}$$

We have

$$\begin{aligned} \begin{bmatrix} \dot{X} \\ \dot{x}_N \end{bmatrix} &= (\begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix}[-K\ K_e])\begin{bmatrix} X \\ x_N \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} R \\ &= \begin{bmatrix} A - BK & BK_e \\ -C & 0 \end{bmatrix}\begin{bmatrix} X \\ x_N \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} R \end{aligned}$$

Why zeros of T(s) don't change

We know

$$\begin{aligned} G(s) &= \frac{C\text{adj}(sI - A)B}{|sI - A|} \\ T(s) &= \frac{C\text{adj}(sI - A + BK)B}{|sI - A + BK|} \end{aligned}$$

Why the numerator of G(S), T(s) is the same, because $\forall C$, so must prove

$$\text{adj}(sI - A)B = \text{adj}(sI - A + BK)B$$

that is mean $\forall A$ (replace $sI - A$ with A)

$$\text{adj}(A)B = \text{adj}(A + BK)B$$

lemma: Cramer's Rule

for $AX = B$, where

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_N \end{bmatrix}$$

We have

$$X = A^{-1}B = \frac{\mathrm{adj}(A)B}{|A|} = \frac{|A \overset{i}{\leftarrow} B|\text{for x\_i}}{|A|}$$
$$x_i|A| = \mathrm{adj}(A)B \quad \text{ith element} = |A \overset{i}{\leftarrow} B|$$

Here

$$A = (\mathbf{a}_1 \cdots \mathbf{a}_n)$$
$$\left(A^i \leftarrow B\right) \overset{\text{def}}{=} \left(\mathbf{a}_1 \quad \cdots \quad \mathbf{a}_{i-1} \quad B \quad \mathbf{a}_{i+1} \quad \cdots \quad \mathbf{a}_n\right)$$

proof

$$\mathrm{adj}(A+BK)B \quad \text{ith element} = |(A+BK) \overset{i}{\leftarrow} B|$$
$$= |\,\mathbf{a}_1 + k_1B \quad \cdots \quad \mathbf{a}_{i-1} + k_{i-1}B \quad B \quad \cdots \quad \mathbf{a}_N + k_NB|$$
$$= |\,\mathbf{a}_1 \quad \cdots \quad \mathbf{a}_{i-1} \quad B \quad \cdots \quad \mathbf{a}_N|$$
$$= |A \overset{i}{\leftarrow} B| = \mathrm{adj}(A)B \quad \text{ith element}$$

Another lemma $|A+BK| = |A| + K\mathrm{adj}(A)B$

$$|A+BK| = |\,\mathbf{a}_1 + k_1B \quad \cdots \quad \mathbf{a}_i + k_iB \quad \cdots \quad \mathbf{a}_N + k_NB|$$
$$= |\,\mathbf{a}_1 \quad \cdots \quad \mathbf{a}_{i-1} \quad \cdots \quad \mathbf{a}_N|$$
$$+ \sum_i k_i\,|\,\mathbf{a}_1 \quad \cdots \quad \mathbf{a}_{i-1} \quad B \quad \cdots \quad \mathbf{a}_N|$$
$$= |A| + \sum_i k_i\;[\mathrm{adj}(A)B \quad \text{ith element}]$$
$$= |A| + K\mathrm{adj}(A)B$$

Conclusion

$$
\begin{aligned}
G(s) &= \frac{C\text{adj}(sI-A)B}{|sI-A|} = \frac{N(s)}{D_1(s)} \\
T(s) &= \frac{C\text{adj}(sI-A+BK)B}{|sI-A+BK|} \\
&= \frac{C\text{adj}(sI-A)B}{|sI-A| + K\text{adj}(sI-A)B} = \frac{N(s)}{D_2(s)}
\end{aligned}
$$

If we introduce K_e

$$
\begin{aligned}
\frac{Y(s)}{R(s)} \equiv T'(s) &= \frac{K_e \frac{T(s)}{s}}{1 + K_e \frac{T(s)}{s}} \\
&= \frac{K_e N(s)}{sD_2(s) + K_e N(s)}
\end{aligned}
$$

So, no matter introduce K or K_e, zeros of $T(s)$ don't change