

# Z transform

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## Sampling

sampling interval T

$$f_T(t) \equiv \sum_{k=0}^{\infty} f(kT)\delta(t - kT)$$

## Laplace Transform

Let's try to derive Z transform from Laplace transform

$$\begin{aligned} F_T(s) &\equiv \int_{0-}^{\infty} f_T(t)e^{-st} dt = \sum_{k=0}^{\infty} f(kT) \int_{0-}^{\infty} \delta(t - kT)e^{-st} dt \\ &= \sum_{k=0}^{\infty} f(kT)[e^{-Ts}]^k \\ f_T^*(t) &\equiv \frac{1}{2\pi j} \int_{\beta-j\infty}^{\beta+j\infty} F_T(s)e^{st} ds = \sum_{k=0}^{\infty} f(kT) \frac{1}{2\pi j} \int_{\beta-j\infty}^{\beta+j\infty} [e^{(t-kT)s}]^s ds \\ &= \sum_{k=0}^{\infty} e^{(t-kT)\beta} f(kT) \left\{ \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-jkT\omega} e^{j\omega t} d\omega \right\} \\ &= \sum_{k=0}^{\infty} e^{(t-kT)\beta} f(kT)\delta(t - kT) \\ &= \sum_{k=0}^{\infty} f(kT)\delta(t - kT) \end{aligned}$$

think about  $\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-jkT\omega} e^{j\omega t} d\omega$ , introduce the parameter  $a$ , then

$$\begin{aligned} \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-jkT\omega} e^{j\omega t} d\omega &= \lim_{a \rightarrow 0} \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{j(t-kT)\omega} e^{-a\omega^2} d\omega \\ &= \lim_{a \rightarrow 0} e^{-\frac{(t-kT)^2}{4a}} \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-a[\omega - j\frac{(t-kT)}{2a}]^2} d\omega \\ &= \lim_{a \rightarrow 0} e^{-\frac{(t-kT)^2}{4a}} \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-a\omega^2} d\omega \\ &= \lim_{a \rightarrow 0} e^{-\frac{(t-kT)^2}{4a}} \frac{1}{2\pi} \pi^{\frac{1}{2}} a^{-\frac{1}{2}} \\ &= \lim_{a \rightarrow 0} \frac{\pi^{-\frac{1}{2}} a^{-\frac{1}{2}}}{2} e^{-\frac{(t-kT)^2}{4a}} \left[ \frac{\pi^{-\frac{1}{2}} a^{-\frac{1}{2}}}{2} \int_{-\infty}^{+\infty} e^{-\frac{(t-kT)^2}{4a}} dt = \frac{\pi^{-\frac{1}{2}} a^{-\frac{1}{2}}}{2} \pi^{\frac{1}{2}} 2a^{\frac{1}{2}} = 1 \right] \\ &= \delta(t - kT) \end{aligned}$$

To sum up, we can choose  $\beta$  wisely, to make sure the convergence of  $F_T(s) = \sum_{k=0}^{\infty} f(kT)[e^{-Ts}]^k$ , thus

$$\begin{aligned} f_T(t) &= f_T^*(t) = \sum_{k=0}^{\infty} f(kT)\delta(t - kT) \\ F_T(s) &= \sum_{k=0}^{\infty} f(kT)[e^{-Ts}]^k \end{aligned}$$

## Keep the signal

$\hat{f}_T(t) = f(kT)$  keep the signal for  $t \in [kT, (k+1)T)$

then, we have

$$\begin{aligned}
\hat{f}_T(t) &\equiv \sum_{k=0}^{\infty} [f(kT) - f((k-1)T)] u(t - kT) \\
\hat{F}_T(s) &\equiv \sum_{k=0}^{\infty} [f(kT) - f((k-1)T)] \int_{0-}^{\infty} u(t - kT) e^{-st} dt \\
&= \sum_{k=0}^{\infty} [f(kT) - f((k-1)T)] \int_{kT}^{\infty} e^{-st} dt \\
&= \sum_{k=0}^{\infty} [f(kT) - f((k-1)T)] \frac{[e^{-Ts}]^k}{s} \\
&= \left\{ \sum_{k=0}^{\infty} f(kT) [e^{-Ts}]^k \right\} \frac{[1 - e^{-Ts}]}{s} \\
&= F_T(s) \frac{[1 - e^{-Ts}]}{s} \\
\hat{f}_T^*(t) &\equiv \frac{1}{2\pi j} \int_{\beta-j\infty}^{\beta+j\infty} \hat{F}(s) e^{st} ds \\
&= \sum_{k=0}^{\infty} [f(kT) - f((k-1)T)] \frac{1}{2\pi j} \int_{\beta-j\infty}^{\beta+j\infty} \frac{[e^{(t-kT)}]^s}{s} ds \\
&= \sum_{k=0}^{\infty} e^{(t-kT)\beta} [f(kT) - f((k-1)T)] \left\{ \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-jkT\omega} e^{j\omega t}}{\beta + j\omega} d\omega \right\} \\
&= \sum_{k=0}^{\infty} e^{(t-kT)\beta} [f(kT) - f((k-1)T)] \{ e^{-(t-kT)\beta} u(t - kT) \} \\
&= \sum_{k=0}^{\infty} [f(kT) - f((k-1)T)] u(t - kT) \\
[u(t)]_{t=0} &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{1}{\beta + j\omega} d\omega = \frac{1}{2\pi j} \lim_{A \rightarrow \infty} \ln \left( \frac{A - j\beta}{-A - j\beta} \right) = \frac{1}{2}
\end{aligned}$$

On the other hand, let's set  $t \rightarrow 0$ , then

$$\begin{aligned}
f(t) &= \lim_{T \rightarrow 0} \hat{f}_T(t) \\
F(s) &= \lim_{T \rightarrow 0} \hat{F}_T(s) = \lim_{T \rightarrow 0} F_T(s) \lim_{T \rightarrow 0} \frac{[1 - e^{-Ts}]}{s} \\
&= T \lim_{T \rightarrow 0} F_T(s) \\
&= T \lim_{T \rightarrow 0} \left\{ \sum_{k=0}^{\infty} f(kT) [e^{-Ts}]^k \right\} \\
&\equiv \int_{0-}^{\infty} f(t) e^{-st} dt \quad [T = dt, kT = t]
\end{aligned}$$

## Z transform and Fourier transform

Think about the flip side, the inverse transform of Z transform and Fourier transform

$$\begin{aligned}
x(n) &= \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz \quad [C \text{ contains poles of } X(z)] \\
X(z) &= \sum_{n=0}^{\infty} x(n) z^{-n}
\end{aligned}$$

Here replace  $z = e^{sT}$ , we have

$$\begin{aligned}
f(nT) &= x(n) = \frac{1}{2\pi j} \oint_C X(z) e^{s(n-1)T} de^{sT} \\
&= \frac{1}{2\pi j} \int_{\beta - \frac{\pi}{T}j}^{\beta + \frac{\pi}{T}j} T X(z) e^{snT} ds \\
X(z) &= \sum_{n=0}^{\infty} x(n) z^{-n} \\
&= \sum_{n=0}^{\infty} f(nT) e^{-snT}
\end{aligned}$$

Think about the following function

$$\begin{aligned}
f_z(t)|_{t=nT} &= f(nT) \\
F_z(s) &= TX(z)[u(\text{Im}(s) + \frac{\pi}{T}) - u(\text{Im}(s) - \frac{\pi}{T})] \\
&= [\sum_{n=0}^{\infty} f(nT)e^{-snT}] \cdot T[u(\text{Im}(s) + \frac{\pi}{T}) - u(\text{Im}(s) - \frac{\pi}{T})] \\
&= F_T(s) \cdot F_u(s)
\end{aligned}$$

If we know  $f_T(t) = \mathbf{L}^{-1}[F_T(s)]$ ,  $f_u(t) = \mathbf{L}^{-1}[F_u(s)]$   
Then we have  $f_z(t) = f_T(t) * f_u(t)$

$$\begin{aligned}
f_T(t) &= \sum_{n=0}^{\infty} f(nT)\delta(t - nT) \\
f_u(t) &= \frac{1}{2\pi j} \int_{\beta - \infty}^{\beta + \infty} T[u(\text{Im}(s) + \frac{\pi}{T}) - u(\text{Im}(s) - \frac{\pi}{T})]e^{st} ds \\
&= \frac{1}{2\pi} e^{\beta t} T \int_{-\frac{\pi}{T}}^{+\frac{\pi}{T}} e^{j\omega t} d\omega \\
&= e^{\beta t} \frac{\sin(\frac{\pi}{T}t)}{(\frac{\pi}{T}t)} \quad [t \in (-\infty, +\infty)] \\
f_z(t) &= f_T(t) * f_u(t) = \sum_{n=0}^{\infty} f(nT)e^{\beta(t-nT)} \frac{\sin(\frac{\pi}{T}(t-nT))}{(\frac{\pi}{T}(t-nT))} \quad [t \in (-\infty, +\infty)] \\
&= e^{\beta t} \sum_{n=0}^{\infty} f(nT)e^{-\beta nT} \frac{\sin(\frac{\pi}{T}(t-nT))}{(\frac{\pi}{T}(t-nT))} \quad [t \in (-\infty, +\infty)]
\end{aligned}$$

$$\begin{aligned}
F_z(s) &= \mathbf{L}(f_z(t)) = \mathbf{F}(f_z(t)e^{-\beta t}) \\
&= \mathbf{F}[\sum_{n=0}^{\infty} f(nT)e^{-\beta nT} \frac{\sin(\frac{\pi}{T}(t-nT))}{(\frac{\pi}{T}(t-nT))}] \\
&= \sum_{n=0}^{\infty} f(nT)e^{-\beta nT} \mathbf{F}[\frac{\sin(\frac{\pi}{T}(t-nT))}{(\frac{\pi}{T}(t-nT))}] \\
&= [\sum_{n=0}^{\infty} f(nT)e^{-(\beta+j\omega)nT}] \int_{-\infty}^{+\infty} \frac{\sin(\frac{\pi}{T}(t-nT))}{(\frac{\pi}{T}(t-nT))} e^{-j\omega(t-nT)} d(t-nT) \\
&= [\sum_{n=0}^{\infty} f(nT)e^{-(\beta+j\omega)nT}] \int_{-\infty}^{+\infty} \frac{\sin(\frac{\pi}{T}x)}{(\frac{\pi}{T}x)} e^{-j\omega x} dx \\
&= [\sum_{n=0}^{\infty} f(nT)e^{-(\beta+j\omega)nT}] \int_{-\infty}^{+\infty} \frac{\sin(\frac{\pi}{T}x)}{(\frac{\pi}{T}x)} e^{j\omega x} dx \\
&= [\sum_{n=0}^{\infty} f(nT)e^{-(\beta+j\omega)nT}] \frac{T}{2\pi j} \int_{-\infty}^{+\infty} \frac{e^{j\frac{\pi}{T}x} - e^{-j\frac{\pi}{T}x}}{x} e^{j\omega x} dx \\
&= [\sum_{n=0}^{\infty} f(nT)e^{-(\beta+j\omega)nT}] \cdot T[u(\omega + \frac{\pi}{T}) - u(\omega - \frac{\pi}{T})] \\
&= [\sum_{n=0}^{\infty} f(nT)e^{-snT}] \cdot T[u(\text{Im}(s) + \frac{\pi}{T}) - u(\text{Im}(s) - \frac{\pi}{T})]
\end{aligned}$$

[  $\int_{-\infty}^{+\infty} \frac{e^{j(\omega+a)x} - e^{j(\omega-a)x}}{x} dx = 2$  ]

The reason is

$$\begin{aligned}
\int_{-\infty}^{+\infty} \frac{e^{j(\omega+a)x} - e^{j(\omega-a)x}}{x} dx &= \lim_{\epsilon \rightarrow 0+} [\int_{+\epsilon}^{+\infty} + \int_{-\infty}^{-\epsilon} + \int_{-\infty}^{-\epsilon}] \\
&= \lim_{\epsilon \rightarrow 0+} [\int_{+\epsilon}^{+\infty} + \int_{-\infty}^{-\epsilon}] + \lim_{\epsilon \rightarrow 0+} \int_{-\epsilon}^{+\epsilon} [j2a + o(1)] dx \\
&= \lim_{\epsilon \rightarrow 0+} [\int_{+\epsilon}^{+\infty} + \int_{-\infty}^{-\epsilon}] + \lim_{\epsilon \rightarrow 0+} j4a\epsilon \\
&= \lim_{\epsilon \rightarrow 0+} j2 \int_{+\epsilon}^{+\infty} \frac{\sin((\omega+a)x) - \sin((\omega-a)x)}{x} dx + 0 \\
&= j\pi[\text{sgn}(\omega+a) - \text{sgn}(\omega-a)] \\
&= 2\pi j[u(\omega+a) - u(\omega-a)]
\end{aligned}$$

we will find poles for  $F_z(s)$ , here

$$s_k = \beta_k + j\omega_k + jm \frac{2\pi}{T} \quad m \in Z$$

In the other way

$$\begin{aligned}
f_z^*(t) &= \frac{1}{2\pi j} \int_{\beta-j\infty}^{\beta+j\infty} F_z(s) e^{st} ds \\
&= \frac{1}{2\pi j} \int_{\beta-j\infty}^{\beta+j\infty} \left[ \sum_{n=0}^{\infty} f(nT) e^{-snT} \right] \cdot T \left[ u(\text{Im}(s) + \frac{\pi}{T}) - u(\text{Im}(s) - \frac{\pi}{T}) \right] e^{st} ds \\
&= \sum_{n=0}^{\infty} f(nT) \left[ \frac{T}{2\pi j} \int_{\beta-j\frac{\pi}{T}}^{\beta+j\frac{\pi}{T}} e^{s(t-nT)} ds \right] \\
&= \sum_{n=0}^{\infty} f(nT) \left[ \frac{T}{2\pi j} e^{\beta(t-nT)} 2j \frac{\sin(\frac{\pi}{T}(t-nT))}{(t-nT)} \right] \\
&= \sum_{n=0}^{\infty} f(nT) e^{\beta(t-nT)} \frac{\sin(\frac{\pi}{T}(t-nT))}{(\frac{\pi}{T}(t-nT))} \\
f_z^*(nT) &= f(nT)
\end{aligned}$$

## Z transform and Laplace transform

define  $x(n)$ ,  $X(z)$ , with  $e^{sT} = z$ ,  $\text{Im}(s) \in (-\frac{\pi}{T}, +\frac{\pi}{T})$ ,  $z \in C$ , here we have

$$\begin{aligned}
x(n) &\equiv f_z(t)|_{t=nT} = f(nT) \\
X(z) &\equiv \frac{F_z(s)}{T} = \left[ \sum_{n=0}^{\infty} f(nT) e^{-snT} \right] = \left[ \sum_{n=0}^{\infty} f(nT) z^{-n} \right]
\end{aligned}$$

So, inverse Z transformation is derived from inverse Laplace transform

$$\begin{aligned}
x(n) &\equiv f_z(t)|_{t=nT} = \mathbf{L}^{-1}[F_z(s)]|_{t=nT} \\
&= \left[ \frac{1}{2\pi j} \int_{\beta-j\infty}^{\beta+j\infty} F_z(s) e^{st} ds \right]_{t=nT} \\
&= \left[ \frac{1}{2\pi j} \int_{\beta-j\frac{\pi}{T}}^{\beta+j\frac{\pi}{T}} F_z(s) e^{st} ds \right]_{t=nT} \\
&= \frac{1}{2\pi j} \oint_{|z|=e^{\beta}} TX(z) z^n d\frac{\ln(z)}{T} \\
&= \frac{1}{2\pi j} \oint_{|z|=e^{\beta}} X(z) z^{n-1} dz
\end{aligned}$$

## Z transform and DTFT

when  $n \in (-\infty, +\infty)$ ,  $\beta = 0$

$$\begin{aligned}
f_{DTFT}(t) &= \sum_{n=-\infty}^{\infty} f(nT) \frac{\sin(\frac{\pi}{T}(t-nT))}{(\frac{\pi}{T}(t-nT))} \\
F_{DTFT}(j\omega) &= F_z(s)|_{\text{Re}(s)=0} \\
&= \left[ \sum_{n=0}^{\infty} f(nT) e^{-snT} \right] \cdot T \left[ u(\text{Im}(s) + \frac{\pi}{T}) - u(\text{Im}(s) - \frac{\pi}{T}) \right] |_{\text{Re}(s)=0} \\
&= \left[ \sum_{n=0}^{\infty} f(nT) e^{-j\omega T n} \right] \cdot T \left[ u(\omega + \frac{\pi}{T}) - u(\omega - \frac{\pi}{T}) \right] \\
&= \left[ \sum_{n=0}^{\infty} f(nT) e^{-j\Omega n} \right] \cdot T \left[ u(\Omega + \pi) - u(\Omega - \pi) \right] \quad [\Omega \equiv \omega T]
\end{aligned}$$

Here we have  $e^{sT} = e^{(\beta+j\omega)T} = e^B e^{j\Omega}$

$$\begin{aligned}
x(n) &\equiv f_{DTFT}(t)|_{t=nT} = f(nT) \\
X(e^{j\Omega}) &\equiv \frac{F_{DTFT}(j\omega)}{T} \\
&= [\sum_{n=0}^{\infty} f(nT)e^{-j\omega nT}] = [\sum_{n=0}^{\infty} x(n)e^{-j\Omega n}] \\
x(n) &= [\frac{1}{2\pi j} \int_{-j\infty}^{+j\infty} F_{DTFT}(j\omega)e^{j\omega t} d j\omega]|_{t=nT} \\
&= \frac{1}{2\pi} \int_{-\pi/T}^{+\pi/T} F_{DTFT}(j\omega)e^{j\omega Tn} d\omega \\
&= \frac{1}{2\pi} \int_{-\pi}^{+\pi} T X(e^{j\Omega})e^{j\Omega n} d\frac{\Omega}{T} \\
&= \frac{1}{2\pi} \int_{-\pi}^{+\pi} X(e^{j\Omega})e^{j\Omega n} d\Omega
\end{aligned}$$