under-water model

The unit vector on the laser plane v_m

$$
\begin{aligned}
v_m &= c_1\pi + c_2 n \\
\pi^T v_m &= \pi^T(c_1\pi + c_2 n) = c_1 + c_2(\pi^T n) = 0 \\
v_m^T v_m &= (c_1^2 + c_2^2) + 2c_1c_2(\pi^T n) = 1
\end{aligned}
$$

Then we can have that

$$
\begin{aligned}
c_1 &= -c_2(\pi^T n) \\
[1 + (\pi^T n)^2]c_2^2 - 2(\pi^T n)^2 c_2^2 &= 1
\end{aligned}
$$

Then we solve

$$
\begin{aligned}
c_2 &= \frac{-1}{\sqrt{1-(\pi^T n)^2}} \\
c_1 &= \frac{(\pi^T n)}{\sqrt{1-(\pi^T n)^2}} \\
v_m &= \pi[\frac{(\pi^T n)}{\sqrt{1-(\pi^T n)^2}}] + n[\frac{-1}{\sqrt{1-(\pi^T n)^2}}]
\end{aligned}
$$

expression of r, r'

Consider replace symbol of refraction:

- replace v_1 with r to represent unit direction vector (FROM)
- replace v_2 with r' to represent unit direction vector (TO)

Here r_θ is the unit vector
after rotating v_m around rotaion axis π for angle θ:

$$
\begin{aligned}
r_\theta &= R_\theta v_m \\
&= [\cos\theta I + \sin\theta(\pi\times) + (1-\cos\theta)\pi\pi^T]v_m \\
&= [\cos\theta I + \sin\theta(\pi\times) + (1-\cos\theta)\pi\pi^T]\Big(\pi[\frac{(\pi^T n)}{\sqrt{1-(\pi^T n)^2}}] + n[\frac{-1}{\sqrt{1-(\pi^T n)^2}}]\Big) \\
&= \cos\theta\Big(\pi[\frac{(\pi^T n)}{\sqrt{1-(\pi^T n)^2}}] + n[\frac{-1}{\sqrt{1-(\pi^T n)^2}}]\Big) \\
&+ \sin\theta(\pi\times n)[\frac{-1}{\sqrt{1-(\pi^T n)^2}}]
\end{aligned}
$$

After refraction, the formula of r'_θ is:

$$
r'_\theta = n\left[(\frac{n_1}{n_2})[-n^T r_\theta] - \sqrt{1-(\frac{n_1}{n_2})^2(1-[r_\theta^T n]^2)}\right] + r_\theta(\frac{n_1}{n_2})
$$

Here, the inner product, is a function of angle θ:

$$
[-n^T r_\theta] = \cos\theta\sqrt{1-(\pi^T n)^2}
$$

Thus

$$
\begin{aligned}
r'_\theta = n&\left[-(\frac{n_1}{n_2})[\cos\theta\frac{(\pi^T n)^2}{\sqrt{1-(\pi^T n)^2}}] - \sqrt{1-(\frac{n_1}{n_2})^2\Big(1-\cos^2\theta[1-(\pi^T n)^2]\Big)}\right] \\
&+ \sin\theta(\pi\times n)(\frac{n_1}{n_2})[\frac{-1}{\sqrt{1-(\pi^T n)^2}}] \\
&+ \cos\theta\,\pi(\frac{n_1}{n_2})[\frac{(\pi^T n)}{\sqrt{1-(\pi^T n)^2}}]
\end{aligned}
$$

find intersection I

set the interface plane

$$
n^T I = h
$$

Here it must on the ray, all start from laser point P:

$$
I_\theta = P + k_\theta r_\theta
$$

Solve k_θ, I_θ respectively

$$n^T P - k_\theta \cos\theta \sqrt{1-(\pi^T n)^2} = h$$

$$k_\theta = \frac{n^T P - h}{\cos\theta \sqrt{1-(\pi^T n)^2}}$$

$$I_\theta = P + [n^T P - h]\left[\pi[\frac{(\pi^T n)}{1-(\pi^T n)^2}] + n[\frac{-1}{1-(\pi^T n)^2}]\right] + [n^T P - h]\tan\theta(\pi \times n)[\frac{-1}{1-(\pi^T n)^2}]$$