refract ray

here is comment in `Triangulate.refractRay()`
of project `underwater-camera-calibration`

```
# - 'rayDir' is the vector of the incoming ray
# - 'planeNormal' is the plane normal of the refracting interface
# - 'n1' is the refraction index of the medium the ray travels
>FROM<
# - 'n2' is the refractio index of the medium the ray travels >TO<
```

table of symbol

SMYBOL	MEANING
v_1	unit direction vector (FROM)
v_2	unit direction vector (TO)
n	unit normal vector of interface plane (FROM / TO)
n_1	refraction index (FROM)
n_2	refraction index (TO)
θ_1	angle (FROM)
θ_2	angle (TO)

analysis

We should notice that n is opposed to the incoming ray direction v_1:

$$v_1^T n < 0, v_1^T v_1 = v_2^T v_2 = n^T n = 1$$

Now that we know all the parameters except v_2,
we need derive the expression of v_2 with other variables v_1, n, n_1, n_2

reconstruct vector

and notice cross product matrix $n \times n \times = nn^T - I$ and $n^T n = 1$:

$$
\begin{aligned}
|v_1| \cos\theta_1 &\equiv \frac{-v_1^T n}{|n|} = -v_1^T n \\
|v_1|^2 \sin^2\theta_1 &\equiv \left| v_1 - (-n)\frac{(-n^T v_1)}{n^T n} \right|^2 = \left| v_1 - \frac{nn^T}{n^T n} v_1 \right|^2 = |(I - nn^T)v_1|^2 \\
&= |-n \times (n \times v_1)|^2 = v_1^T (I - nn^T)^T (I - nn^T) v_1 \\
&= v_1^T (I - nn^T)(I - nn^T) v_1 \\
&= v_1^T (I - nn^T) v_1 = v_1^T [-n \times n \times v_1] v_1 \\
&= v_1^T v_1 - [v_1^T n]^2 \\
v_1 &\equiv nn^T v_1 + (I - nn^T) v_1 = n(n^T v_1) - n \times (n \times v_1) \\
&= (-n)|v_1| \cos\theta_1 + \frac{[-n \times (n \times v_1)]}{|-n \times (n \times v_1)|} |v_1| \sin\theta_1
\end{aligned}
$$

Use $v_1^T v_1 = 1$:

$$
\begin{aligned}
\sin\theta_1 &= \sqrt{\frac{v_1^T v_1 - [v_1^T n]^2}{v_1^T v_1}} = \sqrt{1 - [v_1^T n]^2} \\
v_1 &= (-n)\cos\theta_1 + \frac{[-n \times (n \times v_1)]}{|-n \times (n \times v_1)|} \sin\theta_1
\end{aligned}
$$

For the same reason, $v_2^T v_2 = 1$,
and v_1, v_2, n in the same plane: $n \times v_1, n \times v_2$ are linear related,
and $v_1^T n < 0, v_2^T n < 0$:

$$
\frac{[-n \times (n \times v_1)]}{|-n \times (n \times v_1)|} = \frac{[-n \times (n \times v_2)]}{|-n \times (n \times v_2)|}
$$

We want to reconstruct v_2 with n, v_1, similarly:
here is decomposition of orthogonal basis,
because $n^T [n \times n \times v_1] = n^T [nn^T - I] v_1 = 0^T v_1 = 0$

$$
\begin{aligned}
v_2 &= (-n)\cos\theta_2 + \frac{[-n \times (n \times v_2)]}{|-n \times (n \times v_2)|} \sin\theta_2 \\
&= (-n)\cos\theta_2 + \frac{[-n \times (n \times v_1)]}{|-n \times (n \times v_1)|} \sin\theta_2
\end{aligned}
$$

Snell's Law

From previous formula, only things are missing to reconstruct v_2 is $\sin\theta_2, \cos\theta_2$

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

We want to express θ_1, θ_2 with parameters that we know v_1, n, n_1, n_2,

$$\begin{aligned}
\cos\theta_1 &= -v_1^T n \\
\sin\theta_1 &= \sqrt{\frac{v_1^T v_1 - [v_1^T n]^2}{v_1^T v_1}} = \sqrt{1 - [v_1^T n]^2} \\
\sin\theta_2 &= \frac{n_1}{n_2}\sin\theta_1 = \frac{n_1}{n_2}\sqrt{1 - [v_1^T n]^2} \\
\cos\theta_2 &= \sqrt{1 - (\frac{n_1}{n_2})^2(1 - [v_1^T n]^2)}
\end{aligned}$$

Represent $\frac{[-n\times(n\times v_1)]}{|-n\times(n\times v_1)|}$ with v_1, n:

$$\begin{aligned}
\frac{[-n\times(n\times v_1)]}{|-n\times(n\times v_1)|} &= \frac{v_1 + n\cos\theta_1}{\sin\theta_1} \\
\frac{[-n\times(n\times v_1)]}{|-n\times(n\times v_1)|}\sin\theta_2 &= v_1\frac{\sin\theta_2}{\sin\theta_1} + n\frac{\sin\theta_2}{\sin\theta_1}\cos\theta_1 \\
&= v_1(\frac{n_1}{n_2}) - nn^T v_1(\frac{n_1}{n_2}) \\
&= (I - nn^T)v_1(\frac{n_1}{n_2}) \\
&= -n\times(n\times v_1)(\frac{n_1}{n_2})
\end{aligned}$$

Eventually, represent v_2 with v_1, n, n_1, n_2

$$\begin{aligned}
v_2 &= (-n)\cos\theta_2 + \frac{[-n\times(n\times v_1)]}{|-n\times(n\times v_1)|}\sin\theta_2 \\
&= n\left[-\sqrt{1 - (\frac{n_1}{n_2})^2(1 - [v_1^T n]^2)}\right] + v_1(\frac{n_1}{n_2}) - n\left[n^T v_1(\frac{n_1}{n_2})\right] \\
&= n\left[(\frac{n_1}{n_2})[-n^T v_1] - \sqrt{1 - (\frac{n_1}{n_2})^2(1 - [v_1^T n]^2)}\right] + v_1(\frac{n_1}{n_2})
\end{aligned}$$

Consider all the possible values of v1

If the family of vector v_1 are always on the same plane,
the unit normal vector of this plane is π, always holds:

$$\pi^T v_1 = 0$$

We want to prove always exist $A, B \neq 0$ for any v_1 that holds $\pi^T v_1 = 0$, make sure

$$\begin{aligned}
(A\pi + Bn)^T v_2 &= 0 \\
&= A[\pi^T n]\left[(\frac{n_1}{n_2})[-n^T v_1] - \sqrt{1 - (\frac{n_1}{n_2})^2 (1 - [v_1^T n]^2)}\right] \\
&- B\sqrt{1 - (\frac{n_1}{n_2})^2 (1 - [v_1^T n]^2)} \\
&= A[\pi^T n](\frac{n_1}{n_2})[-n^T v_1] \\
&- \left[A[\pi^T n] + B\right]\sqrt{1 - (\frac{n_1}{n_2})^2 (1 - [v_1^T n]^2)} \\
\frac{A[\pi^T n] + B}{A[\pi^T n]} &= \frac{(\frac{n_1}{n_2})[-n^T v_1]}{\sqrt{1 - (\frac{n_1}{n_2})^2 (1 - [v_1^T n]^2)}}
\end{aligned}$$

So, constant A, B don't exist when v_1 keeps changing