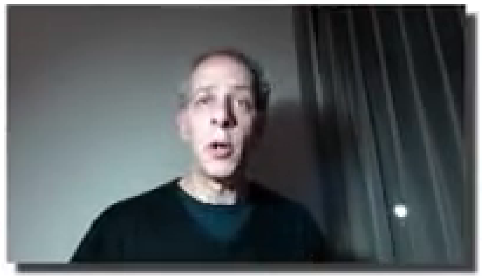
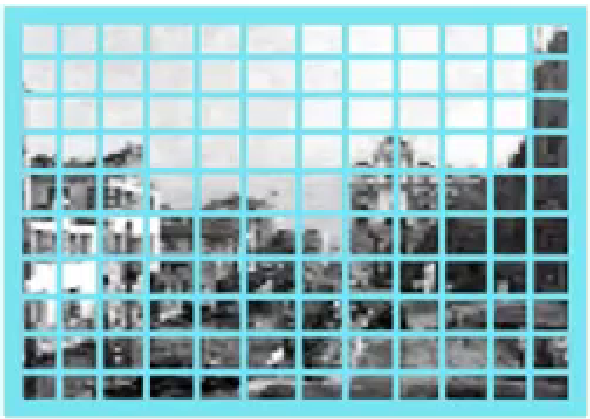
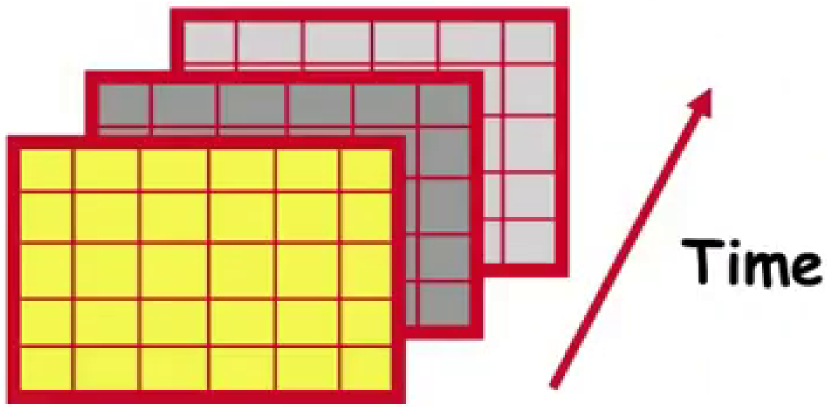
What and why a discrete computer image?

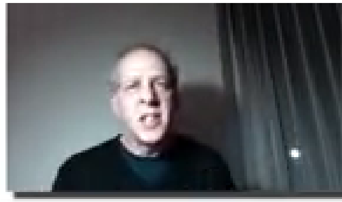




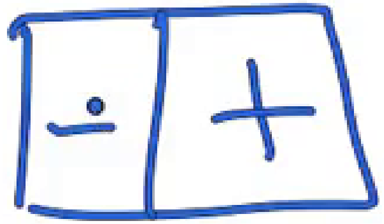


Movie courtesy "Sleepers" by W. Allen

Discrete image representations

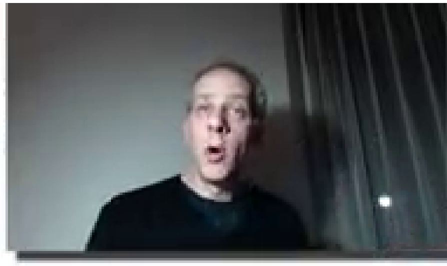


- Classical image processing is based on discrete mathematics (most of it)
  - Sums instead of integrals



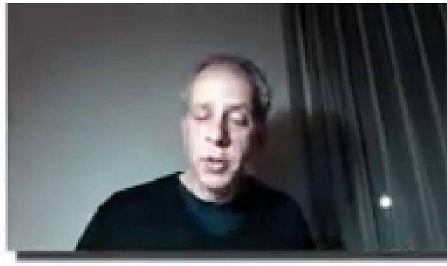
  - Re-definition of classical continuous operators such as gradients, Laplacian, etc

The PDEs approach

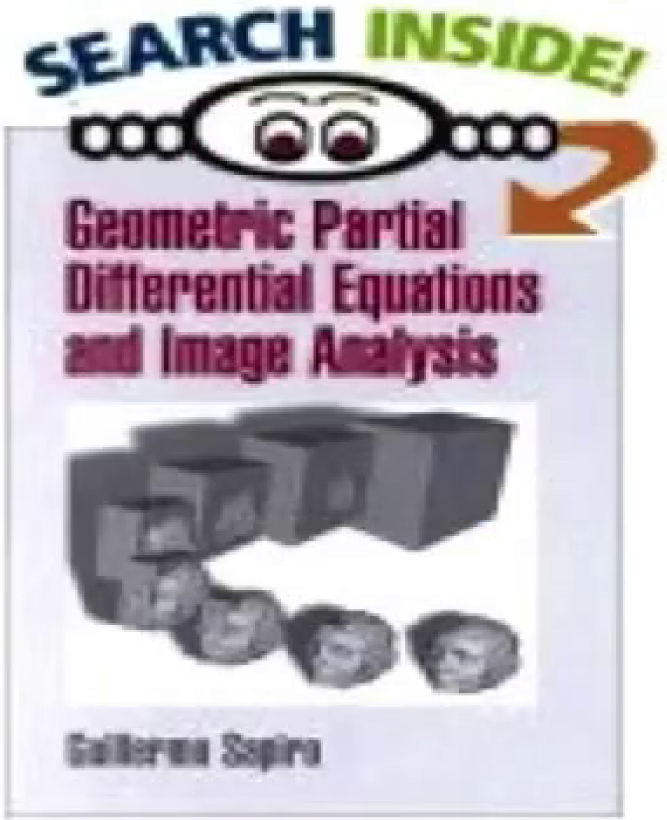


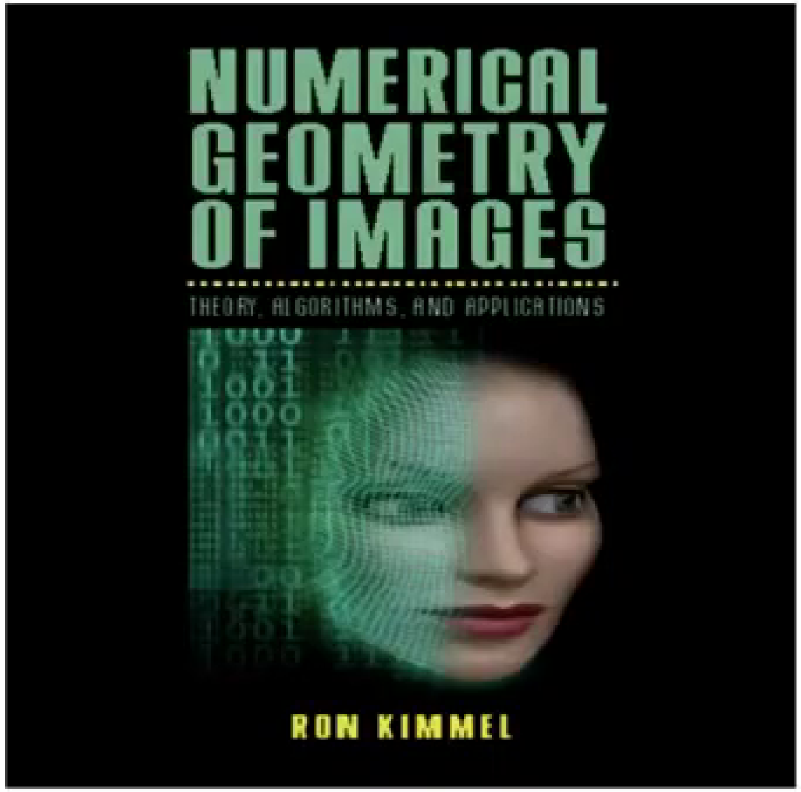
- Images are continuous objects
- Image processing is the results of iteration of infinitesimal operations: PDEs
- Differential geometry on images
- Computer image processing is based on numerical analysis

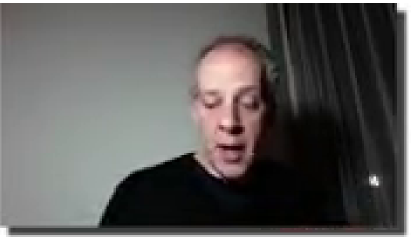
Why? Why Now? Who?



- Why now:
  - Computers!!!
  - People
- Why:
  - New concepts
  - Accuracy
  - Formal analysis (existence, uniqueness, etc)
- Consequences:
  - Many state of the art results
  - New tools in the bookshelf


SEARCH INSIDE!
Geometric Partial Differential Equations and Image Analysis
Guillermo Sapiro

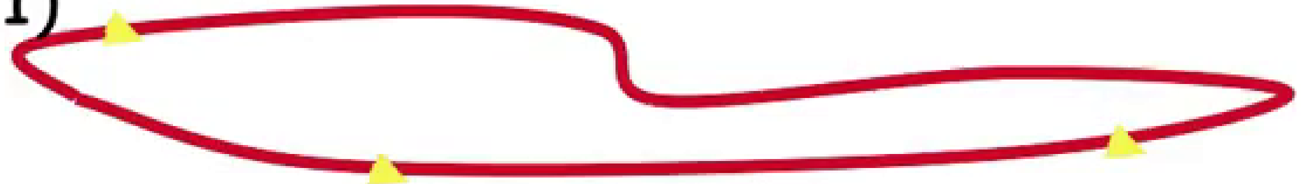

NUMERICAL GEOMETRY OF IMAGES
THEORY, ALGORITHMS, AND APPLICATIONS
RON KIMMEL

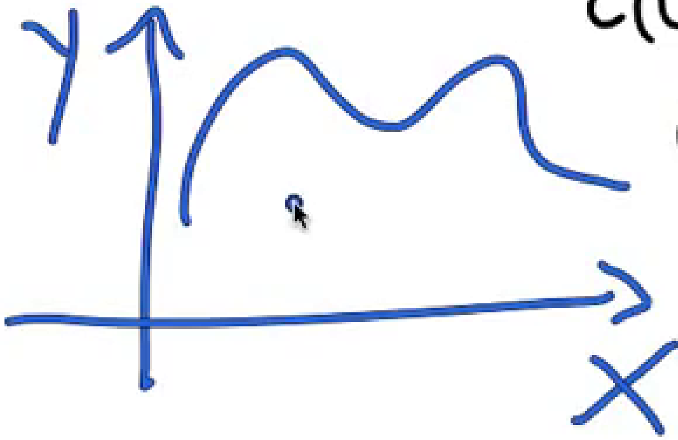


Planar Curves

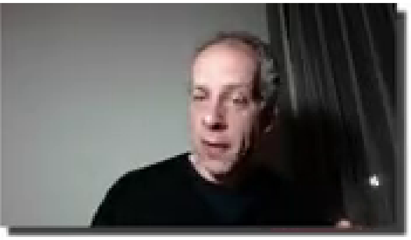
- $C(p)=\{x(p),y(p)\}, \quad p \in [0,1]$

$C(0)=C(1)$



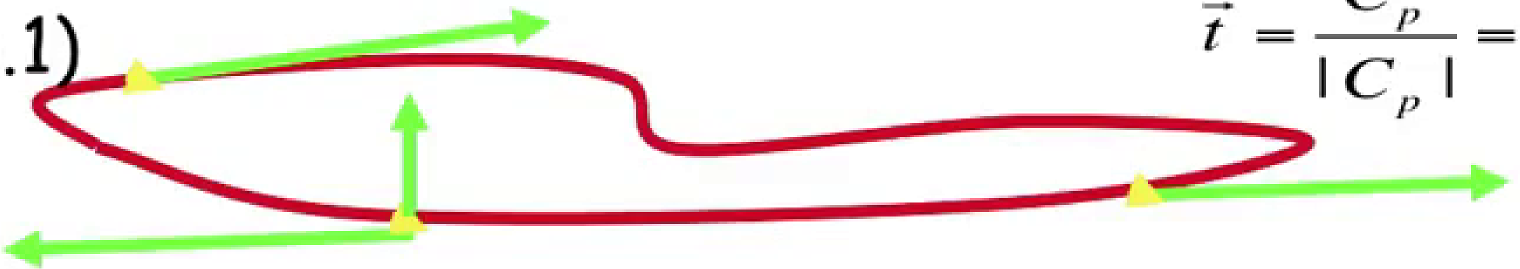


$C(x)=\{x,f(x)\}$



Planar Curves

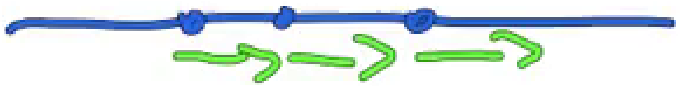
- $C(p)=\{x(p),y(p)\}, \quad p \in [0,1]$



$$\vec{t} = \frac{C_p}{|C_p|} = C_s$$

$$C_{ss} = \kappa \vec{n}$$

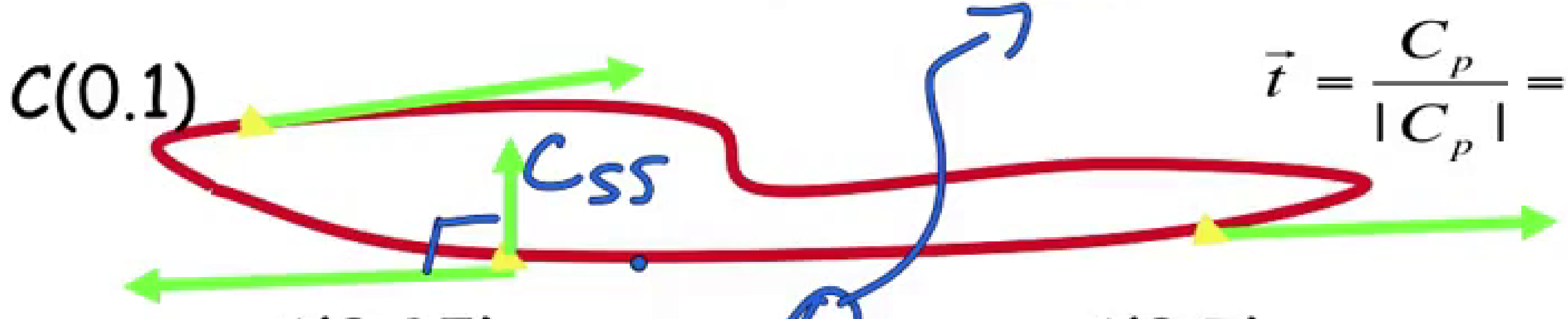
$$C_p = \frac{\partial C}{\partial p} = [x_p, y_p]$$



Planar Curves

- $C(p)=\{x(p),y(p)\}, \qquad p \in [0,1]$

Curvature

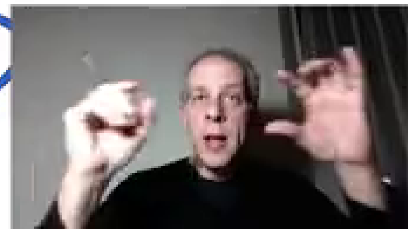


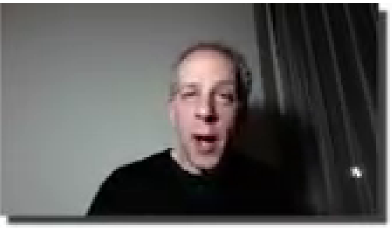
$$\vec{t} = \frac{C_p}{|C_p|} = C_s$$

$$C_{ss} = \kappa \vec{n}$$

$$|C_s| = 1 \qquad \frac{\partial}{\partial s}\langle C_s, C_s\rangle = \frac{\partial}{\partial s} 1$$

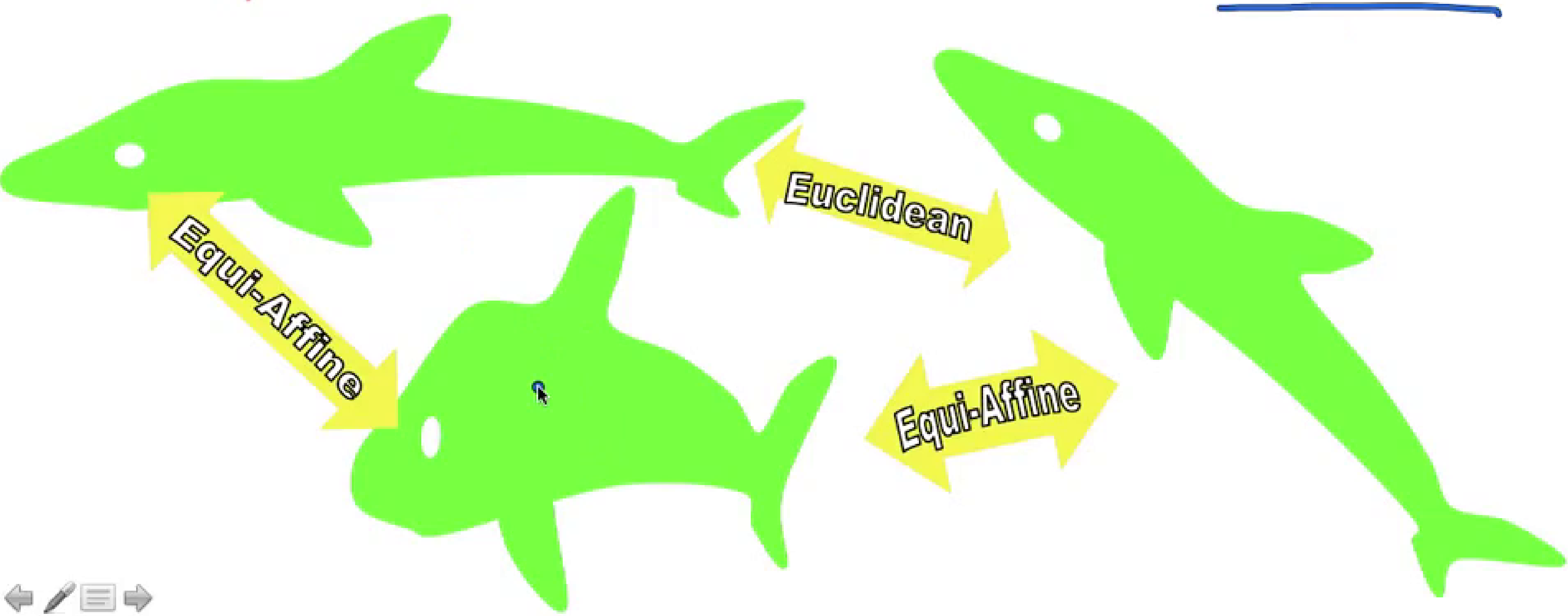
$$C_s \perp C_{ss} \Leftarrow \cancel{2}\langle C_s, C_{ss}\rangle = 0$$

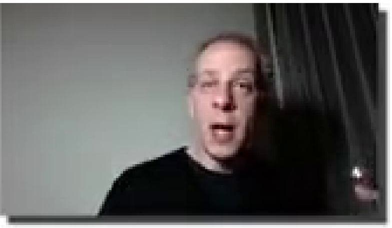




Linear Transformations

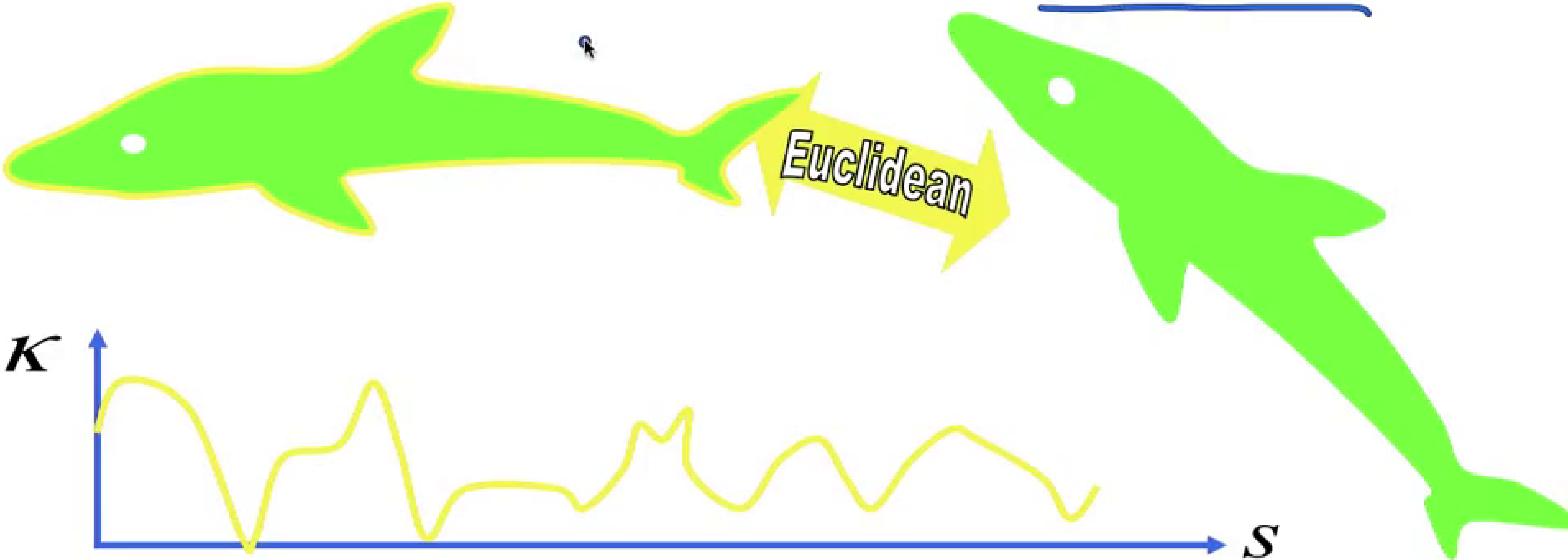
Equi-Affine: $\{\tilde{x}, \tilde{y}\}^T = A\{x, y\}^T + \bar{b}, \quad \det(A) = 1.$

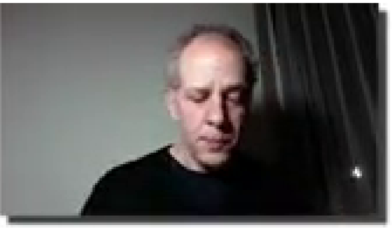




Differential Signatures

- Euclidean invariant signature $\{s, \kappa(s)\}$





Differential Signatures

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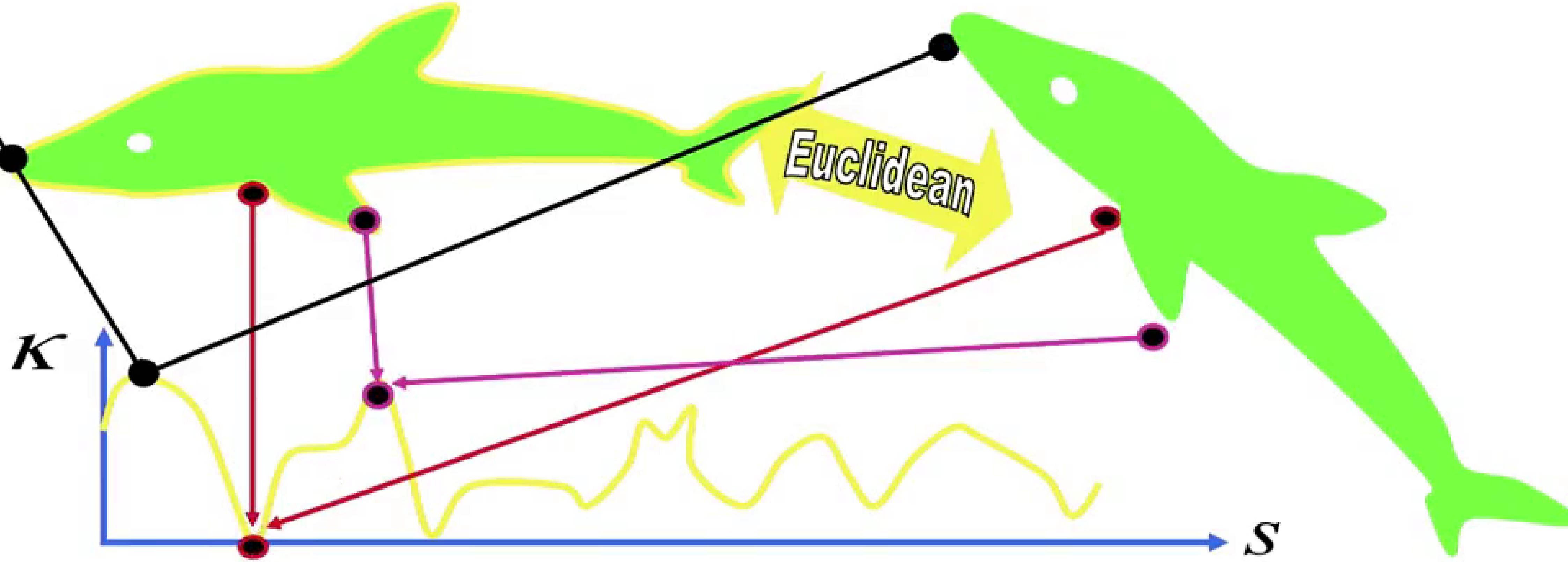

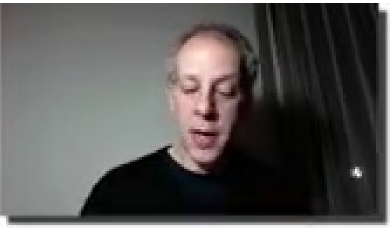

Differential Signatures

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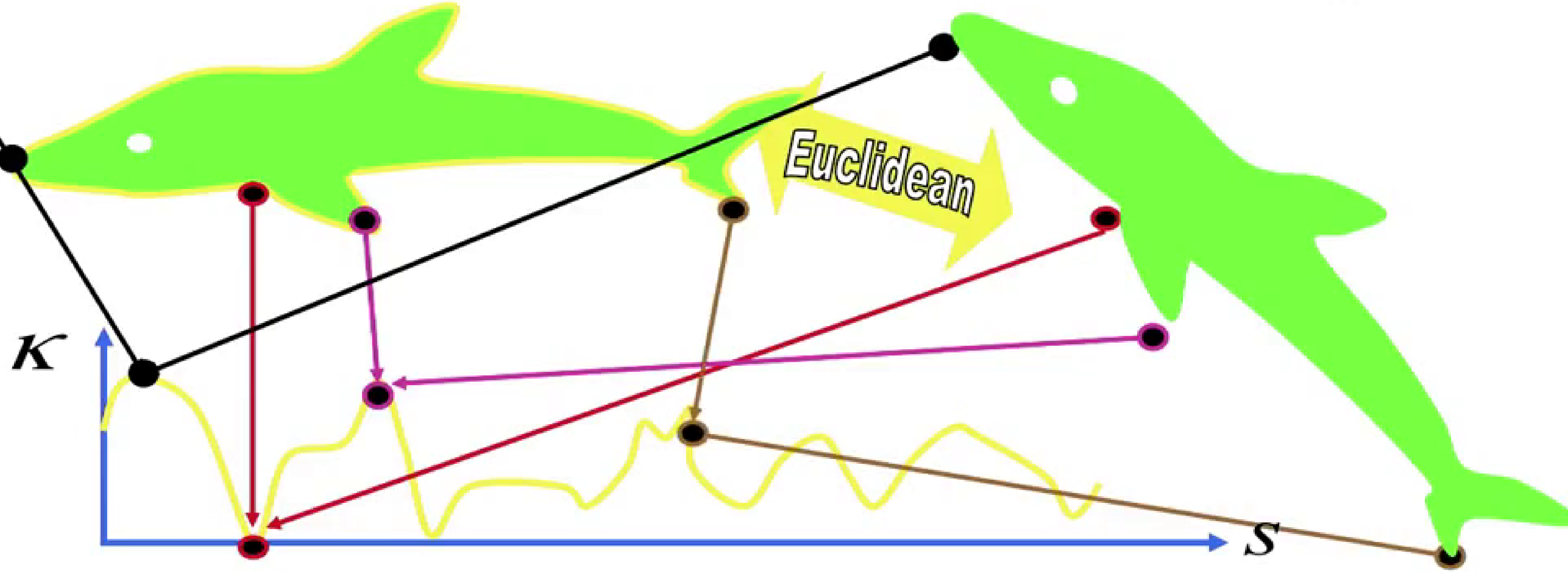
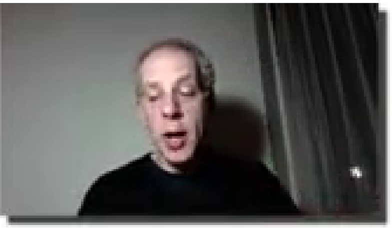
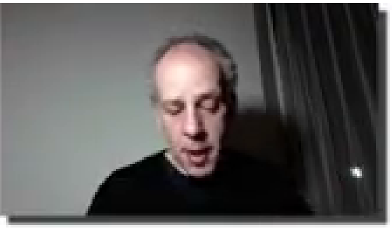




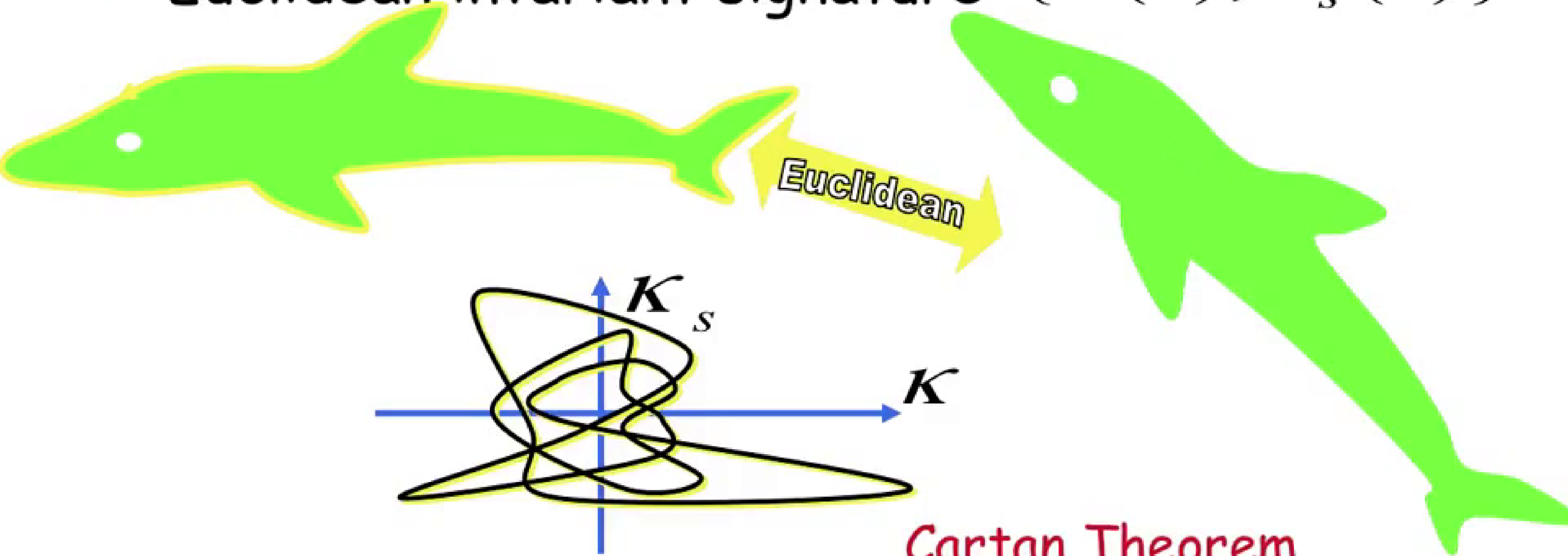
Differential Signatures

- Euclidean invariant signature $\boxed{\{s, \kappa(s)\}}$

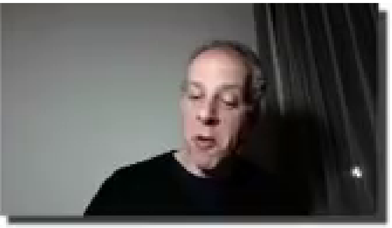


Differential Signatures

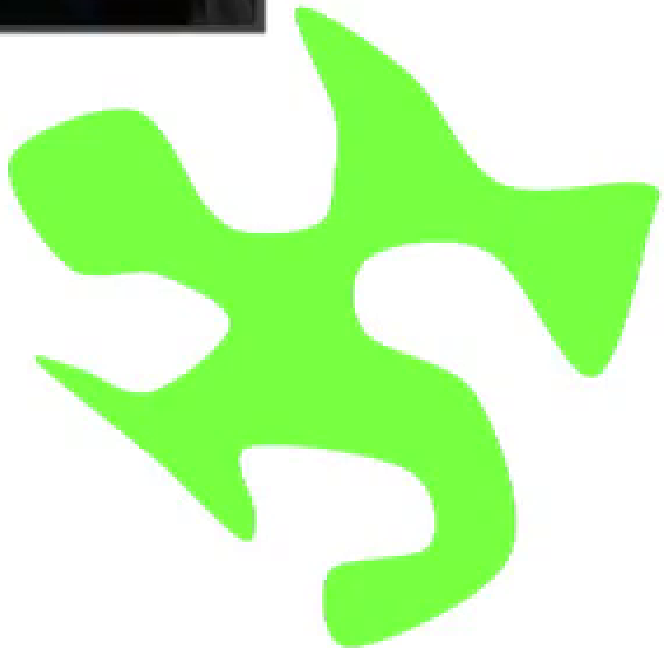
- Euclidean invariant signature $\{\kappa(s), \kappa_s(s)\}$

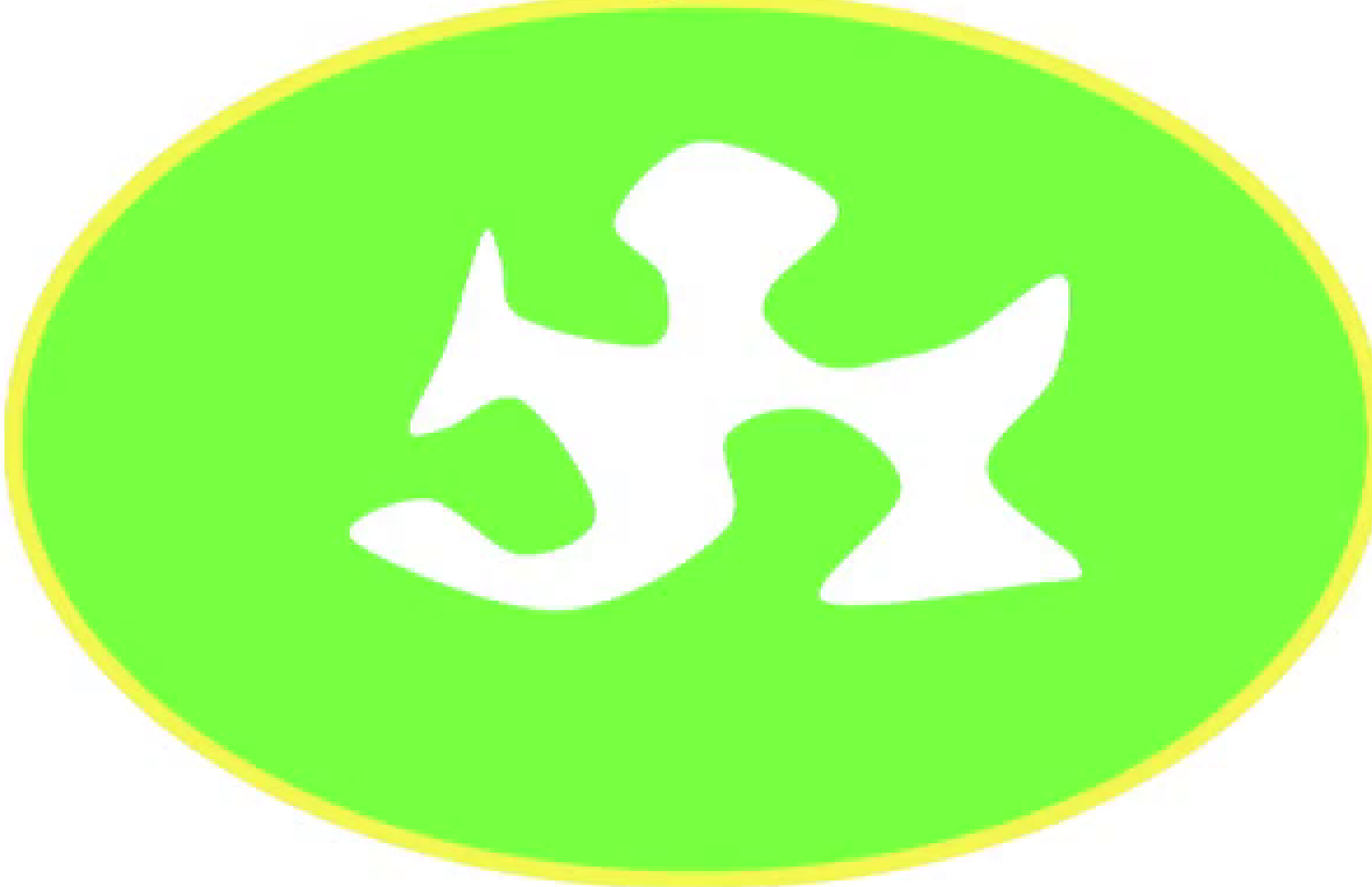


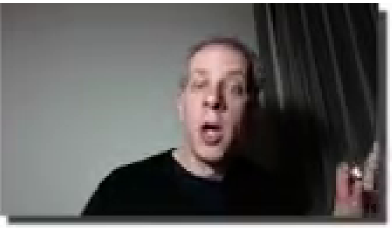
Cartan Theorem



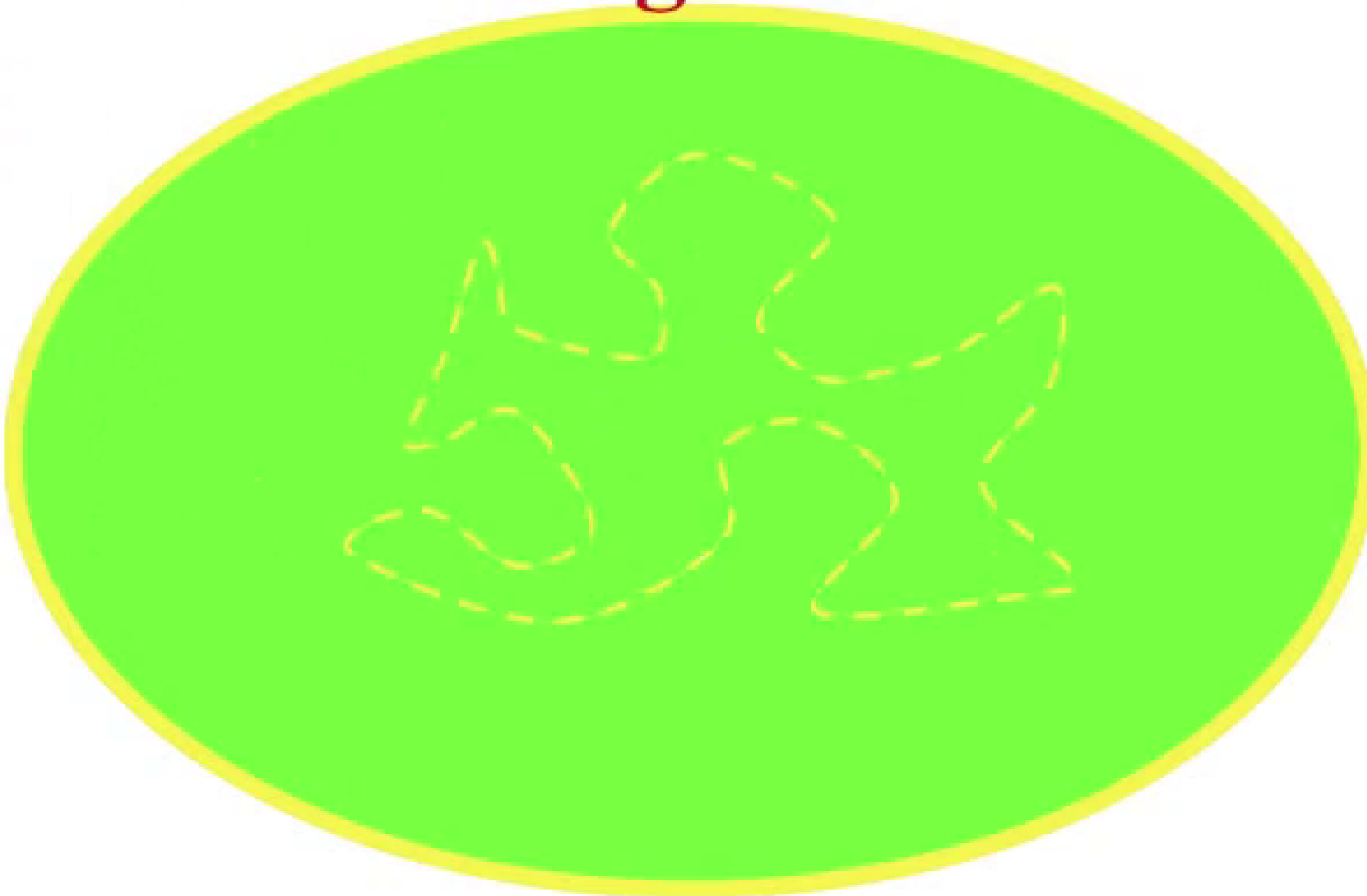
Differential Signatures



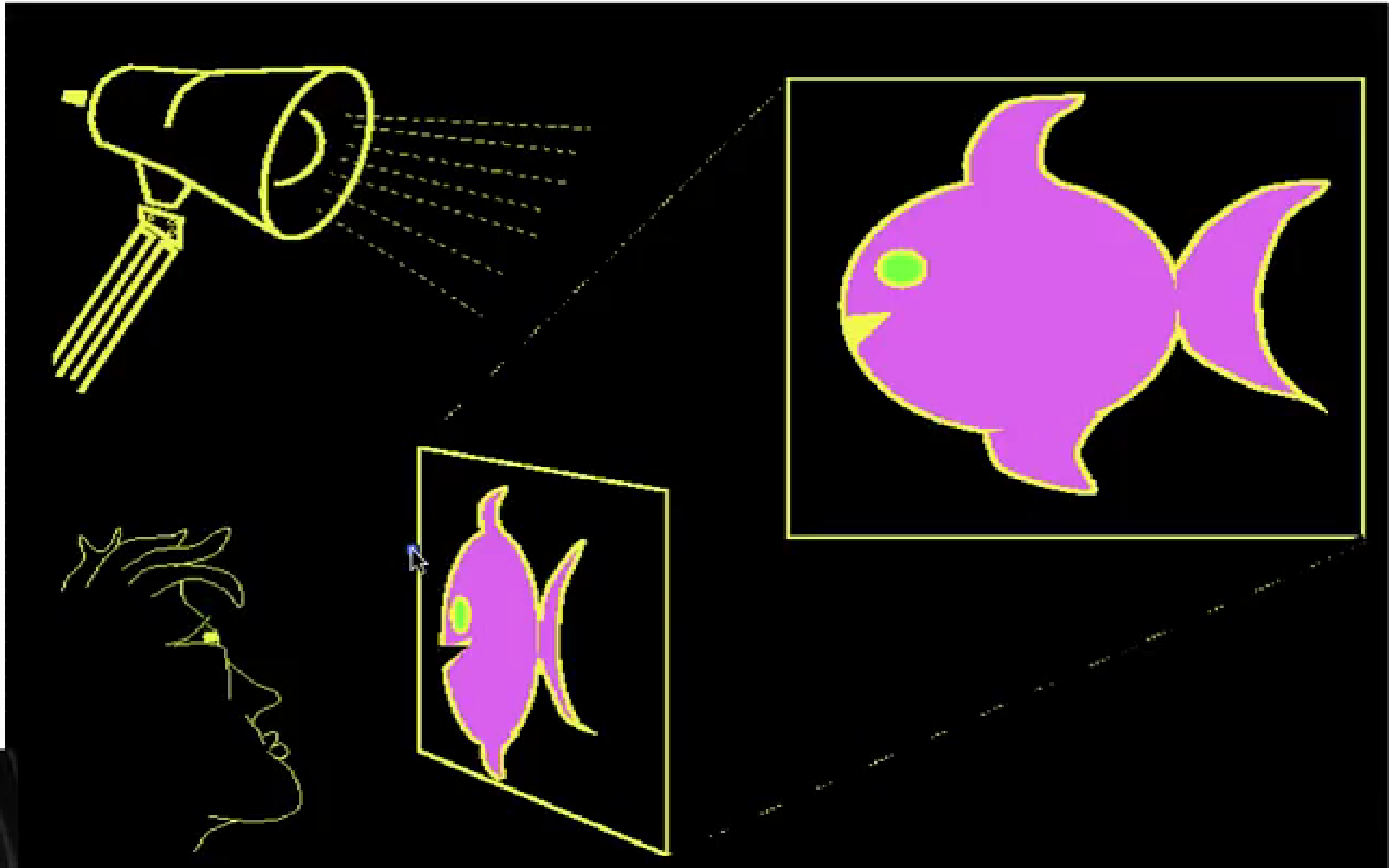


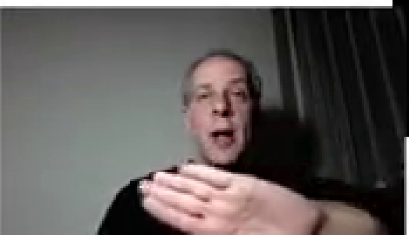


Differential Signatures

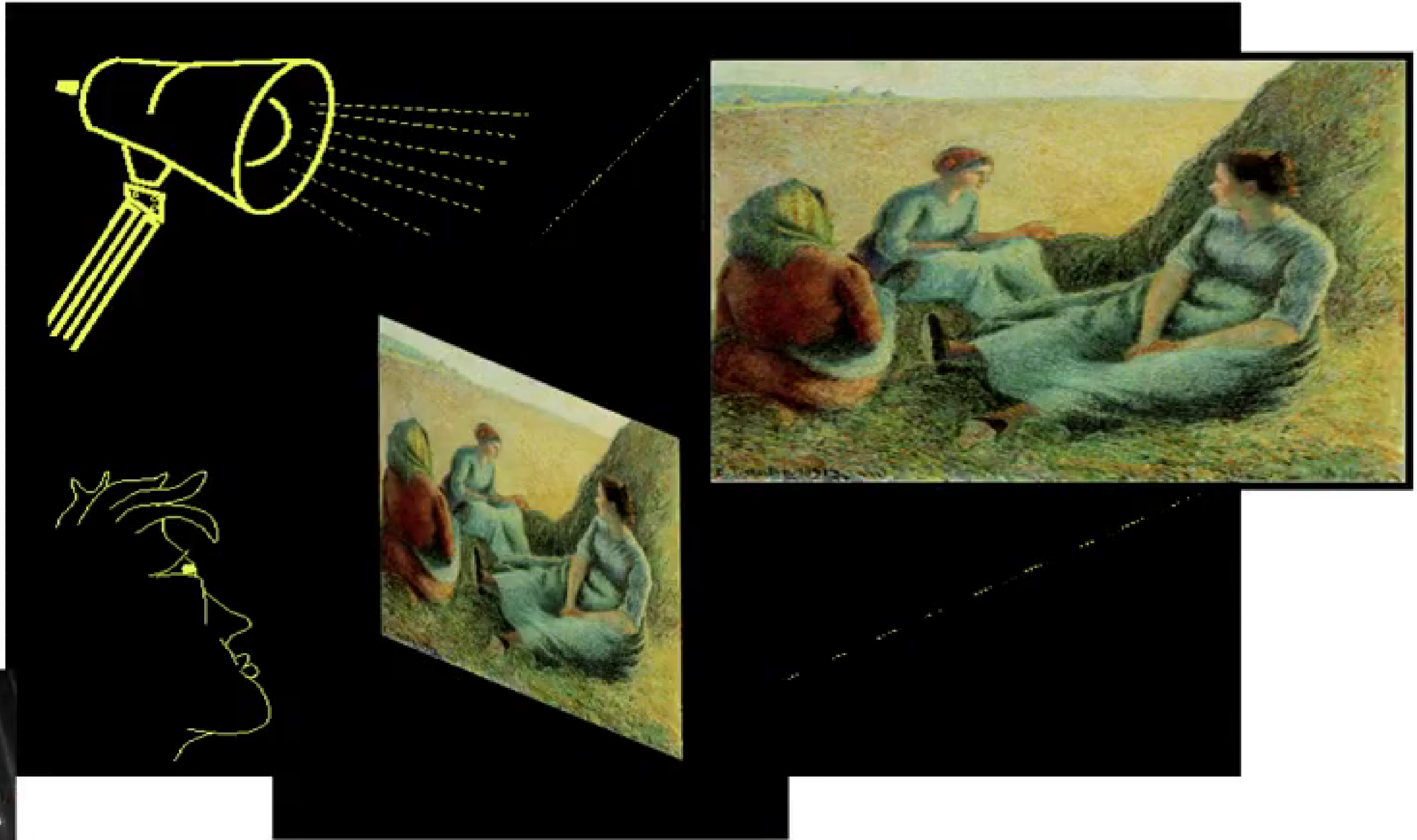


~Affine





~Affine



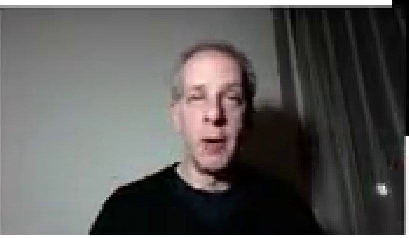
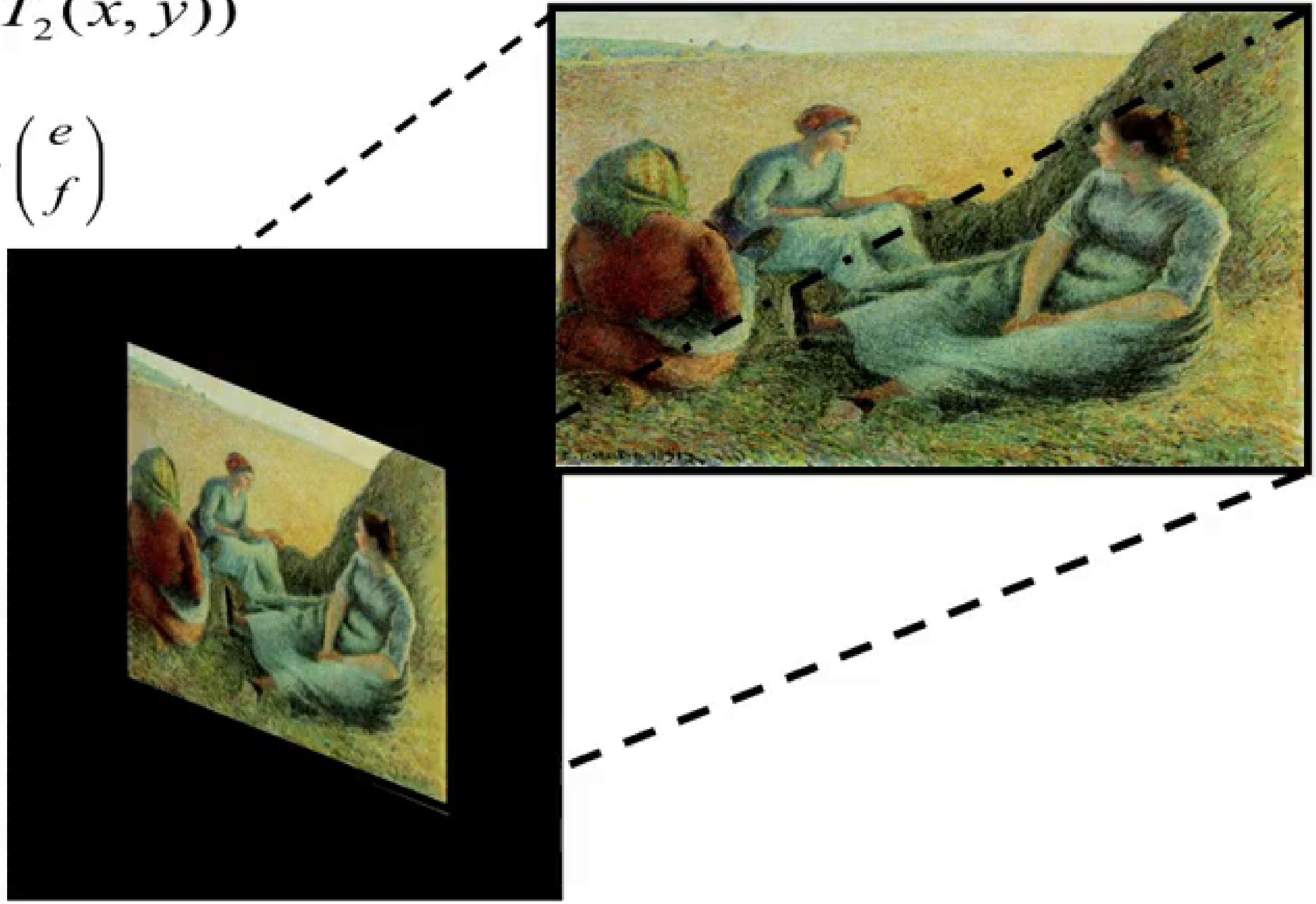


Image transformation

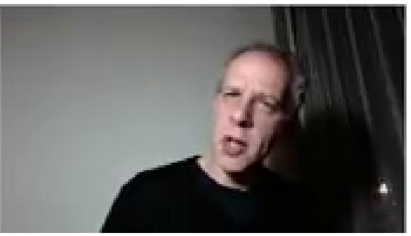
$$I_2(x,y) = I_1(T_1(x,y), T_2(x,y))$$

$$\begin{pmatrix} T_1(x,y) \\ T_2(x,y) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$$



- **Equi-affine:**

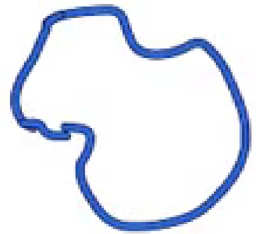
$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1$$



$C(p)$ $C(r)$

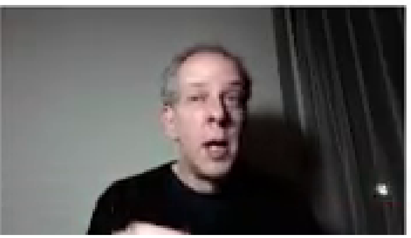
Invariant arclength should be

1. Re-parameterization invariant



$$w = \int F(C, C_p, C_{pp}, \ldots)\,dp = \int F(C, C_r, C_{rr}, \ldots)\,dr$$

2. Invariant under the group of transformations



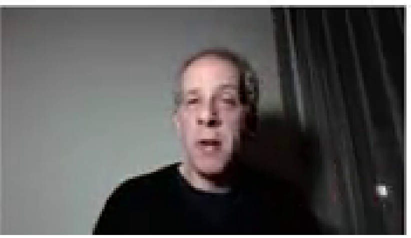
Invariant arclength should be

1. Re-parameterization invariant

Geometric measure

$$w = \int F(C, C_p, C_{pp}, \ldots)\,dp = \int \boxed{F(C, C_r, C_{rr}, \ldots)}\,dr$$

2. Invariant under the group of transformations



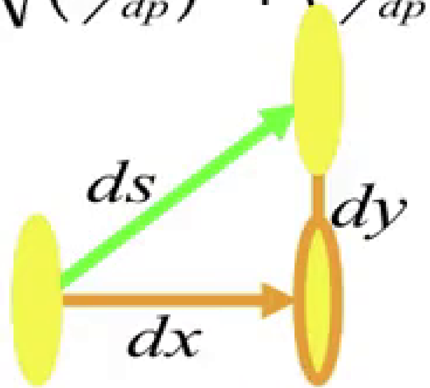
Euclidean arclength

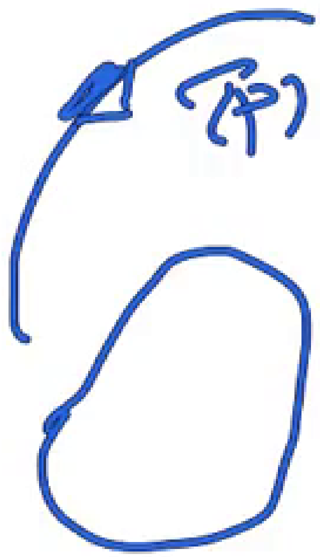
- Length is preserved, thus

$$ds = \sqrt{dx^2 + dy^2} = \frac{dp}{dp}\sqrt{dx^2 + dy^2} = dp\sqrt{\left({}^{dx}\!/\!_{dp}\right)^2 + \left({}^{dy}\!/\!_{dp}\right)^2} = |C_p|\,dp$$

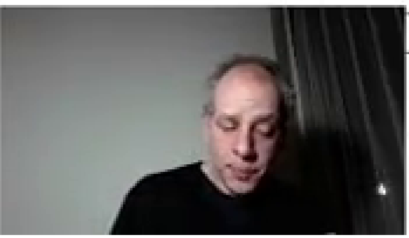
$$s = \int |C_p|\,dp$$

$$|C_s| = 1$$





$$\text{Length } L = \int_0^1 |C_p|\,dp = \int_0^1 \langle C_p, C_p \rangle^{1/2}\,dp = \int_0^L ds$$



Equi-affine arclength

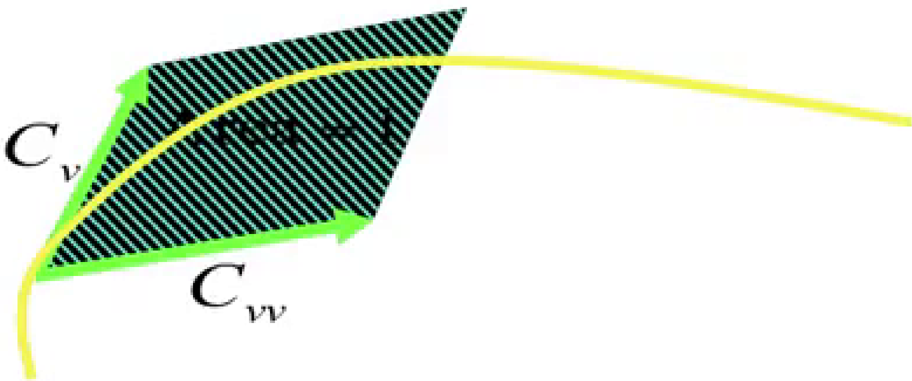
- Area is preserved, thus

$$\begin{bmatrix} x_v & x_{vv} \\ y_v & y_{vv} \end{bmatrix}$$

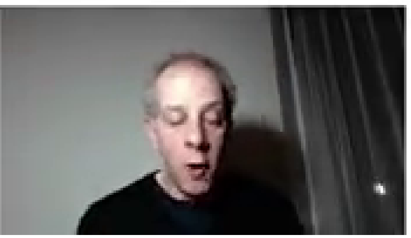
$$(C_v, C_{vv}) = 1$$

$$v = \int (C_p, C_{pp})^{1/3} dp$$

$$v = \int (C_s, C_{ss})^{1/3} ds = \int \kappa^{1/3} ds$$



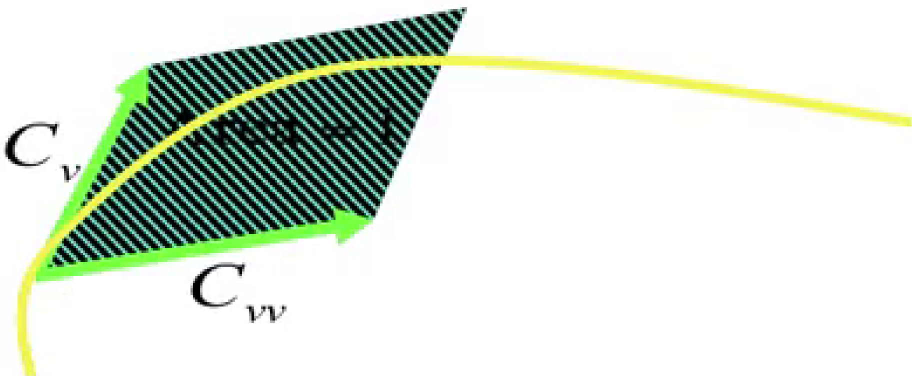
$$dv = \kappa^{1/3} ds$$



Equi-affine arclength

- Area is preserved, thus

re-parameterization invariance

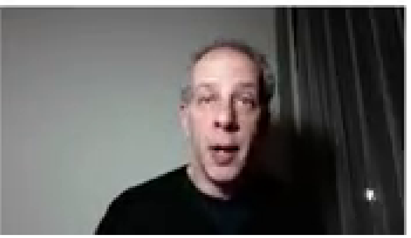


$$(C_v, C_{vv}) = 1$$

$$v = \int (C_p, C_{pp})^{1/3} dp$$

$$v = \int (C_s, C_{ss})^{1/3} ds = \int \kappa^{1/3} ds$$

$$dv = \kappa^{1/3} ds$$

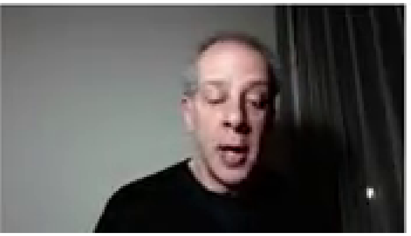


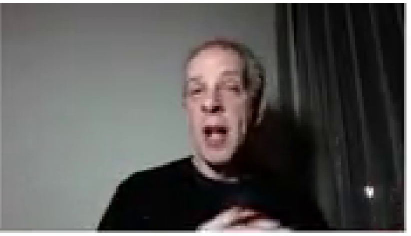
$\langle C_s, C_s \rangle = 1$

Equi-affine curvature

$$
\begin{aligned}
(C_v, C_{vv}) = 1 \quad &\Rightarrow \quad \tfrac{d}{dv}(C_v, C_{vv}) = 0 \\
&\Rightarrow \quad \cancel{(C_{vv}, C_{vv})} + (C_v, C_{vvv}) = 0 \\
&\Rightarrow \quad (C_v, C_{vvv}) = 0 \\
&\Rightarrow \quad C_v \| C_{vvv} \quad \Rightarrow \quad C_{vvv} = \mu C_v
\end{aligned}
$$

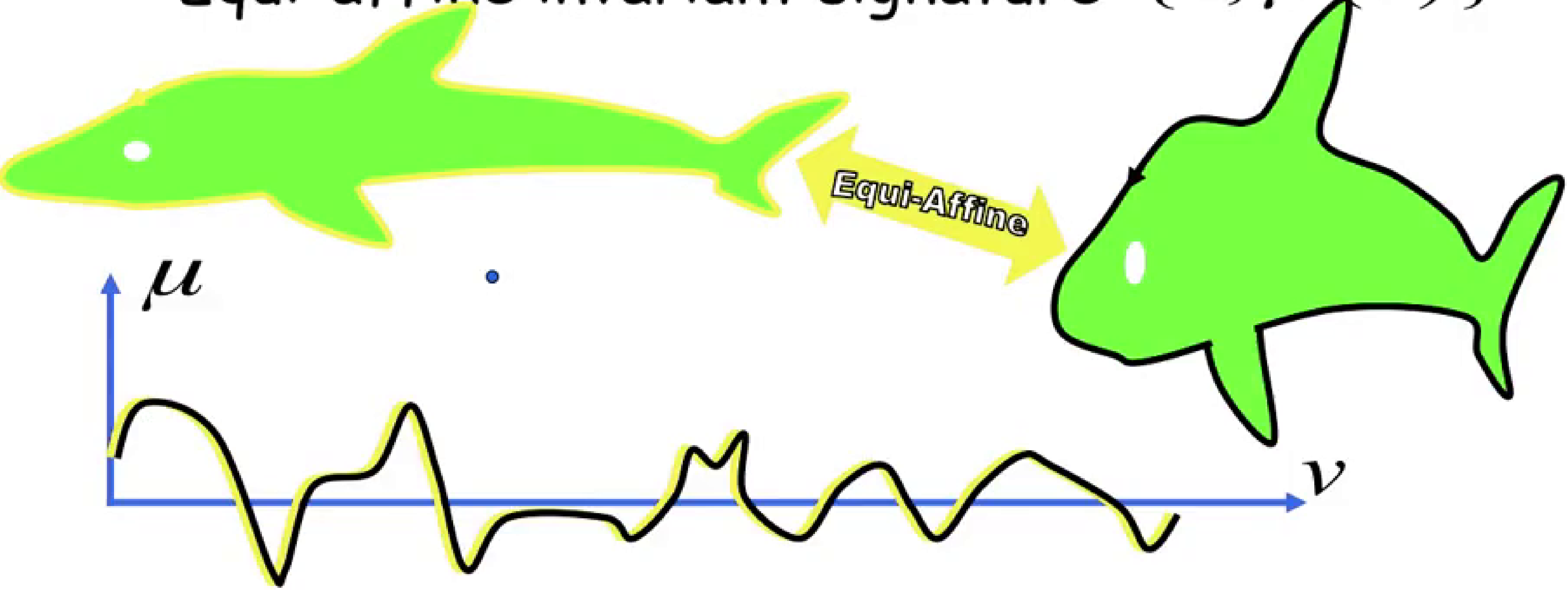
μ is the affine invariant curvature





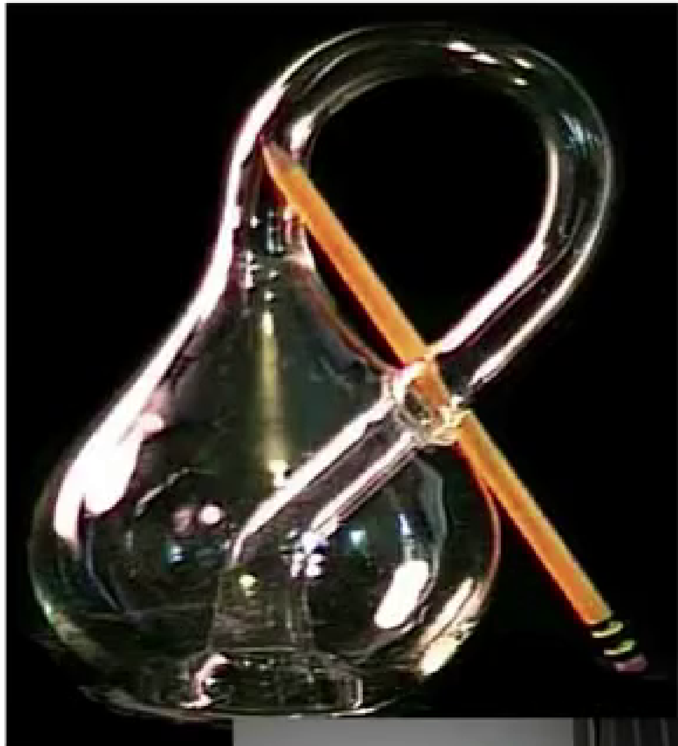
Differential Signatures

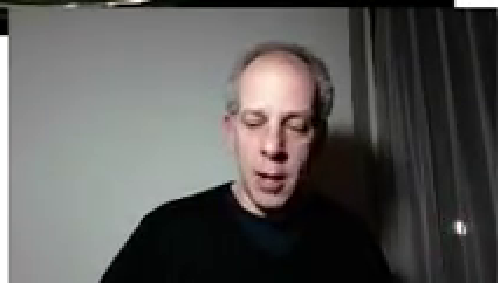
- Equi-affine invariant signature $\{\nu, \mu(\nu)\}$



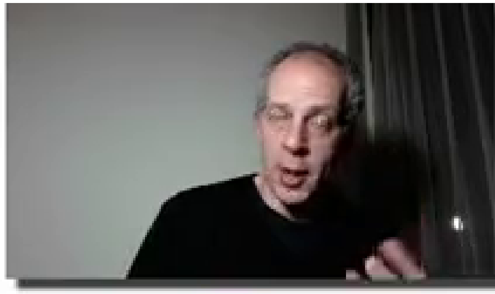
Surfaces

- Topology (Klein Bottle)





Surface

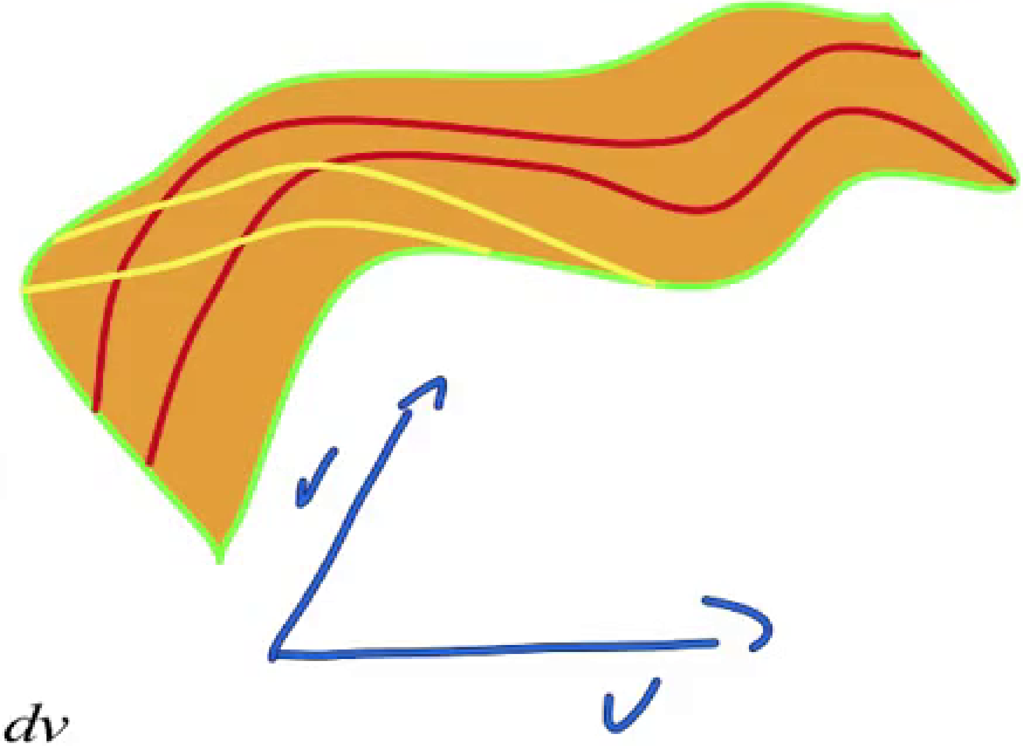


$$S(u,v) = \{x(u,v), y(u,v), z(u,v)\}$$

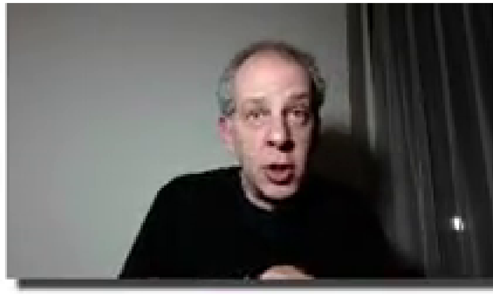
- Normal $\vec{N} = \dfrac{S_u \times S_v}{|S_u \times S_v|}$

- Area element $dA = |S_u \times S_v|$

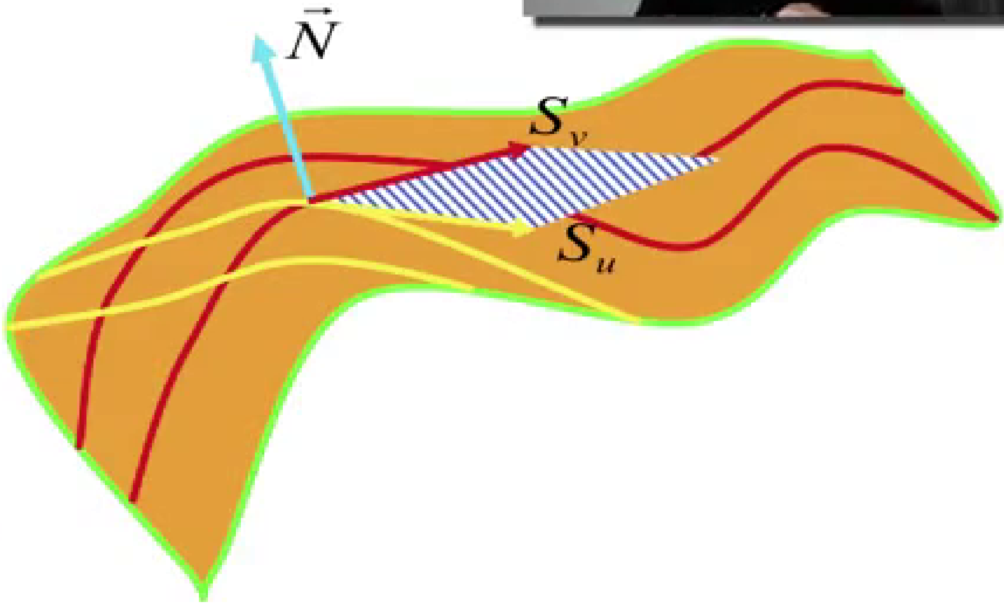
- Total area $A = \iint |S_u \times S_v| \, du\,dv$



Surface



$$S(u,v)=\{x(u,v),y(u,v),z(u,v)\}$$

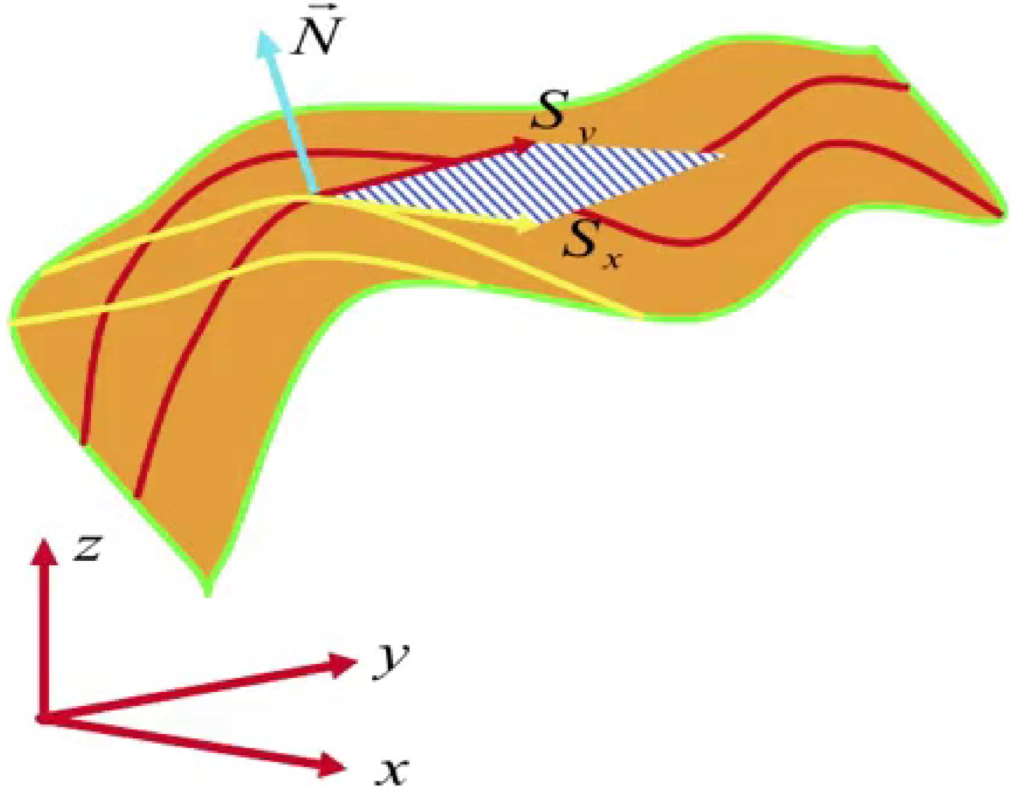


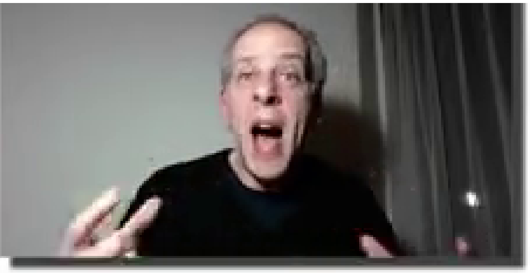
- Normal $\vec{N}=\dfrac{S_u\times S_v}{|S_u\times S_v|}$
- Area element $dA=|S_u\times S_v|$
- Total area $A=\iint|S_u\times S_v|\,du\,dv$

Example: Surface as graph of function

- A surface, $S:\mathbf{R}^2 \rightarrow \mathbf{R}^3$

$$S(u,v)=\left\{x=u,\, y=v,\, z(u,v)\right\}$$

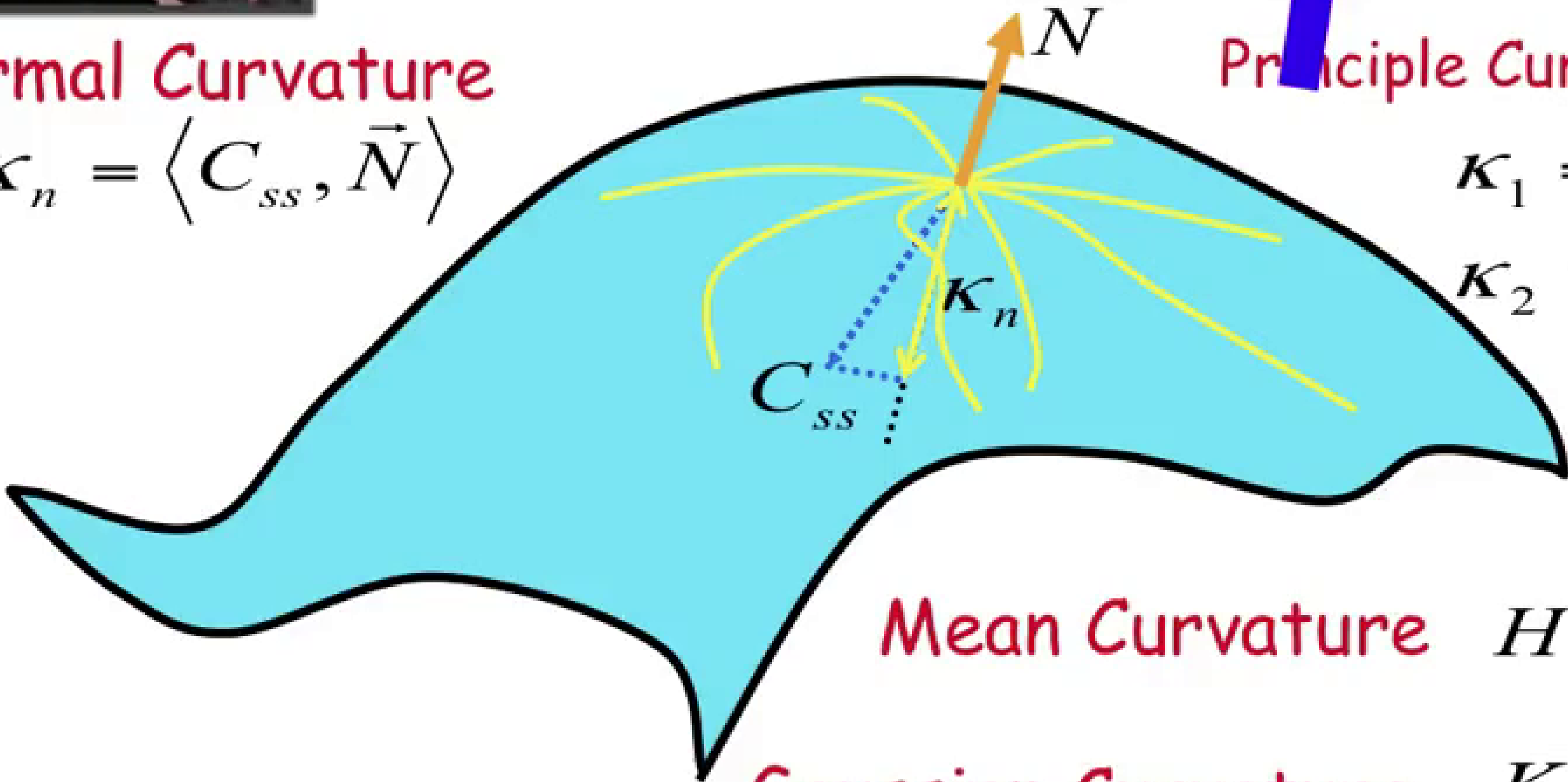




Principal

Normal Curvature

$$\kappa_n = \left\langle C_{ss}, \vec{N} \right\rangle$$



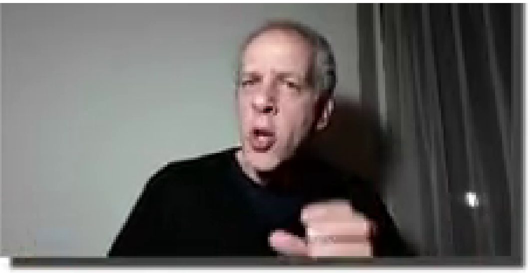
Principle Curvatures

$$\kappa_1 = \max_\theta(\kappa)$$

$$\kappa_2 = \min_\theta(\kappa)$$

Mean Curvature $H = \dfrac{\kappa_1 + \kappa_2}{2}$

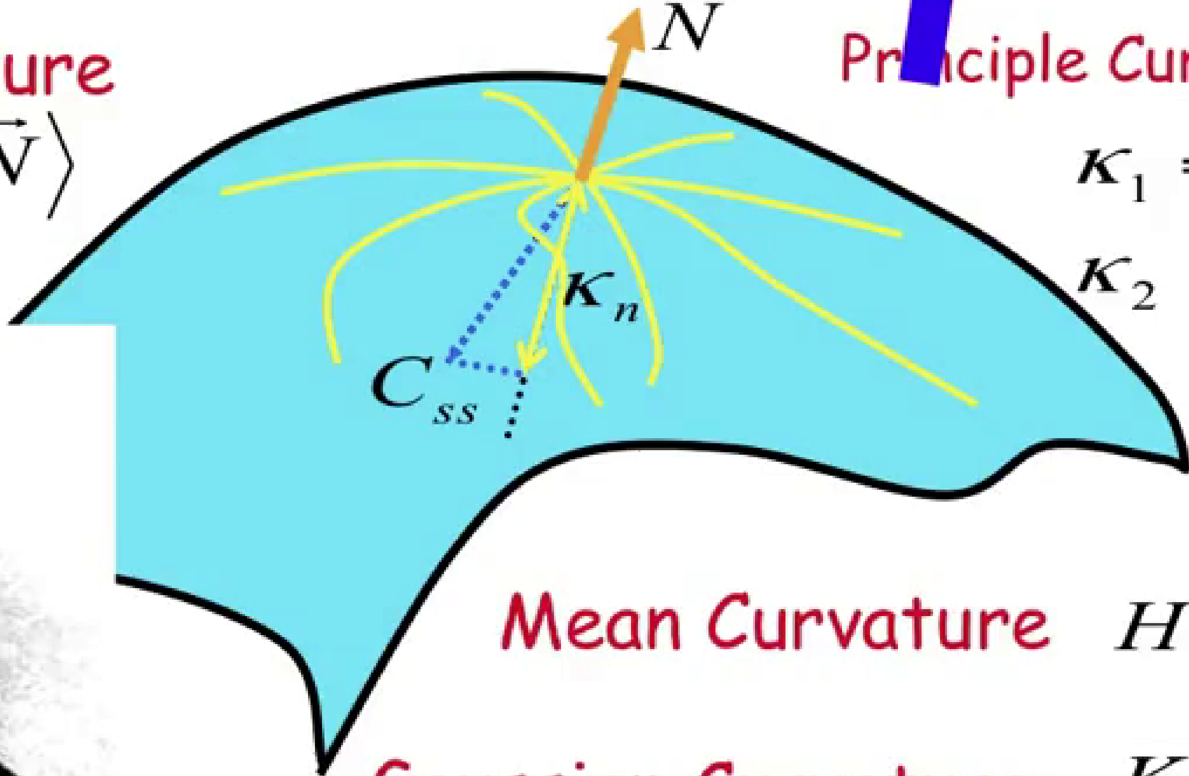
Gaussian Curvature $K = \kappa_1 \kappa_2$



Principal

Normal Curvature

$$\kappa_n = \left\langle C_{ss}, \vec{N} \right\rangle$$



Principle Curvatures

$$\kappa_1 = \max_\theta(\kappa)$$

$$\kappa_2 = \min_\theta(\kappa)$$

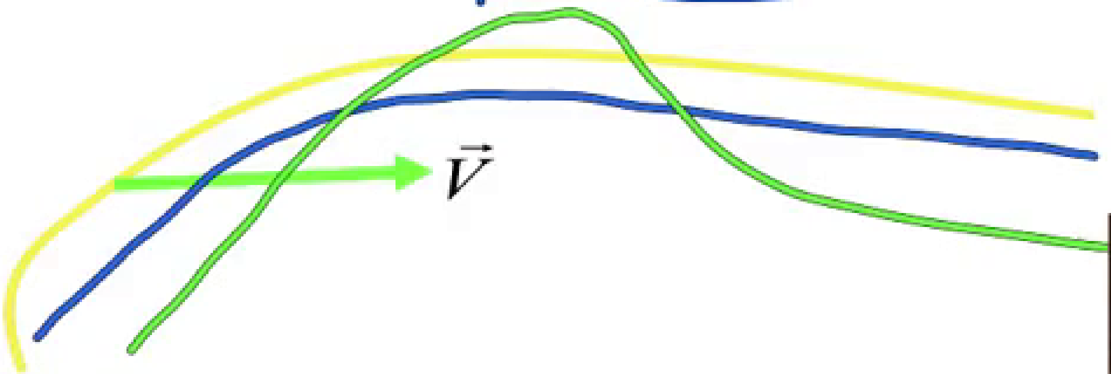
Gauss

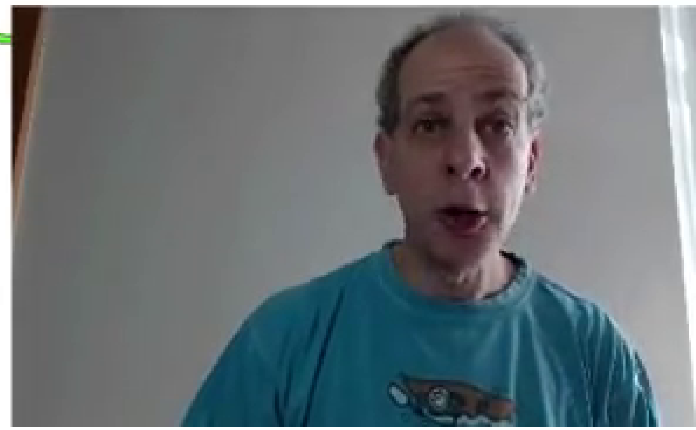
Mean Curvature $H = \dfrac{\kappa_1 + \kappa_2}{2}$

Gaussian Curvature $K = \kappa_1 \kappa_2$

$$\frac{\partial C(p)}{\partial t} = \vec{V}(p,t)$$

$$\boxed{C_t = \vec{V}}$$

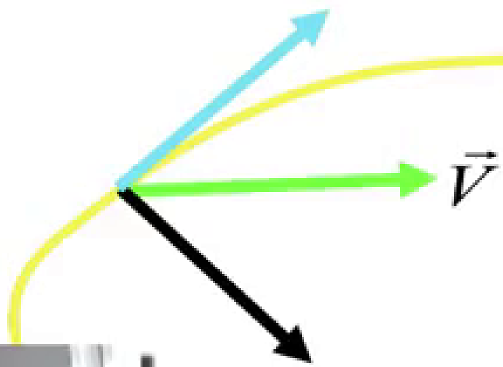




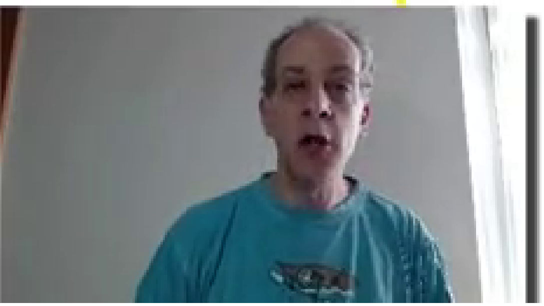
Important property

- Tangential components do not affect the geometry of an evolving curve

$$C_t = \vec{V} \Leftrightarrow C_t = \langle \vec{V}, \vec{n} \rangle \vec{n}$$



$$C_t = \alpha \vec{t}$$

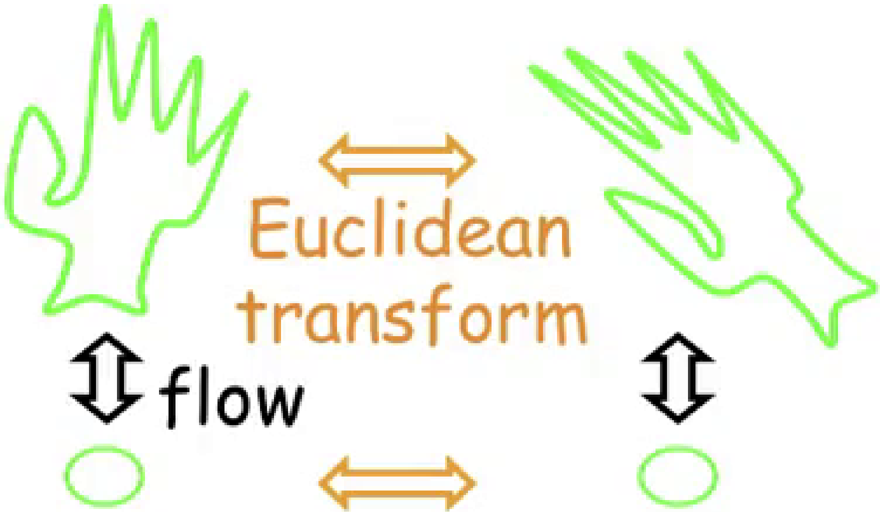


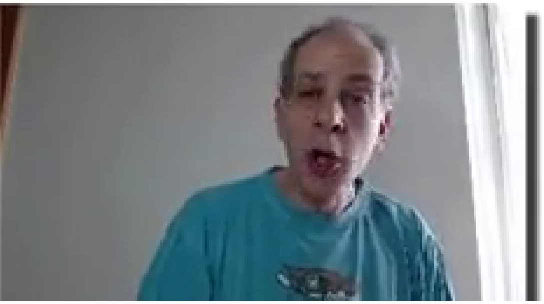
Curvature flow

- Euclidean geometric heat equation $C_t = \kappa \vec{n}$

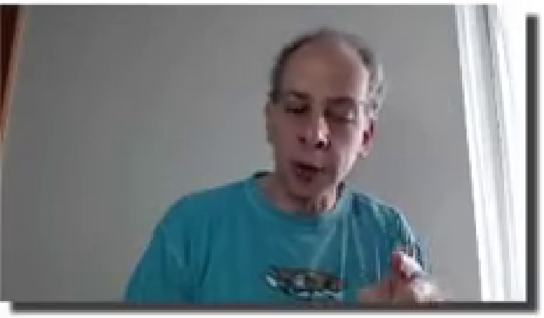
$$C_t = C_{ss} \quad \text{where } C_{ss} = \kappa \vec{n}$$

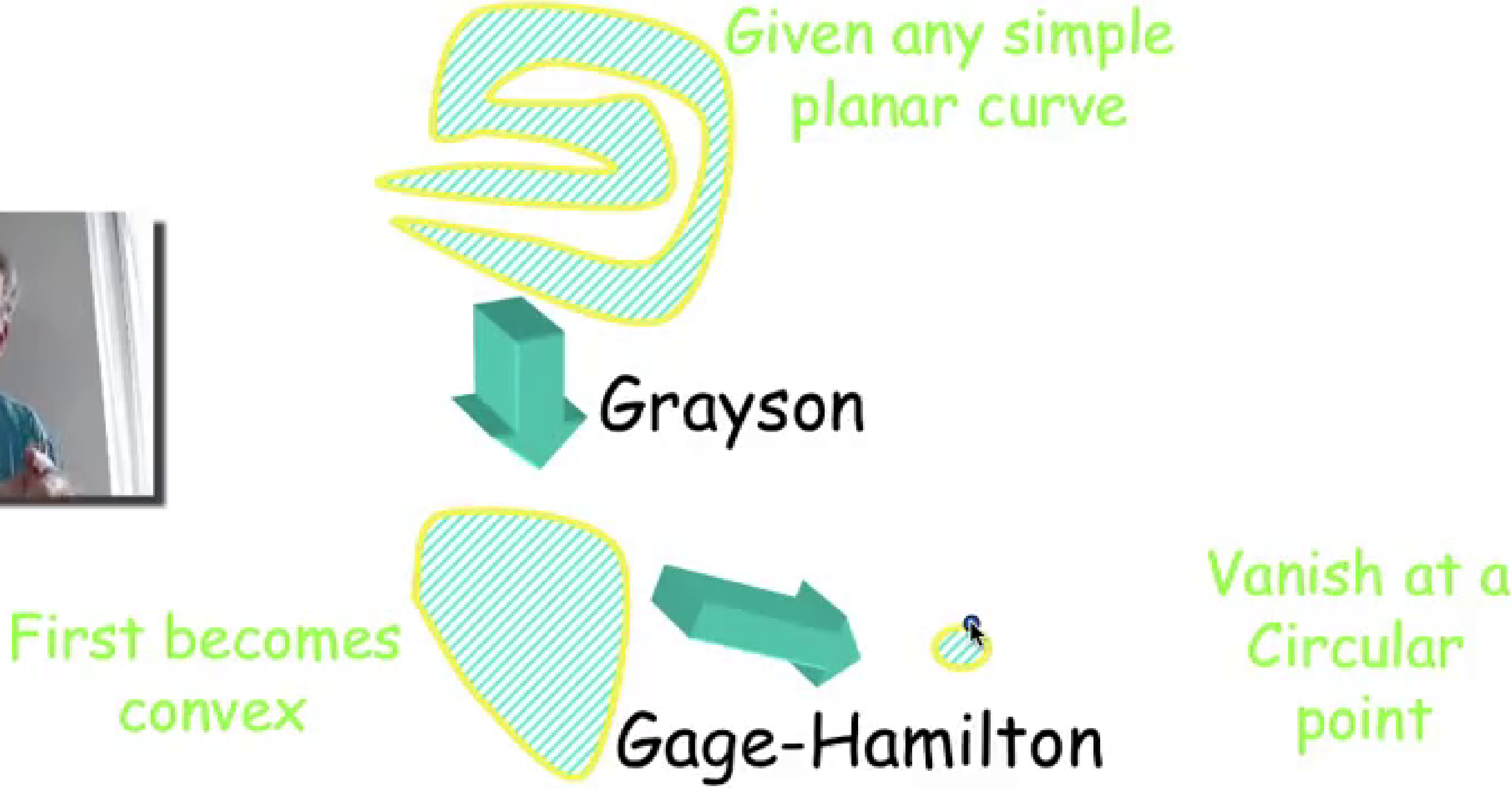
$$\frac{\partial C}{\partial t} = \frac{\partial^2 C}{\partial s^2}$$

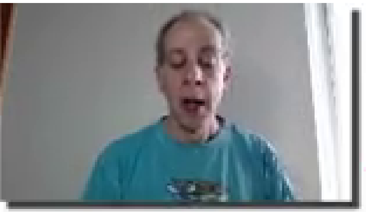




Curvature flow $C_t = \kappa \vec{n}$





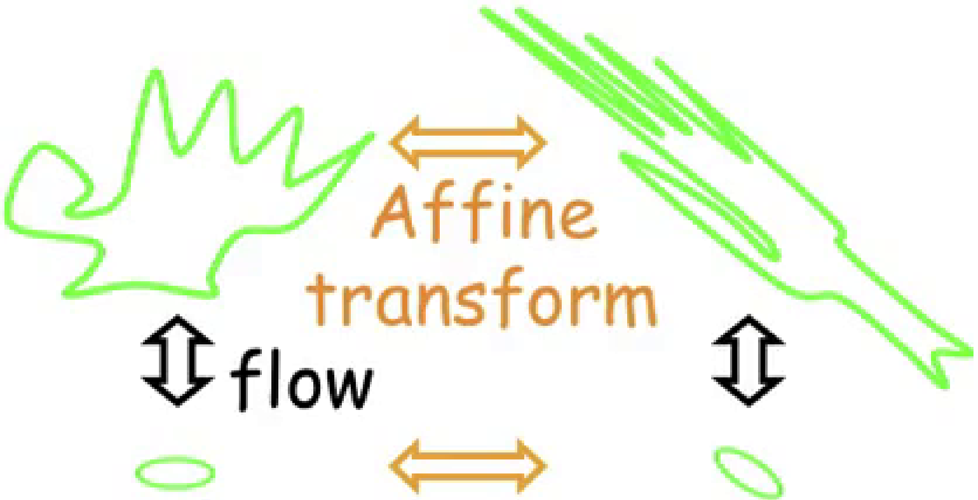


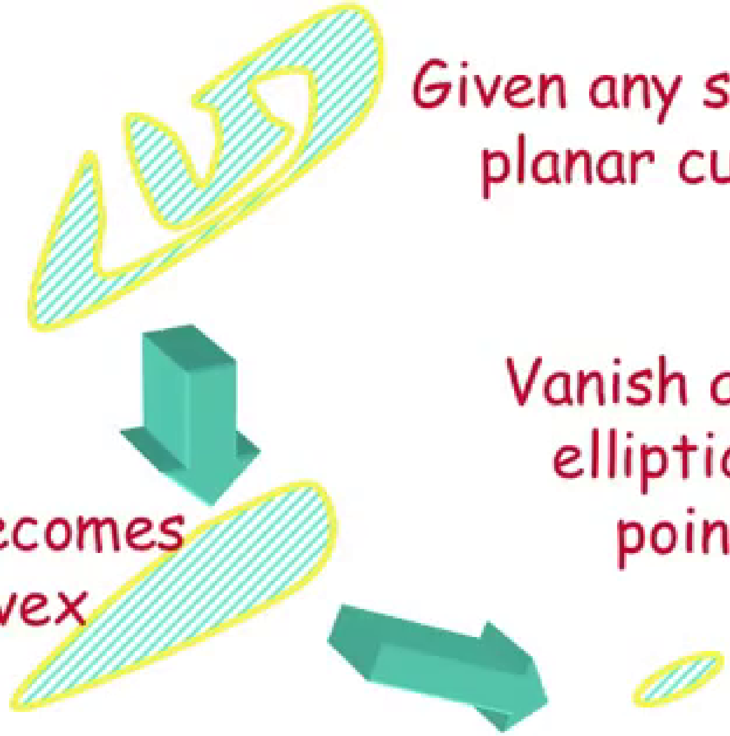
Affine heat equation

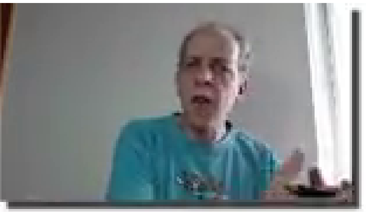
$C_t = C_{vv}$

- Special (equi-)affine heat flow $C_t = \kappa^{1/3}\vec{n}$

$$C_t = \langle C_{vv}, \vec{n} \rangle \vec{n} \quad \text{where } \langle C_{vv}, \vec{n} \rangle = \kappa^{1/3}$$



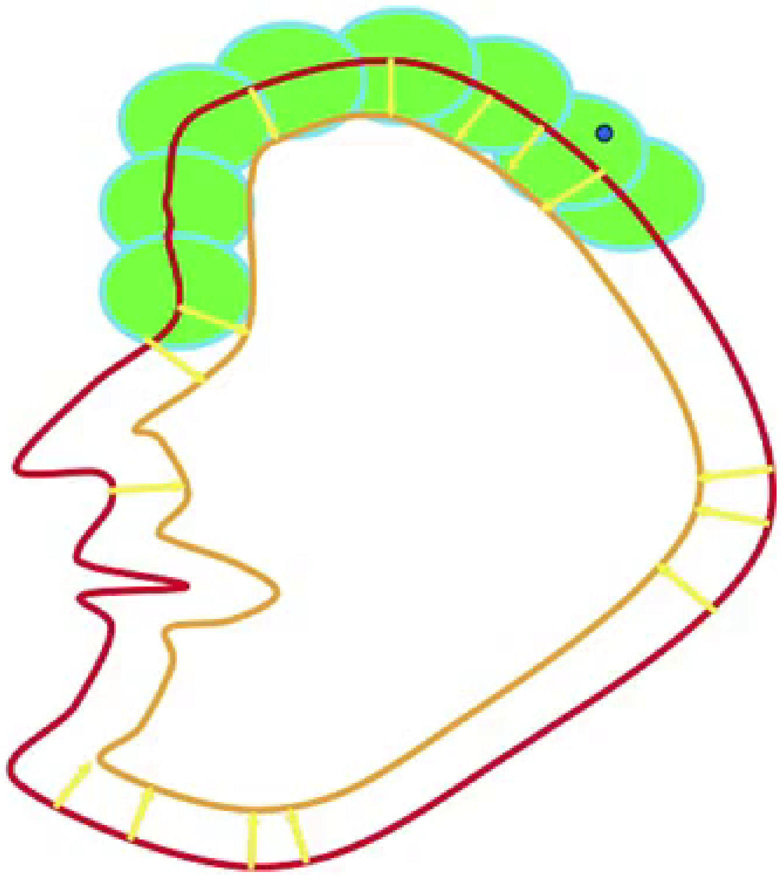


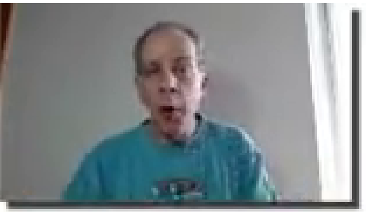


Constant flow

$$C_t = \vec{n}$$

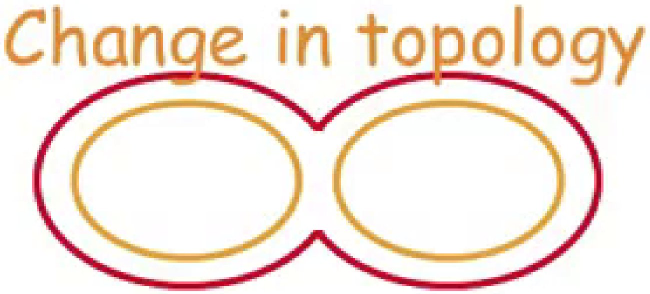
- Offset curves
- Equal-height contours of the distance transform
- Envelope of all disks of equal radius centered along the curve (Huygens principle)

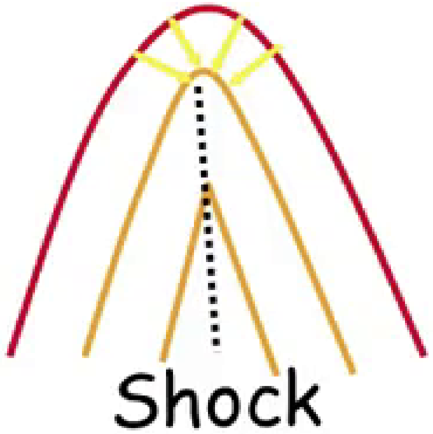


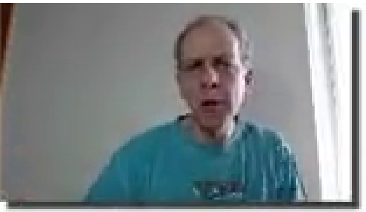


Constant flow $C_t = \vec{n}$

- Offset curves





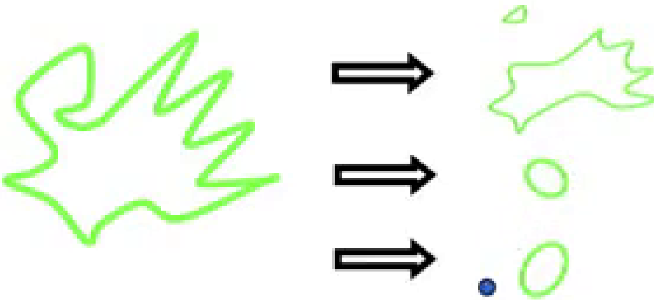


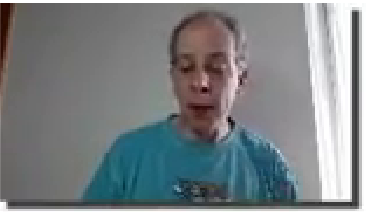
So far we defined

Constant flow $C_t = \vec{n}$

Curvature flow $C_t = \kappa \vec{n}$

Equi-affine flow $C_t = \kappa^{1/3} \vec{n}$



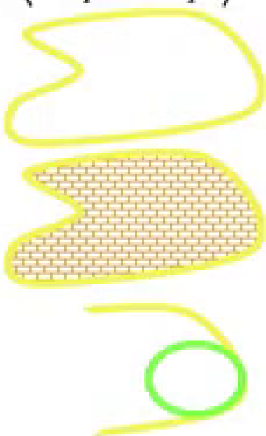


$$C_t = V\vec{n}$$

$$\frac{\partial}{\partial t}L = \frac{\partial}{\partial t}\oint \left\langle C_p, C_p \right\rangle^{1/2} dp = 2\oint \left\langle \frac{\partial}{\partial t}C_p, C_p \right\rangle dp = \ldots = -\int_0^L \kappa V ds$$

$$\frac{\partial}{\partial t}A = \frac{1}{2}\frac{\partial}{\partial t}\oint \left(C, C_p \right) dp = \oint \left(\frac{\partial}{\partial t}C, C_p \right) dp + \oint \left(C, \frac{\partial}{\partial t}C_p \right) dp = \ldots = -\int_0^L V ds$$

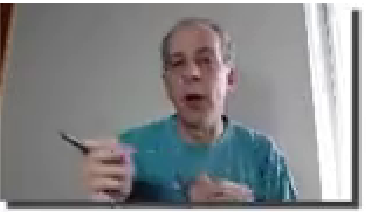
$$\frac{\partial}{\partial t}\kappa = \frac{\partial}{\partial t}\left(\frac{\left(C_p, C_{pp} \right)}{\left\langle C_p, C_p \right\rangle^{3/2}} \right) = \ldots = V_{ss} + \kappa^2 V$$



Length $\quad L_t = -\int_0^L \kappa V ds$

Area $\quad A_t = \int_0^L V ds$

Curvature $\quad \kappa_t = V_{ss} + \kappa^2 V$



Constant flow $(V = 1)$

Length $L_t = -\int_0^L \kappa V ds = -\int_0^L \kappa ds = -2\pi$

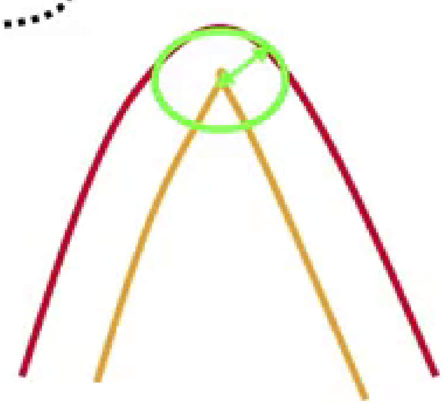
Area $A_t = -\int_0^L V ds = -\int_0^L ds = -L$

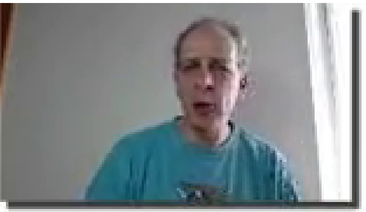
Curvature $\kappa_t = V_{ss} + \kappa^2 V = \kappa^2$

The curve vanishes at $t = \frac{L(0)}{2\pi}$

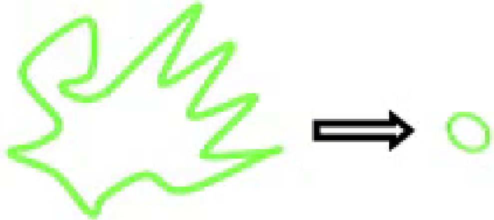
Riccati eq. $\kappa(p,t) = \frac{\kappa(p,0)}{1 - t\kappa(p,0)}$

Singularity (`shock') at $t = \rho(p,0)$





Curvature flow ($V = \kappa$)

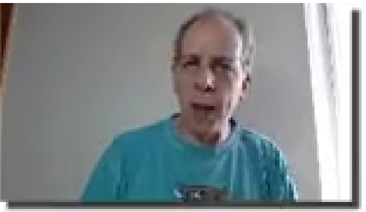


Length $$L_t = -\int_0^L \kappa V ds = -\int_0^L \kappa^2 ds$$

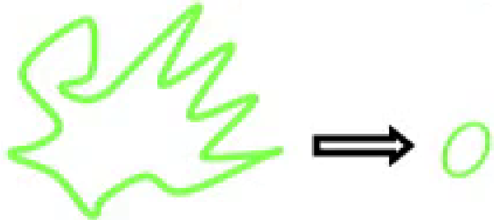
Area $$A_t = -\int_0^L V ds = -\int_0^L \kappa ds = -2\pi$$

Curvature $$\kappa_t = V_{ss} + \kappa^2 V = \kappa_{ss} + \kappa^3$$

The curve vanishes at $t = \frac{A(0)}{2\pi}$



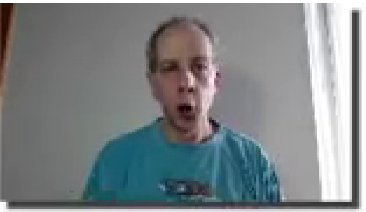
Equi-Affine flow $\left(V = \kappa^{1/3}\right)$



Length $$L_t = -\int_0^L \kappa V ds = -\int_0^L \kappa^{4/3} ds$$

Area $$A_t = -\int_0^L V ds = -\int_0^L \kappa^{1/3} ds$$

Curvature $$\kappa_t = V_{ss} + \kappa^2 V = \tfrac{1}{3}\kappa^{-2/3}\kappa_{ss} - \tfrac{2}{9}\kappa^{-5/3}{\kappa_s}^2 + \kappa^{7/3}$$

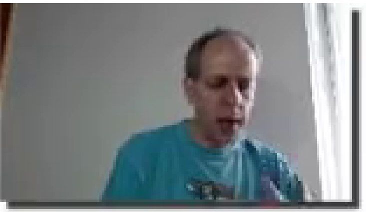


Geodesic active contours

$$C_t = \overbrace{\left(g(x,y)\kappa - \langle \nabla g(x,y), \vec{n} \rangle\right)}^{\checkmark}\vec{n}$$

$$C_t = g\,\kappa\,\vec{n}$$

$$g \sim \frac{1}{\nabla I}$$

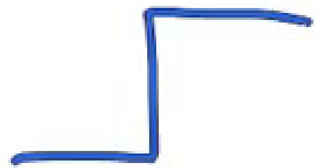


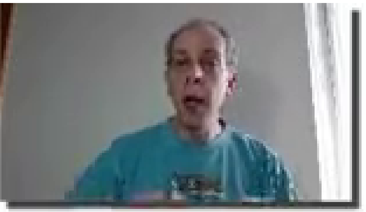
Geodesic active contours

$$C_t = \overbrace{\left(g(x,y)\kappa - \langle \nabla g(x,y), \vec{n} \rangle\right)}^{\checkmark}\vec{n}$$

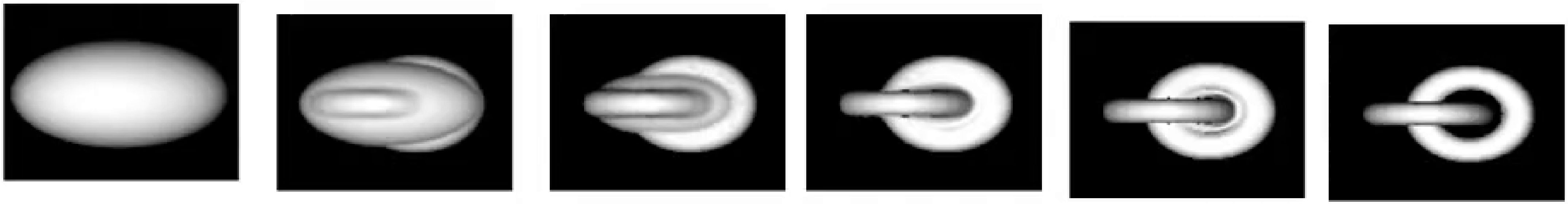
$$C_t = g\,\kappa\vec{n} + g\vec{n}$$

$$g \sim \frac{1}{\nabla I}$$



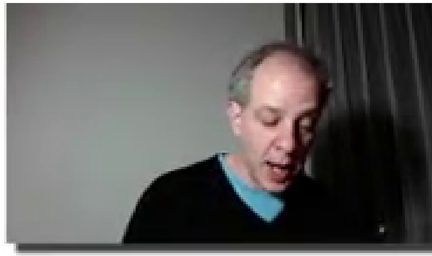


Surface evolution…



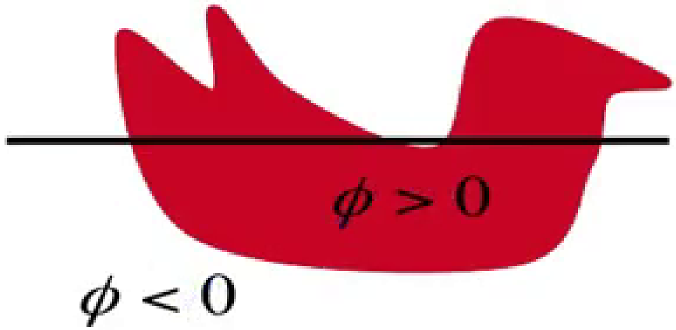
$$\frac{\partial S}{\partial t} = g\, k \vec{N}$$

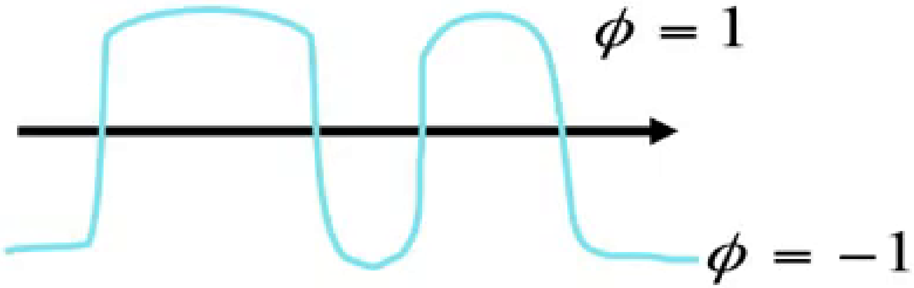
Implicit representation



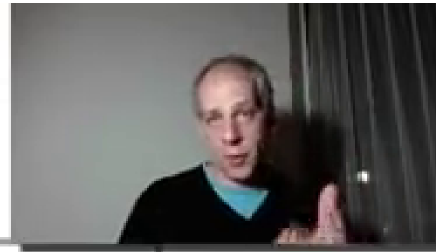
Consider a closed planar curve $C(p): \mathbf{S}^1 \to \mathbf{R}^2$

The geometric trace of the curve can be alternatively represented implicitly as $C = \{(x, y) \mid \phi(x, y) = 0\}$

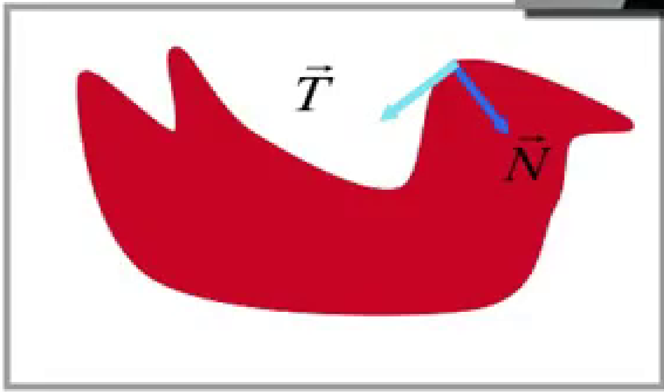




Properties of level sets



The level set normal ϕ



$$\vec{N} = -\frac{\nabla\phi}{|\nabla\phi|} \qquad \left(\vec{T} = \frac{\overline{\nabla}\phi}{|\nabla\phi|}\right)$$

Proof. Along the level sets we have zero change, that is $\phi_s = 0$, but by the chain rule

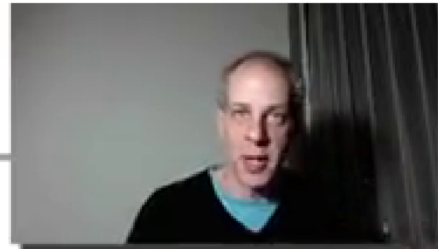
$$\phi_s(x,y) = \phi_x x_s + \phi_y y_s = \left\langle \nabla\phi, \vec{T} \right\rangle$$

$$\begin{bmatrix} a, & c \\ b, & d \end{bmatrix}$$

So,

$$\left\langle \frac{\nabla\phi}{|\nabla\phi|}, \vec{T} \right\rangle = 0 \Rightarrow \frac{\nabla\phi}{|\nabla\phi|} \perp \vec{T} \Rightarrow \vec{N} = -\frac{\nabla\phi}{|\nabla\phi|}$$

Properties of level sets

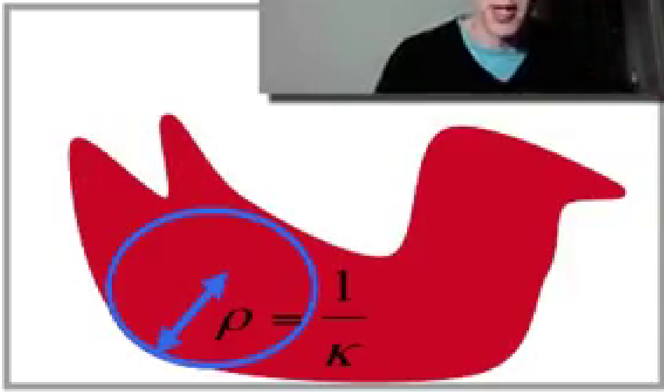


The level set curvature

$C_{ss} = \kappa \vec{N}$

$$\kappa = \operatorname{div}\left(\frac{\nabla\phi}{|\nabla\phi|}\right)$$

$\operatorname{div}(\alpha, \beta) = \alpha_x + \beta_y$



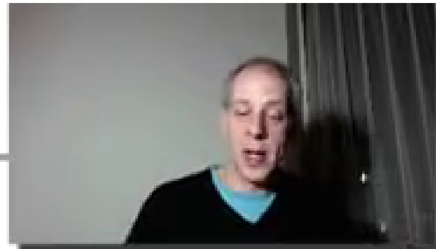
Proof: zero change along the level sets, $\phi_{ss} = 0$, also

$$\phi_{ss}(x,y) = \frac{d}{ds}(\phi_x x_s + \phi_y y_s) = \frac{d}{ds}\left\langle \nabla\phi, \vec{T} \right\rangle = \left\langle \frac{d}{ds}\nabla\phi, \vec{T} \right\rangle + \left\langle \nabla\phi, \kappa\vec{N} \right\rangle = 0$$

$$\kappa\left\langle \nabla\varphi, \frac{\nabla\varphi}{|\nabla\varphi|} \right\rangle = \kappa\,|\nabla\varphi| = -\left\langle [\varphi_{xx}x_s + \varphi_{xy}y_s, \varphi_{xy}x_s + \varphi_{yy}y_s], \frac{\nabla\varphi}{|\nabla\varphi|} \right\rangle$$

ϕ

Properties of level sets

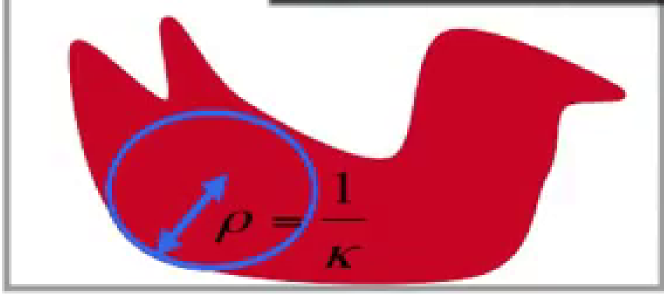


The level set curvature

$C_{ss} = \kappa \vec{n}$

$$\kappa = \operatorname{div}\left(\frac{\nabla \phi}{|\nabla \phi|}\right)$$

$\operatorname{div}(\alpha, \beta) = \alpha_x + \beta_y$



Proof: zero change along the level sets, $\phi_{ss} = 0$, also

$$\phi_{ss}(x, y) = \frac{d}{ds}(\phi_x x_s + \phi_y y_s) = \frac{d}{ds}\left\langle \nabla \phi, \vec{T} \right\rangle = \left\langle \frac{d}{ds} \nabla \phi, \vec{T} \right\rangle + \left\langle \nabla \phi, \kappa \vec{N} \right\rangle = 0$$

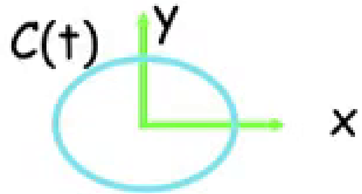
$$\kappa \left\langle \nabla \varphi, \frac{\nabla \varphi}{|\nabla \varphi|} \right\rangle = \kappa |\nabla \varphi| = -\left\langle [\varphi_{xx} x_s + \varphi_{xy} y_s, \varphi_{xy} x_s + \varphi_{yy} y_s], \frac{\bar{\nabla} \varphi}{|\nabla \varphi|} \right\rangle$$

ϕ

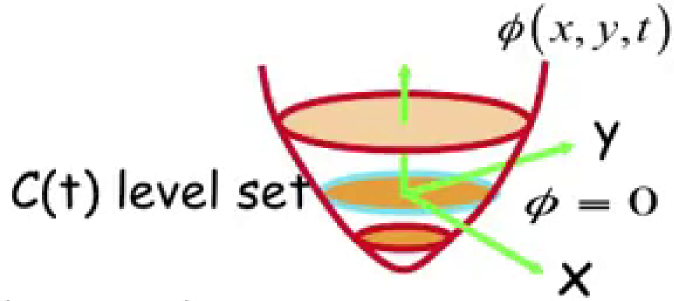
Level Set Formulation

(Osher-Sethian)

$$\phi(x,y): \boldsymbol{R}^2 \rightarrow \boldsymbol{R} \qquad C = \{(x,y): \phi(x,y) = 0\}$$



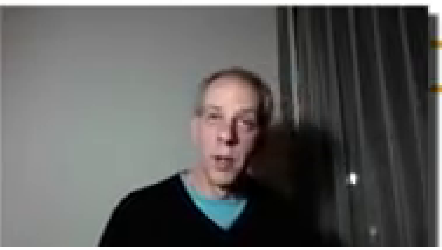
$$\frac{dC}{dt} = V\vec{N} \Leftrightarrow \boxed{\frac{d\phi}{dt} = V\left|\nabla\phi\right|}$$



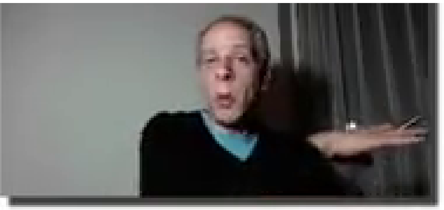
$$0 = \frac{\partial\phi(x,y;t)}{\partial t} = \phi_x x_t + \phi_y y_t + \phi_t$$

$$-\phi_t = \phi_x x_t + \phi_y y_t = \langle\nabla\phi, C_t\rangle = \langle\nabla\phi, V\vec{N}\rangle = V\langle\nabla\phi, \vec{N}\rangle$$

$$\vec{N} = -\frac{\nabla\phi}{|\nabla\phi|} \qquad -V\langle\nabla\phi, \vec{N}\rangle = V\left\langle\nabla\phi, \frac{\nabla\phi}{|\nabla\phi|}\right\rangle = V\,|\nabla\phi|$$

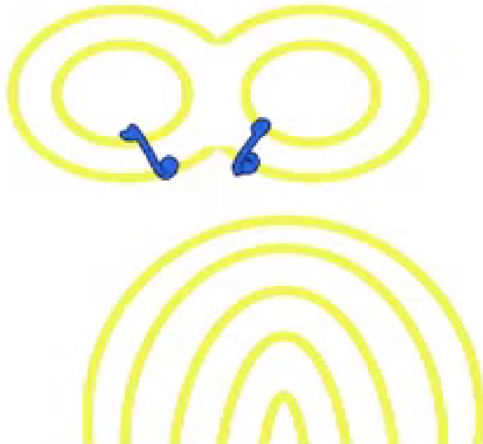


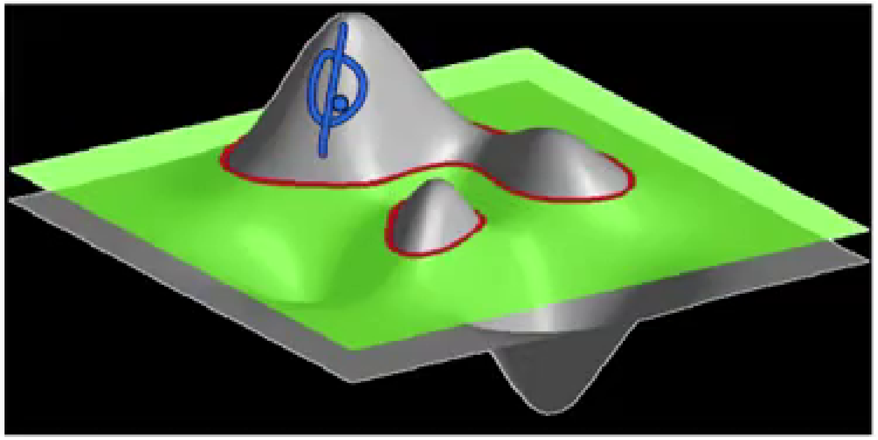
$$\Rightarrow \boxed{\phi_t = V\,|\nabla\phi|}$$

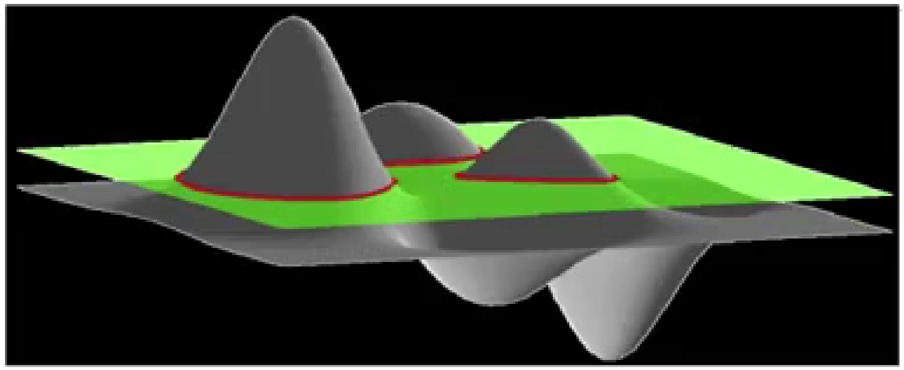


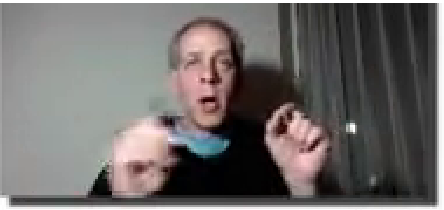
Level Set Formulation

- Handles changes in topology
- Numeric grid points never collide or drift apart.
- Natural philosophy for dealing with gray level images.



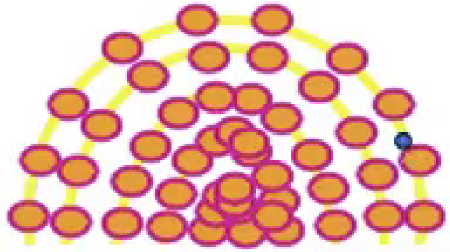


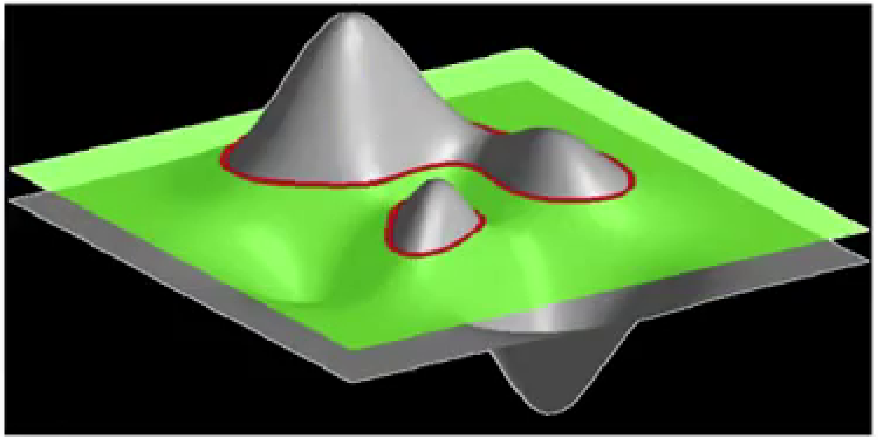


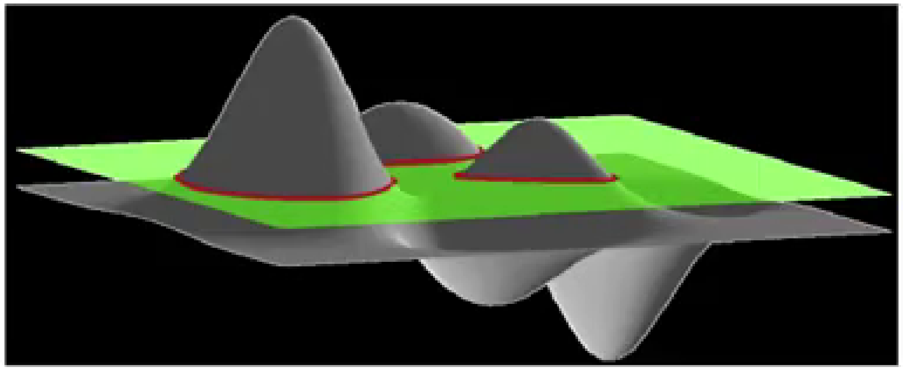


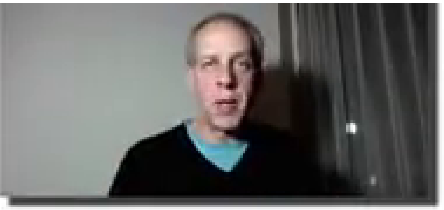
Level Set Formulation

- Handles changes in topology
- Numeric grid points never collide or drift apart.
- Natural philosophy for dealing with gray level images.









Level Set Formulation

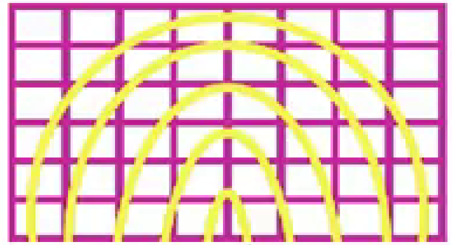
$V = g\kappa$

- Handles changes in topology

$\phi_t = g\kappa|\nabla\phi|$

- Numeric grid points never collide or drift apart.

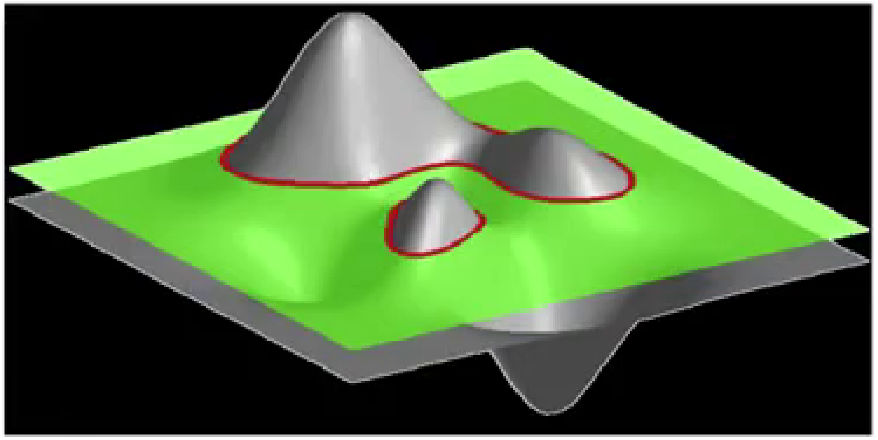
$\phi_t = |\nabla\phi|$

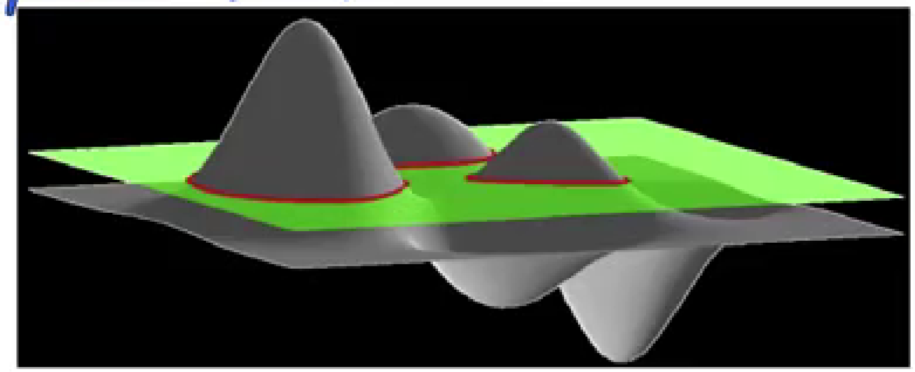


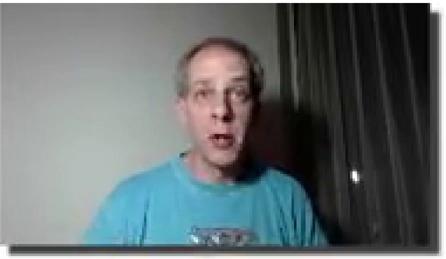
- Natural philosophy for dealing with gray level images.

$\phi_t = V\nabla\phi$

$\phi = 0$







Calculus of Variations

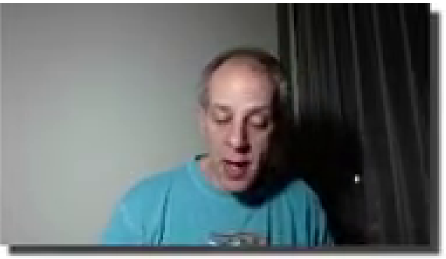
Generalization of Calculus that seeks to find the path, curve, surface, etc., for which a *given* Functional has a minimum or maximum.

Goal: find extrema values of integrals of the form

$$\int F(u, u_x)\,dx$$

It has an extremum only if the Euler-Lagrange Differential Equation is satisfied,

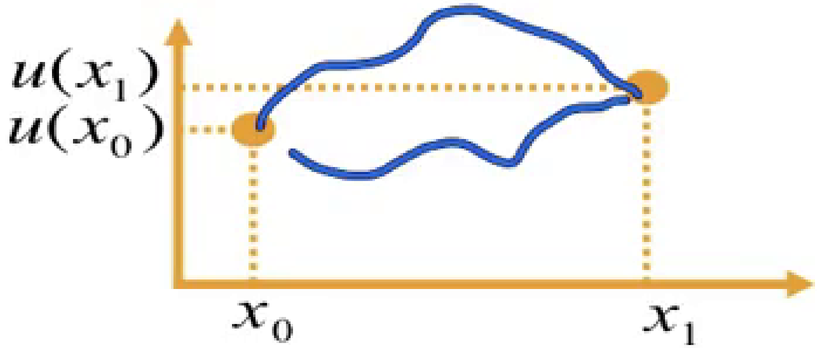
$$\left(\frac{\partial}{\partial u} - \frac{d}{dx}\frac{\partial}{\partial u_x}\right)F(u, u_x) = 0$$



Calculus of Variations

Example: Find the shape of the curve {x,u(x)} with shortest length:

$$\int_{x_0}^{x_1} \sqrt{1+u_x^2}\, dx$$

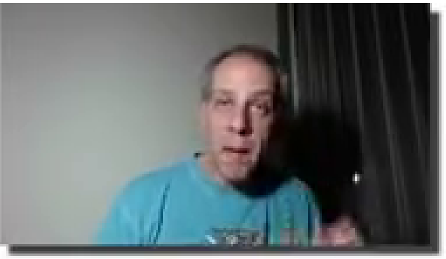


given *u(x0),u(x1)*

Solution: a differential equation that *u(x)* must satisfy,

$$\left(\frac{\partial}{\partial u} - \frac{d}{dx}\frac{\partial}{\partial u_x}\right) F(u, u_x) = 0$$

$$\frac{u_{xx}}{\cancel{\left(1+u_x^2\right)^{3/2}}} = 0 \quad \Rightarrow \quad u_x = a \quad \Rightarrow \quad u(x) = ax + b$$



Calculus of Variations

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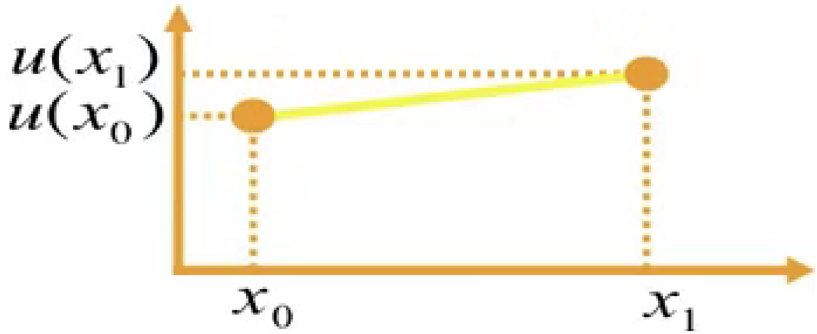

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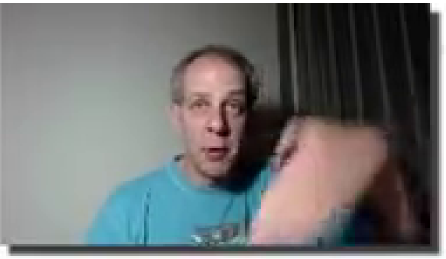
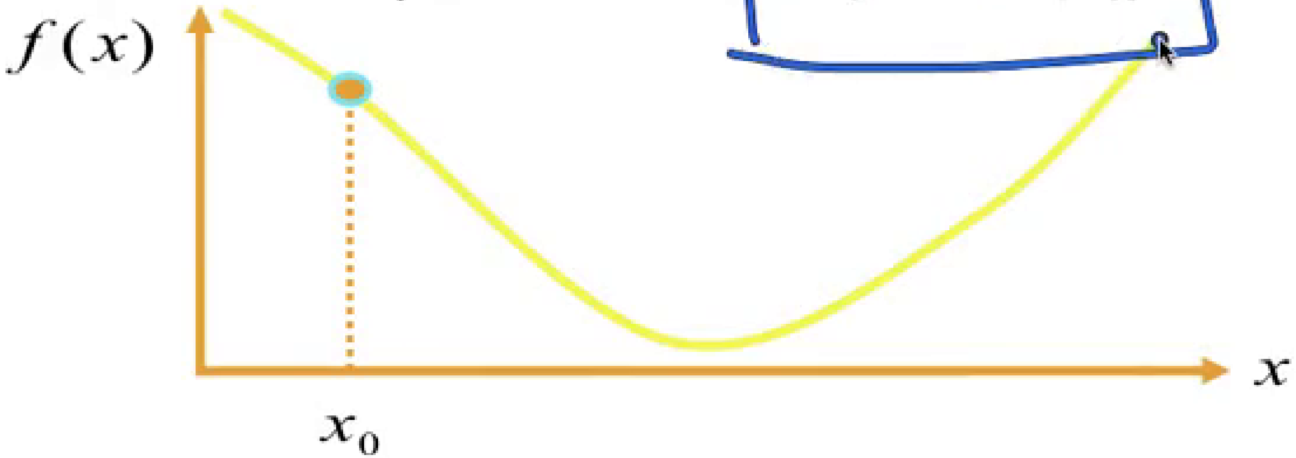
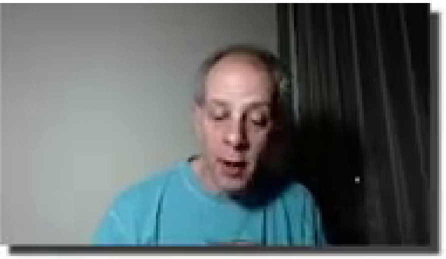


Extrema points in calculus

$$\forall \eta : \lim_{\varepsilon \to 0}\left(\frac{df(x+\varepsilon\eta)}{d\varepsilon}\right) = 0 \Leftrightarrow \forall \eta : f_x(x)\eta = 0 \Leftrightarrow \underline{f_x(x) = 0}$$

Gradient descent process $\boxed{x_t = -f_x}$





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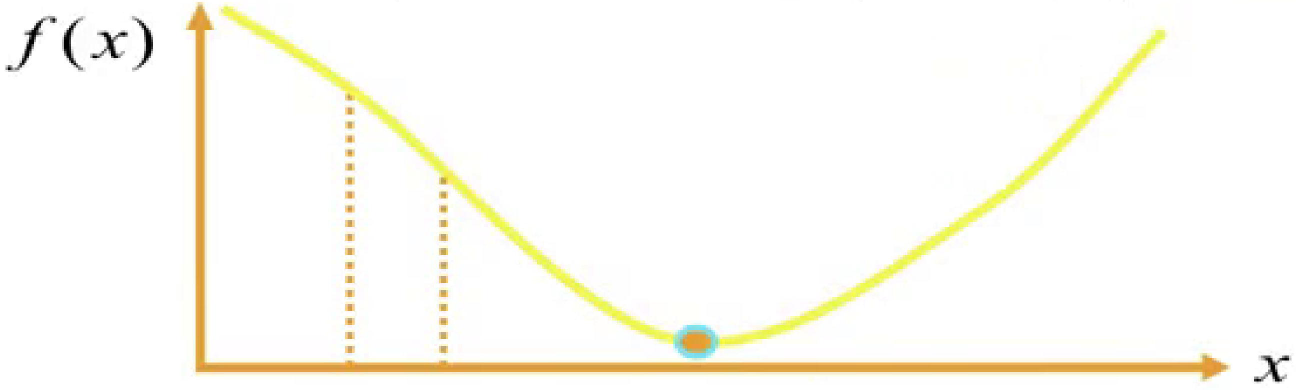
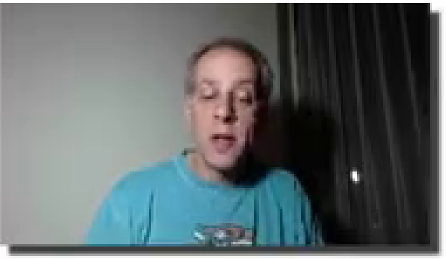
Gradient descent process $\quad x_t = -f_x \quad \frac{x(t+\Delta t) - x(t)}{\Delta t}$



$x_0 \quad x_1 = x_0 - dt f_x(x_0)$



Extrema points in calculus

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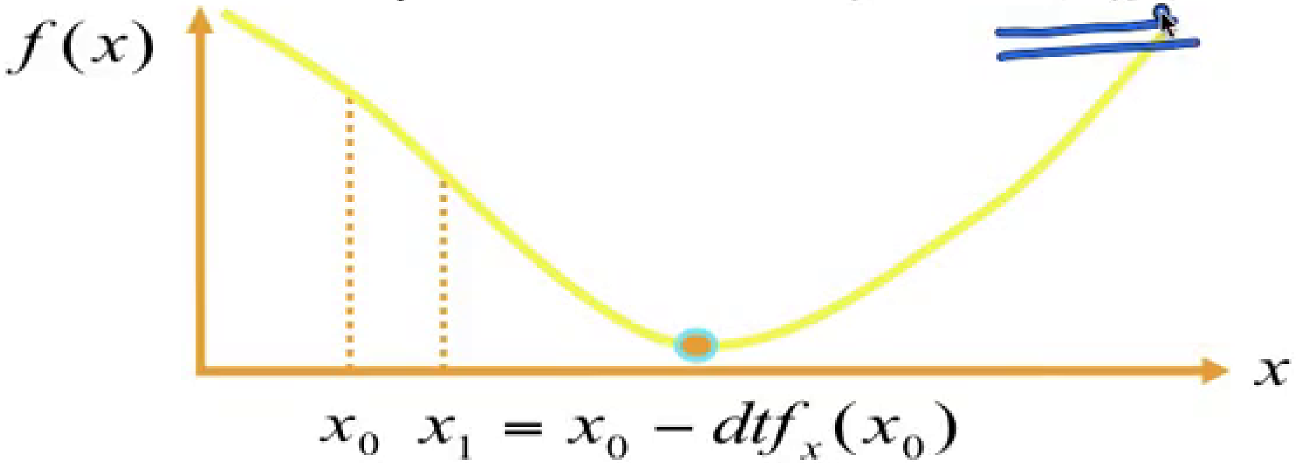
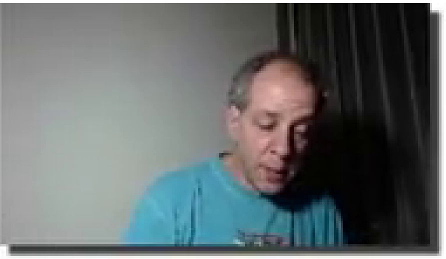
Gradient descent process

$$x_t = -f_x$$

$$x_t = 0$$





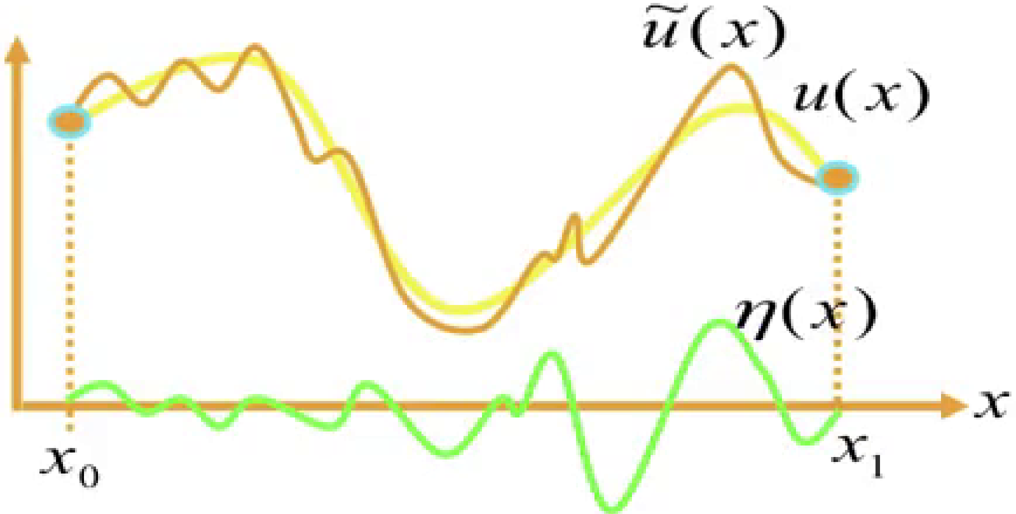
Calculus of variations

$$E(u(x)) = \int F(u, u_x)\,dx$$

$$\tilde{u}(x) = u(x) + \varepsilon \eta(x)$$

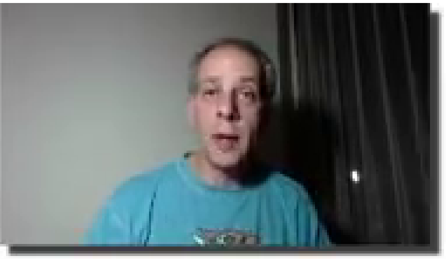
$$\forall \eta(x) : \lim_{\varepsilon \to 0}\left(\frac{d}{d\varepsilon} \int F(\tilde{u}, \tilde{u}_x)\,dx \right) \stackrel{?}{=} 0$$

$$\frac{\delta E(u)}{\delta u} = \left(\frac{\partial}{\partial u} - \frac{d}{dx}\frac{\partial}{\partial u_x} \right) F(u, u_x)$$



Gradient descent process

$$u_t = -\frac{\delta E(u)}{\delta u}$$



Calculus of variations

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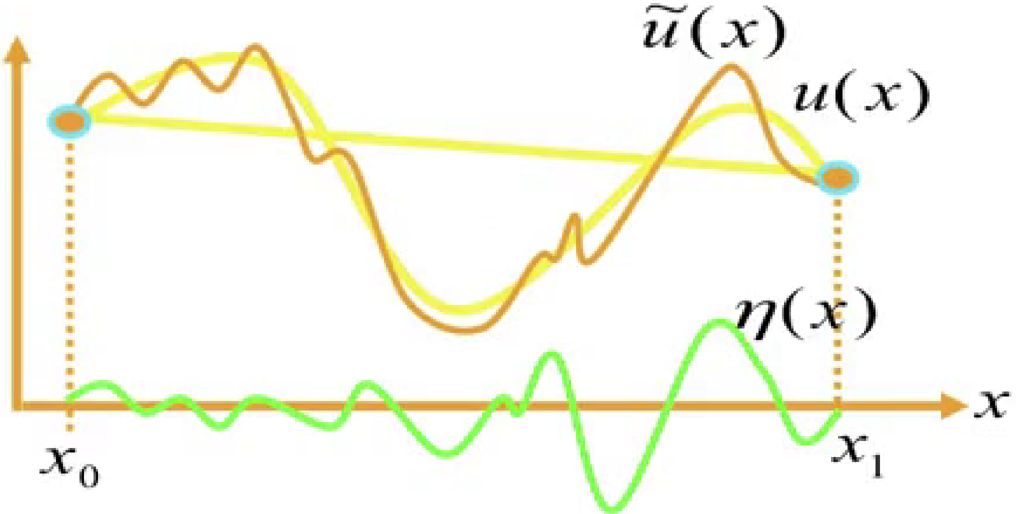
$$\tilde{u}(x) = u(x) + \varepsilon\eta(x)$$

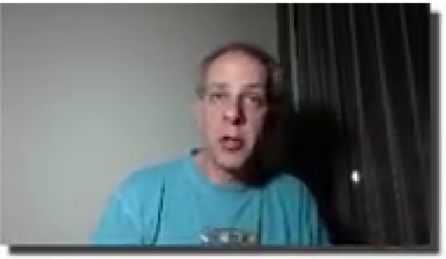
$$\forall\,\eta(x) : \lim_{\varepsilon \to 0}\left(\frac{d}{d\varepsilon}\int F(\tilde{u}, \tilde{u}_x)\,dx\right) \stackrel{?}{=} 0$$

$$\Downarrow$$

$$\frac{\delta E(u)}{\delta u} = \left(\frac{\partial}{\partial u} - \frac{d}{dx}\frac{\partial}{\partial u_x}\right)F(u, u_x)$$



Gradient descent process $\quad u_t = -\dfrac{\delta E(u)}{\delta u}$

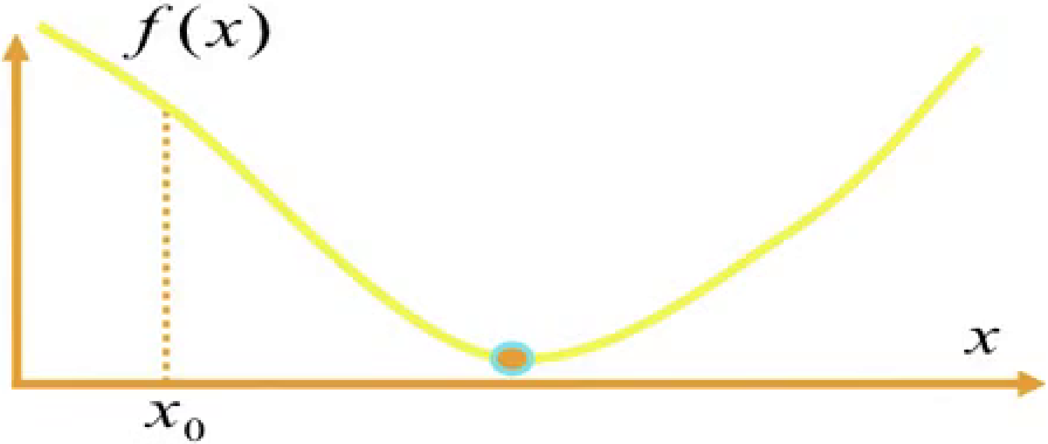


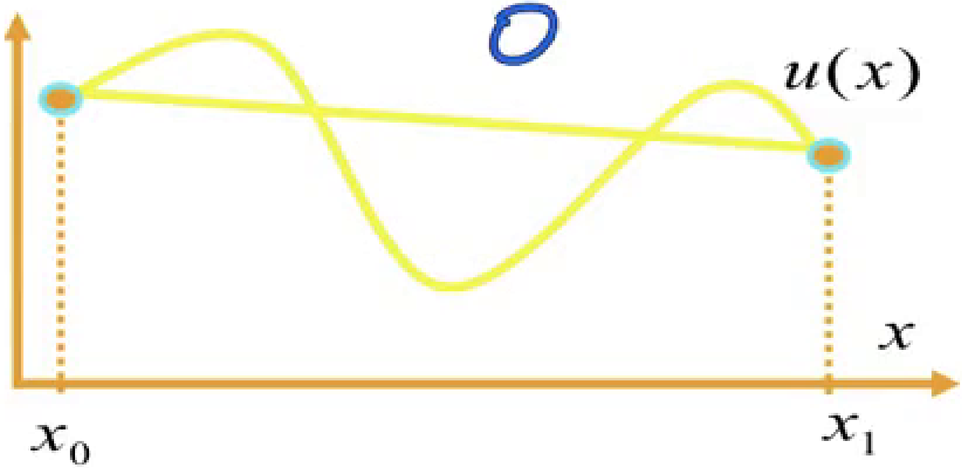
Conclusions

- Gradient descent process

Calculus $\arg\min\limits_{x} f(x) \Rightarrow x_t = -f_x$

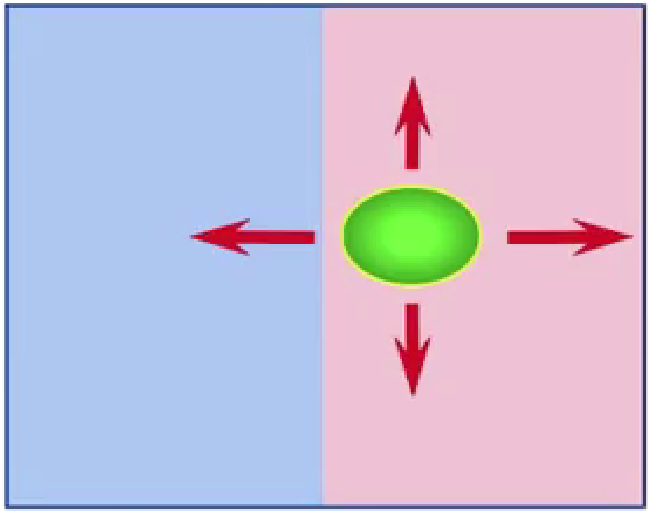
Calculus of variations $\arg\min\limits_{u(x)} \underbrace{\boxed{\int F(u,u_x)dx}}_{E(u)} \Rightarrow \underbrace{u_t}_{=0} = -\overbrace{\boxed{\frac{\delta E(u)}{\delta u}}}^{\text{Euler-Lagrange}}$



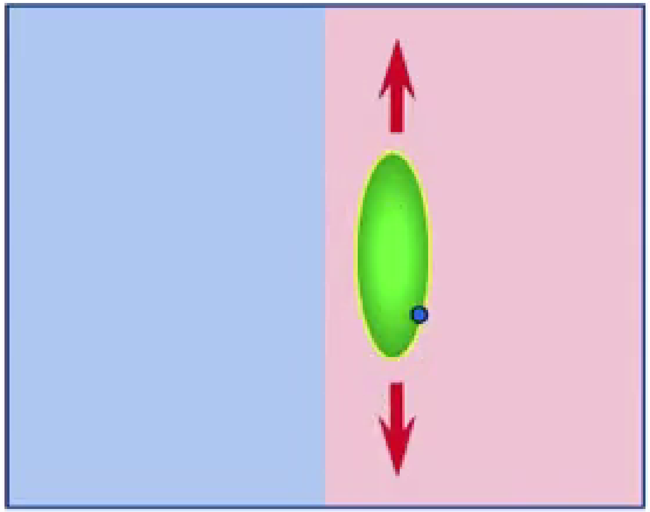


Anisotropic diffusion

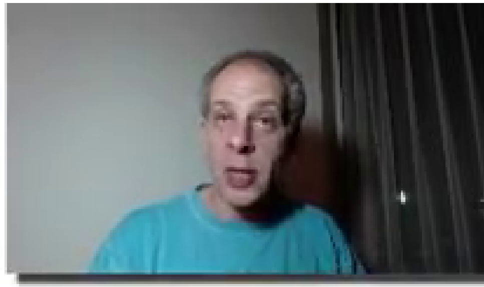
Isotropic vs. Anisotropic Smoothing

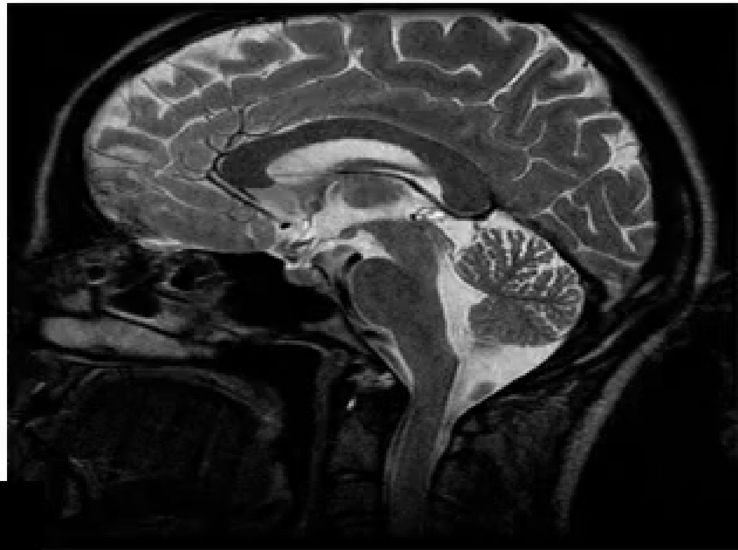


Isotropic smoothing



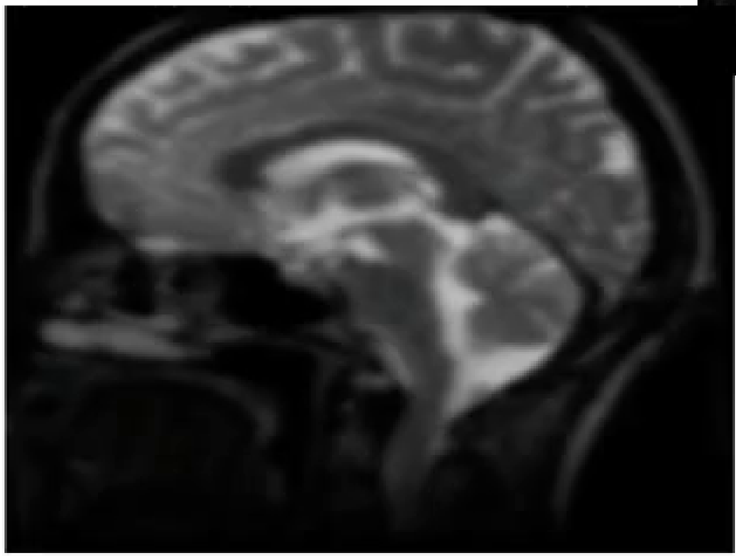
Anisotropic smoothing



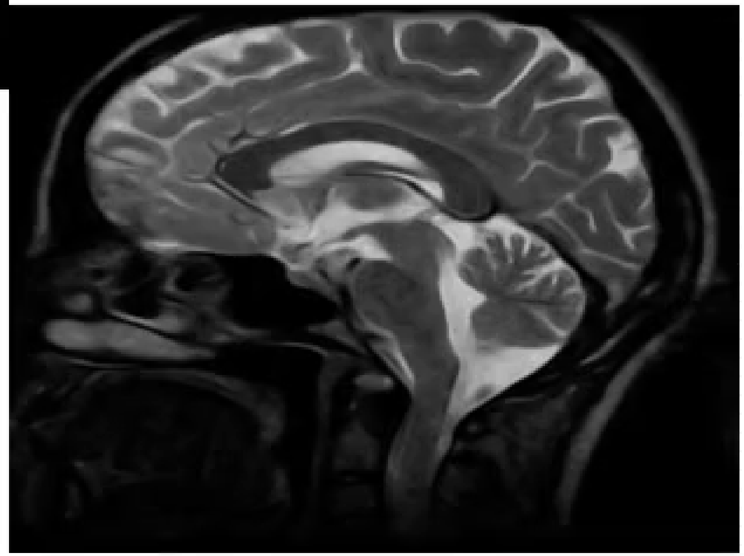


Isotropic (Heat equation)

Anisotropic



$$\Delta I = div(g \nabla I)$$

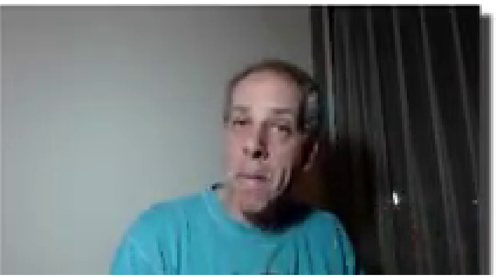


$$\frac{\partial I(x,y,t)}{\partial t} = \Delta I$$

$$\frac{\partial I(x,y,t)}{\partial t} = div(g(|\nabla I|)\nabla I)$$

$$\min_I \int_\Omega \rho(|\nabla I|)\,d\Omega \quad \Rightarrow \quad \frac{\partial I(x,y,t)}{\partial t} = div\left(\rho' \frac{\nabla I}{|\nabla I|}\right)$$

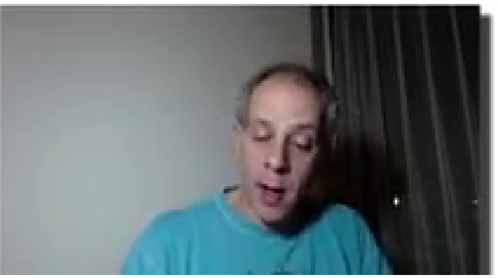
$$\int F(u, u_x)$$



$$\min_I \int_\Omega \rho(|\nabla I|)\,d\Omega \quad \Rightarrow \quad \frac{\partial I(x,y,t)}{\partial t} = div\left(\rho' \frac{\nabla I}{|\nabla I|}\right)$$

$$\rho(a) = a^2 \qquad \rho' = 2a$$

$$\int |\nabla I|^2 \qquad I_t = div\left(\overbrace{|\nabla I|}^{\rho'} \frac{\nabla I}{|\nabla I|}\right)$$



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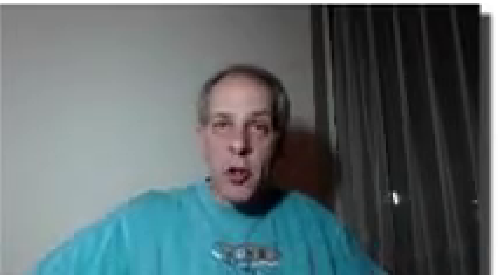
$$I_t = div\left(\overbrace{\cancel{|\nabla I|}}^{\rho'} \frac{\nabla I}{\cancel{|\nabla I|}}\right)$$

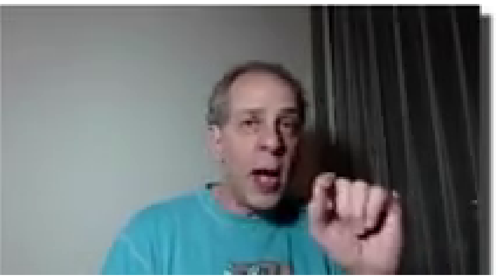
$$= \Delta I$$

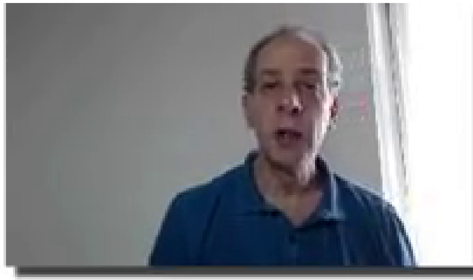


$$\min_I \int_\Omega \rho(|\nabla I|)\,d\Omega \quad \Rightarrow \quad \frac{\partial I(x,y,t)}{\partial t} = div\left(\rho' \frac{\nabla I}{|\nabla I|}\right)$$

$\rho(a) = a \Rightarrow \rho' = 1$ Total Variation

$$\int |\nabla I| \qquad I_t = div\left(\frac{\nabla I}{|\nabla I|}\right)$$

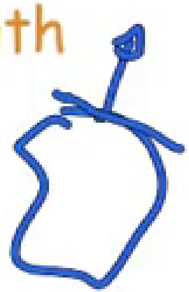


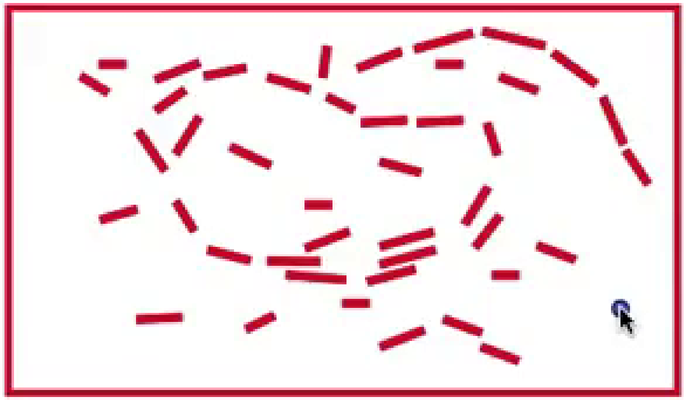


Edge Detection

- Edge Detection:
  - The process of labeling the locations in the image where the gray level's "rate of change" is high.
    - OUTPUT: "edgels" locations, direction, strength

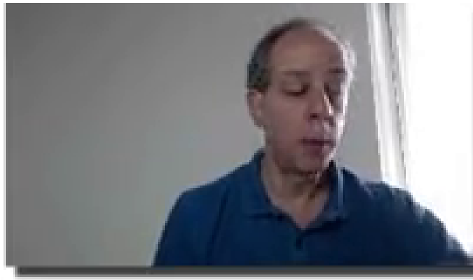
∇I





- Edge Integration:
  - The process of combining "local" and perhaps sparse and non-contiguous "edgel"-data into meaningful, long edge curves (or closed contours) for segmentation
    - OUTPUT: edges/curves consistent with the local data

Edge Detection



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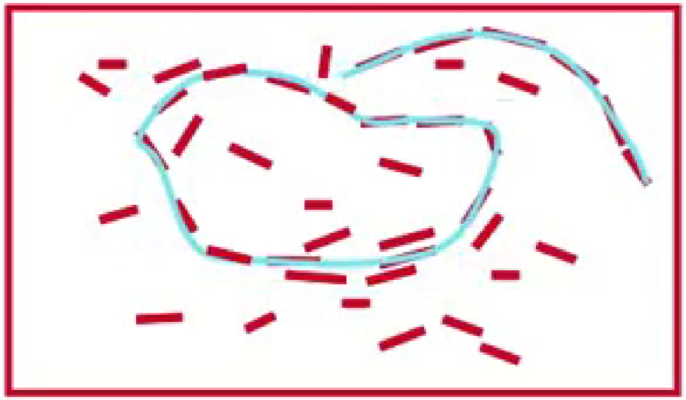

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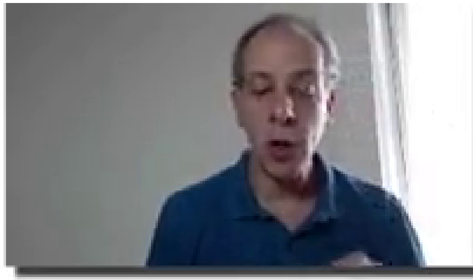


Active Contours

Image

Edge Indicator Function

$$g(x, y) = \frac{1}{1 + |\nabla(G_\sigma * I)|^2}$$

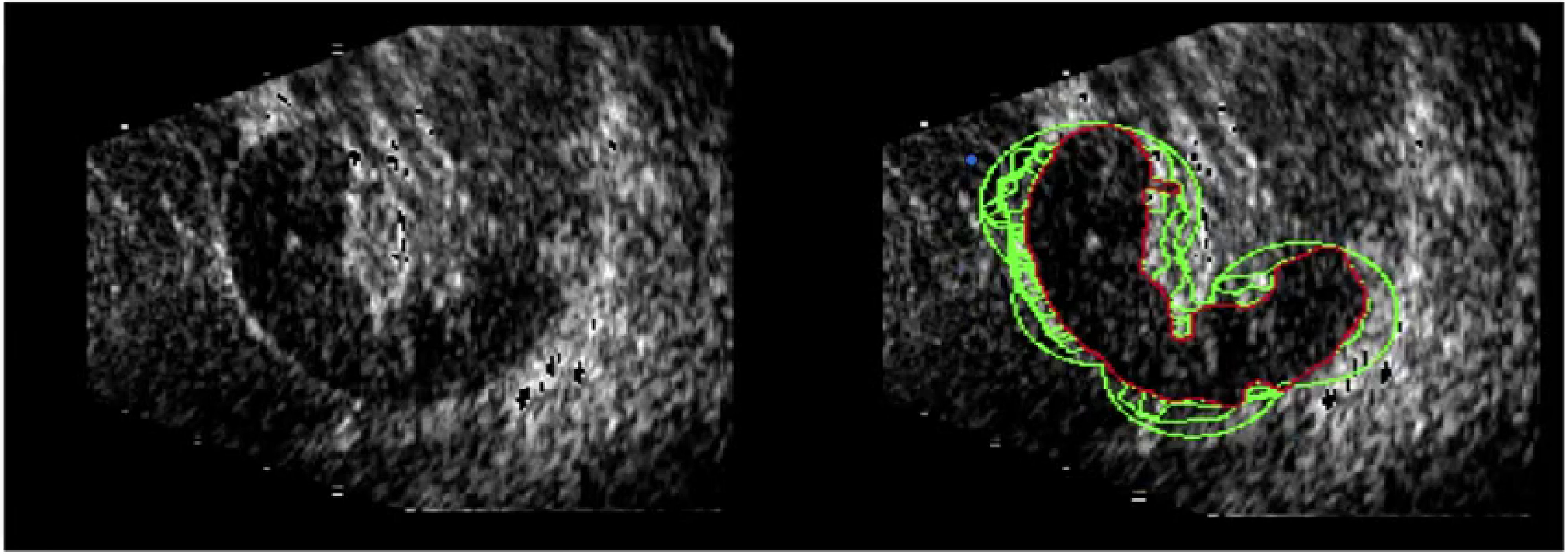
"nice" curves that optimize a functional of g( ), i.e.

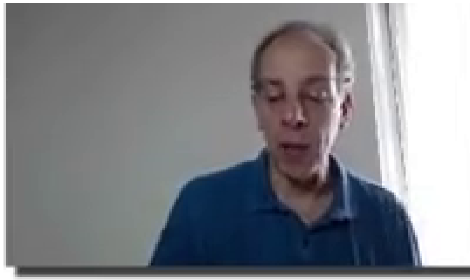
$$\int_{curve} g(\,)ds$$

nice: "regularized", smooth, fit some prior information

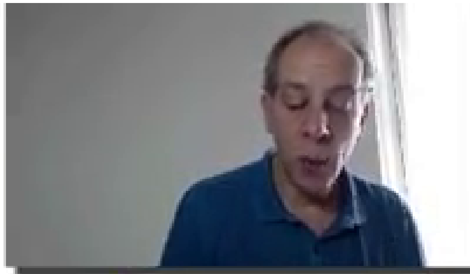
Edge Curves

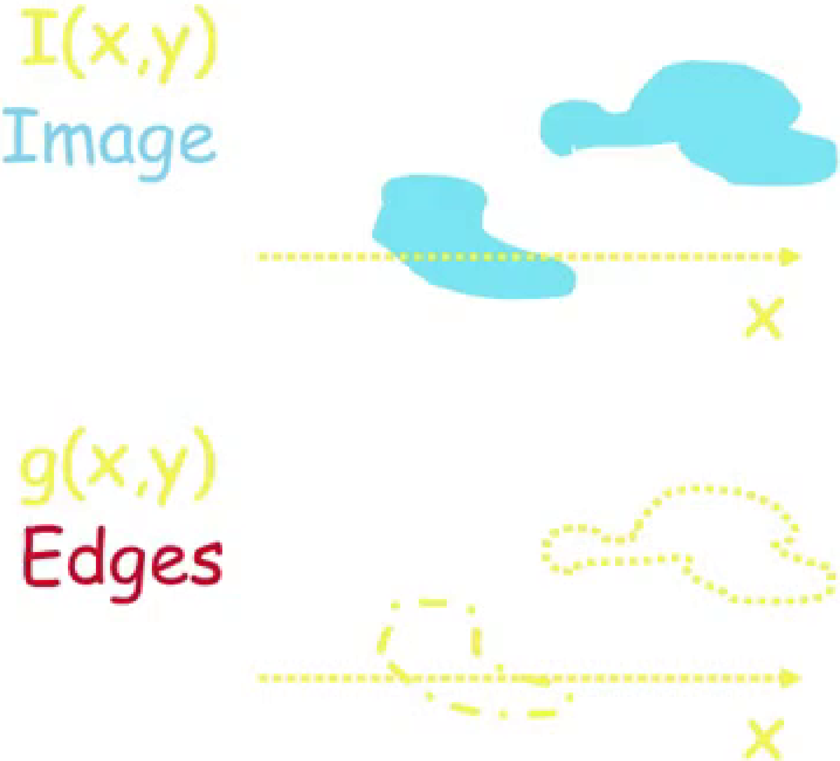
Segmentation

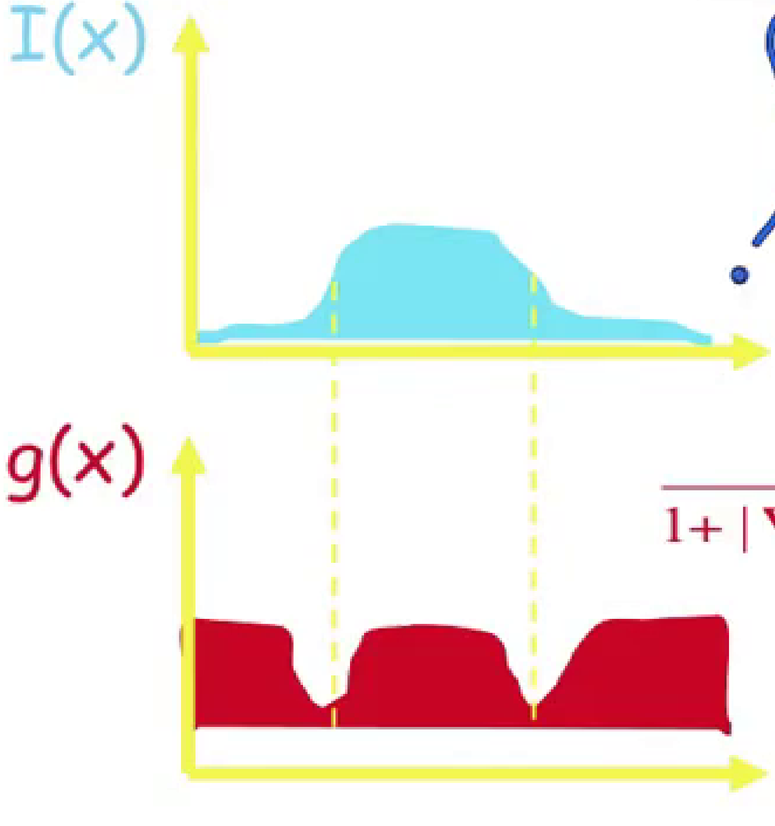




Potential Functions

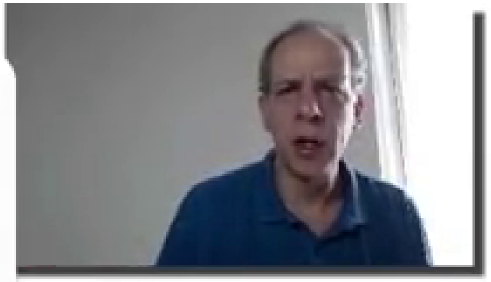






$\int g$

$$\frac{1}{1+|\nabla(G_\sigma * I)|^2}$$

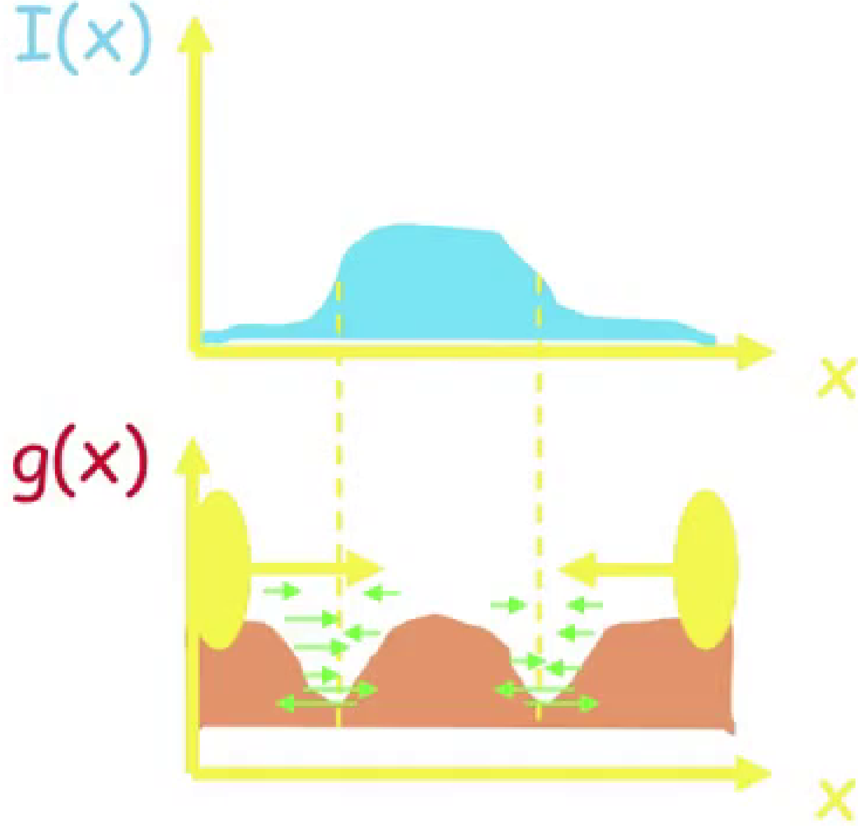


Geodesic Active Contours in 1D

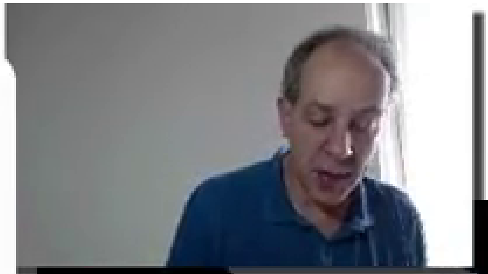
Geodesic active contours are reparameterization invariant

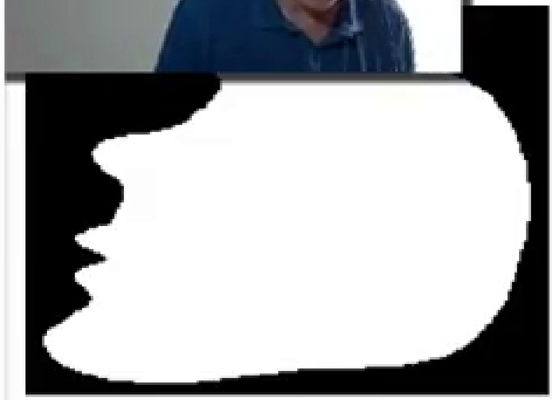
$$\boxed{\frac{dC}{dt} = \left(g(C)\kappa - \left\langle \nabla g(C), \vec{N} \right\rangle\right)\vec{N}}$$

$\int g$



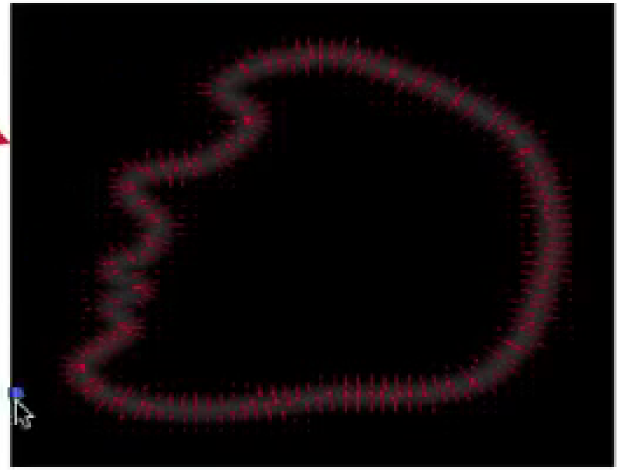
Geodesic Active Contours in 2D



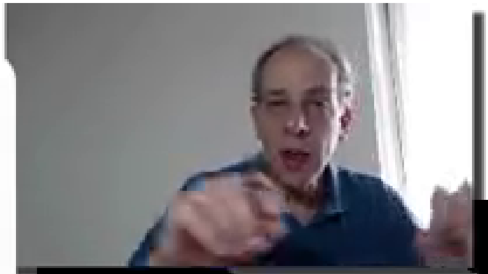


$G_s * I$

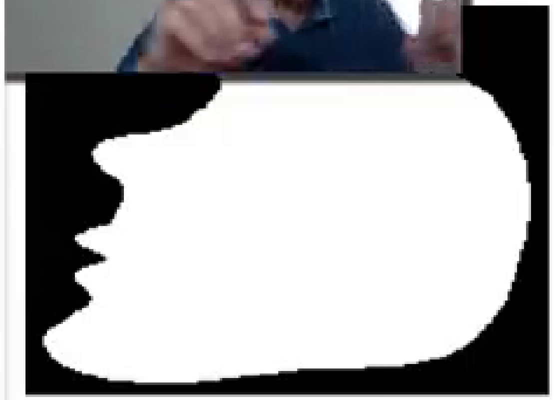
$$g(x) = \frac{1}{1 + |\nabla(G_\sigma * I)|^2}$$



$$\boxed{\frac{dC}{dt} = \left(g(C)\kappa - \left\langle \nabla g(C), \vec{N} \right\rangle\right)\vec{N}}$$

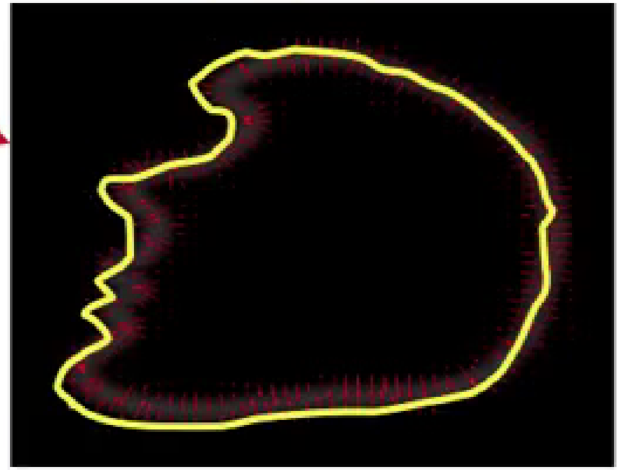


Geodesic Active Contours in 2D

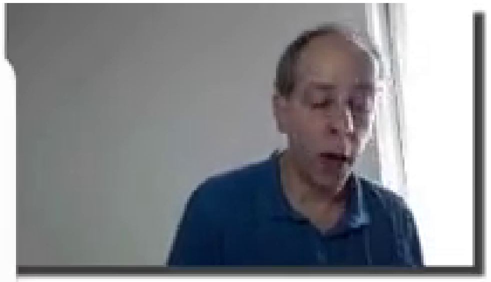


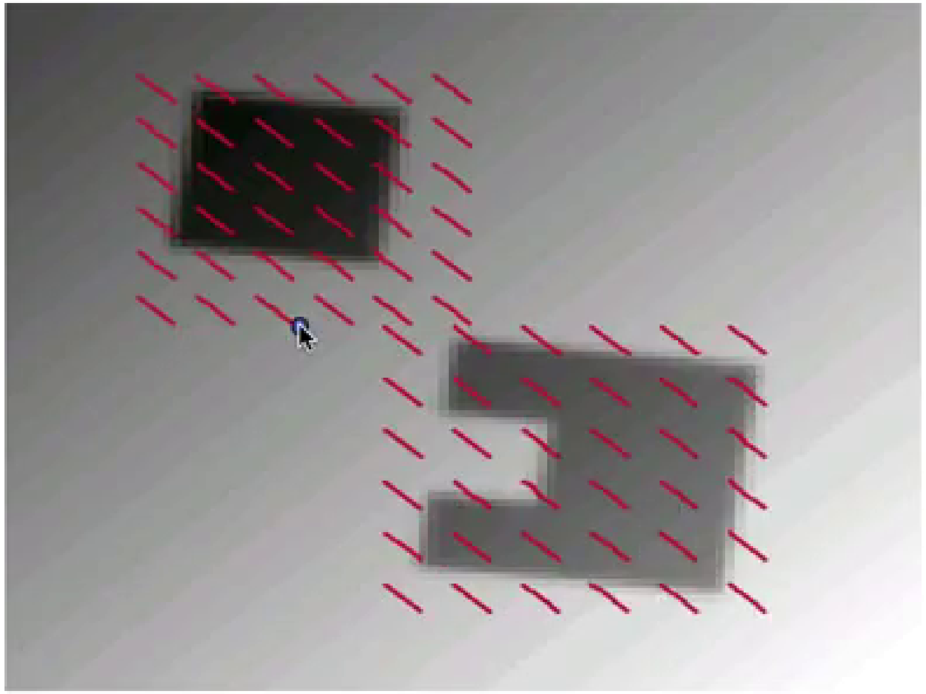
$G_s * I$

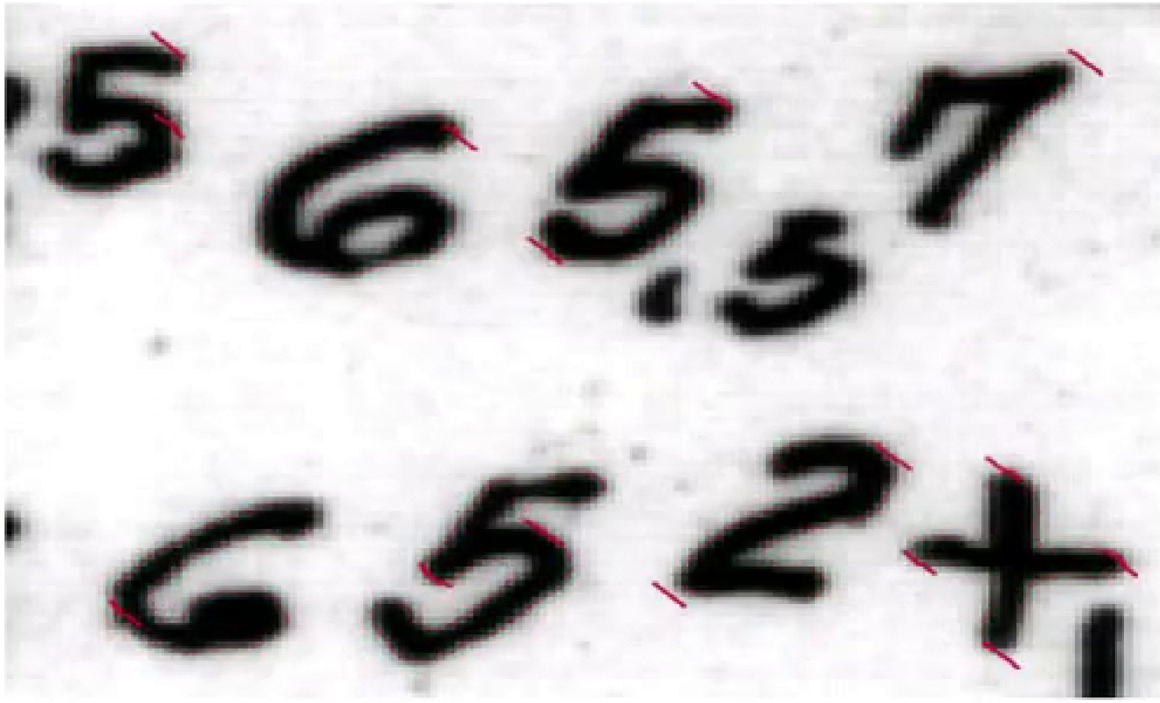
$g(x) = \frac{1}{1 + |\nabla(G_\sigma * I)|^2}$



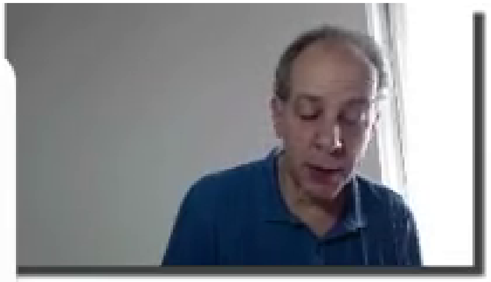
$$\boxed{\frac{dC}{dt} = \left(g(C)\kappa - \left\langle \nabla g(C), \vec{N} \right\rangle\right)\vec{N}}$$

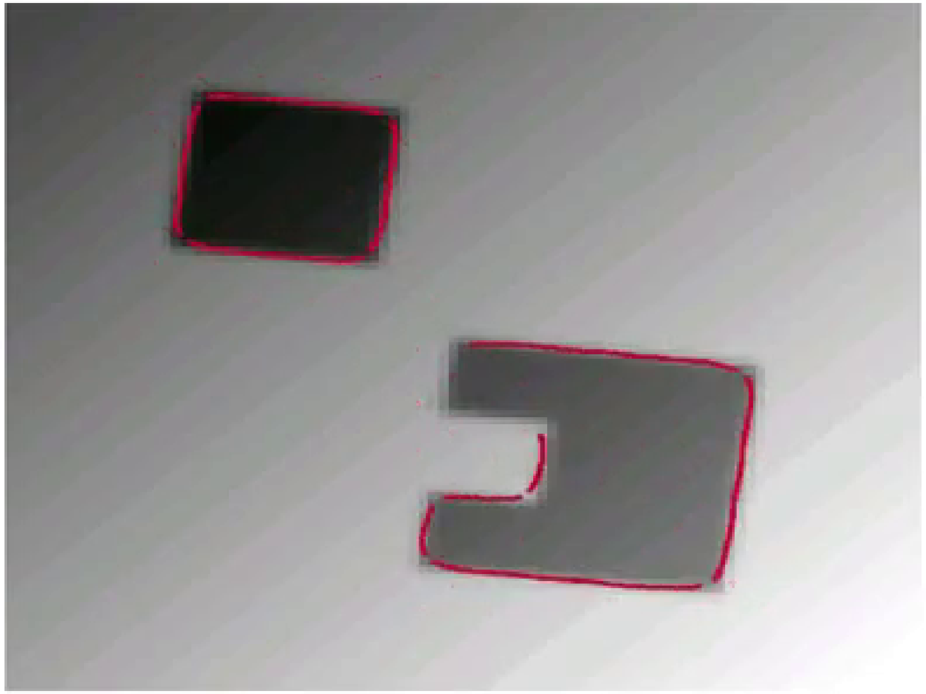


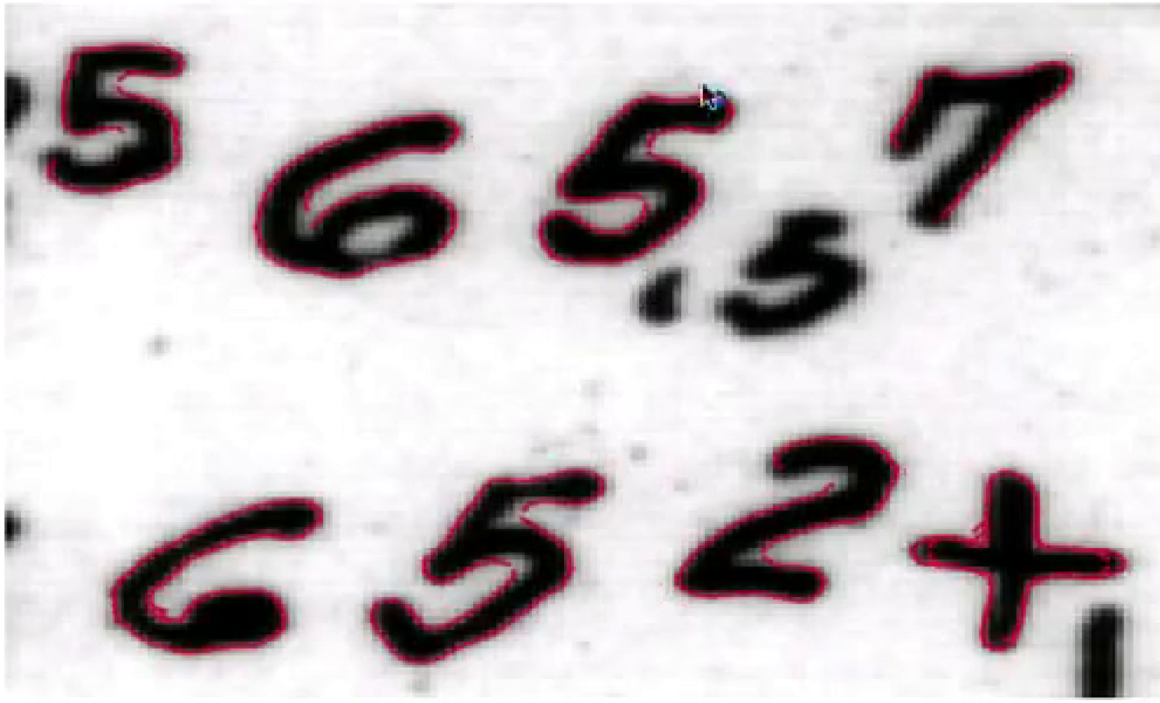




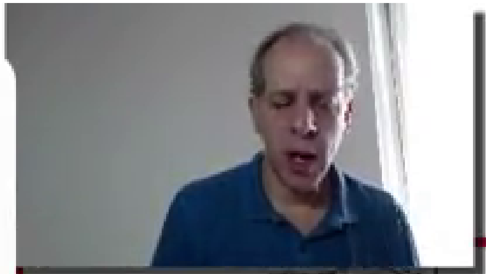
Images courtesy of Kimmel-Bruckstein







Images courtesy of Kimmel-Bruckstein



Gray Matter Segmentation

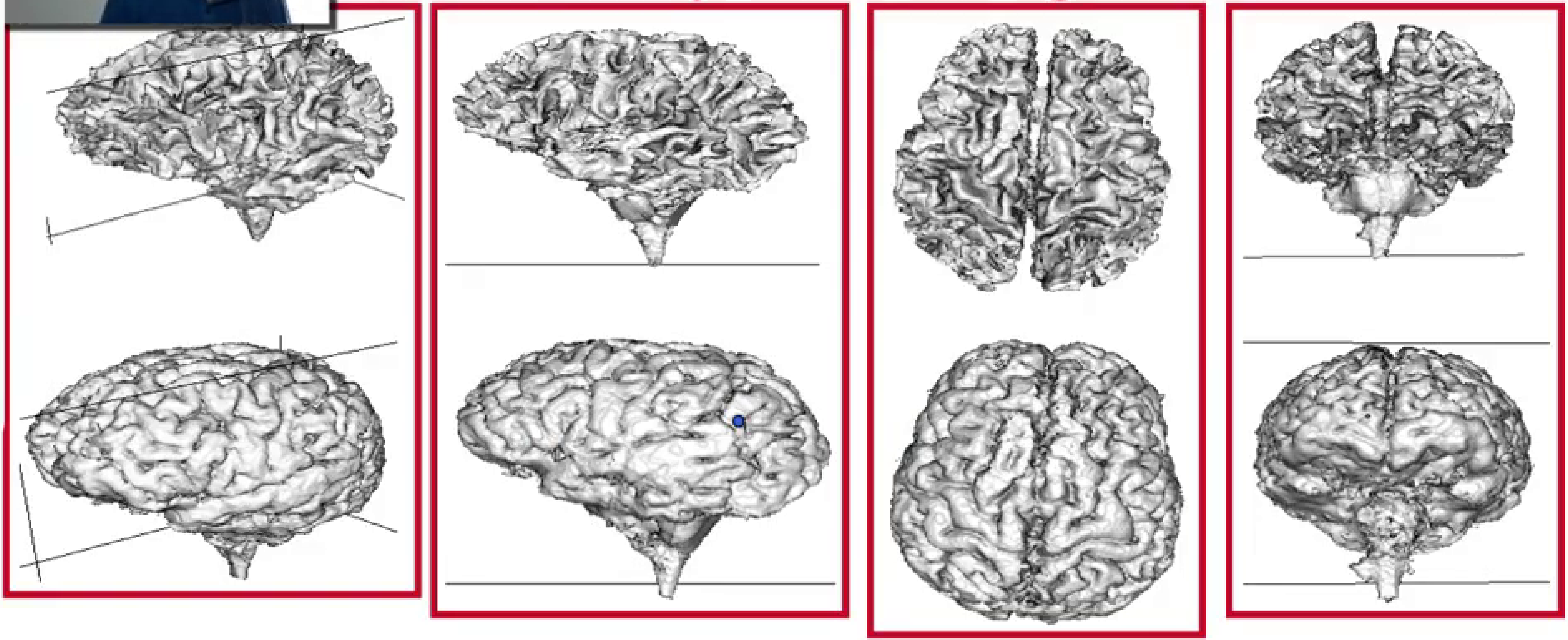
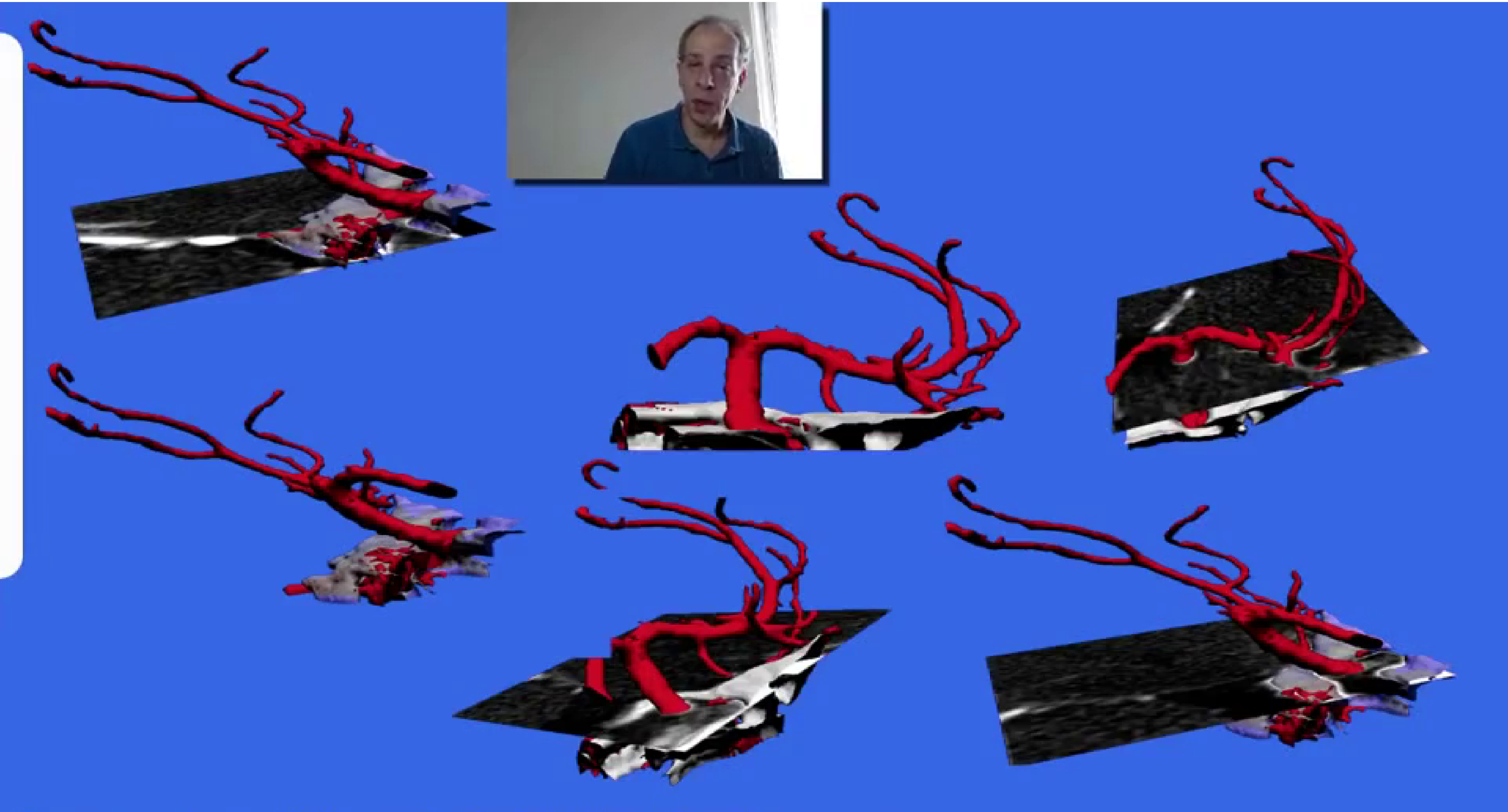


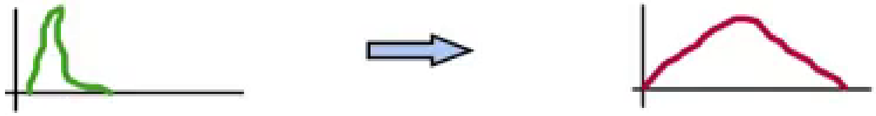
Image courtesy of Goldenberg Kimmel Rivlin Rudzsky,



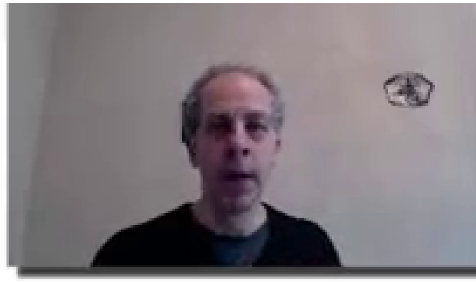
Images courtesy of Holzman-Gazit, Goldshier, Kimmel

Contrast Enhancement

- Contrast enhancement via image deformations
  - Approach: Histogram modification

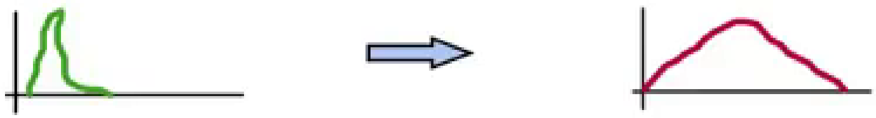


$$0 = \frac{\partial I(x,y)}{\partial t} = I(x,y) - (\#\, pixels\ of\ value \geq I(x,y)\ )$$



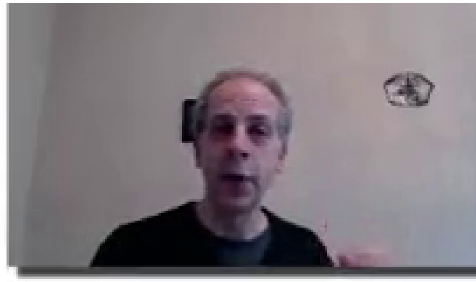
Contrast Enhancement

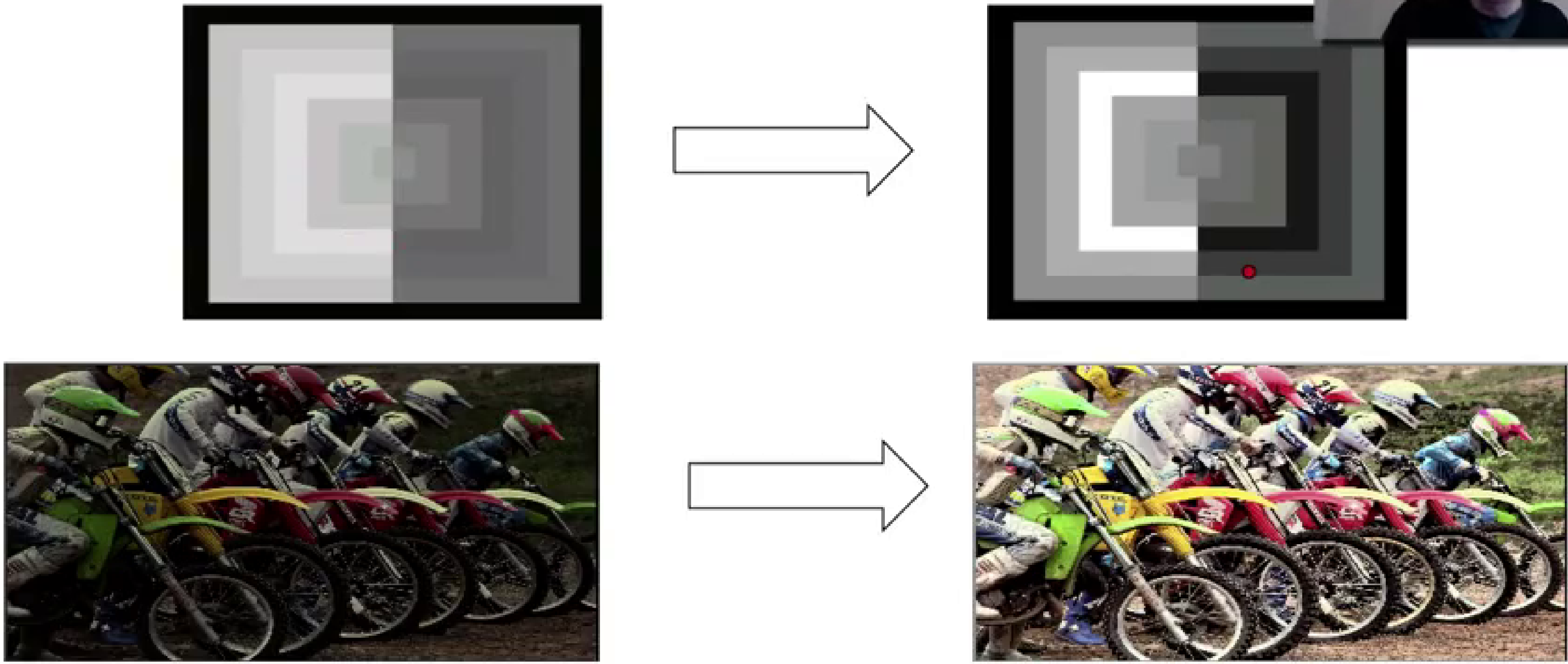
- **Contrast enhancement via image deformations**
  - **Approach: Histogram modification**

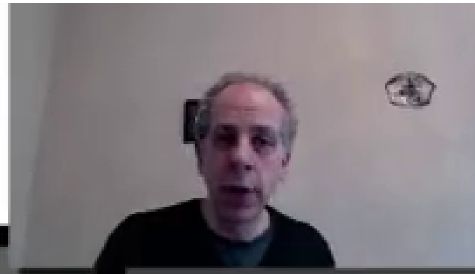


$$\frac{\partial I(x,y)}{\partial t} = I(x,y) - (\#\,pixels\ of\ value \geq I(x,y)\ )$$

$$U(I) = \frac{1}{2}\int [I(\vec{x}) - 1/2]^2\, d\vec{x} - \frac{1}{4}\iint [I(\vec{x}) - I(\vec{z})]\, d\vec{x}\, d\vec{z}$$







• **Images courtesy JDE and IEEE**