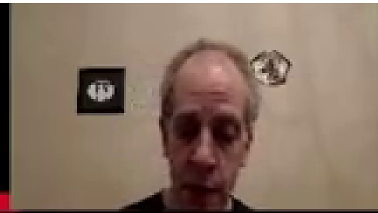
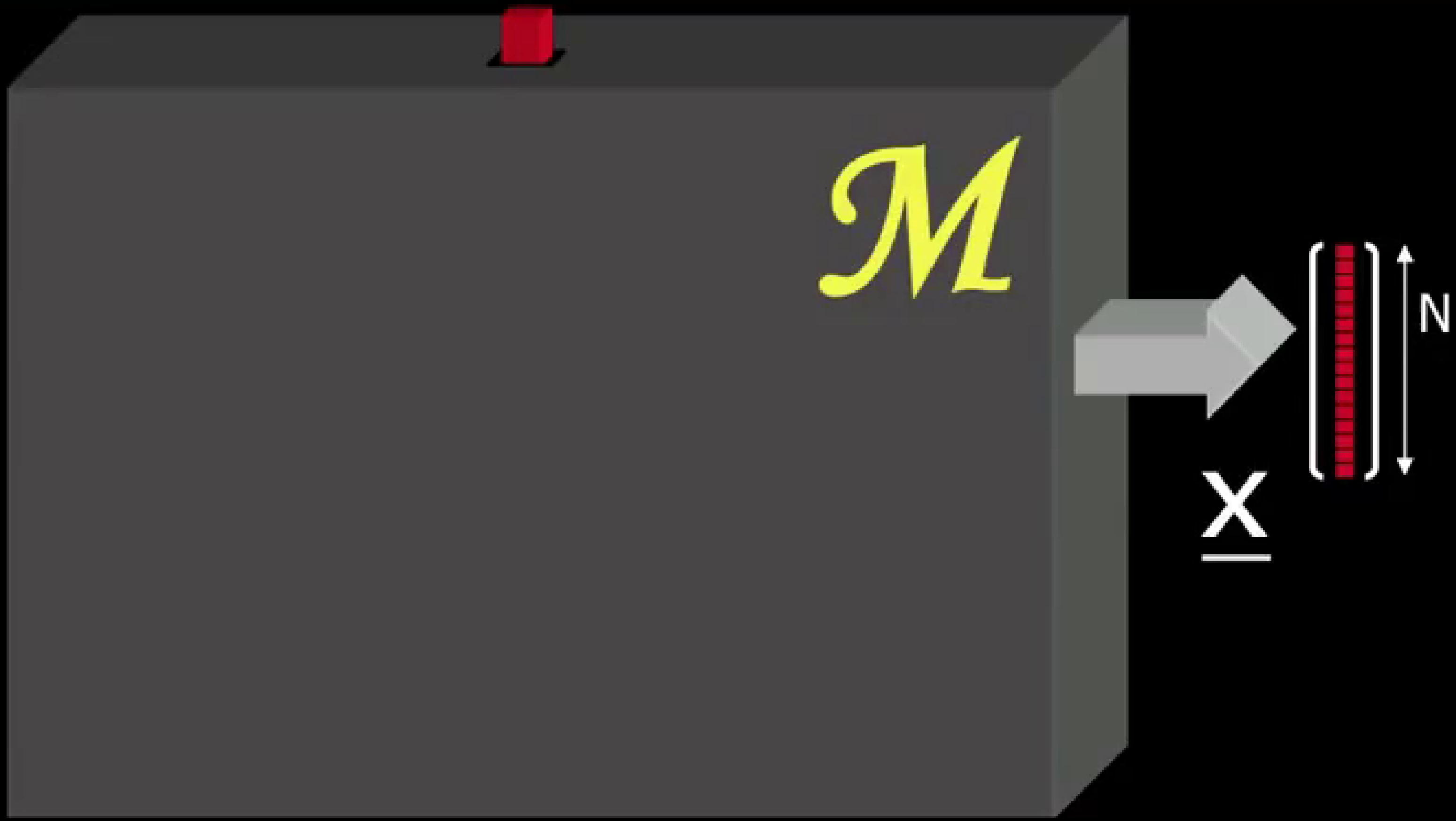
Sparse Modeling of Signals



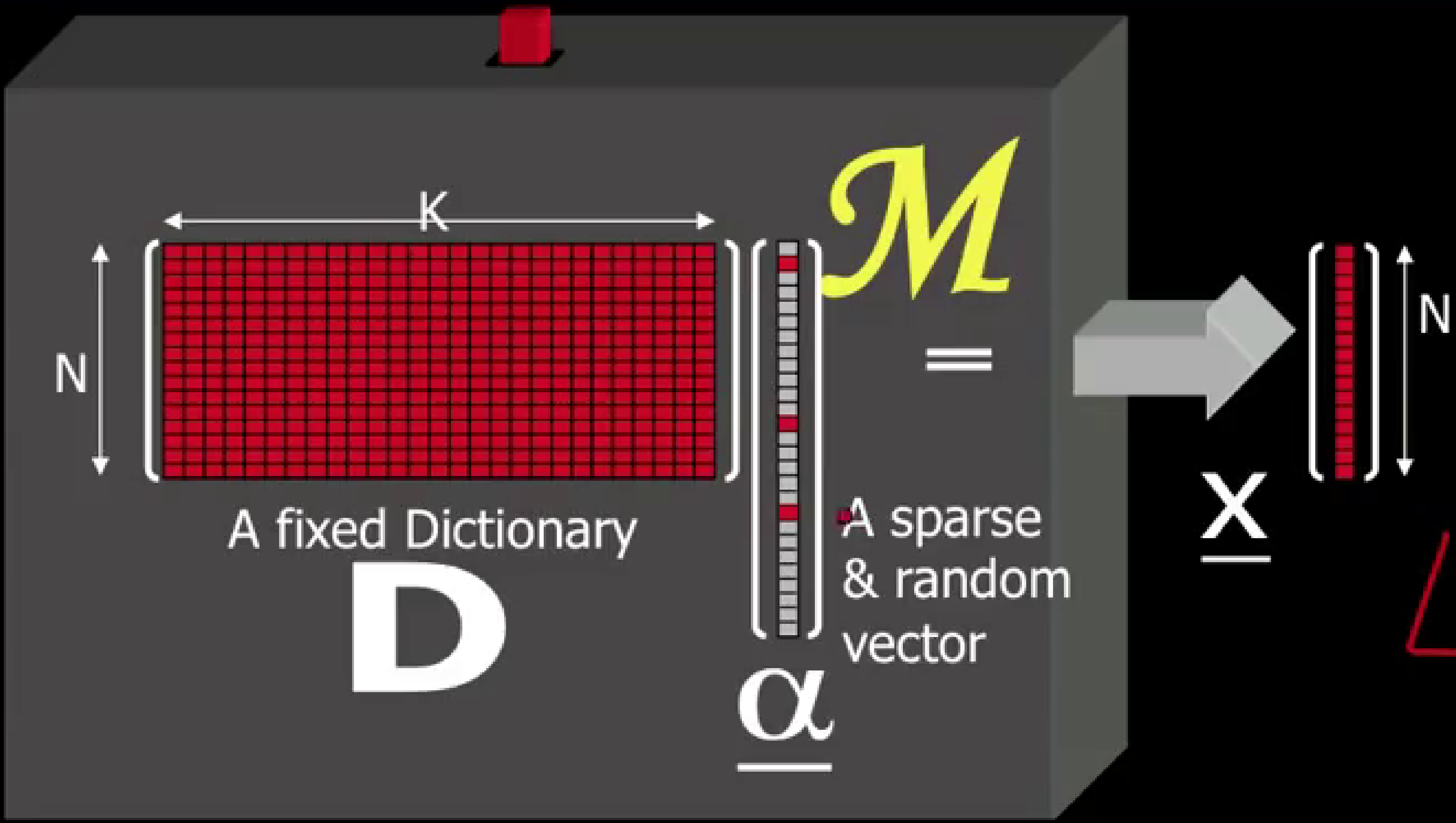


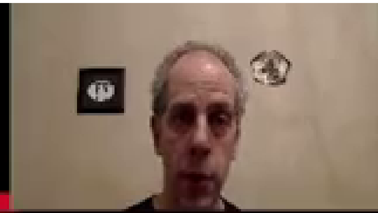
8

8

$N = 64$

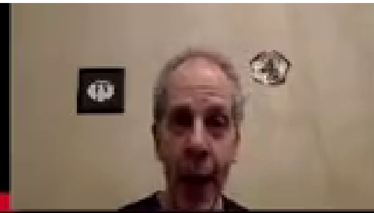
Sparse Modeling of Signals

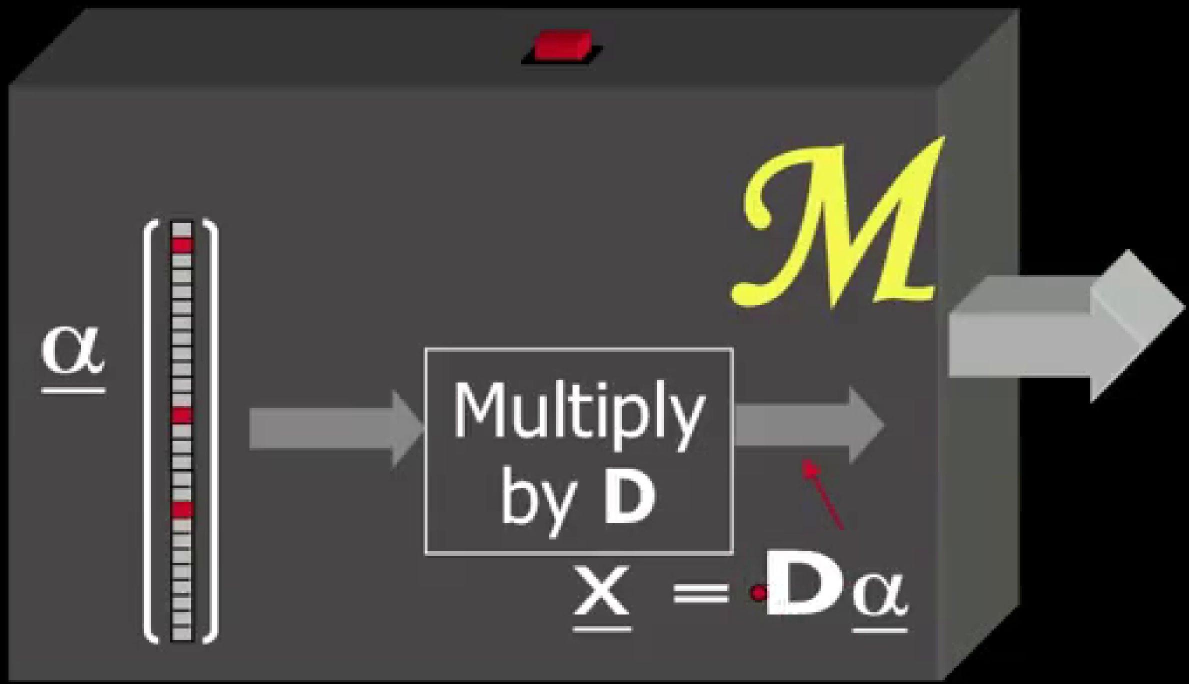




- **D** (dictionary) is a set of prototype signals (atom).
- The vector $\underline{\alpha}$ is generated with few (L) non-zeros.

Sparseland Signals are Special



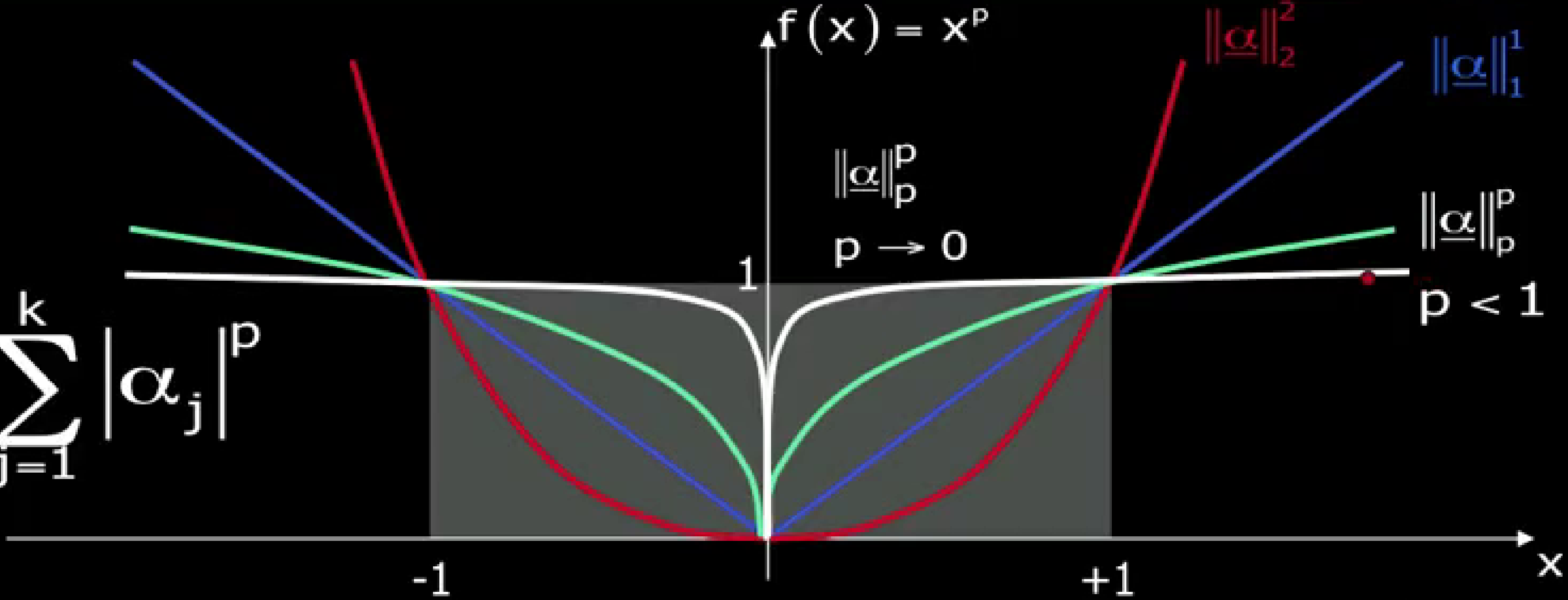


- **Simple:** Every generated signal is built as a linear combination of **<u>few</u>** atoms from our dictionary **D**
- **Rich:** A general model: the obtained signals are a union of many low-dimensional spaces.
- **Familiar:** We have been using this model in other context for a while now

JPEG

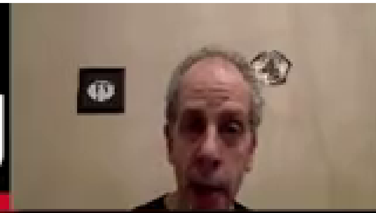
Sparse & Redundant Rep. Modeling

$$\|\underline{\alpha}\|_p^p = \sum_{j=1}^{k} |\alpha_j|^p$$

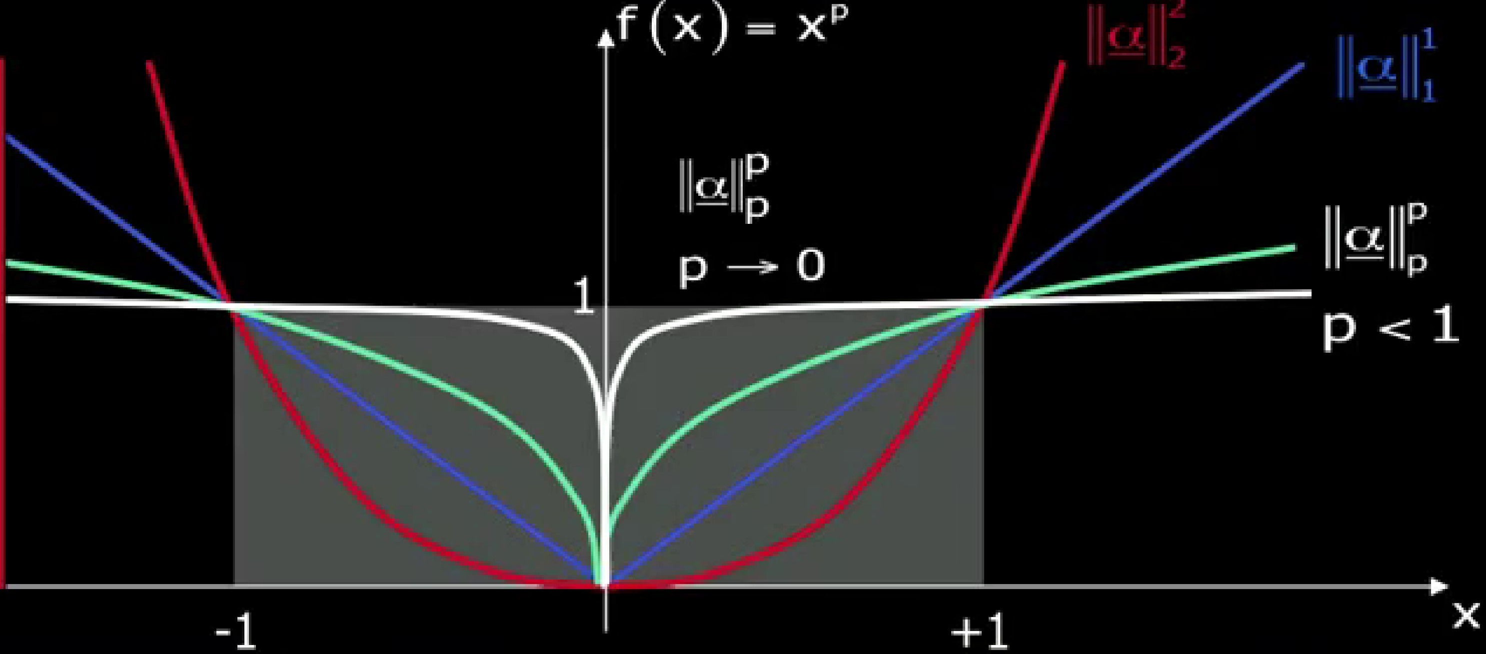


Our signal model $\underline{x} = \mathbf{D}\underline{\alpha}$ where $\underline{\alpha}$ is sparse

Sparse & Redundant Rep. Modeling

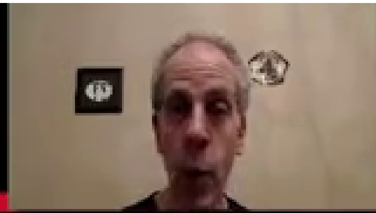


As $p \rightarrow 0$ we get a count of the non-zeros in the vector $\Rightarrow \|\underline{\alpha}\|_0^0$



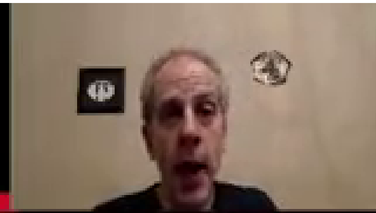
Our signal model $\underline{x} = \mathbf{D}\underline{\alpha}$ where $\|\underline{\alpha}\|_0^0 \leq L$

Back to Our MAP Energy Function

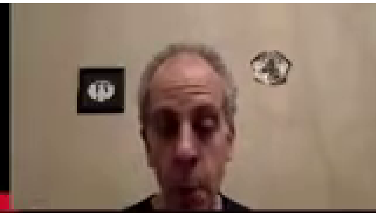


$$\frac{1}{2}\| \underline{x} - \underline{y} \|_2^2$$

Back to Our MAP Energy Function



$$\frac{1}{2}\left\| \mathbf{D}\underline{\alpha} - \underline{y} \right\|_2^2$$



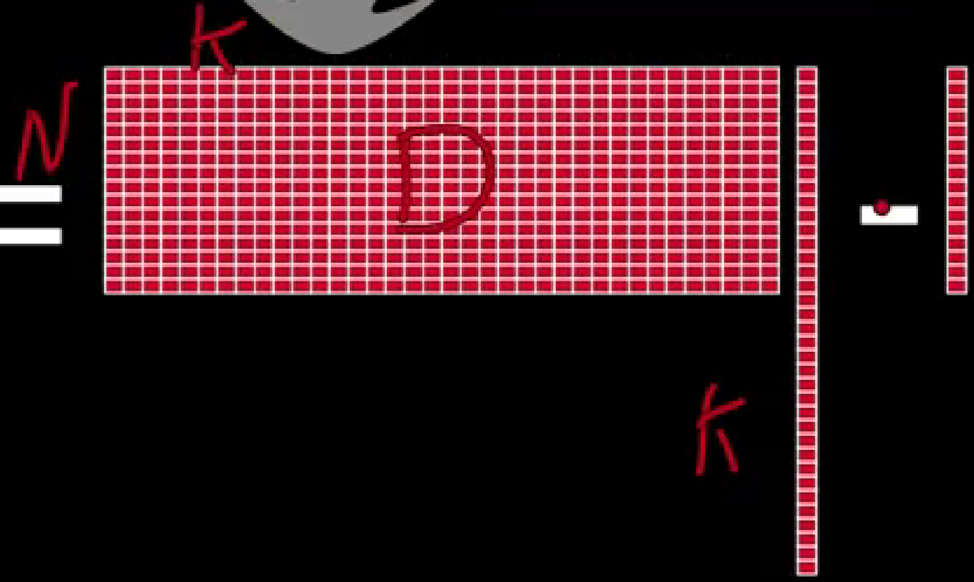
Back to Our MAP Energy Function

- L$_0$ “norm” is effectively counting the number of non-zeros in $\underline{\alpha}$.

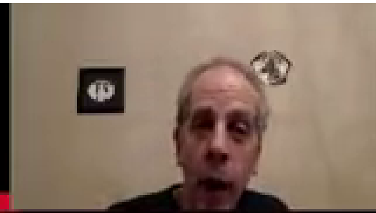
$$\hat{\underline{\alpha}} = \arg\min_{\underline{\alpha}} \frac{1}{2}\left\|\mathbf{D}\underline{\alpha}-\underline{y}\right\|_2^2 \text{ s.t. } \left\|\underline{\alpha}\right\|_0^0 \le L$$

$$\hat{\underline{x}} = \mathbf{D}\hat{\underline{\alpha}}$$

$$\mathbf{D}\underline{\alpha}\text{-}y =$$



Back to Our MAP Energy Function



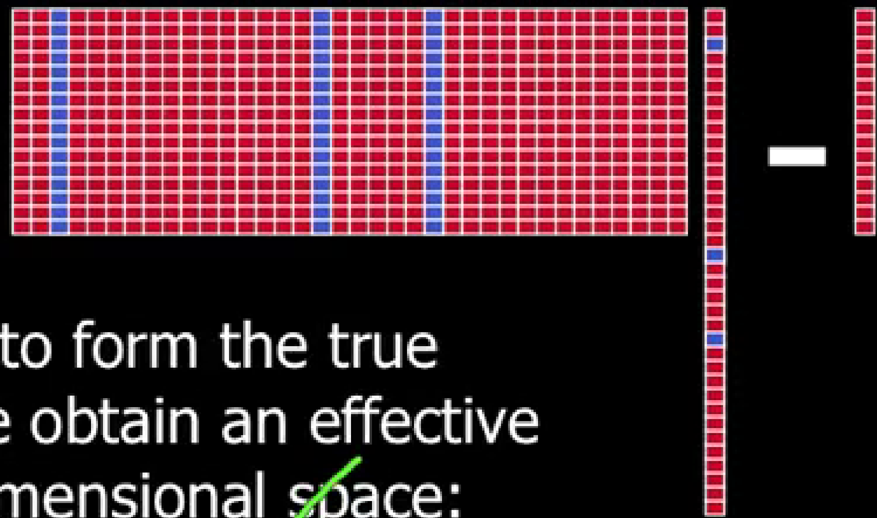
- L_0 "norm" is effectively counting the number of non-zeros in $\underline{\alpha}$.

$$\hat{\underline{\alpha}} = \arg\min_{\underline{\alpha}} \frac{1}{2}\left\| \mathbf{D}\underline{\alpha} - \underline{y} \right\|_2^2 \text{ s.t. } \left\|\underline{\alpha}\right\|_0^0 \le L$$

$$\hat{\underline{x}} = \mathbf{D}\hat{\underline{\alpha}}$$

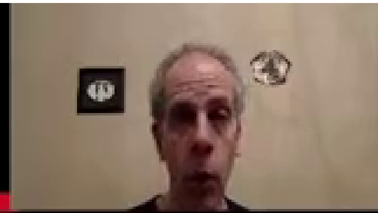
- The vector $\underline{\alpha}$ is the representation signal x.

$$\mathbf{D}\underline{\alpha} - \underline{y} =$$



- Few (L out of K) atoms can be combined to form the true signal, the noise cannot be fitted well. We obtain an effective projection of the noise onto a very low-dimensional space: Denoising

Wait! There are Some Issues



- **Numerical Problems:** How should we solve or approximate the solution of the problem

$$\min_{\underline{\alpha}} \left\| \mathbf{D}\underline{\alpha} - \underline{y} \right\|_2^2 \ \text{ s.t. } \ \left\| \underline{\alpha} \right\|_0^0 \le L \qquad\qquad \min_{\underline{\alpha}} \left\| \underline{\alpha} \right\|_0^0 \ \text{ s.t. } \ \left\| \mathbf{D}\underline{\alpha} - \underline{y} \right\|_2^2 \le \varepsilon^2$$

$$\min_{\underline{\alpha}} \lambda \left\| \underline{\alpha} \right\|_0^0 + \left\| \mathbf{D}\underline{\alpha} - \underline{y} \right\|_2^2$$

- **Theoretical Problems:** Is there a unique sparse representation? If we are to approximate the solution somehow, how close will we get?

- **Practical Problems:** What dictionary **D** should we use, such that all this leads to effective denoising? Will all this work in applications?

To Summarize So Far …

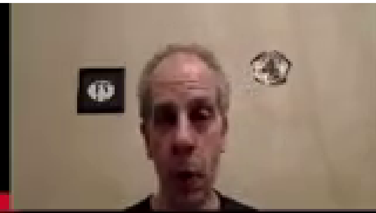


Image denoising (and many other problems in image processing) requires a model for the desired image

What do we do?

We proposed a model for signals/ images based on sparse and redundant representatio

Great! No?

There are some issues:

1. Theoretical
2. How to approximate?
3. What about **D**?