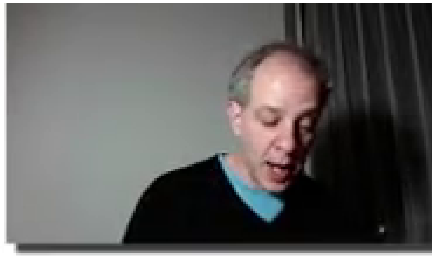
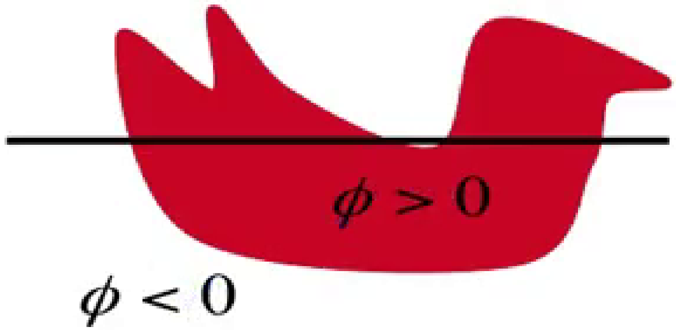
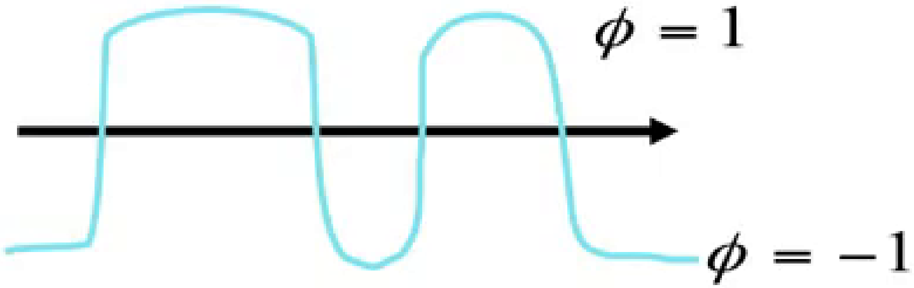
Implicit representation



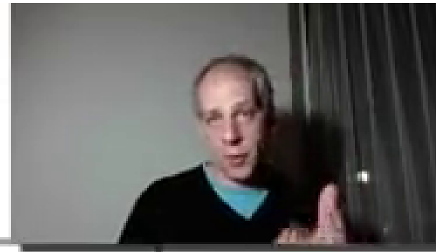
Consider a closed planar curve $C(p): \mathbf{S}^1 \to \mathbf{R}^2$

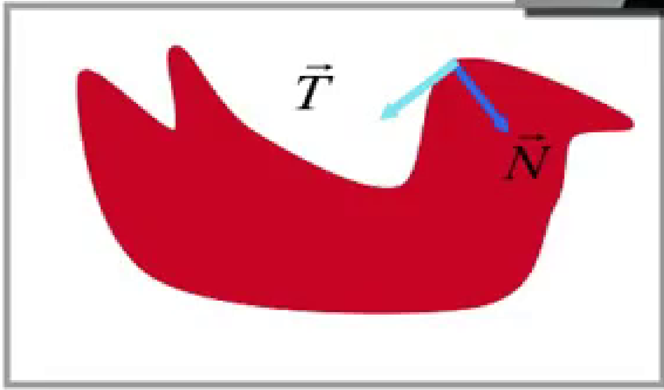
The geometric trace of the curve can be alternatively represented implicitly as $C = \{(x, y) \mid \phi(x, y) = 0\}$





Properties of level sets





The level set normal ϕ

$$\vec{N} = -\frac{\nabla\phi}{|\nabla\phi|} \qquad \left(\vec{T} = \frac{\overline{\nabla}\phi}{|\nabla\phi|}\right)$$

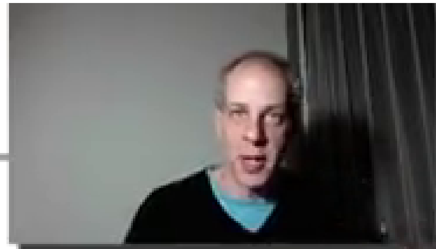
Proof. Along the level sets we have zero change, that is $\phi_s = 0$, but by the chain rule

$$\phi_s(x,y) = \phi_x x_s + \phi_y y_s = \left\langle \nabla\phi, \vec{T} \right\rangle \qquad \begin{bmatrix} a, & c \\ b, & d \end{bmatrix}$$

So,

$$\left\langle \frac{\nabla\phi}{|\nabla\phi|}, \vec{T} \right\rangle = 0 \Rightarrow \frac{\nabla\phi}{|\nabla\phi|} \perp \vec{T} \Rightarrow \vec{N} = -\frac{\nabla\phi}{|\nabla\phi|} \qquad \blacksquare$$

Properties of level sets

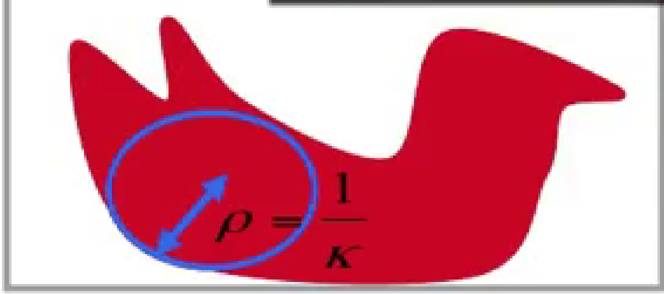


The level set curvature

$C_{ss} = \kappa \vec{n}$

$$\kappa = \mathrm{div}\left(\frac{\nabla\phi}{|\nabla\phi|}\right)$$

$\mathrm{div}(\alpha, \beta) = \alpha_x + \beta_y$



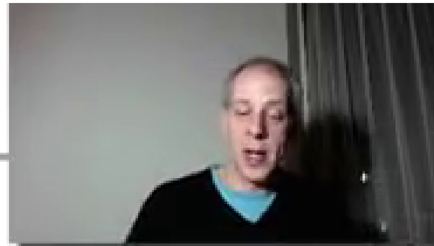
Proof: zero change along the level sets, $\phi_{ss} = 0$, also

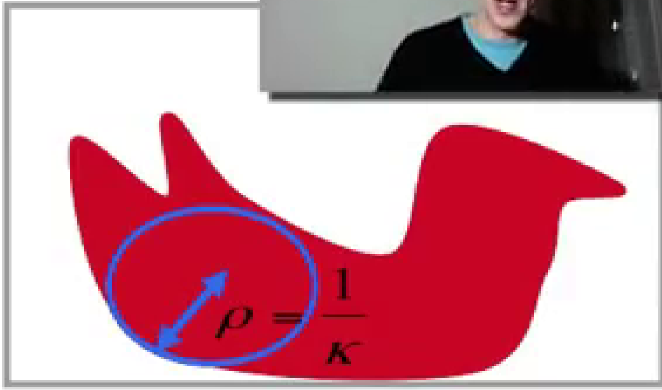
$$\phi_{ss}(x,y) = \frac{d}{ds}\underbrace{(\phi_x x_s + \phi_y y_s)} = \frac{d}{ds}\left\langle \nabla\phi, \vec{T} \right\rangle = \left\langle \frac{d}{ds}\nabla\phi, \vec{T} \right\rangle + \left\langle \nabla\phi, \kappa\vec{N} \right\rangle = 0$$

$$\kappa\left\langle \nabla\varphi, \frac{\nabla\varphi}{|\nabla\varphi|} \right\rangle = \kappa\,|\nabla\varphi| = -\left\langle [\varphi_{xx}x_s + \varphi_{xy}y_s, \varphi_{xy}x_s + \varphi_{yy}y_s], \frac{\bar{\nabla}\varphi}{|\nabla\varphi|} \right\rangle$$

ϕ

Properties of level sets





The level set curvature

$C_{ss} = k\vec{n}$

$$\kappa = \operatorname{div}\left(\frac{\nabla\phi}{|\nabla\phi|}\right)$$

$\operatorname{div}(\alpha, \beta) = \alpha_x + \beta_y$

Proof: zero change along the level sets, $\phi_{ss} = 0$, also

$$\phi_{ss}(x,y) = \frac{d}{ds}(\phi_x x_s + \phi_y y_s) = \frac{d}{ds}\left\langle \nabla\phi, \vec{T} \right\rangle = \left\langle \frac{d}{ds}\nabla\phi, \vec{T} \right\rangle + \left\langle \nabla\phi, \kappa\vec{N} \right\rangle = 0$$

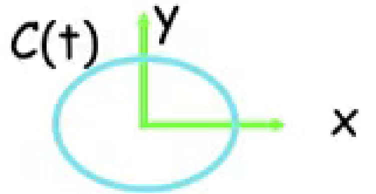
$$\kappa\left\langle \nabla\varphi, \frac{\nabla\varphi}{|\nabla\varphi|} \right\rangle = \kappa\,|\nabla\varphi| = -\left\langle [\varphi_{xx}x_s + \varphi_{xy}y_s, \varphi_{xy}x_s + \varphi_{yy}y_s], \frac{\bar{\nabla}\varphi}{|\nabla\varphi|} \right\rangle \quad \blacksquare$$

ϕ

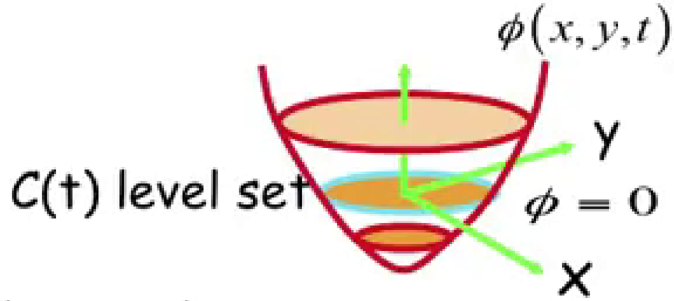
Level Set Formulation

(Osher-Sethian)

$$\phi(x,y): \boldsymbol{R}^2 \to \boldsymbol{R} \qquad C = \{(x,y): \phi(x,y) = 0\}$$



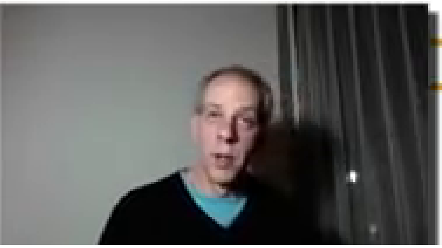
$$\frac{dC}{dt} = V\vec{N} \Leftrightarrow \boxed{\frac{d\phi}{dt} = V\left|\nabla\phi\right|}$$



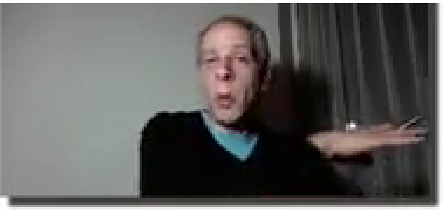
$$0 = \frac{\partial\phi(x,y;t)}{\partial t} = \phi_x x_t + \phi_y y_t + \phi_t$$

$$-\phi_t = \phi_x x_t + \phi_y y_t = \left\langle \nabla\phi, C_t \right\rangle = \left\langle \nabla\phi, V\vec{N} \right\rangle = V\left\langle \nabla\phi, \vec{N} \right\rangle$$

$$\vec{N} = -\frac{\nabla\phi}{|\nabla\phi|} \qquad -V\left\langle \nabla\phi, \vec{N} \right\rangle = V\left\langle \nabla\phi, \frac{\nabla\phi}{|\nabla\phi|} \right\rangle = V\,|\nabla\phi|$$

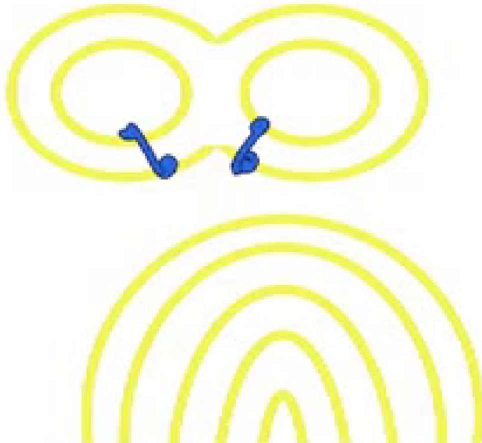


$$\boxed{\phi_t = V\,|\nabla\phi|}$$

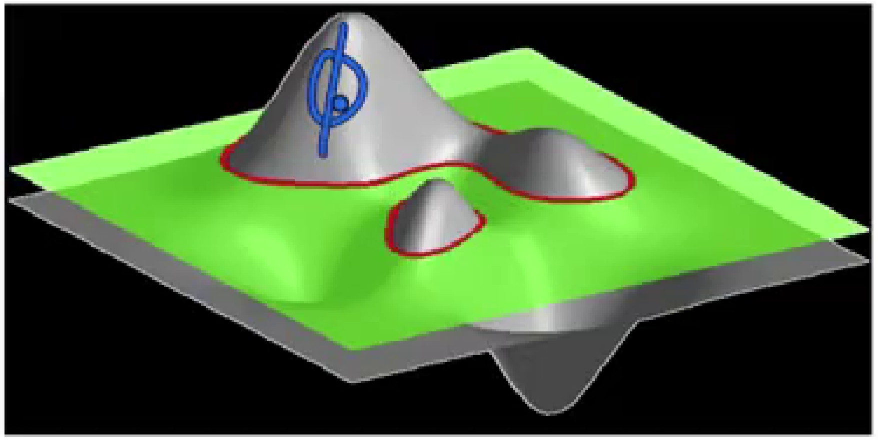


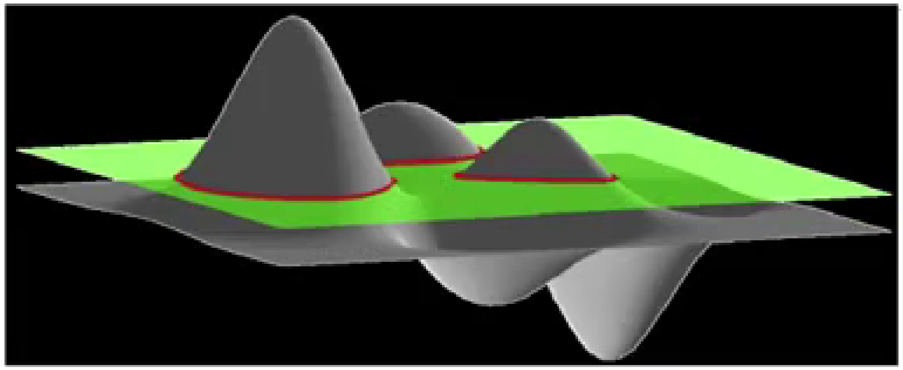
Level Set Formulation

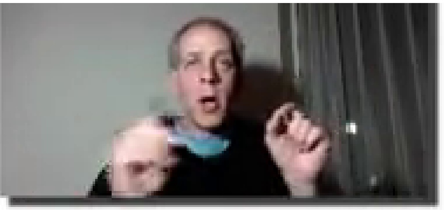
- Handles changes in topology



- Numeric grid points never collide or drift apart.
- Natural philosophy for dealing with gray level images.

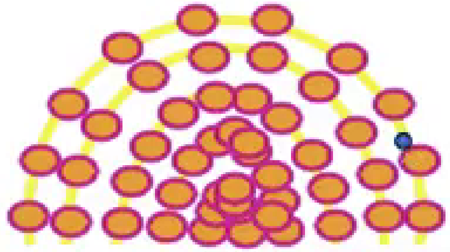


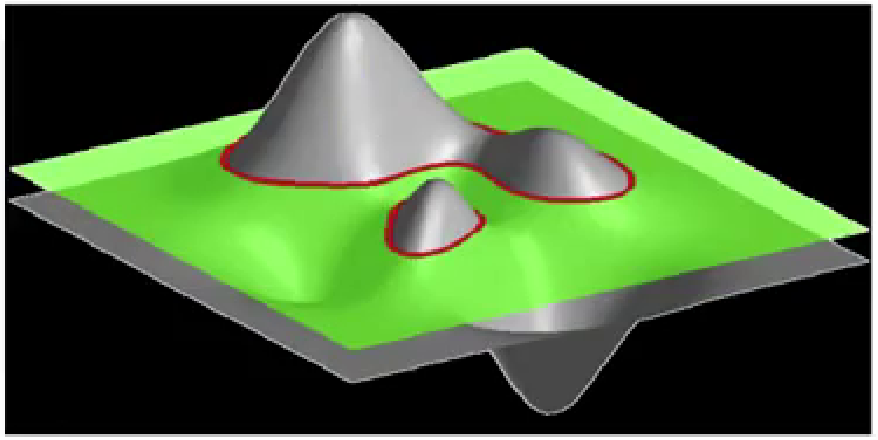


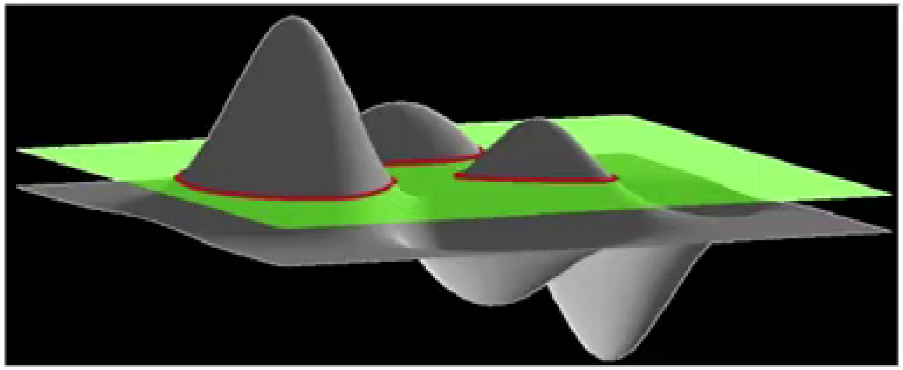


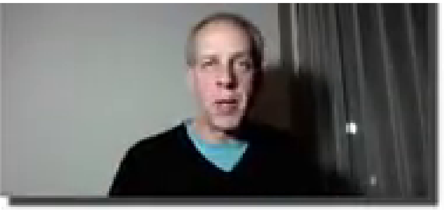
Level Set Formulation

- Handles changes in topology
- Numeric grid points never collide or drift apart.
- Natural philosophy for dealing with gray level images.









Level Set Formulation

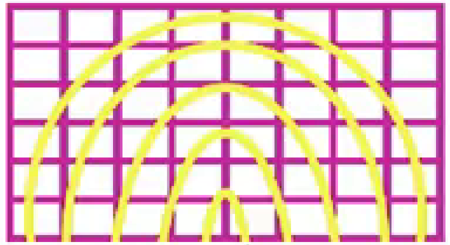
$V = g\kappa$

- Handles changes in topology

$\phi_t = g\kappa\,|\nabla\phi|$

- Numeric grid points never collide or drift apart.

$\phi_t = |\nabla\phi|$



- Natural philosophy for dealing with gray level images.

$\phi_t = V\nabla\phi$

$\phi = 0$