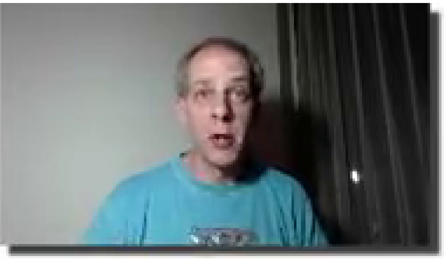
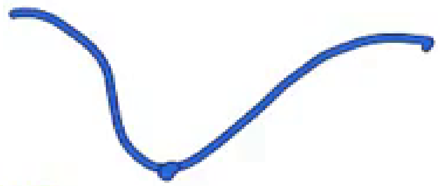


Calculus of Variations

Generalization of Calculus that seeks to find the path, curve, surface, etc., for which a *given* Functional has a minimum or maximum.

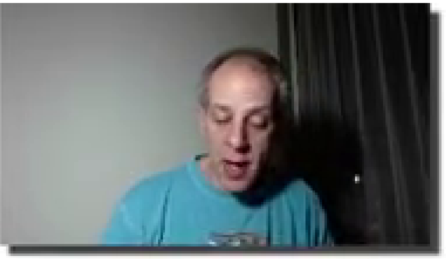
Goal: find extrema values of integrals of the form

$$\int F(u, u_x)\,dx$$



It has an extremum only if the Euler-Lagrange Differential Equation is satisfied,

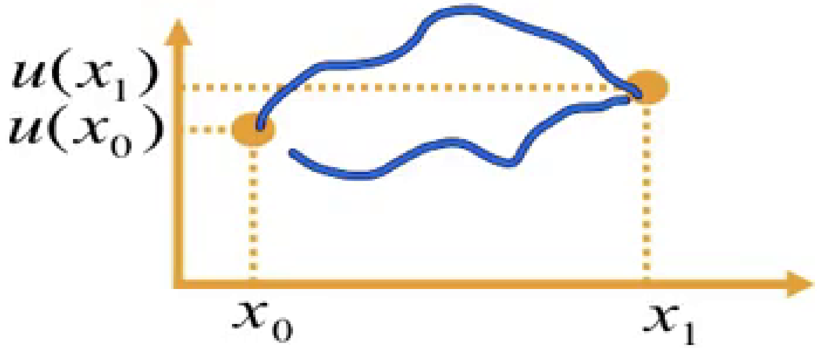
$$\left(\frac{\partial}{\partial u} - \frac{d}{dx}\frac{\partial}{\partial u_x}\right)F(u, u_x) = 0$$



Calculus of Variations

Example: Find the shape of the curve $\{x,u(x)\}$ with shortest length:

$$\int_{x_0}^{x_1} \sqrt{1+u_x^2}\, dx$$

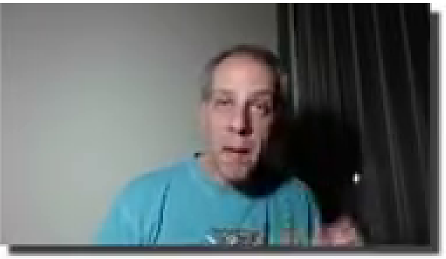


given $u(x0),u(x1)$

Solution: a differential equation that $u(x)$ must satisfy,

$$\left(\frac{\partial}{\partial u} - \frac{d}{dx}\frac{\partial}{\partial u_x}\right) F(u,u_x) = 0$$

$$\frac{u_{xx}}{\cancel{\left(1+u_x^2\right)^{3/2}}} = 0 \quad \Rightarrow \quad u_x = a \quad \Rightarrow \quad u(x) = ax + b$$



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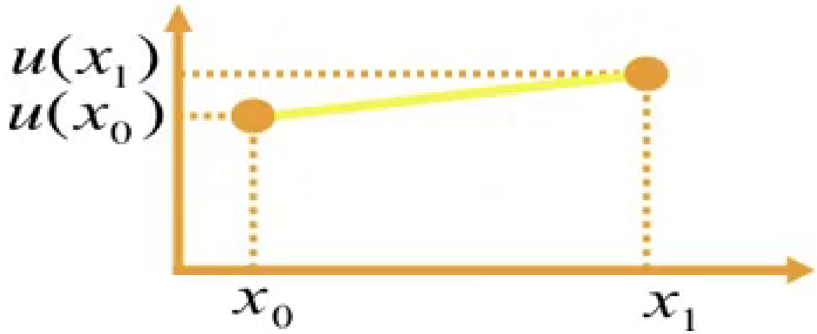

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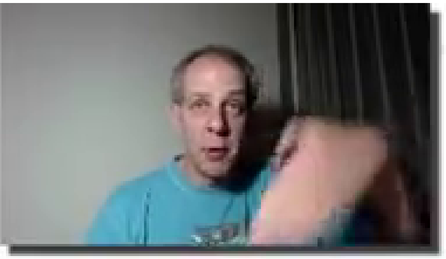
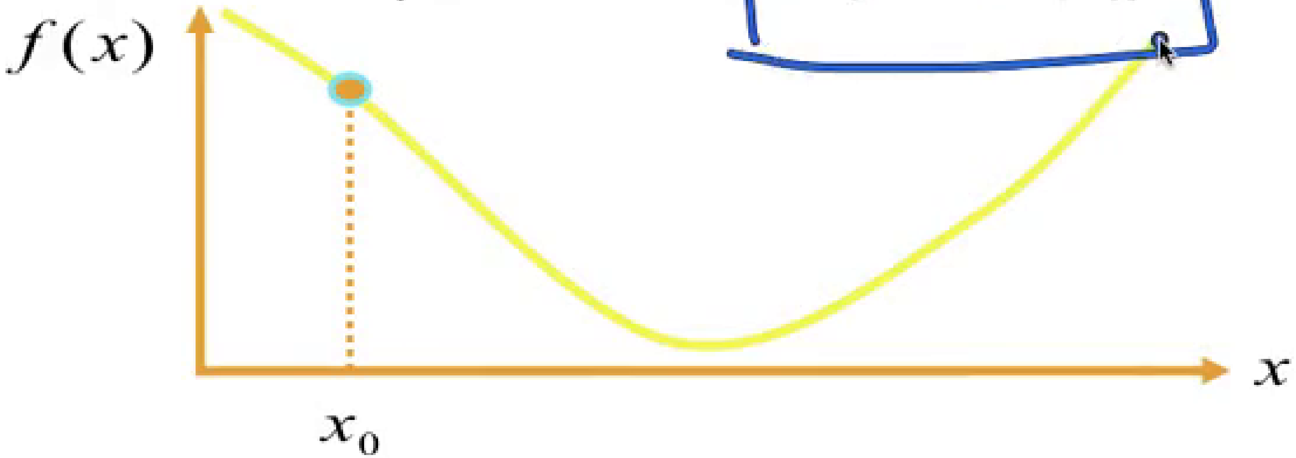
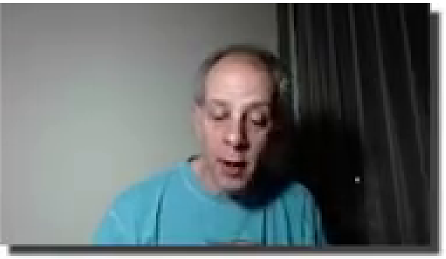


Extrema points in calculus

$$\forall \eta : \lim_{\varepsilon \to 0}\left(\frac{df(x+\varepsilon\eta)}{d\varepsilon} \right) = 0 \Leftrightarrow \forall \eta : f_x(x)\eta = 0 \Leftrightarrow f_x(x) = 0$$

Gradient descent process $\boxed{x_t = -f_x}$





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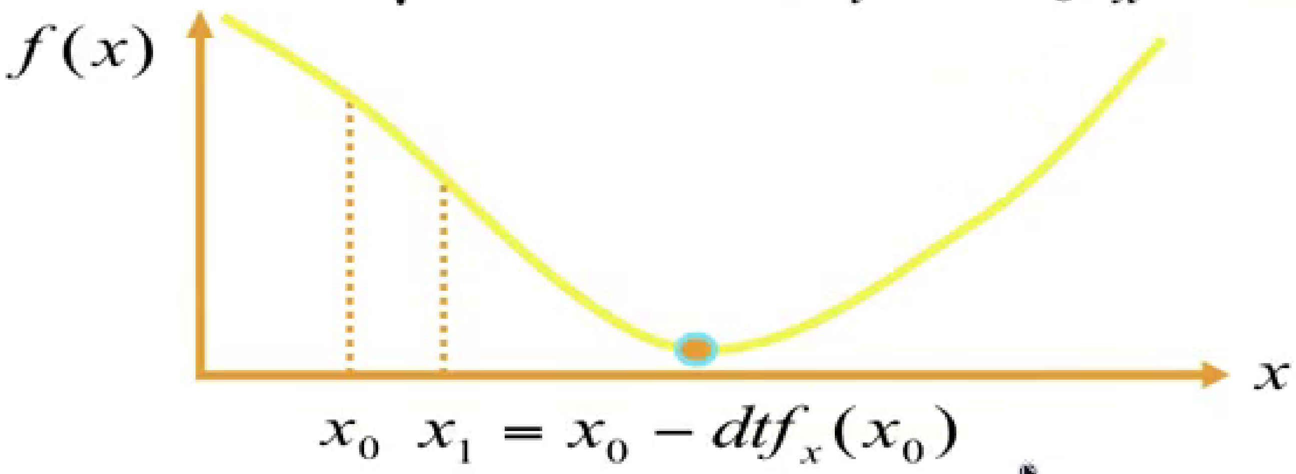
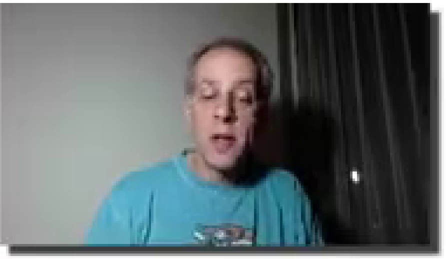
Gradient descent process $\quad x_t = -f_x \quad \frac{x(t+\Delta t)-x(t)}{\Delta t}$





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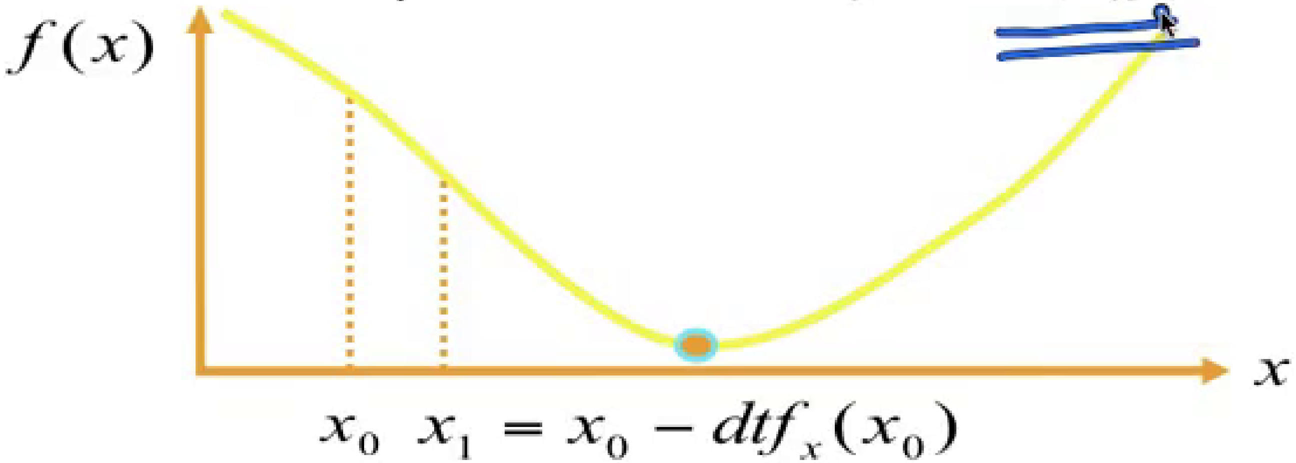
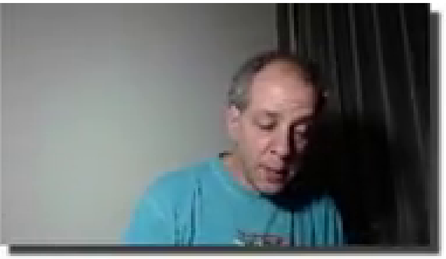
Gradient descent process $x_t = -f_x$

$x_t = 0$





Calculus of variations

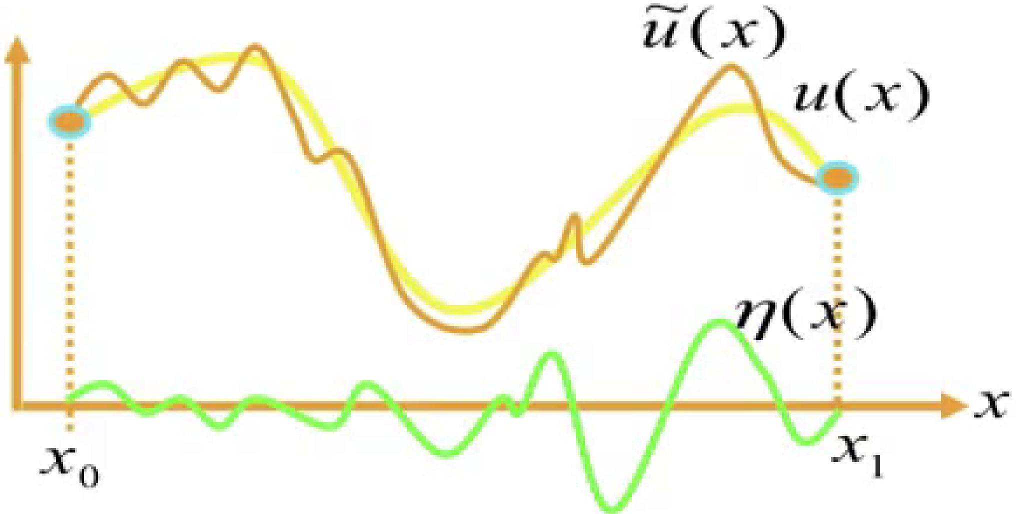
$$E(u(x)) = \int F(u, u_x)\,dx$$

$$\tilde{u}(x) = u(x) + \varepsilon \eta(x)$$

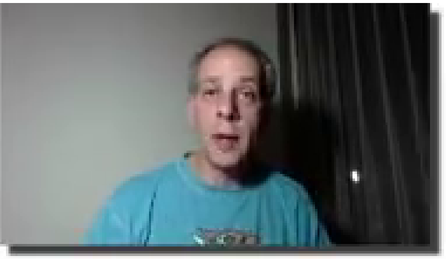
$$\forall\, \eta(x) : \lim_{\varepsilon \to 0}\left(\frac{d}{d\varepsilon} \int F(\tilde{u}, \tilde{u}_x)\,dx \right) \stackrel{?}{=} 0$$

$$\Downarrow$$

$$\frac{\delta E(u)}{\delta u} = \left(\frac{\partial}{\partial u} - \frac{d}{dx}\frac{\partial}{\partial u_x} \right) F(u, u_x)$$



Gradient descent process $$u_t = -\frac{\delta E(u)}{\delta u}$$



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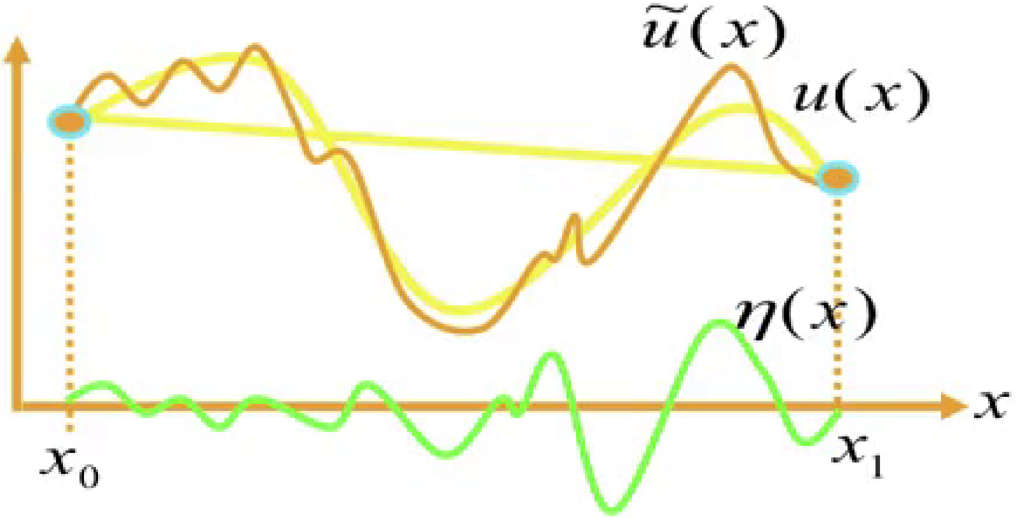

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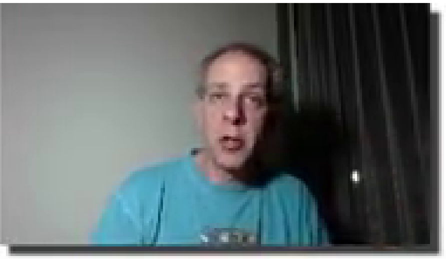

Conclusions

- Gradient descent process

Calculus $\arg\min\limits_{x} f(x) \quad \Longrightarrow \quad x_t = -f_x$

Calculus of variations $\arg\min\limits_{u(x)} \underbrace{\boxed{\int F(u,u_x)dx}}_{E(u)} \quad \Longrightarrow \quad \underbrace{u_t}_{=0} = -\overbrace{\boxed{\frac{\delta E(u)}{\delta u}}}^{\text{Euler-Lagrange}}$