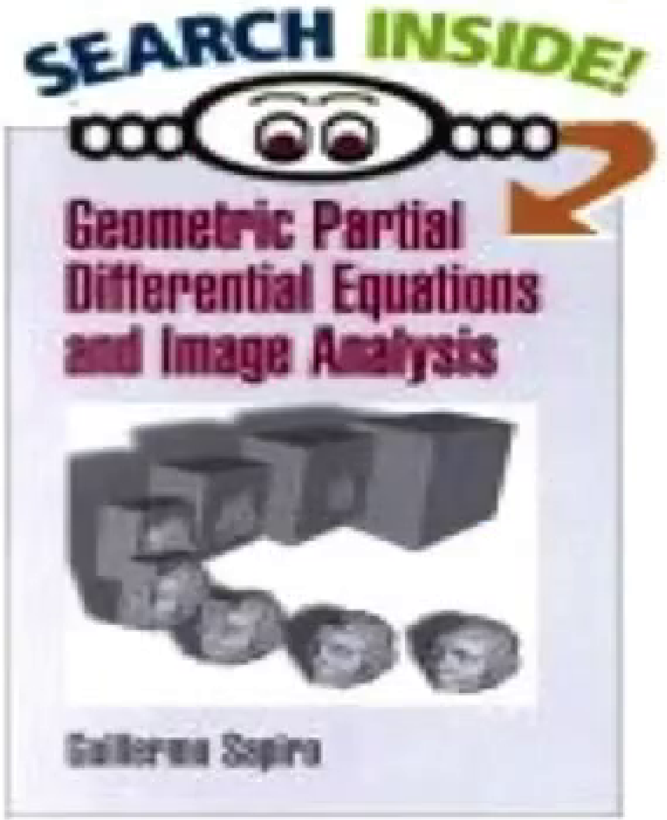
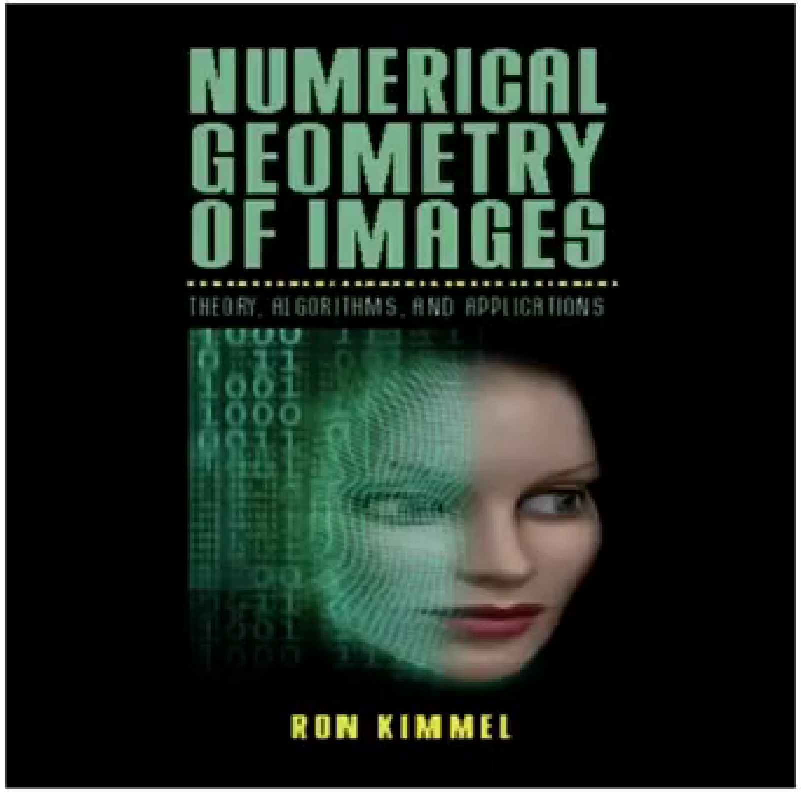
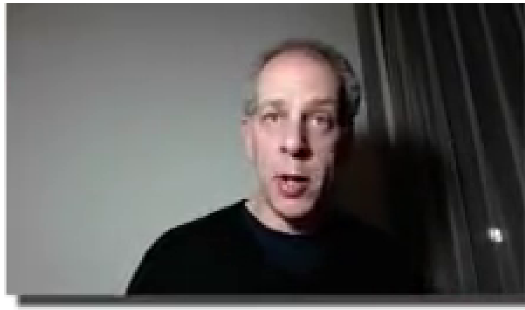
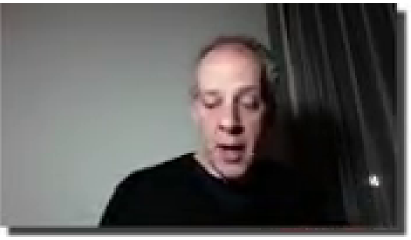

SEARCH INSIDE!
Geometric Partial Differential Equations and Image Analysis
Guillermo Sapiro


NUMERICAL GEOMETRY OF IMAGES
THEORY, ALGORITHMS, AND APPLICATIONS
RON KIMMEL

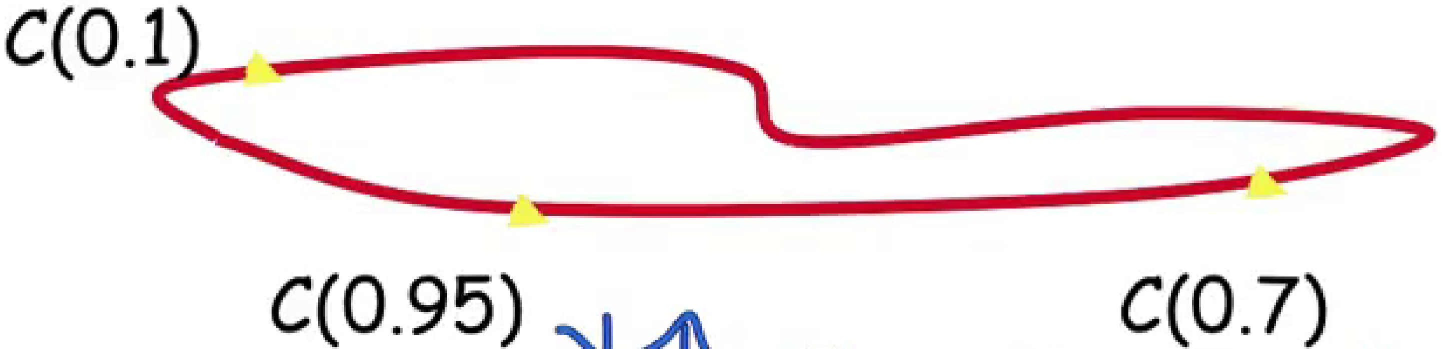


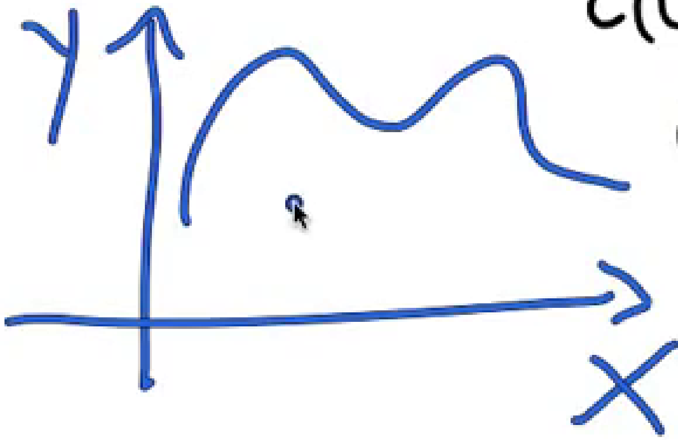


Planar Curves

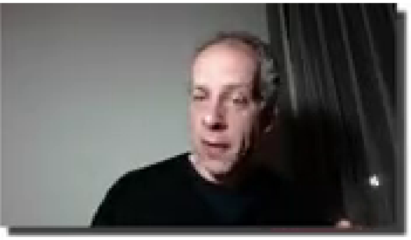
- $C(p)=\{x(p),y(p)\}, \quad p \in [0,1]$

$C(0)=C(1)$



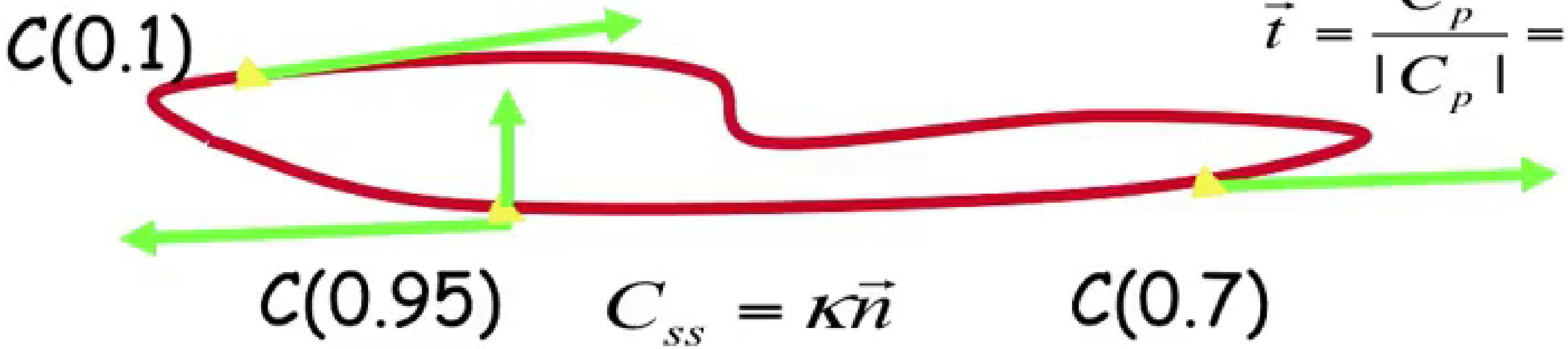


$C(x)=\{x,f(x)\}$



Planar Curves

- $C(p)=\{x(p),y(p)\}, \quad p \in [0,1]$



$$\vec{t} = \frac{C_p}{|C_p|} = C_s$$

$$C_{ss} = \kappa \vec{n}$$

$$C_p = \frac{\partial C}{\partial p} = [x_p, y_p]$$

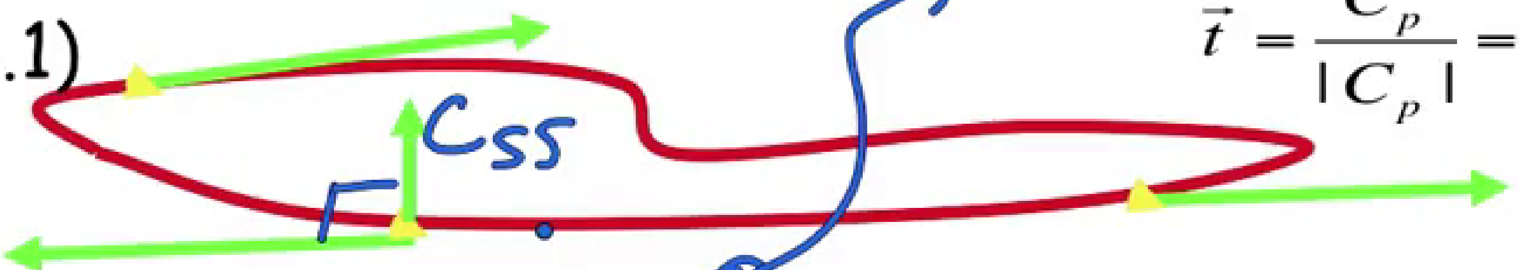
Planar Curves

- $C(p)=\{x(p),y(p)\},\quad p\in[0,1]$

Curvature

$$\vec{t}=\frac{C_p}{|C_p|}=C_s$$

$C(0.1)$

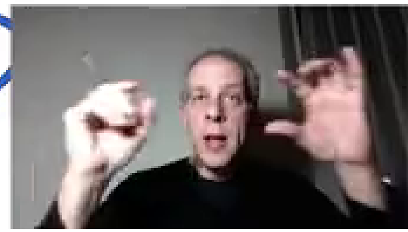


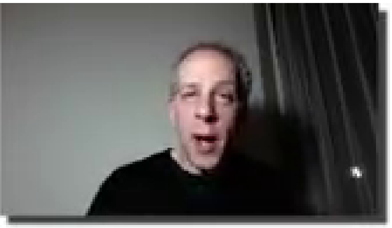
$1/r$

$C(0.95)$ $\quad C_{ss}=\kappa\vec{n}$ $\quad C(0.7)$

$$|C_s|=1 \qquad \frac{\partial}{\partial s}\langle C_s,C_s\rangle=\frac{\partial 1}{\partial s}$$

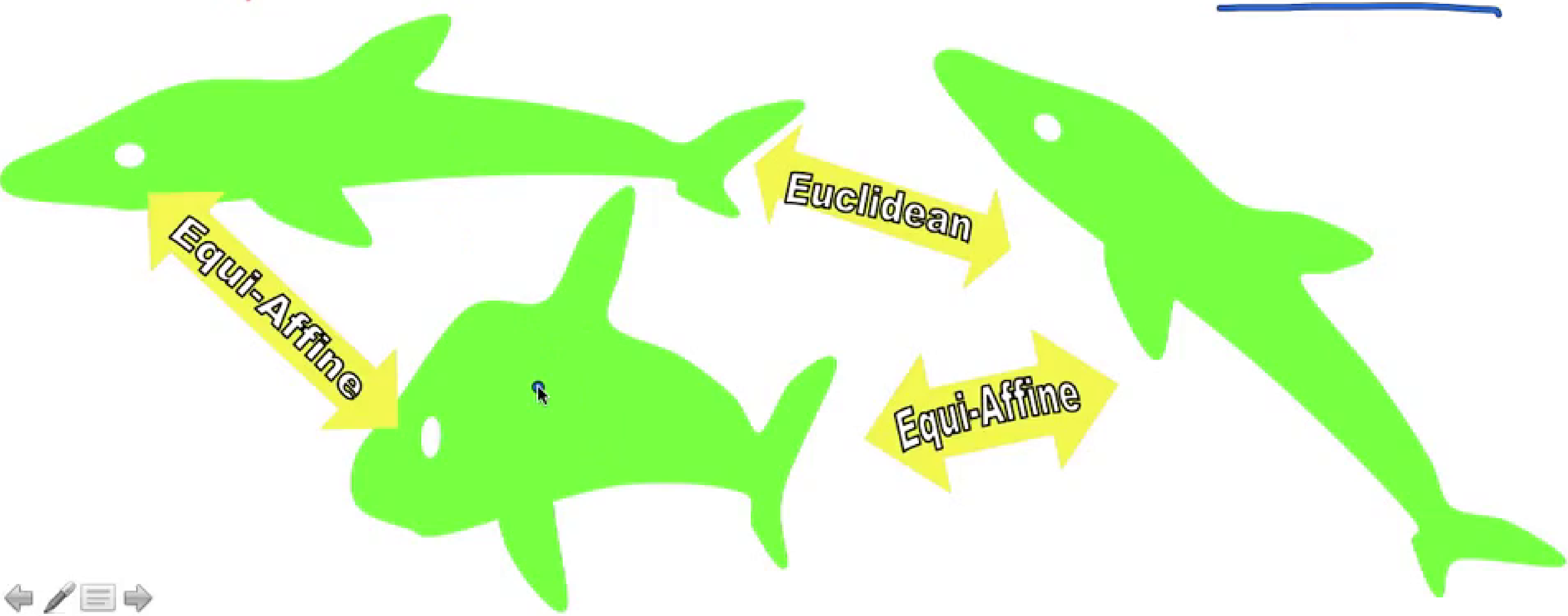
$$C_s\perp C_{ss} \Leftarrow \cancel{2}\langle C_s,C_{ss}\rangle=0$$

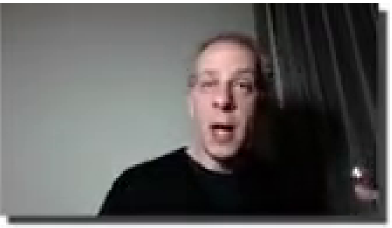




Linear Transformations

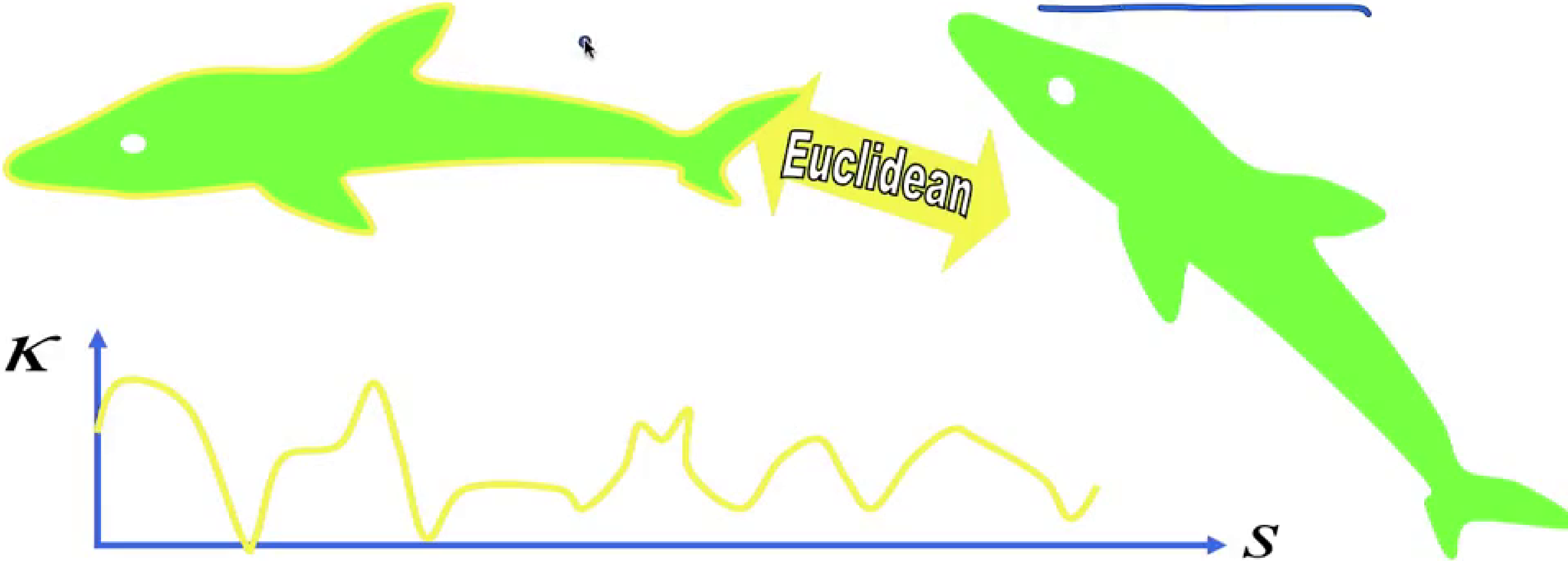
Equi-Affine: $\{\tilde{x}, \tilde{y}\}^T = A\{x, y\}^T + \bar{b}, \quad \det(A) = 1.$

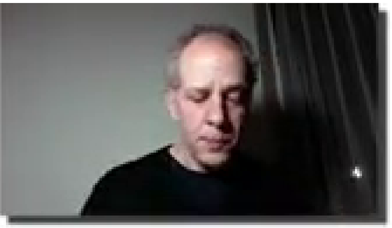




Differential Signatures

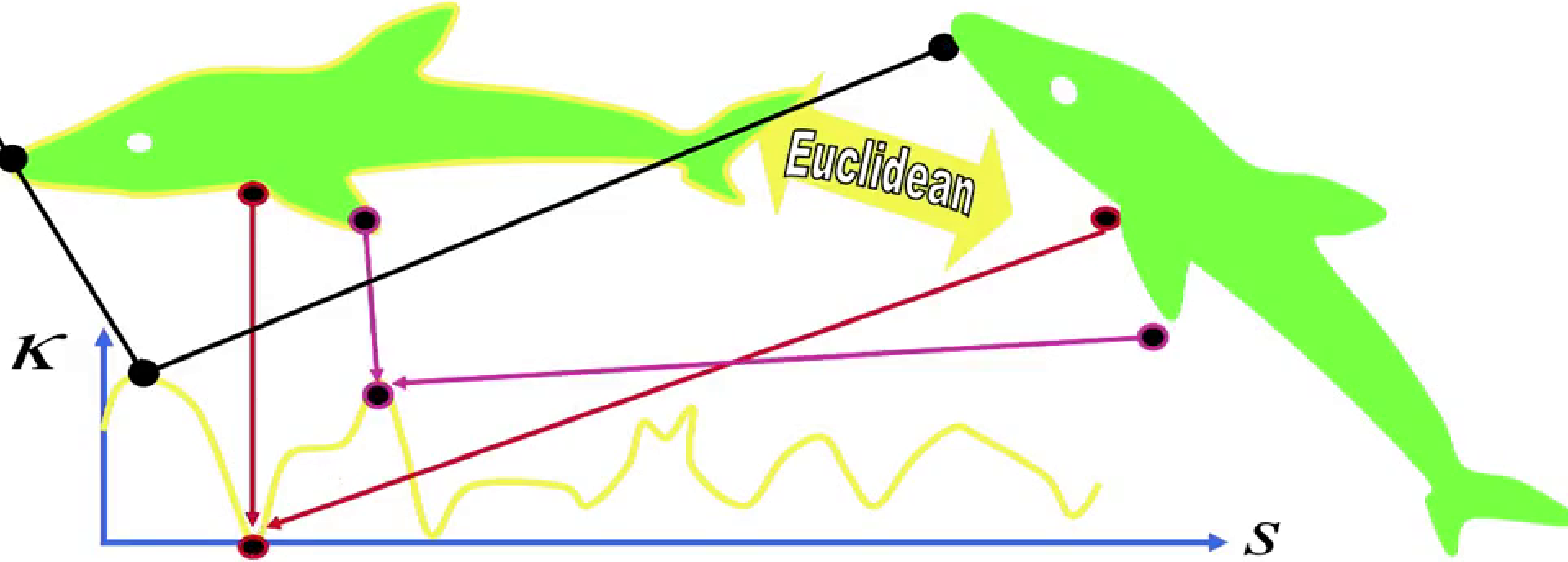
- Euclidean invariant signature $\{s, \kappa(s)\}$

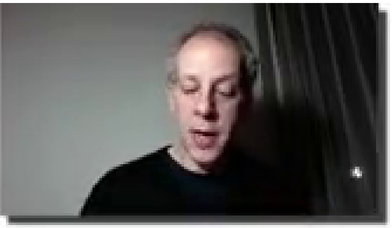




Differential Signatures

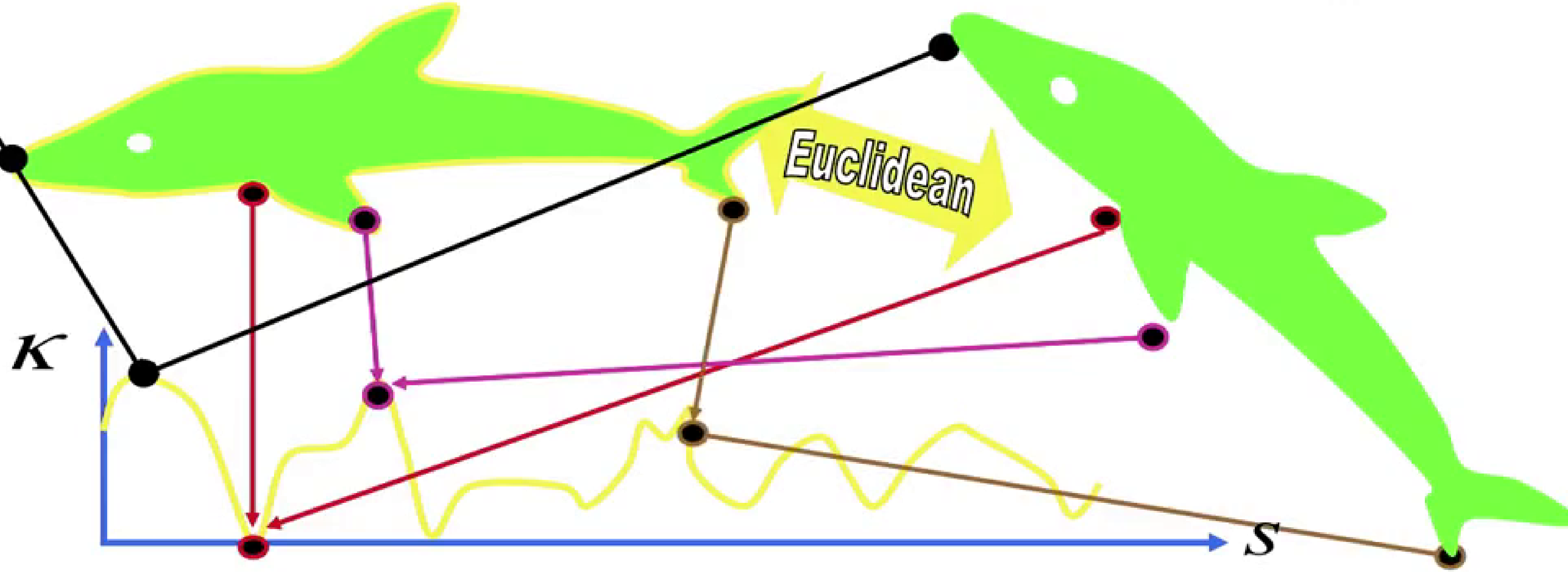
- Euclidean invariant signature $\{s, \kappa(s)\}$

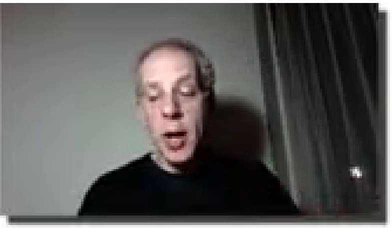




Differential Signatures

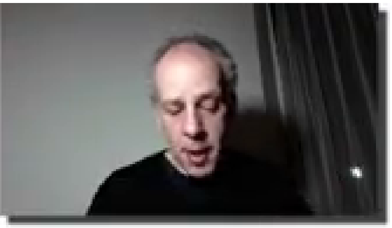
- Euclidean invariant signature $\{s, \kappa(s)\}$





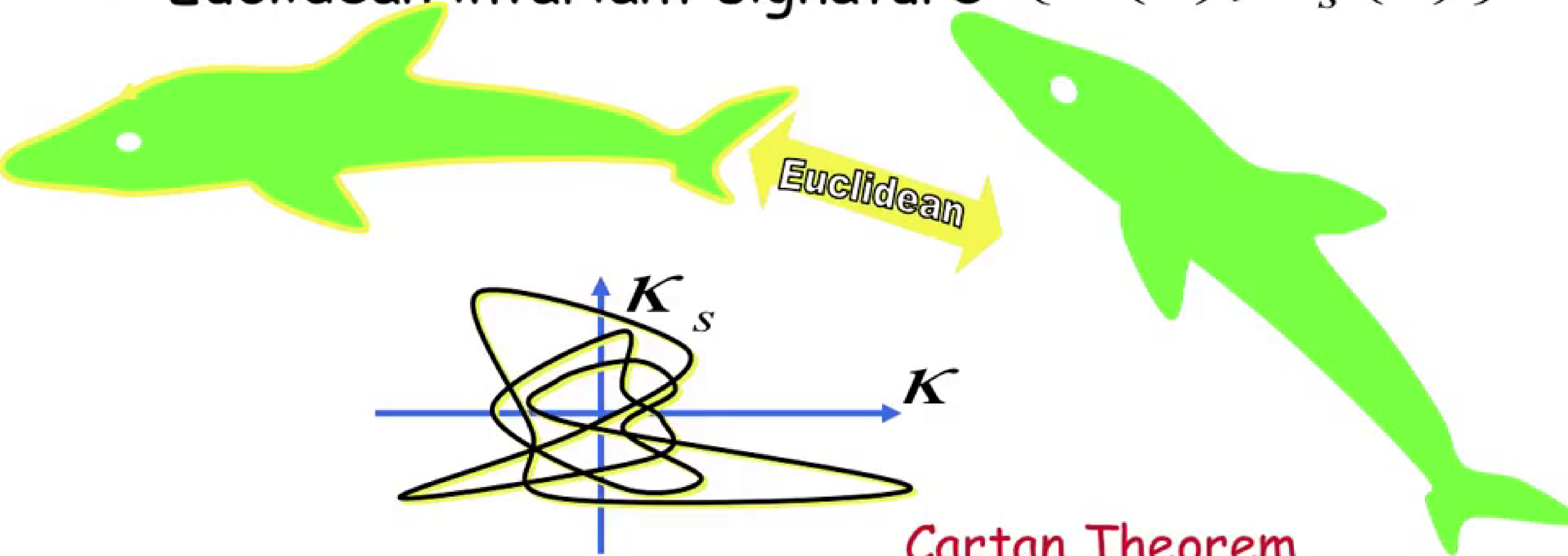
Differential Signatures

- Euclidean invariant signature $\boxed{\{s, \kappa(s)\}}$

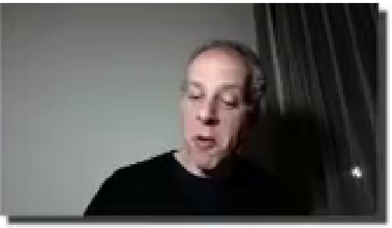


Differential Signatures

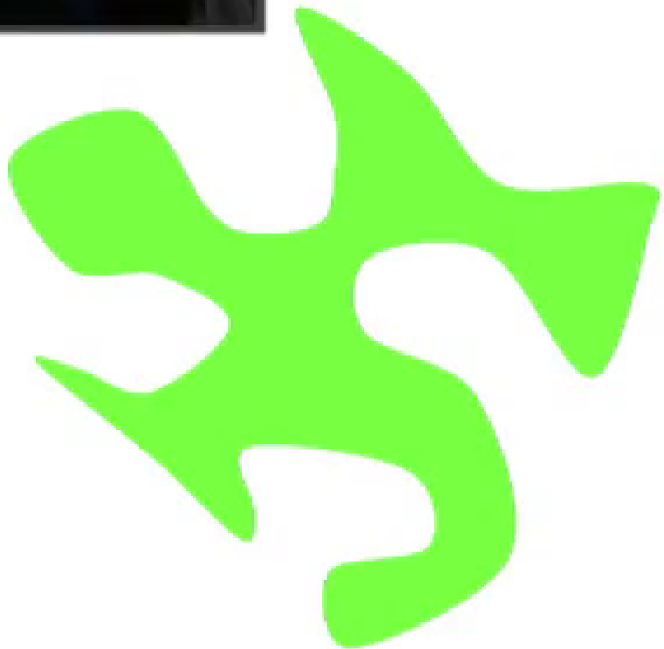
- Euclidean invariant signature $\{\kappa(s), \kappa_s(s)\}$

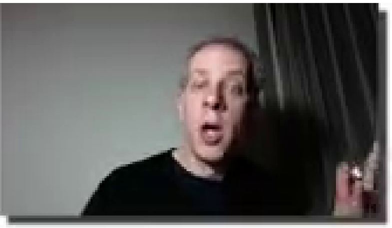


Cartan Theorem

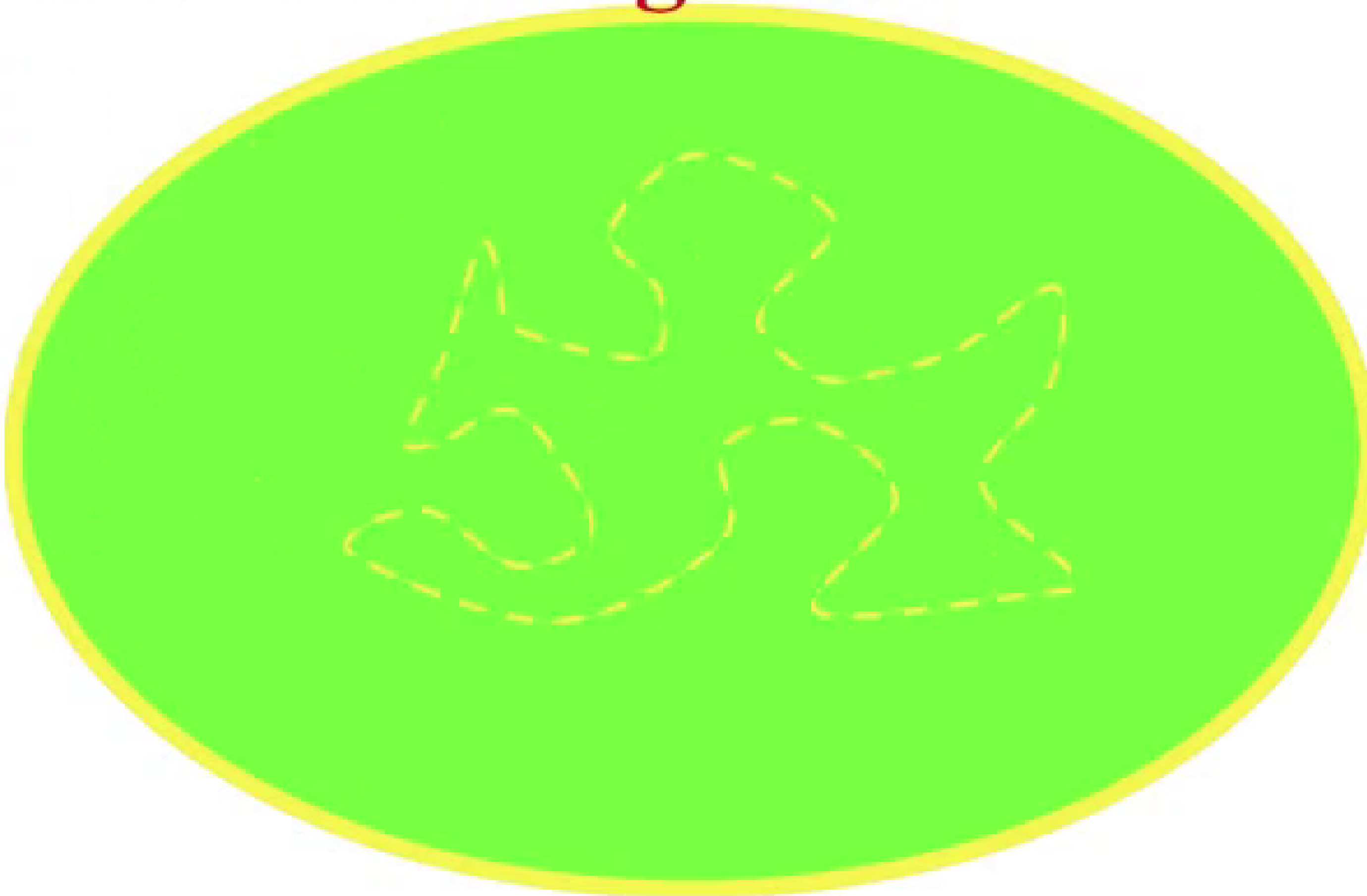


Differential Signatures

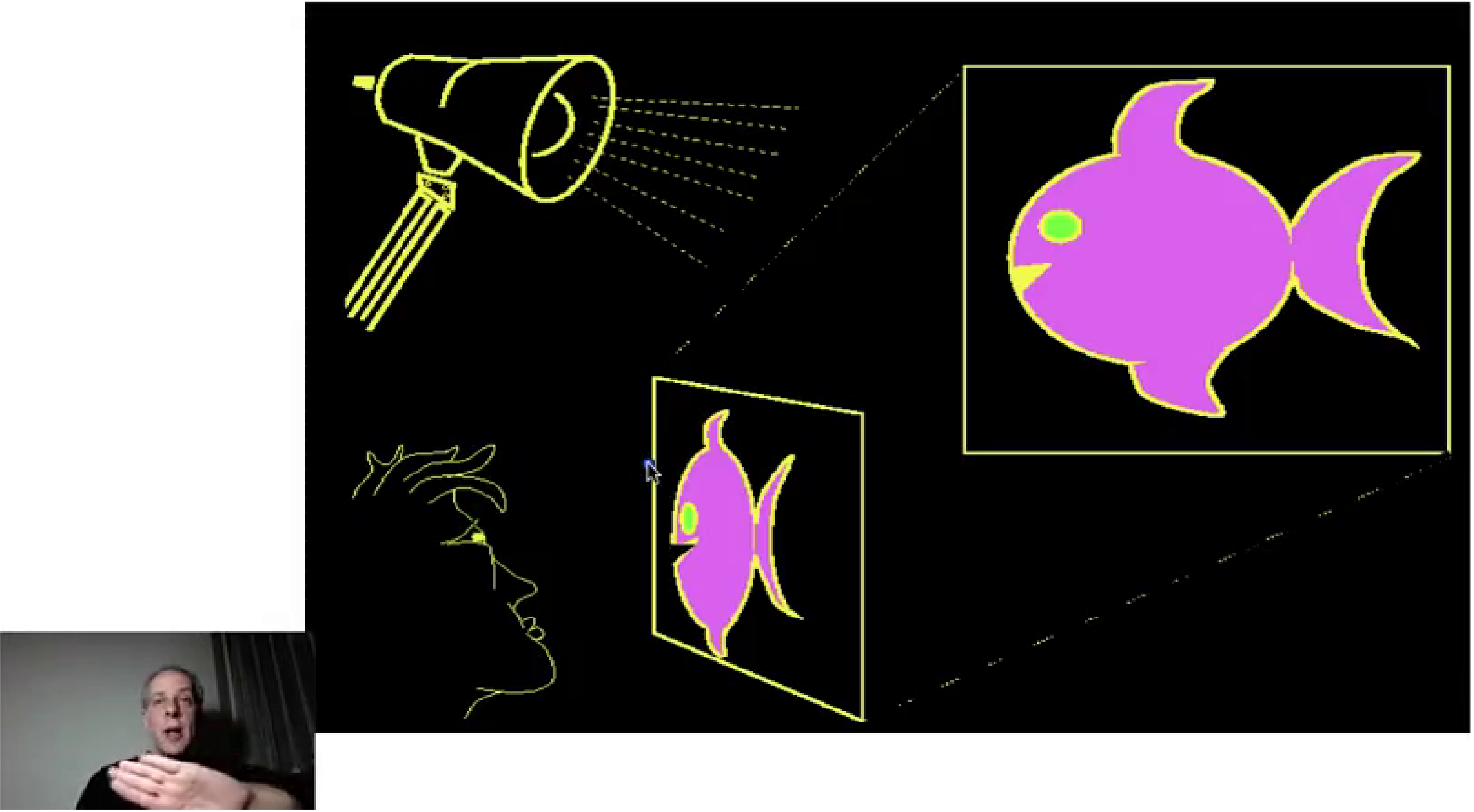




Differential Signatures



~Affine



~Affine

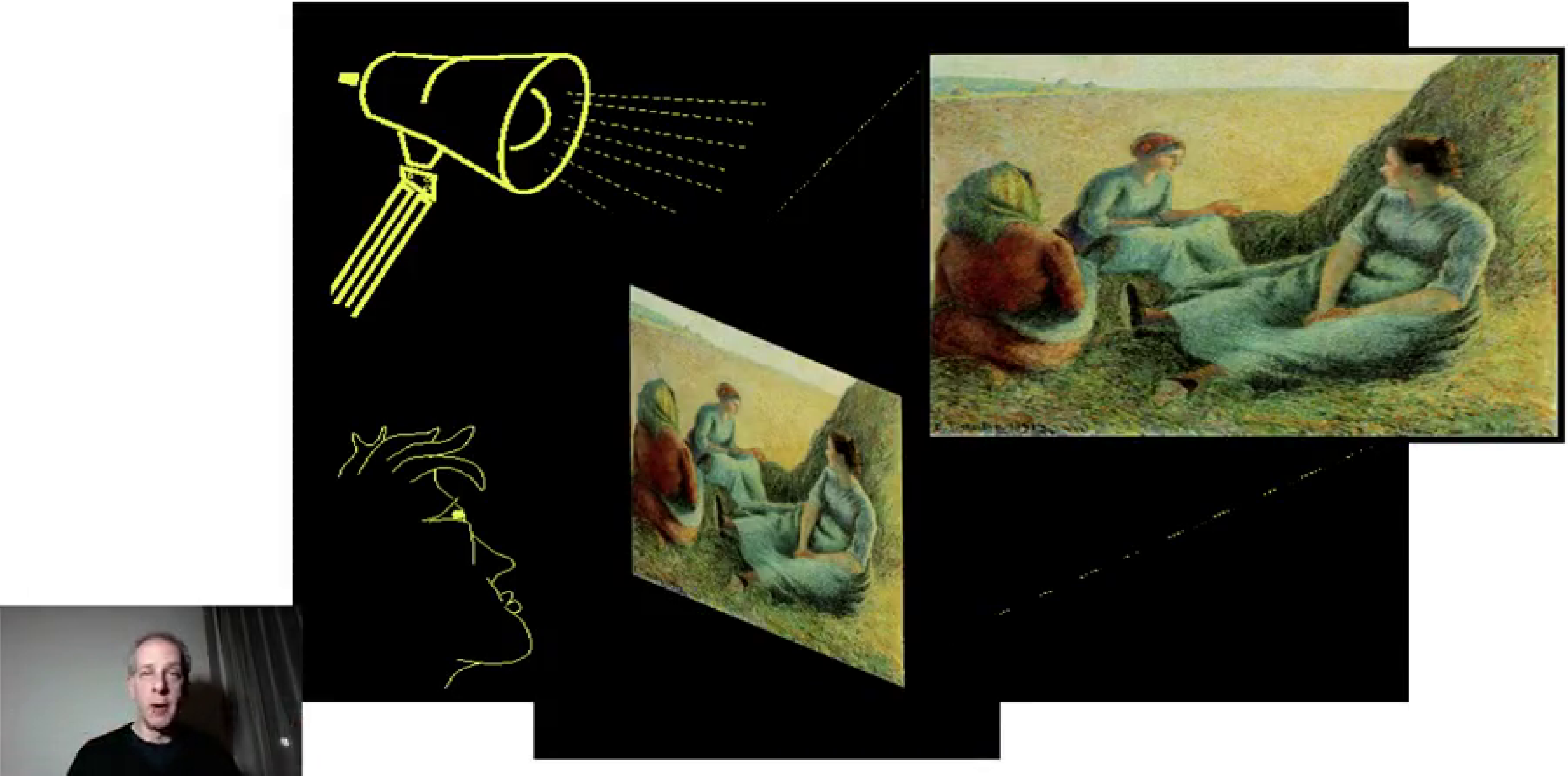
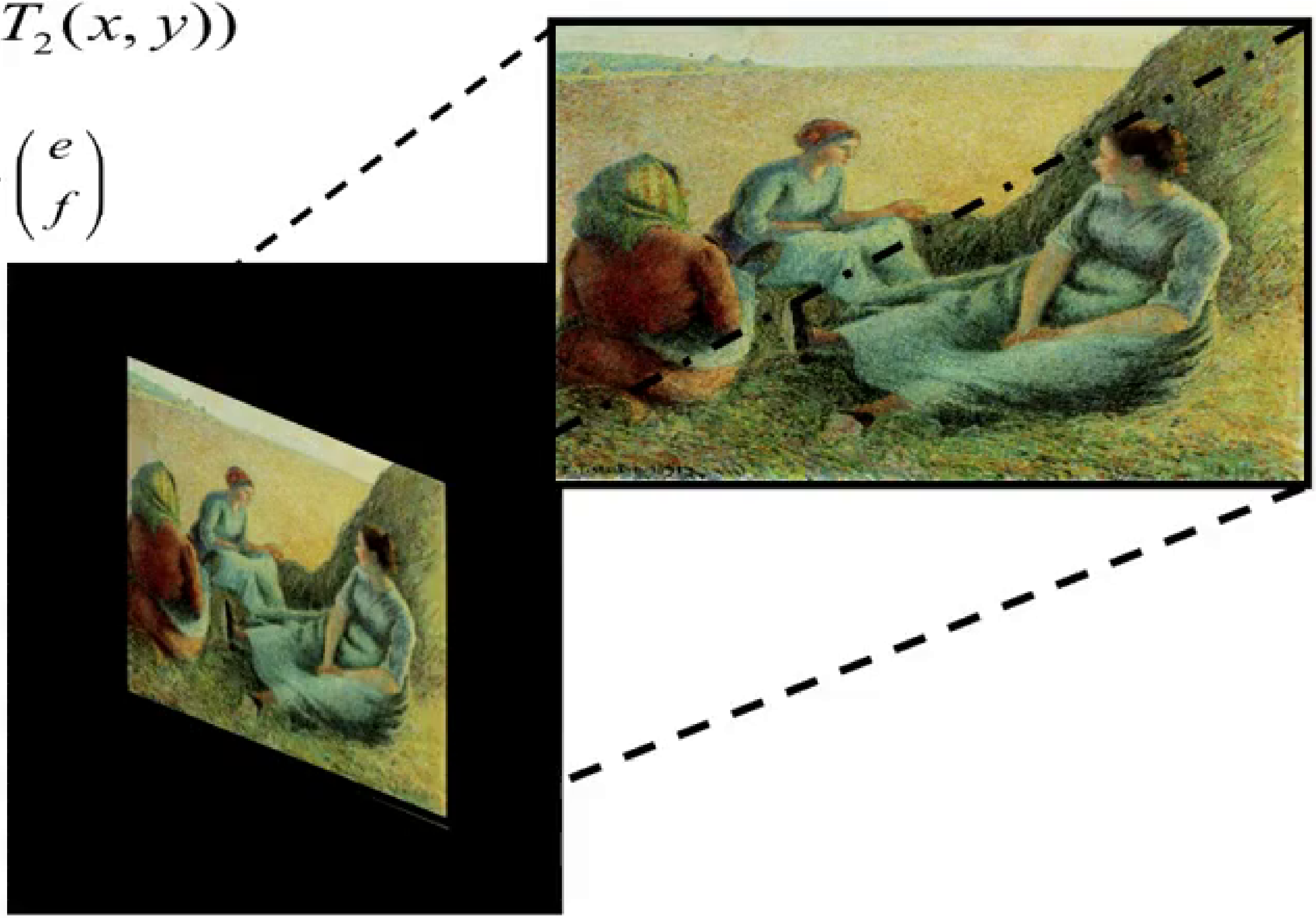


Image transformation

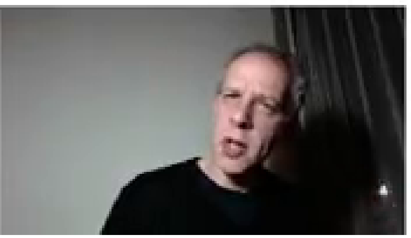
$$I_2(x,y) = I_1(T_1(x,y), T_2(x,y))$$

$$\begin{pmatrix} T_1(x,y) \\ T_2(x,y) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$$



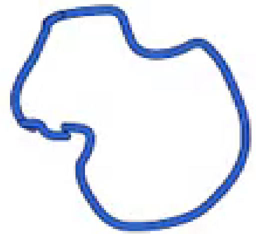
- **Equi-affine:**

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1$$



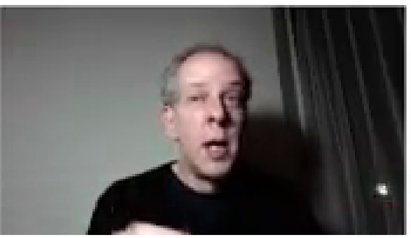
$C(p)$ $C(r)$ **Invariant arclength should be**

1. Re-parameterization invariant



$$w=\int F(C,C_p,C_{pp},\ldots)dp=\int F(C,C_r,C_{rr},\ldots)dr$$

2. Invariant under the group of transformations

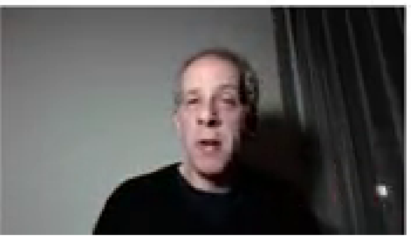


Invariant arclength should be

1. Re-parameterization invariant

$$w=\int F\left(C,C_p,C_{pp},\ldots\right)dp=\int \overset{\text{Geometric measure}}{\boxed{F\left(C,C_r,C_{rr},\ldots\right)}}dr$$

2. Invariant under the group of transformations



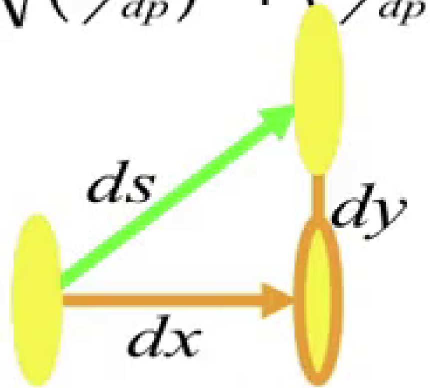
Euclidean arclength

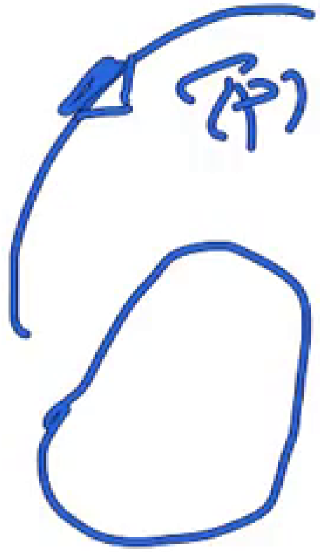
- Length is preserved, thus

$$ds=\sqrt{dx^2+dy^2}=\frac{dp}{dp}\sqrt{dx^2+dy^2}=dp\sqrt{\left(dx/dp\right)^2+\left(dy/dp\right)^2}=\left|C_p\right|dp$$

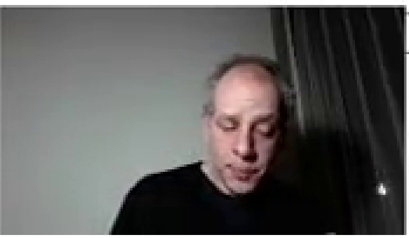
$$s=\int\left|C_p\right|dp$$

$$\left|C_s\right|=1$$





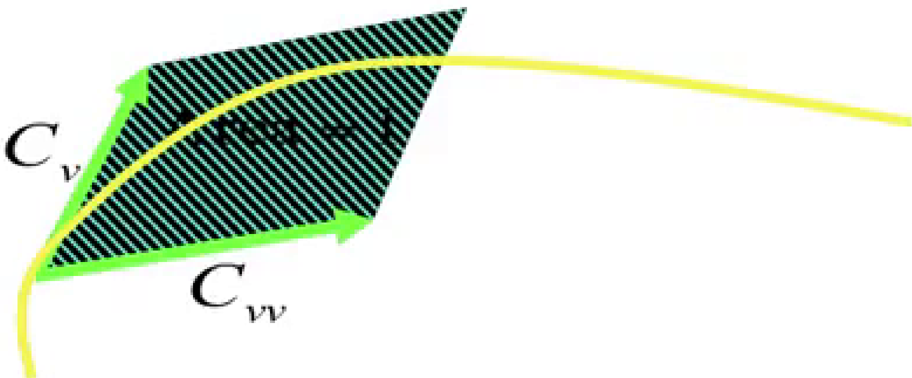
$$\text{Length } L=\int_0^1\left|C_p\right|dp=\int_0^1\left\langle C_p,C_p\right\rangle^{1/2}dp=\int_0^L ds$$



Equi-affine arclength

- Area is preserved, thus

$$\begin{bmatrix} x_v & x_{vv} \\ y_v & y_{vv} \end{bmatrix}$$

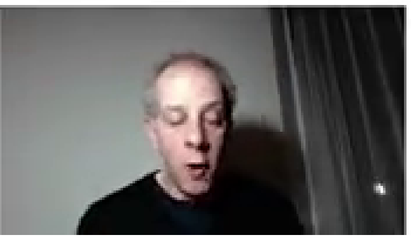


$$(C_v, C_{vv}) = 1$$

$$v = \int (C_p, C_{pp})^{1/3} dp$$

$$v = \int (C_s, C_{ss})^{1/3} ds = \int \kappa^{1/3} ds$$

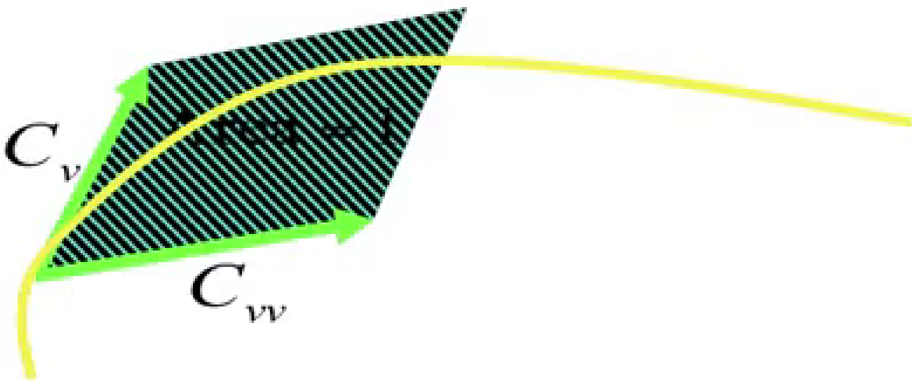
$$dv = \kappa^{1/3} ds$$



Equi-affine arclength

- Area is preserved, thus

re-parameterization invariance

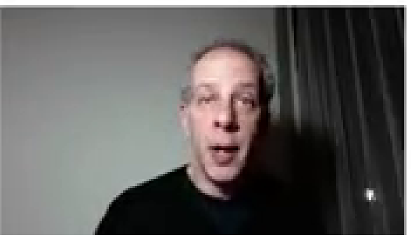


$$(C_v, C_{vv}) = 1$$

$$v = \int (C_p, C_{pp})^{1/3}\, dp$$

$$v = \int (C_s, C_{ss})^{1/3}\, ds = \int \kappa^{1/3}\, ds$$

$$dv = \kappa^{1/3}\, ds$$



$\langle C_s, C_s \rangle = 1$

Equi-affine curvature

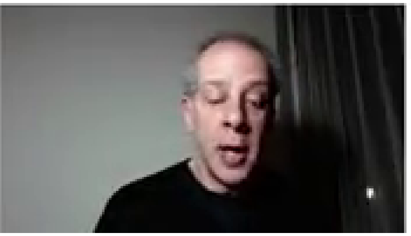
$$(C_v, C_{vv}) = 1 \implies \tfrac{d}{dv}(C_v, C_{vv}) = 0$$

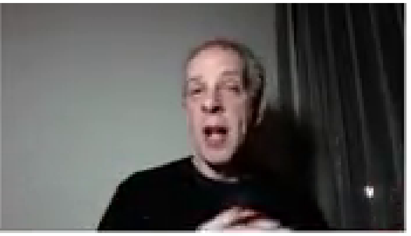
$$\implies \cancel{(C_{vv}, C_{vv})} + (C_v, C_{vvv}) = 0$$

$$\implies (C_v, C_{vvv}) = 0$$

$$\implies C_v \| C_{vvv} \implies C_{vvv} = \mu C_v$$

μ is the affine invariant curvature





Differential Signatures

- Equi-affine invariant signature $\{\nu, \mu(\nu)\}$