Supplementary Guide for Lab 1: Gaussian Distributions & Covariance

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1 Univariate Gaussian Density

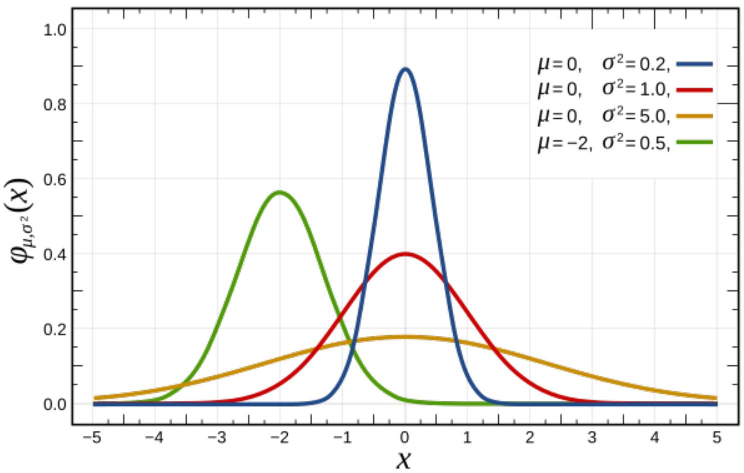

Figure 1: Univariate Gaussian Density

A *univariate Gaussian density* (also called the *normal distribution*) is a continuous PDF on the real line, parameterized by mean μ and variance $\sigma^2 > 0$.

Definition:

$$f(x) \;=\; \frac{1}{\sqrt{2\pi\,\sigma^2}}\exp\Bigl(-\tfrac{(x-\mu)^2}{2\,\sigma^2}\Bigr), \quad x \in \mathbb{R}.$$

- μ is the location (center of the bell).
- σ^2 is the spread (width of the bell).
- The normalizing factor ensures $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

Properties:

- **Shape & Symmetry:** Bell curve centered at $x = \mu$, symmetric: $f(\mu + t) = f(\mu - t)$.
- **Moments:**
  - Mean: $E[X] = \mu$.
  - Variance: $\mathrm{Var}(X) = \sigma^2$.

- **Standard Gaussian:** When $\mu = 0$, $\sigma^2 = 1$,

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} \exp\!\Big(-\tfrac{x^2}{2}\Big),$$

and any $X \sim \mathcal{N}(\mu, \sigma^2)$ can be standardized via $Z = (X - \mu)/\sigma \sim \mathcal{N}(0, 1)$.

- **Visual Intuition:**
  - High $\sigma \to$ wide, flat bell.
  - Low $\sigma \to$ tall, narrow bell.

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2 Multivariate Gaussian Distribution

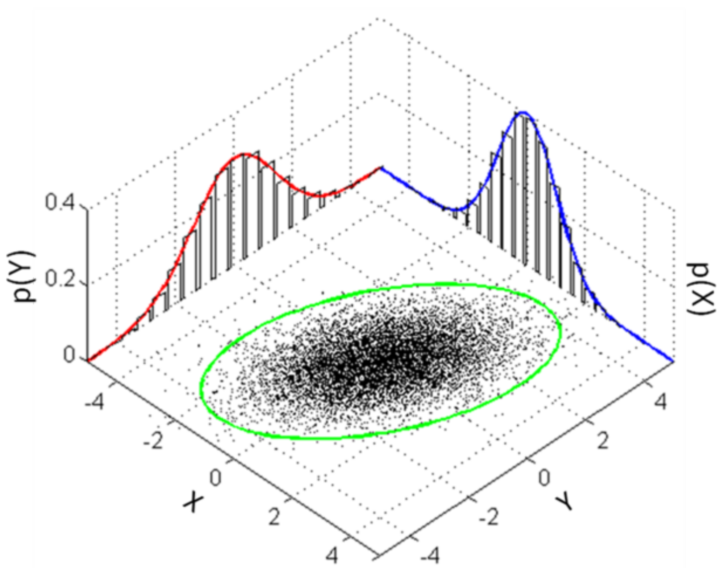

Figure 2: Multivariate Gaussian Distribution

A *multivariate Gaussian distribution* generalizes the normal distribution to d-dimensional vectors. It is defined by a mean vector $\mu \in \mathbb{R}^d$ and a covariance matrix $\Sigma \in \mathbb{R}^{d \times d}$ (symmetric, positive semidefinite).

For $d = 2$, let

$$X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim \mathcal{N}(\mu, \Sigma), \quad \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \sigma_1^2 & \rho\,\sigma_1\sigma_2 \\ \rho\,\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}.$$

- Each marginal $X_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$.
- The correlation is $\rho = \dfrac{\mathrm{Cov}(X_1, X_2)}{\sigma_1\sigma_2}$.

- Contours of the joint density are ellipses oriented and scaled by the eigenvectors and eigenvalues of Σ.

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3 Central Limit Theorem (CLT)

The CLT explains why sums (or averages) of many small, independent random effects tend toward a Gaussian distribution.

1. **Informal Statement:**
   Taking a large number of i.i.d. random variables (finite mean and variance) and summing them (with rescaling) yields an approximately Gaussian distribution, regardless of the originals' shape.

2. **Formal Version:**
   Let $X_1, \ldots, X_n$ be i.i.d. with $E[X_i] = \mu$, $\mathrm{Var}(X_i) = \sigma^2 < \infty$. Define

   $$S_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (X_i - \mu).$$

   Then as $n \to \infty$,

   $$S_n \xrightarrow{d} \mathcal{N}(0, \sigma^2) \quad \text{or} \quad \frac{S_n}{\sigma} \xrightarrow{d} \mathcal{N}(0, 1).$$

3. **Key Points:**
   - *Distribution-free:* Original X_i can be uniform, exponential, etc.
   - *Universality:* Explains prevalence of bell-shaped noise/averages.
   - *Inference Foundation:* Justifies normal-based confidence intervals/tests for large n.

4. **Illustration:**
   - Sum of 1 $\mathrm{Uniform}(0, 1)$: flat histogram.
   - Sum of 2 uniforms: triangular.
   - Sum of 4, 12 uniforms: increasingly bell-shaped.

5. **Lab Connection:**
   In Section 1 of the lab, summing $\mathrm{Uniform}(0, 1)$ samples and plotting histograms for increasing n demonstrates the CLT in action.

If you take a bunch of random samples from any population (no matter how weird or skewed), the averages of those samples will start to form a bell curve (normal distribution) — as long as the sample size is big enough.

4 Covariance

Covariance generalizes variance to pairs of variables, measuring how they vary together.

1. **Variance (univariate):**

$$\mathrm{Var}(X) = E\big[(X-\mu)^2\big],$$

   always nonnegative.

2. **Covariance (two variables):**

$$\mathrm{Cov}(X,Y) = E\big[(X-\mu_X)(Y-\mu_Y)\big].$$

   - Positive: X, Y tend to move together.
   - Negative: they move oppositely.
   - Zero: uncorrelated (not necessarily independent).

3. **Variance as Special Case:**

$$\mathrm{Cov}(X,X) = \mathrm{Var}(X).$$

4. **Covariance Matrix (multivariate):**
   For $\mathbf{X} = (X_1, \dots, X_d) \sim \mathcal{N}(\mu, \Sigma)$,

$$\Sigma_{ij} = \mathrm{Cov}(X_i, X_j).$$

   - Diagonals: variances.
   - Off-diagonals: covariances.
   - Σ must be positive semidefinite.

Let

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sim \mathcal{N}\Big(\mu = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ \Sigma = \begin{bmatrix} 1 & 0.8 \\ 0.8 & 2 \end{bmatrix}\Big).$$

1. **Variances & Covariance:**

$$\mathrm{Var}(X_1) = 1, \quad \mathrm{Var}(X_2) = 2, \quad \mathrm{Cov}(X_1, X_2) = 0.8 > 0.$$

2. **Joint Density:**

$$f(\mathbf{x}) = \frac{1}{2\pi\sqrt{\det\Sigma}} \exp\Big(-\tfrac{1}{2}(\mathbf{x}-\mu)^T \Sigma^{-1} (\mathbf{x}-\mu)\Big),$$

   with $\det\Sigma = 1.36$ and

$$\Sigma^{-1} = \frac{1}{1.36}\begin{bmatrix} 2 & -0.8 \\ -0.8 & 1 \end{bmatrix}.$$

3. **Geometric Interpretation:**
   - Contours are ellipses.
   - Axis lengths $= \sqrt{\lambda_1}, \sqrt{\lambda_2}$ (eigenvalues).
   - Tilt toward the line $x_1 = x_2$ due to positive covariance.
4. **Relation to Univariate:**
   If $\Sigma_{12} = 0$, the bivariate factors into two independent univariate Gaussians.
5. **Marginals & Conditionals:**

$$X_1 \sim \mathcal{N}(0,1), \quad X_2 \sim \mathcal{N}(0,2), \quad X_1 \mid X_2 = x_2 \sim \mathcal{N}(0.4\, x_2,\ 1 - 0.32).$$

Use this guide alongside your lab exercises for quick reference on definitions, properties, and geometric insights.

Covariance tells you how two variables change together.

If they increase together, covariance is positive.

If one increases while the other decreases, covariance is negative.

If they don't follow any clear pattern together, covariance is near zero.