

THE MAGIC OF PASCAL'S TRIANGLE

PASCAL'S TRIANGLE

This represents a way to write down the "early" binomial coefficients $\binom{n}{r}$ easily.

- Each row begins and ends with "1". (We have a "tent" of "1"s.)
- Every other entry equals the sum of the two entries immediately above it.

Here it is:

1	Row 0: Contains $\binom{0}{0}$
1 1	Row 1: Contains $\binom{1}{0}, \binom{1}{1}$
1 2 1	Row 2: Contains $\binom{2}{0}, \binom{2}{1}, \binom{2}{2}$
1 3 3 1	Row 3: Contains $\binom{3}{0}, \binom{3}{1}, \binom{3}{2}, \binom{3}{3}$
1 4 6 4 1	Row 4: Contains $\binom{4}{0}, \binom{4}{1}, \binom{4}{2}, \binom{4}{3}, \binom{4}{4}$

etc. - The "histograms" of the rows approach a bell-shaped "normal" distribution!

We can observe some basic properties of binomial coefficients:

$$\text{Symmetry about the center: } \binom{n}{r} = \binom{n}{n-r}$$

(The process of choosing r winners is equivalent to the process of choosing $n-r$ losers.)

$$\text{The "tent" of "1"s: } \binom{n}{0} = \binom{n}{n} = 1$$

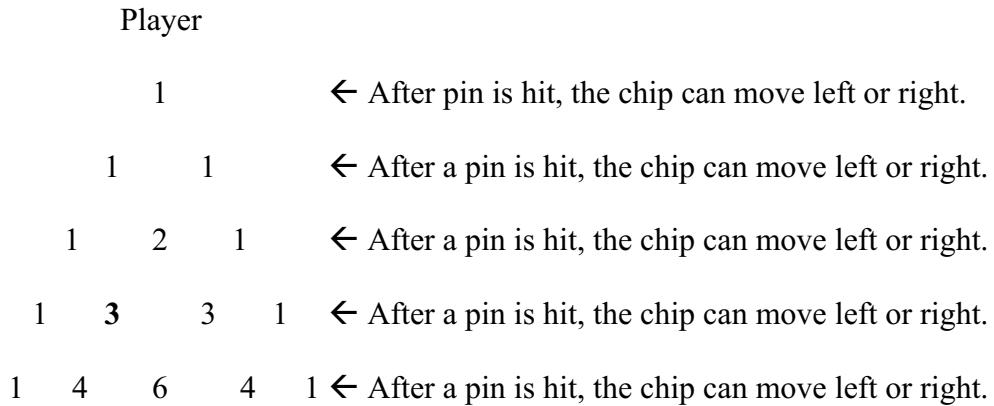
$$\text{The "inner tent" of natural numbers: } \binom{n}{1} = \binom{n}{n-1} = n$$

(There are n ways to get one winner and $n-1$ losers from a group of n people.)

THE PLINKO / PACHINKO APPROACH TO PASCAL'S TRIANGLE

In the game of Plinko on the CBS game show "The Price is Right", contestants won money by standing over a board full of pins and dropping chips. The chips then fell into bins at the bottom of the board; each bin was labeled with a dollar amount that was added to the contestant's winnings.

Now imagine a pinboard shaped like Pascal's triangle:

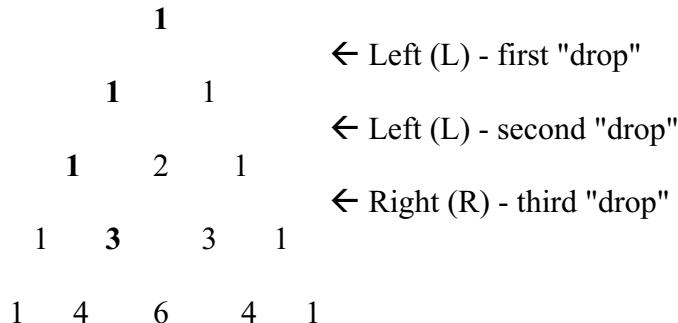


It turns out that the number of ways to hit a pin is equal to the number on the pin!

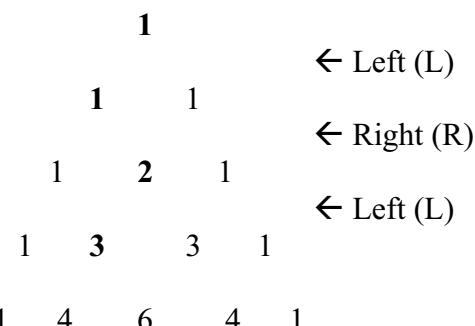
Example

There are 3 ways to hit the boldfaced pin labeled "3" above.

One way: LLR



A second way: LRL



A third way: RLL



The only way to reach the "3" is if there are exactly one "R" and two "L"s in any order. How many ways are there to choose exactly one of the three drops to be an "R"? $\binom{3}{1}$, which the 3 represents!

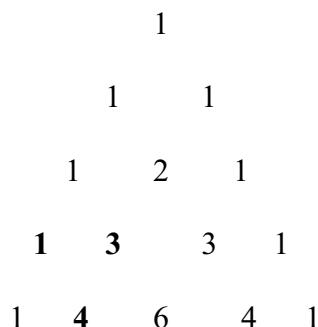
Likewise, there are 6 ways to hit the "6" pin, because out of four drops, we need exactly two of them to be "R"s. There are $\binom{4}{2} = 6$ ways to do this.

Remember the construction of Pascal's triangle:

- Each row begins and ends with "1". (We have a "tent" of "1"s.)
 - Every other entry equals the sum of the two entries immediately above it.

How does the Plinko model exhibit this?

Example



There is 1 way to hit the "1" (all "L"s).
Then, you can move right to hit the "4".

There are 3 ways to hit the "3".
Then, you can move left to hit the "4".

So, there are 4 total ways to hit the "4".

Note: If you go to a science museum that has a "Plinko-type" board with numerous balls being dropped from the top-center, the bottom of the board should look like a bell curve!

EXPANDING POWERS OF BINOMIALS: THE BINOMIAL THEOREM (MATH 96)

Remember that Row n of Pascal's Triangle provides the coefficients for the expansion of $(a + b)^n$:

$$(a + b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \dots + \binom{n}{n}b^n$$

Example

$$(a + b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$$

Here's why this expansion of $(a + b)^3$ is correct:

You can start by writing down 8 terms, each corresponding to a sequence of choices.

$$\begin{aligned}(a + b)^3 &= \underbrace{(a + b)}_{\text{Choose "a" or "b"}} \underbrace{(a + b)}_{\text{Choose "a" or "b"}} \underbrace{(a + b)}_{\text{Choose "a" or "b"}} \\(a + b)^3 &= \underbrace{aaa}_{a^3} \quad \leftarrow \begin{cases} \binom{3}{0} = 1 \text{ way to never} \\ \text{choose "b"} \end{cases} \\&\quad + \underbrace{aab + aba + baa}_{3a^2b} \quad \leftarrow \begin{cases} \binom{3}{1} = 3 \text{ ways to choose "b"} \\ \text{exactly once among the 3 factors} \end{cases} \\&\quad + \underbrace{abb + bab + bba}_{3ab^2} \quad \leftarrow \begin{cases} \binom{3}{2} = 3 \text{ ways to choose "b"} \\ \text{exactly 2 times among the 3 factors} \end{cases} \\&\quad + \underbrace{bbb}_{b^3} \quad \leftarrow \begin{cases} \binom{3}{3} = 1 \text{ way to always} \\ \text{choose "b"} \end{cases} \\&= a^3 + 3a^2b + 3ab^2 + b^3\end{aligned}$$

Notice that choosing the "a" or the "b" from each factor is analogous to dropping "left" or "right" at each step in our Plinko model!