University of Toronto

CSC418 Assignment 1 Part A

Nicholas Dujay
999194900

February 4, 2015

1 Question 1

a) The tangent vector is the derivative of the parametric form with respect to t.

Tangent vector:

$$\begin{aligned} \frac{\delta p(t)}{\delta t} &= \left(\frac{\delta x(t)}{\delta t}, \frac{\delta y(t)}{\delta t}\right) \\ &= \left(\frac{\delta (at)}{\delta t}, \frac{\delta(-\frac{1}{2}gt^2 + bt + h)}{\delta t}\right) \\ &= (a, b - gt) \end{aligned}$$

The normal vector is just any vector that is perpendicular to the tangent vector. $\frac{\delta p(t)}{\delta t} \cdot \vec{n} = 0$, $\vec{n} = (b - gt, -a)$.

b) To find the time of impact t_i, we must solve the quadratic equation for $y(t) = 0$.

$$\begin{aligned} y(t_i) &= 0 \\ -\frac{1}{2}gt_i^2 + bt_i + h &= 0 \\ t_i &= \frac{-b \pm \sqrt{b^2 - 4(-\frac{1}{2}g)(h)}}{2(-\frac{1}{2}g)} \\ t_i &= \frac{b \mp \sqrt{b^2 + 2gh}}{g} \end{aligned}$$

Since $\sqrt{b^2 + 2gh}$ is always positive, we choose $t_i = \frac{b+\sqrt{b^2+2gh}}{g}$.

The velocity at t_i:

$$\frac{\delta p(t_i)}{\delta t} = (a, b - gt_i)$$

$$= \left(a, b - g\left(\frac{b + \sqrt{b^2 + 2gh}}{g}\right)\right)$$
$$= \left(a, -\sqrt{b^2 + 2gh}\right)$$

The location at t_i:

$$p(t_i) = (x(t_i), y(t_i))$$
$$= \left(\frac{a(b + \sqrt{b^2 + 2gh})}{g}, 0\right)$$

2 Question 2

a)

$$(\text{Translate in x and then shear in x}) = \begin{bmatrix} 1 & s_x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & s_x & t_x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(\text{Shear in x and then translate in x}) = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & s_x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & s_x & t_x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= (\text{Translate in x and then shear in x })$$

As you can see, translation and shearing in x is commutative.

b)

$$(\text{Rotate by } \phi \text{ and then by } \theta) = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix} \cdot \begin{bmatrix} cos(\phi) & -sin(\phi) \\ sin(\phi) & cos(\phi) \end{bmatrix}$$
$$= \begin{bmatrix} cos(\theta)cos(\phi) - sin(\theta)sin(\phi) & -cos(\theta)sin(\phi) - sin(\theta)cos(\phi) \\ sin(\theta)cos(\phi) + cos(\theta)sin(\phi) & -sin(\theta)sin(\phi) + cos(\theta)cos(\phi) \end{bmatrix}$$

$$(\text{Rotate by } \theta \text{ and then by } \phi) = \begin{bmatrix} cos(\phi) & -sin(\phi) \\ sin(\phi) & cos(\phi) \end{bmatrix} \cdot \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix}$$
$$= \begin{bmatrix} cos(\phi)cos(\theta) - sin(\phi)sin(\theta) & -cos(\phi)sin(\theta) - sin(\phi)cos(\theta) \\ sin(\phi)cos(\theta) + cos(\phi)sin(\theta) & -sin(\phi)sin(\theta) + cos(\phi)cos(\theta) \end{bmatrix}$$
$$= \begin{bmatrix} cos(\theta)cos(\phi) - sin(\theta)sin(\phi) & -cos(\theta)sin(\phi) - sin(\theta)cos(\phi) \\ sin(\theta)cos(\phi) + cos(\theta)sin(\phi) & -sin(\theta)sin(\phi) + cos(\theta)cos(\phi) \end{bmatrix}$$
$$= \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix} \cdot \begin{bmatrix} cos(\phi) & -sin(\phi) \\ sin(\phi) & cos(\phi) \end{bmatrix}$$
$$= (\text{Rotate by } \phi, \text{ and then by } \theta.)$$

As you can see, two rotations are commutative.

c)

$$\text{(Rotate by } \phi \text{ and then scale by s)} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} \cdot \begin{bmatrix} cos(\phi) & -sin(\phi) \\ sin(\phi) & cos(\phi) \end{bmatrix}$$
$$= \begin{bmatrix} s \cdot cos(\phi) & -s \cdot sin(\phi) \\ s \cdot sin(\phi) & s \cdot cos(\phi) \end{bmatrix}$$

$$\text{(Scale by s and then rotate by } \phi) = \begin{bmatrix} cos(\phi) & -sin(\phi) \\ sin(\phi) & cos(\phi) \end{bmatrix} \cdot \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}$$
$$= \begin{bmatrix} s \cdot cos(\phi) & -s \cdot sin(\phi) \\ s \cdot sin(\phi) & s \cdot cos(\phi) \end{bmatrix}$$
$$= \text{(Rotate by } \phi \text{ and then scale by s)}$$

As you can see, uniform scaling and rotation are commutative.

d)

$$\text{(Rotate by } \phi \text{ and then scale by } s_x, s_y) = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \cdot \begin{bmatrix} cos(\phi) & -sin(\phi) \\ sin(\phi) & cos(\phi) \end{bmatrix}$$
$$= \begin{bmatrix} s_x \cdot cos(\phi) & -s_x \cdot sin(\phi) \\ s_y \cdot sin(\phi) & s_y \cdot cos(\phi) \end{bmatrix}$$

$$\text{(Scale by } s_x, s_y \text{ and then rotate by } \phi) = \begin{bmatrix} cos(\phi) & -sin(\phi) \\ sin(\phi) & cos(\phi) \end{bmatrix} \cdot \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$
$$= \begin{bmatrix} s_x \cdot cos(\phi) & -s_y \cdot sin(\phi) \\ s_x \cdot sin(\phi) & s_y \cdot cos(\phi) \end{bmatrix}$$
$$\neq \text{(Rotate by } \phi \text{ and then scale by s)}$$

As you can see, non uniform scaling and rotation are **not** commutative.

3 Question 3

a) First, we get the direction vector from $\vec{p_i}$ to $\vec{p}_{i+1}$: $(x_{i+1} - x_i, y_{i+1} - y_i)$. Then, since the inward facing normal is a 90 degree counter clockwise rotation of this vector, the result is $(y_i - y_{i+1}, x_{i+1} - x_i)$.

Counter clockwise rotation:

$$\begin{bmatrix} cos(90) & -sin(90) \\ sin(90) & cos(90) \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$$

b) Let d_q be a vector from q to p_i, ie $d_q = p_i - q$. The following function tells you if the point is on the inward facing line:

$$sameside(q, p_i, n_i) = \begin{cases} true & \text{if } d_q \cdot n_i > 0 \\ false & \text{if } d_q \cdot n_i \leq 0 \end{cases}$$

If the angle between d_q and n_i is less than 90 degrees, we know q is on the inward side. Since $\forall \theta \in [0, 90) \cdot cos(\theta) > 0$, then the dot product between d_q and n_i is positive if q is on the inward side, and negative if q is on the outward side.

c) The algorithm will use the above two parts to the question to solve this. The high level steps are the following:

i) For each pair of points (p_i, p_{i+1}) calculate the **inward facing normal** n_{p_i}.

ii) For each pair of points (r_i, r_{i+1}) calculate the **outward facing normal** n_{r_i}. Note: in the actual implementation, we can just reverse the order of points (r_{i+1}, r_i) and calculate the inward facing normal.

iii) For all of the above normals, ensure that **sameside** (from part b) is true for the point.

Some pseudo code:

```
function NORMAL(p_{i+1}, p_i)
    return (y_i − y_{i+1}, x_{i+1} − x_i)
end function

function InConvexPolgyon(point)
    for all p_i, p_{i+1} in the outer polygon do          ▷ check if the point is inside the outer polygon
        if not sameside(point, p_i, Normal(p_{i+1},p_i)) then
            return False
        end if
    end for
    for all r_j, r_{j+1} in the inner polygon do          ▷ check if the point is outside the inner polygon
        if not sameside(point, r_i, Normal(r_j,r_{j+1})) then
            return False                  ▷ reversing the order gives you an outward facing normal here
        end if
    end for
    return True
end function
```

4 Question 4

To calculate the Homography H, we start with the following matrix equation:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ h & k & 1 \end{bmatrix} \cdot \begin{bmatrix} x_k \\ y_k \\ 1 \end{bmatrix} \cong \begin{bmatrix} u_k \\ v_k \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} ax_k + by_k + c \\ dx_k + ey_k + f \\ hx_k + ky_k + 1 \end{bmatrix} \cong \begin{bmatrix} u_k \\ v_k \\ 1 \end{bmatrix}$$

Since these two vectors are homogeneously equivalent, we divide through by the z coordinate to get two equations:

$$\begin{bmatrix} \frac{ax_k+by_k+c}{hx_k+ky_k+1} \\ \frac{dx_k+ey_k+f}{hx_k+ky_k+1} \\ 1 \end{bmatrix} = \begin{bmatrix} u_k \\ v_k \\ 1 \end{bmatrix}$$

$$ax_k + by_k + c - u_k(hx_k + ky_k + 1) = 0$$

$$dx_k + ey_k + f - v_k(hx_k + ky_k + 1) = 0$$

From these two equations, we just plug in x_k, y_k, u_k, v_k and get the following system of equations:

$$a0 + b0 + c + 4(h0 + k0 + 1) = 0$$

$$d0 + e0 + f - 1(h0 + k0 + 1) = 0$$

$$a1 + b0 + c + 1.5(h1 + k0 + 1) = 0$$

$$
\begin{aligned}
d1 + e0 + f + 0.5(h1 + k0 + 1) &= 0 \\
a0 + b1 + c - 0.5(h0 + k1 + 1) &= 0 \\
d0 + e1 + f + 0.5(h0 + k1 + 1) &= 0 \\
a1 + b1 + c - 0(h1 + k1 + 1) &= 0 \\
d1 + e1 + f - 1(h1 + k1 + 1) &= 0
\end{aligned}
$$

Solving all of these equations, we get $a = 1, b = 3, c = -4, d = -2, e = 0, f = 1, h = 1, k = -3$.

5 Question 5

First, lets express the transformation as a translation and then followed by a scale and finally followed by a rotation. Let the given matrix $\begin{bmatrix} 8 & 3 & -7 \\ 6 & -4 & -24 \\ 0 & 0 & 1 \end{bmatrix}$ be A, Let $R(\theta)$ be the rotation matrix, $S(s_x, s_y)$ be the non uniform scale matrix, $T(t_x, t_y)$ be the translation matrix.

$$
\begin{aligned}
R(\theta) \cdot S(s_x, s_y) \cdot T(t_x, t_y) &= R(\theta) \cdot S(s_x, s_y) \cdot \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \\
&= R(\theta) \cdot \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \\
&= R(\theta) \cdot \begin{bmatrix} s_x & 0 & s_x t_x \\ 0 & s_y & s_y t_y \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} s_x & 0 & s_x t_x \\ 0 & s_y & s_y t_y \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} s_x cos(\theta) & -s_y sin(\theta) & s_x cos(\theta) t_x - s_y sin(\theta) t_y \\ s_x sin(\theta) & s_y cos(\theta) & s_x sin(\theta) t_x + s_y cos(\theta) t_y \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

Setting $A = R(\theta) \cdot S(s_x, s_y) \cdot T(t_x, t_y)$ will give us the following system of equations.

$$
\begin{bmatrix} 8 & 3 & -7 \\ 6 & -4 & -24 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} s_x cos(\theta) & -s_y sin(\theta) & s_x cos(\theta) t_x - s_y sin(\theta) t_y \\ s_x sin(\theta) & s_y cos(\theta) & s_x sin(\theta) t_x + s_y cos(\theta) t_y \\ 0 & 0 & 1 \end{bmatrix}
$$

$$
\begin{aligned}
s_x cos(\theta) &= 8 \\
s_x sin(\theta) &= 6 \\
-s_y sin(\theta) &= 3 \\
s_y cos(\theta) &= -4 \\
s_x cos(\theta) t_x - s_y sin(\theta) t_y &= -7 \\
s_x sin(\theta) t_x + s_y cos(\theta) t_y &= -24
\end{aligned}
$$

Solving these system of equations, we get a rotation of $\theta = tan^{-1}(\frac{3}{4})$, a scale of $s_x = 10, s_y = -5$, and a translation of $t_x = -2, t_y = 3$. However, the scaling values are not unique. The steps to solving these equations are left out to keep this answer short.