University of Toronto

# CSC418 Assignment 1 Part A

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## 1 Question 1

Given that $\vec{e} = (1, 2, 2)$ $\vec{g} = (1, 1, 3)$ and $\vec{r} = (0, 1, 0)$, $\vec{s}$ $\vec{u}$ and $\vec{v}$ are calculated like this:

$$\vec{s} = -\frac{\vec{g}}{\|\vec{g}\|} = \left(-\frac{1}{\sqrt{11}}, -\frac{1}{\sqrt{11}}, -\frac{3}{\sqrt{11}}\right)$$
$$\vec{u} = \frac{\vec{r} \times \vec{s}}{\|\vec{r} \times \vec{s}\|} = \left(-\frac{1}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}}\right)$$
$$\vec{v} = \frac{\vec{s} \times \vec{u}}{\|\vec{s} \times \vec{u}\|} = \left(-\frac{1}{3\sqrt{2}}, -\frac{4}{3\sqrt{2}}, -\frac{1}{3\sqrt{2}}\right)$$

From here, we can calculate the world to camera transformation matrix as follows:

$$M_{wc} = \left[\begin{array}{ccc|c} -\frac{1}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{3\sqrt{2}} & -\frac{4}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} & \frac{11}{3\sqrt{2}} \\ -\frac{1}{\sqrt{11}} & -\frac{1}{\sqrt{11}} & -\frac{3}{\sqrt{11}} & \frac{9}{\sqrt{11}} \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Where the $M_{wc}$ matrix has the following form:

$$M_{wc} = \left[\begin{array}{ccc|c} u_x & u_y & u_z & \vec{u} \cdot \vec{e} \\ v_x & v_y & v_z & \vec{v} \cdot \vec{e} \\ s_x & s_y & s_z & \vec{s} \cdot \vec{e} \\ 0 & 0 & 0 & 1 \end{array}\right]$$

## 2 Question 2

First lets calculate $m$ then project it to $m'$.

$$m = \frac{1}{2}(p + q) = \frac{1}{2}\left(\begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} + \begin{bmatrix} q_x \\ q_y \\ q_z \\ 1 \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2}(p_x + q_x) \\ \frac{1}{2}(p_y + q_y) \\ \frac{1}{2}(p_z + q_z) \\ 1 \end{bmatrix}$$

$$m' = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & -f & 0\end{bmatrix} \cdot \begin{bmatrix}\frac{1}{2}(p_x+q_x)\\ \frac{1}{2}(p_y+q_y)\\ \frac{1}{2}(p_z+q_z)\\ 1\end{bmatrix} \cong \begin{bmatrix}-\frac{1}{f}\left(\frac{p_x+q_x}{p_z+q_z}\right)\\ -\frac{1}{f}\left(\frac{p_y+q_y}{p_z+q_z}\right)\\ -\frac{1}{f}\\ 1\end{bmatrix}$$

Next lets calculate $p'$ and $q'$ then calculate the midpoint between them.

$$p' = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & -f & 0\end{bmatrix} \cdot \begin{bmatrix}p_x\\ p_y\\ p_z\\ 1\end{bmatrix} \cong \begin{bmatrix}-\frac{p_x}{p_z f}\\ -\frac{p_y}{p_z f}\\ -\frac{1}{f}\\ 1\end{bmatrix}$$

$q'$ is calculated similarly.

$$\frac{1}{2}(p'+q') = \frac{1}{2}\left(\begin{bmatrix}-\frac{p_x}{p_z f}\\ -\frac{p_y}{p_z f}\\ -\frac{1}{f}\\ 1\end{bmatrix} + \begin{bmatrix}-\frac{q_x}{q_z f}\\ -\frac{q_y}{q_z f}\\ -\frac{1}{f}\\ 1\end{bmatrix}\right) \cong \begin{bmatrix}-\frac{1}{2f}\left(\frac{p_x}{p_z}+\frac{q_x}{q_z}\right)\\ -\frac{1}{2f}\left(\frac{p_y}{p_z}+\frac{q_y}{q_z}\right)\\ -\frac{1}{f}\\ 1\end{bmatrix}$$

Therefore, $m' \neq \frac{1}{2}(p'+q')$ for this projection. However, if we use an orthographic projection, then $m' = \frac{1}{2}(p'+q')$:

$$m' = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} \cdot \begin{bmatrix}\frac{1}{2}(p_x+q_x)\\ \frac{1}{2}(p_y+q_y)\\ \frac{1}{2}(p_z+q_z)\\ 1\end{bmatrix} = \begin{bmatrix}\frac{1}{2}(p_x+q_x)\\ \frac{1}{2}(p_y+q_y)\\ 1\end{bmatrix}$$

$$p' = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} \cdot \begin{bmatrix}p_x\\ p_y\\ p_z\\ 1\end{bmatrix} = \begin{bmatrix}p_x\\ p_y\\ 1\end{bmatrix}$$

$$\begin{aligned}\frac{1}{2}(p'+q') &= \frac{1}{2}\left(\begin{bmatrix}p_x\\ p_y\\ 1\end{bmatrix} + \begin{bmatrix}q_x\\ q_y\\ 1\end{bmatrix}\right)\\ &= \begin{bmatrix}\frac{1}{2}(p_x+q_x)\\ \frac{1}{2}(p_y+q_y)\\ 1\end{bmatrix} = m'\end{aligned}$$

# 3 Question 3

a) First, we get the direction vector from $\vec{p}_i$ to $\vec{p}_{i+1}$: $(x_{i+1}-x_i, y_{i+1}-y_i)$. Then, since the inward facing normal is a 90 degree counter clockwise rotation of this vector, the result is $(y_i - y_{i+1}, x_{i+1}-x_i)$.

   Counter clockwise rotation:

$$\begin{bmatrix}cos(90) & -sin(90)\\ sin(90) & cos(90)\end{bmatrix} \cdot \begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} \cdot \begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}-y\\ x\end{bmatrix}$$

b) Let $d_q$ be a vector from $q$ to $p_i$, ie $d_q = p_i - q$. The following function tells you if the point is on the inward facing line:

$$sameside(q, p_i, n_i) = \begin{cases} true & \text{if } d_q \cdot n_i > 0\\ false & \text{if } d_q \cdot n_i \leq 0\end{cases}$$

   If the angle between $d_q$ and $n_i$ is less than 90 degrees, we know $q$ is on the inward side. Since $\forall\theta \in [0, 90) \cdot cos(\theta) > 0$, then the dot product between $d_q$ and $n_i$ is positive if $q$ is on the inward side, and negative if $q$ is on the outward side.

c) The algorithm will use the above two parts to the question to solve this. The high level steps are the following:

i) For each pair of points $(p_i, p_{i+1})$ calculate the **inward facing normal** $n_{p_i}$.

ii) For each pair of points $(r_i, r_{i+1})$ calculate the **outward facing normal** $n_{r_i}$. Note: in the actual implementation, we can just reverse the order of points $(r_{i+1}, r_i)$ and calculate the inward facing normal.

iii) For all of the above normals, ensure that **sameside** (from part b) is true for the point.

Some pseudo code:

```
function NORMAL(p_{i+1}, p_i)
    return (y_i − y_{i+1}, x_{i+1} − x_i)
end function

function InConvexPolgyon(point)
    for all p_i, p_{i+1} in the outer polygon do        ▷ check if the point is inside the outer polygon
        if not sameside(point, p_i, Normal(p_{i+1},p_i)) then
            return False
        end if
    end for
    for all r_j, r_{j+1} in the inner polygon do        ▷ check if the point is outside the inner polygon
        if not sameside(point, r_i, Normal(r_j,r_{j+1})) then
            return False                 ▷ reversing the order gives you an outward facing normal here
        end if
    end for
    return True
end function
```

# 4 Question 4

To calculate the Homography H, we start with the following matrix equation:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ h & k & 1 \end{bmatrix} \cdot \begin{bmatrix} x_k \\ y_k \\ 1 \end{bmatrix} \cong \begin{bmatrix} u_k \\ v_k \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} ax_k + by_k + c \\ dx_k + ey_k + f \\ hx_k + ky_k + 1 \end{bmatrix} \cong \begin{bmatrix} u_k \\ v_k \\ 1 \end{bmatrix}$$

Since these two vectors are homogeneously equivalent, we divide through by the z coordinate to get two equations:

$$\begin{bmatrix} \frac{ax_k+by_k+c}{hx_k+ky_k+1} \\ \frac{dx_k+ey_k+f}{hx_k+ky_k+1} \\ 1 \end{bmatrix} = \begin{bmatrix} u_k \\ v_k \\ 1 \end{bmatrix}$$
$$ax_k + by_k + c - u_k(hx_k + ky_k + 1) = 0$$
$$dx_k + ey_k + f - v_k(hx_k + ky_k + 1) = 0$$

From these two equations, we just plug in $x_k, y_k, u_k, v_k$ and get the following system of equations:

$$a0 + b0 + c + 4(h0 + k0 + 1) = 0$$

$$d0+e0+f-1(h0+k0+1)=0$$
$$a1+b0+c+1.5(h1+k0+1)=0$$
$$d1+e0+f+0.5(h1+k0+1)=0$$
$$a0+b1+c-0.5(h0+k1+1)=0$$
$$d0+e1+f+0.5(h0+k1+1)=0$$
$$a1+b1+c-0(h1+k1+1)=0$$
$$d1+e1+f-1(h1+k1+1)=0$$

Solving all of these equations, we get $a=1, b=3, c=-4, d=-2, e=0, f=1, h=1, k=-3$.

# 5 Question 5

First, lets express the transformation as a translation and then followed by a scale and finally followed by a rotation. Let the given matrix $\begin{bmatrix} 8 & 3 & -7 \\ 6 & -4 & -24 \\ 0 & 0 & 1 \end{bmatrix}$ be $A$, Let $R(\theta)$ be the rotation matrix, $S(s_x, s_y)$ be the non uniform scale matrix, $T(t_x, t_y)$ be the translation matrix.

$$\begin{aligned}
R(\theta)\cdot S(s_x,s_y)\cdot T(t_x,t_y) &= R(\theta)\cdot S(s_x,s_y)\cdot \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \\
&= R(\theta)\cdot \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}\cdot \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \\
&= R(\theta)\cdot \begin{bmatrix} s_x & 0 & s_xt_x \\ 0 & s_y & s_yt_y \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}\cdot \begin{bmatrix} s_x & 0 & s_xt_x \\ 0 & s_y & s_yt_y \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} s_xcos(\theta) & -s_ysin(\theta) & s_xcos(\theta)t_x - s_ysin(\theta)t_y \\ s_xsin(\theta) & s_ycos(\theta) & s_xsin(\theta)t_x + s_ycos(\theta)t_y \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

Setting $A = R(\theta)\cdot S(s_x,s_y)\cdot T(t_x,t_y)$ will give us the following system of equations.

$$\begin{bmatrix} 8 & 3 & -7 \\ 6 & -4 & -24 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} s_xcos(\theta) & -s_ysin(\theta) & s_xcos(\theta)t_x - s_ysin(\theta)t_y \\ s_xsin(\theta) & s_ycos(\theta) & s_xsin(\theta)t_x + s_ycos(\theta)t_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned}
s_xcos(\theta) &= 8 \\
s_xsin(\theta) &= 6 \\
-s_ysin(\theta) &= 3 \\
s_ycos(\theta) &= -4 \\
s_xcos(\theta)t_x - s_ysin(\theta)t_y &= -7 \\
s_xsin(\theta)t_x + s_ycos(\theta)t_y &= -24
\end{aligned}$$

Solving these system of equations, we get a rotation of $\theta = tan^{-1}(\frac{3}{4})$, a scale of $s_x = 10, s_y = -5$, and a translation of $t_x = -2, t_y = 3$. The steps to solving these equations are left out to keep this answer short.