University of Toronto

# CSC418 Assignment 1 Part A


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## 1 Question 1

Given that $\vec{e} = (1, 2, 2)$ $\vec{g} = (1, 1, 3)$ and $\vec{r} = (0, 1, 0)$, $\vec{s}$ $\vec{u}$ and $\vec{v}$ are calculated like this:

$$\begin{aligned}
\vec{s} &= -\frac{\vec{g}}{\|\vec{g}\|} = \left(-\frac{1}{\sqrt{11}}, -\frac{1}{\sqrt{11}}, -\frac{3}{\sqrt{11}}\right) \\
\vec{u} &= \frac{\vec{r} \times \vec{s}}{\|\vec{r} \times \vec{s}\|} = \left(-\frac{3}{\sqrt{10}}, 0, \frac{1}{\sqrt{10}}\right) \\
\vec{v} &= \frac{\vec{s} \times \vec{u}}{\|\vec{s} \times \vec{u}\|} = \left(-\frac{1}{\sqrt{110}}, \sqrt{\frac{10}{11}}, -\frac{3}{\sqrt{110}}\right)
\end{aligned}$$

From here, we can calculate the world to camera transformation matrix as follows:

$$M_{wc} = \left[\begin{array}{ccc|c}
-\frac{3}{\sqrt{10}} & 0 & \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\
-\frac{1}{\sqrt{110}} & \sqrt{\frac{10}{11}} & -\frac{3}{\sqrt{110}} & -\frac{13}{\sqrt{110}} \\
-\frac{1}{\sqrt{11}} & -\frac{1}{\sqrt{11}} & -\frac{3}{\sqrt{11}} & \frac{9}{\sqrt{11}} \\
0 & 0 & 0 & 1
\end{array}\right]$$

Where the $M_{wc}$ matrix has the following form:

$$M_{wc} = \left[\begin{array}{ccc|c}
u_x & u_y & u_z & -\vec{u} \cdot \vec{e} \\
v_x & v_y & v_z & -\vec{v} \cdot \vec{e} \\
s_x & s_y & s_z & -\vec{s} \cdot \vec{e} \\
0 & 0 & 0 & 1
\end{array}\right]$$

## 2 Question 2

First lets calculate $m$ then project it to $m'$.

$$m = \frac{1}{2}(p + q) = \frac{1}{2}\left(\begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} + \begin{bmatrix} q_x \\ q_y \\ q_z \\ 1 \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2}(p_x + q_x) \\ \frac{1}{2}(p_y + q_y) \\ \frac{1}{2}(p_z + q_z) \\ 1 \end{bmatrix}$$

$$m' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{f} & 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2}(p_x + q_x) \\ \frac{1}{2}(p_y + q_y) \\ \frac{1}{2}(p_z + q_z) \\ 1 \end{bmatrix} \cong \begin{bmatrix} -f\left(\frac{p_x+q_x}{p_z+q_z}\right) \\ -f\left(\frac{p_y+q_y}{p_z+q_z}\right) \\ -f \\ 1 \end{bmatrix}$$

Next lets calculate $p'$ and $q'$ then calculate the midpoint between them.

$$p' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{f} & 0 \end{bmatrix} \cdot \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} \cong \begin{bmatrix} -f\frac{p_x}{p_z} \\ -f\frac{p_y}{p_z} \\ -f \\ 1 \end{bmatrix}$$

$q'$ is calculated similarly.

$$\frac{1}{2}(p' + q') = \frac{1}{2}\left( \begin{bmatrix} -f\frac{p_x}{p_z} \\ -f\frac{p_y}{p_z} \\ -f \\ 1 \end{bmatrix} + \begin{bmatrix} -f\frac{q_x}{q_z} \\ -f\frac{q_y}{q_z} \\ -f \\ 1 \end{bmatrix} \right) \cong \begin{bmatrix} -\frac{f}{2}\left(\frac{p_x}{p_z} + \frac{q_x}{q_z}\right) \\ -\frac{f}{2}\left(\frac{p_y}{p_z} + \frac{q_y}{q_z}\right) \\ -f \\ 1 \end{bmatrix}$$

In order for $m' = \frac{1}{2}(p' + q')$, we need $-f\left(\frac{p_x+q_x}{p_z+q_z}\right) = -\frac{f}{2}\left(\frac{p_x}{p_z} + \frac{q_x}{q_z}\right)$ and $-f\left(\frac{p_y+q_y}{p_z+q_z}\right) = -\frac{f}{2}\left(\frac{p_y}{p_z} + \frac{q_y}{q_z}\right)$. There's only one case where this is true, and it is when $p_z = q_z$.
If we instead use an orthographic projection, then $m' = \frac{1}{2}(p' + q')$ for all cases:

$$m' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2}(p_x + q_x) \\ \frac{1}{2}(p_y + q_y) \\ \frac{1}{2}(p_z + q_z) \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(p_x + q_x) \\ \frac{1}{2}(p_y + q_y) \\ 1 \end{bmatrix}$$

$$p' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} = \begin{bmatrix} p_x \\ p_y \\ 1 \end{bmatrix}$$

$$\begin{aligned} \frac{1}{2}(p' + q') &= \frac{1}{2}\left( \begin{bmatrix} p_x \\ p_y \\ 1 \end{bmatrix} + \begin{bmatrix} q_x \\ q_y \\ 1 \end{bmatrix} \right) \\ &= \begin{bmatrix} \frac{1}{2}(p_x + q_x) \\ \frac{1}{2}(p_y + q_y) \\ 1 \end{bmatrix} = m' \end{aligned}$$

# 3 Question 3

The projected lines have the following form

$$l_i(u_i)' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{f} & 0 \end{bmatrix} \cdot \begin{bmatrix} p_{ix} + u_i d_{ix} \\ p_{iy} + u_i d_{iy} \\ p_{iz} + u_i d_{iz} \\ 1 \end{bmatrix} = \begin{bmatrix} p_{ix} + u_i d_{ix} \\ p_{iy} + u_i d_{iy} \\ p_{iz} + u_i d_{iz} \\ \frac{1}{f}(p_{iz} + u_i d_{iz}) \end{bmatrix} \cong \begin{bmatrix} f\frac{p_{ix}+u_i d_{ix}}{p_{iz}+u_i d_{iz}} \\ f\frac{p_{iy}+u_i d_{iy}}{p_{iz}+u_i d_{iz}} \\ f \\ 1 \end{bmatrix}$$

The projection of each line will intersect in the canonical view space, so the final expression is independent of the original lines. However, since the original lines never intersect in camera space, the point of intersection is at $u_i = \pm\infty$. Therefore, we must take the limit of $l_i'(u_i)$ as $u_i \to \pm\infty$.

$$\lim_{u_i \to \pm\infty} l_i(u_i) = \lim_{u_i \to \pm\infty} \begin{bmatrix} f\frac{p_{ix}+u_i d_{ix}}{p_{iz}+u_i d_{iz}} \\ f\frac{p_{iy}+u_i d_{iy}}{p_{iz}+u_i d_{iz}} \\ f \\ 1 \end{bmatrix} = \begin{bmatrix} \lim_{u_i \to \pm\infty} f\frac{p_{ix}+u_i d_{ix}}{p_{iz}+u_i d_{iz}} \\ \lim_{u_i \to \pm\infty} f\frac{p_{iy}+u_i d_{iy}}{p_{iz}+u_i d_{iz}} \\ f \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \lim_{u_i \to \pm\infty} f \frac{p_{ix}+u_i d_{ix}}{\frac{p_{iz}}{u_i}+d_{iz}} \\ \lim_{u_i \to \pm\infty} f \frac{p_{iy}+u_i d_{iy}}{\frac{p_{iz}}{u_i}+d_{iz}} \\ f \\ 1 \end{bmatrix} = \begin{bmatrix} \lim_{u_i \to \pm\infty} f \frac{\frac{p_{ix}}{u_i}+d_{ix}}{\frac{p_{iz}}{u_i}+d_{iz}} \\ \lim_{u_i \to \pm\infty} f \frac{\frac{p_{iy}}{u_i}+d_{iy}}{\frac{p_{iz}}{u_i}+d_{iz}} \\ f \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} f\frac{d_{ix}}{d_{iz}} \\ f\frac{d_{iy}}{d_{iz}} \\ f \\ 1 \end{bmatrix}$$

# 4 Question 4

The canonical view transform using $L = -1, R = 1, B = -1, T = 1, f = 1, F = 1001$ is this:

$$M_{cv} = \begin{bmatrix} \frac{2f}{L-R} & 0 & \frac{R+L}{L-R} & 0 \\ 0 & \frac{2f}{B-T} & \frac{B+T}{B-T} & 0 \\ 0 & 0 & \frac{f+F}{F-f} & \frac{2Ff}{F-f} \\ 0 & 0 & -\frac{1}{f} & 0 \end{bmatrix} = \begin{bmatrix} \frac{2}{-1-1} & 0 & \frac{1+-1}{-1-1} & 0 \\ 0 & \frac{2}{-1-1} & \frac{-1+1}{-1-1} & 0 \\ 0 & 0 & \frac{1+1001}{1001-1} & \frac{2(1001)}{1001-1} \\ 0 & 0 & -\frac{1}{f} & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1.002 & 2.002 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

Now transforming a point from camera space to canonical view space:

$$\begin{bmatrix} x_v \\ y_v \\ z_v \\ 1 \end{bmatrix} = M_{cv} \begin{bmatrix} x_c \\ y_c \\ z_c \\ 1 \end{bmatrix} = \begin{bmatrix} -x_c \\ -y_c \\ 1.002z_c + 2.002 \\ z_c \end{bmatrix} \cong \begin{bmatrix} -\frac{x_c}{z_c} \\ -\frac{y_c}{z_c} \\ 1.002 + \frac{2.002}{z_c} \\ 1 \end{bmatrix}$$

Since we are only interested in the pseudo depth value here, we now have a formula for the pseudo depth of each point. $z_{c1} = -1, z_{c2} = -10, z_{c3} = -100, z_{c4} = -1000$

$$z_{v1} = 1.002 + \frac{2.002}{z_{c1}} = -1$$
$$z_{v2} = 1.002 + \frac{2.002}{z_{c2}} = 0.8018$$
$$z_{v3} = 1.002 + \frac{2.002}{z_{c3}} = 0.98198$$
$$z_{v4} = 1.002 + \frac{2.002}{z_{c4}} = 0.999998$$

The relationship is not linear. Plot $z_{ci}$ and $z_{vi}$ on a graph and calculate slopes between each pair of points $(-1, -1), (-10, 0.8018), (-100, 0.98198), (-1000, 0.999998)$ and the slopes between each point should be the same if the relationship is linear. However, the slopes are different between each point so therefore the relationship is not linear.

# 5 Question 5

a) The general equation of an ellipse in 2d is $x(t) = cos(t), y(t) = sin(t)$, for some $t \in [0, n]$. In the $xy$-plane we must hold hold $u$ constant since $t$ is the planar parameter of the equation. Similarly for the cross section of the ellipsoid and the $x = 0$ plane, we must hold $t$ constant.

Let $u = \frac{1}{4}$. The equation of the ellipse that is the cross section of the ellipsoid and the plane $z = 0$ is given by $e_1(t)$.

$$\begin{aligned}
x(t) &= x\left(t, \frac{1}{4}\right) = a\cos(2\pi t)\sin\left(\frac{\pi}{2}\right) = a\cos(2\pi t) \\
y(t) &= y\left(t, \frac{1}{4}\right) = b\sin(2\pi t)\sin\left(\frac{\pi}{2}\right) = b\sin(2\pi t) \\
z(t) &= z\left(t, \frac{1}{4}\right) = 0 \\
e_1(t) &= (a\cos(2\pi t), b\sin(2\pi t), 0)
\end{aligned}$$

Let $t = \frac{1}{4}$. The equation of the ellipse that is the cross section of the ellipsoid and the plane $x = 0$ is given by $e_2(t)$.

$$\begin{aligned}
x(u) &= x\left(\frac{1}{4}, u\right) = a\cos\left(\frac{\pi}{2}\right)\sin(2\pi u) = 0 \\
y(u) &= y\left(\frac{1}{4}, u\right) = b\sin\left(\frac{\pi}{2}\right)\sin(2\pi u) = b\sin(2\pi u) \\
z(u) &= z\left(\frac{1}{4}, u\right) = c\cos(2\pi u) \\
e_2(u) &= (0, b\sin(2\pi u), c\cos(2\pi u))
\end{aligned}$$

b) The tangent to these ellipses are simply the derivative of their equations.

$$\begin{aligned}
\frac{\delta e_1(t)}{\delta t} &= (a\cos(2\pi t), b\sin(2\pi t), 0) = (-a2\pi\sin(2\pi t), b2\pi\cos(2\pi t), 0) \\
\frac{\delta e_2(u)}{\delta u} &= (0, b\sin(2\pi u), c\cos(2\pi u)) = (0, b2\pi\cos(2\pi u), -c2\pi\sin(2\pi u))
\end{aligned}$$

The normal to the ellipses is the cross product of these two tangent vectors.

$$\begin{aligned}
\frac{\delta e_1(t)}{\delta t} \times \frac{\delta e_2(u)}{\delta u} &= (-4\pi^2 bc\cos(2\pi t)\sin(2\pi u), -4\pi^2 ac\sin(2\pi t)\sin(2\pi u), -4\pi^2 ab\sin(2\pi t)\cos(2\pi u)) \\
&= -4\pi^2 abc\left(\frac{\cos(2\pi t)\sin(2\pi u)}{a}, \frac{\sin(2\pi t)\sin(2\pi u)}{b}, \frac{\sin(2\pi t)\cos(2\pi u)}{c}\right)
\end{aligned}$$

However, these normals are only valid when $e_1(t) = e_2(u)$. Setting $e_1(t) = e_2(u)$ and solving for $t$ and $u$, we get $t = u$ and $t = \frac{1}{4}, \frac{3}{4}$. Subbing in these values of $t$ and $u$, we get the final expression for the normal to be: $-4\pi^2 abc(0, \frac{1}{b}, 0)$, and $4\pi^2 abc(0, \frac{1}{b}, 0)$ (for the other side).

c) First, lets find $x^2$, $y^2$, $z^2$.

$$\begin{aligned}
x^2 &= (a\cos(2\pi t)\sin(2\pi u))^2 = a^2\cos^2(2\pi t)\sin^2(2\pi u) \\
y^2 &= (b\sin(2\pi t)\sin(2\pi u))^2 = b^2\sin^2(2\pi t)\sin^2(2\pi u) \\
z^2 &= (c\cos(2\pi u))^2 = c^2\cos^2(2\pi u)
\end{aligned}$$

Then lets calculate $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$.

$$\begin{aligned}
\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} &= \frac{a^2\cos^2(2\pi t)\sin^2(2\pi u)}{a^2} + \frac{b^2\sin^2(2\pi t)\sin^2(2\pi u)}{b^2} + \frac{c^2\cos^2(2\pi u)}{c^2} \\
&= \cos^2(2\pi t)\sin^2(2\pi u) + \sin^2(2\pi t)\sin^2(2\pi u) + \cos^2(2\pi u) \\
&= \sin^2(2\pi u)(\cos^2(2\pi t) + \sin^2(2\pi t)) + \cos^2(2\pi u) \\
&= \sin^2(2\pi u) + \cos^2(2\pi u) \\
&= 1
\end{aligned}$$

Therefore, the implicit equation of a ellipsoid is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

d) The normal to a point on the ellipse is simply the gradient of the equation derived in part c.

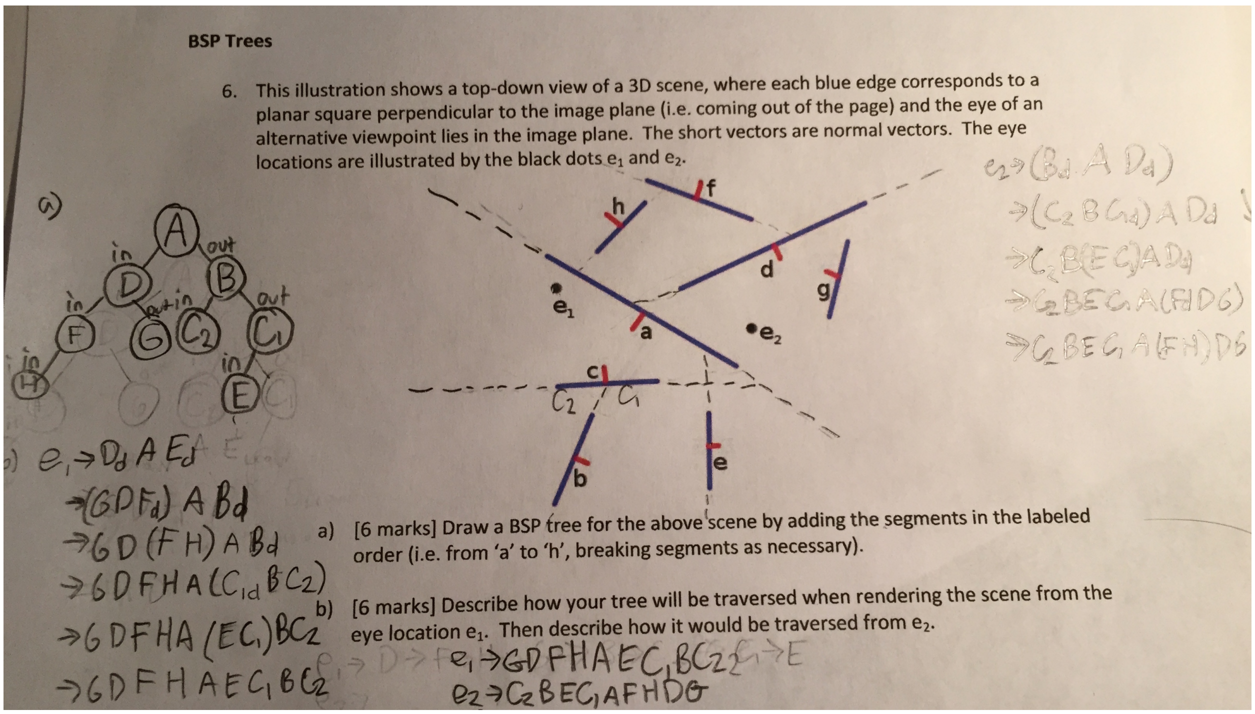
$$\nabla f(x, y, z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

$$= \left(\frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2}\right)$$

If we sub in $x(t,u)$ for $x$, (and similarly for $y$ and $z$) then we get the normal vector in terms of $t, u$

$$= 2\left(\frac{\cos(2\pi t)\sin(2\pi u)}{a}, \frac{\sin(2\pi t)\sin(2\pi u)}{b}, \frac{\cos(2\pi u)}{c}\right)$$

# 6 Question 6



**BSP Trees**

6. This illustration shows a top-down view of a 3D scene, where each blue edge corresponds to a planar square perpendicular to the image plane (i.e. coming out of the page) and the eye of an alternative viewpoint lies in the image plane. The short vectors are normal vectors. The eye locations are illustrated by the black dots $e_1$ and $e_2$.

a) [6 marks] Draw a BSP tree for the above scene by adding the segments in the labeled order (i.e. from 'a' to 'h', breaking segments as necessary).

b) [6 marks] Describe how your tree will be traversed when rendering the scene from the eye location $e_1$. Then describe how it would be traversed from $e_2$.