MSc Program

Mathematics for Engineers

Course project- 'M2. Inverted Pendulum'

Week 2 Task

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Tasks for week 2

Now consider the double inverted pendulum from Fig. 1 and perform a similar analysis,

1. Describe the mathematical model of the system from Fig. 1.
2. Draw the free body/kinematic diagram of the system from Fig. 1.
3. Write the equations of motion for the system from Fig. 1.
4. Write down the asymptotic solution of the system from Fig. 1.
5. Provide a comprehensive conclusion.

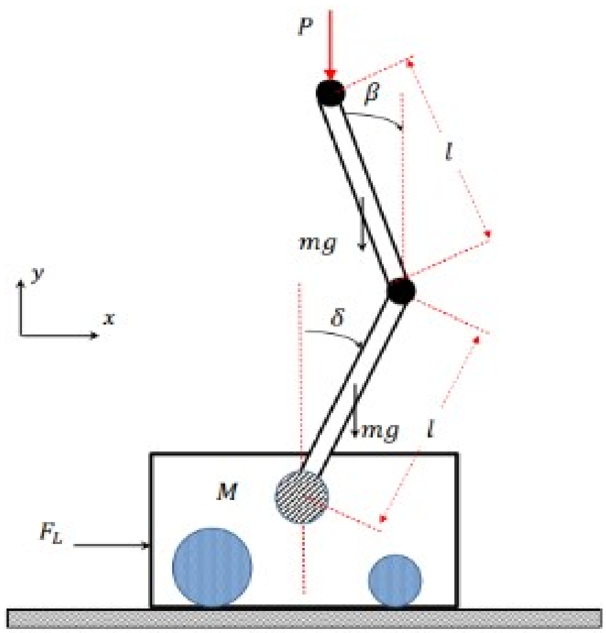

Figure 1: Inverted Pendulum System

The diagram in Fig. 1 depicts a double inverted pendulum.

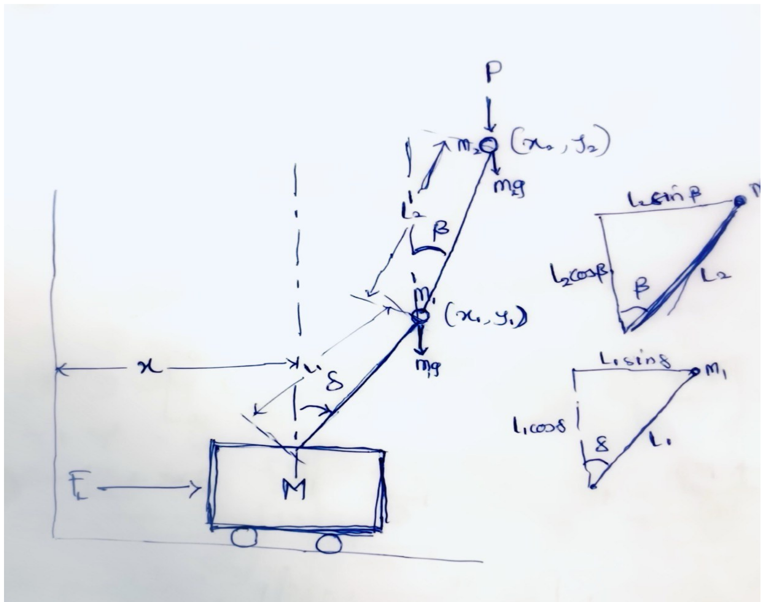

Figure 2: Free Body Diagram

1 Mathematical model of the double inverted pendulum system

1.1 Variables:

- m_1: mass of the first pendulum.
- m_2: mass of the second pendulum.
- M: mass of the cart.
- l_1: length of the first rod.
- l_2: length of the second rod.
- δ: angle of the first pendulum with the vertical.
- β: angle of the second pendulum with the vertical.
- F: horizontal force applied to the cart.
- P: external force (as needed).
- x: horizontal position of the cart.

1.2 Equations of motion:

Using the Lagrangian method, we derive the equations of motion for the system. This results in three differential equations, one for the cart and one for each pendulum.

1. For the cart:

$$
\begin{aligned}
&(M + m_1 + m_2)\ddot{x} + (m_1 l_1 + m_2 l_1)\ddot{\delta}\cos(\delta) \\
&\quad + m_2 l_2 \ddot{\beta}\cos(\beta) \\
&= F + (m_1 l_1 + m_2 l_1)\dot{\delta}^2 \sin(\delta) \\
&\quad + m_2 l_2 \dot{\beta}^2 \sin(\beta)
\end{aligned}
$$

2. For the first pendulum:

$$
\begin{aligned}
&(m_1 l_1^2 + m_2 l_1^2)\ddot{\delta} + (m_1 l_1 + m_2 l_1)\ddot{x}\cos(\delta) \\
&\quad + m_2 l_1 l_2 \ddot{\beta}\cos(\delta - \beta) \\
&= m_1 g l_1 \sin(\delta) + m_2 g l_1 \sin(\delta) \\
&\quad + P l_1 \cos(\delta) + m_2 l_1 l_2 \dot{\beta}^2 \sin(\delta - \beta)
\end{aligned}
$$

3. For the second pendulum:

$$
\begin{aligned}
&m_2 l_2^2 \ddot{\beta} + m_2 l_1 l_2 \ddot{\delta}\cos(\delta - \beta) \\
&\quad + m_2 l_2 \ddot{x}\cos(\beta) \\
&= m_2 g l_2 \sin(\beta) + 2 m_2 l_2 \dot{x}\dot{\beta}\sin(\beta) \\
&\quad + m_2 l_1 l_2 \dot{\delta}\dot{\beta}\sin(\delta - \beta) \\
&\quad + P l_2 \cos(\beta)
\end{aligned}
$$

Where:

- g is the acceleration due to gravity.
- $\ddot{x}$, $\ddot{\delta}$, and $\ddot{\beta}$ are the second time derivatives of x, δ, and β, respectively.
- $\dot{\delta}$ and $\dot{\beta}$ are the first time derivatives of δ and β, respectively, i.e., their angular velocities.

1.3 Assumptions:

1. The pendulum rods are massless and rigid.
2. The system operates in a plane.
3. All movements are frictionless.
4. External disturbances are modeled as a force P.

1.4 Coordinate Definitions:

Let:

– x be the horizontal position of the cart.

– δ be the angle pendulum 1 makes with the vertical.

– β be the angle pendulum 2 makes with the vertical.

1.5 Kinematics:

Using polar coordinates: For pendulum 1:

$$x_1 = x + l_1 \sin(\delta) \tag{1}$$
$$y_1 = l_1 \cos(\delta) \tag{2}$$

For pendulum 2:

$$\begin{aligned} x_2 = x + l_1 \sin(\delta) \\ + l_2 \sin(\beta) \end{aligned} \tag{3}$$
$$y_2 = l_1 \cos(\delta) + l_2 \cos(\beta) \tag{4}$$

1.6 Velocities:

For pendulum 1:

$$\dot{x}_1 = \dot{x} + l_1 \dot{\delta} \cos(\delta) \tag{5}$$
$$\dot{y}_1 = -l_1 \dot{\delta} \sin(\delta) \tag{6}$$

For pendulum 2, velocities are:

$$\begin{aligned} \dot{x}_2 = \dot{x} + l_1 \dot{\delta} \cos(\delta) \\ + l_2 \dot{\beta} \cos(\beta) \end{aligned} \tag{7}$$
$$\begin{aligned} \dot{y}_2 = -l_1 \dot{\delta} \sin(\delta) \\ - l_2 \dot{\beta} \sin(\beta) \end{aligned} \tag{8}$$

kinetic energy (T):

For the cart:

$$T_c = \frac{1}{2} M \dot{x}^2 \tag{9}$$

For pendulum 1:

$$T_1 = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) \tag{10}$$
$$T_1 = \frac{1}{2} m_1 [(\dot{x} + l_1 \dot{\delta} \cos(\delta))^2 + (-l_1 \dot{\delta} \sin(\delta))^2] \tag{11}$$

For pendulum 2:

$$T_2 = \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2) \tag{12}$$

$$\begin{aligned} T_2 = \frac{1}{2}m_2[&(\dot{x} + l_1\dot{\delta}\cos(\delta) + l_2\dot{\beta}\cos(\beta))^2 \\ &+ (-l_1\dot{\delta}\sin(\delta) - l_2\dot{\beta}\sin(\beta))^2] \end{aligned} \tag{13}$$

$$\begin{aligned} T_2 = &\frac{1}{2}m_2\dot{x}^2 + \frac{1}{2}m_2 l_1^2\dot{\delta}^2 + \frac{1}{2}m_2 l_2^2\dot{\beta}^2 \\ &+ m_2 l_1\dot{x}\dot{\delta}\cos(\delta) + m_2 l_1 l_2\dot{\delta}\dot{\beta}\cos(\delta - \beta) \\ &+ m_2 l_2\dot{x}\dot{\beta}\cos(\beta) \end{aligned} \tag{14}$$

Total kinetic energy:

$$T = T_c + T_1 + T_2 \tag{15}$$

$$\begin{aligned} T = &\frac{1}{2}M\dot{x}^2 + \frac{1}{2}m_1[(\dot{x} + l_1\dot{\delta}\cos(\delta))^2 \\ &+ (-l_1\dot{\delta}\sin(\delta))^2] \\ &+ \frac{1}{2}m_2\dot{x}^2 + \frac{1}{2}m_2 l_1^2\dot{\delta}^2 + \frac{1}{2}m_2 l_2^2\dot{\beta}^2 \\ &+ m_2 l_1\dot{x}\dot{\delta}\cos(\delta) + m_2 l_1 l_2\dot{\delta}\dot{\beta}\cos(\delta - \beta) \\ &+ m_2 l_2\dot{x}\dot{\beta}\cos(\beta) \end{aligned} \tag{16}$$

Potential Energy (U):

For pendulum 1:

$$U_1 = m_1 g y_1 \tag{17}$$

$$U_1 = m_1 g l_1\cos(\delta) \tag{18}$$

For pendulum 2:

$$U_2 = m_2 g y_2 \tag{19}$$

$$U_2 = m_2 g(l_1\cos(\delta) + l_2\cos(\beta)) \tag{20}$$

Force P's work:

$$U_P = P y_2 \tag{21}$$

$$U_P = P(l_1\cos(\delta) + l_2\cos(\beta)) \tag{22}$$

Total potential energy:

$$U = U_1 + U_2 + U_P \tag{23}$$

$$\begin{aligned} U = &m_1 g l_1\cos(\delta) + m_2 g(l_1\cos(\delta) + l_2\cos(\beta)) \\ &+ P(l_1\cos(\delta) + l_2\cos(\beta)) \end{aligned} \tag{24}$$

Lagrangian (L):

$$L = T - U \tag{25}$$

To find the Lagrangian L, we subtract the total potential energy U from the total kinetic energy T.
Given:

$$\begin{aligned}
T = &\frac{1}{2}M\dot{x}^2 + \frac{1}{2}m_1[(\dot{x} + l_1\dot{\delta}\cos(\delta))^2 \\
&+ (-l_1\dot{\delta}\sin(\delta))^2] \\
&+ \frac{1}{2}m_2[(\dot{x} + l_1\dot{\delta}\cos(\delta) + l_2\dot{\beta}\cos(\beta))^2 \\
&+ (-l_1\dot{\delta}\sin(\delta) - l_2\dot{\beta}\sin(\beta))^2] \qquad (26) \\
U = &m_1gl_1\cos(\delta) \\
&+ m_2g(l_1\cos(\delta) + l_2\cos(\beta)) \\
&+ P(l_1\cos(\delta) + l_2\cos(\beta)) \qquad (27)
\end{aligned}$$

Substitute in the values for T and U:

$$\begin{aligned}
L = &\frac{1}{2}M\dot{x}^2 + \frac{1}{2}m_1\Big[(\dot{x} + l_1\dot{\delta}\cos(\delta))^2 \\
&+ (-l_1\dot{\delta}\sin(\delta))^2\Big] \\
&+ \frac{1}{2}m_2\Big[(\dot{x} + l_1\dot{\delta}\cos(\delta) + l_2\dot{\beta}\cos(\beta))^2 \\
&+ (-l_1\dot{\delta}\sin(\delta) - l_2\dot{\beta}\sin(\beta))^2\Big] \\
&- (m_1gl_1\cos(\delta) + m_2g(l_1\cos(\delta) + l_2\cos(\beta)) \\
&+ P(l_1\cos(\delta) + l_2\cos(\beta))) \qquad (28) \\
L = &T - U \qquad (29)
\end{aligned}$$

Plugging in the expressions for T and U from above, we have:

$$\begin{aligned}
L = &\frac{1}{2}M\dot{x}^2 + \frac{1}{2}m_1[(\dot{x} + l_1\dot{\delta}\cos(\delta))^2 \\
&+ (-l_1\dot{\delta}\sin(\delta))^2] \\
&+ \frac{1}{2}m_2\dot{x}^2 + \frac{1}{2}m_2l_1^2\dot{\delta}^2 \\
&+ \frac{1}{2}m_2l_2^2\dot{\beta}^2 \\
&+ m_2l_1\dot{x}\dot{\delta}\cos(\delta) \\
&+ m_2l_1l_2\dot{\delta}\dot{\beta}\cos(\delta - \beta) \\
&+ m_2l_2\dot{x}\dot{\beta}\cos(\beta) \\
&- (m_1gl_1\cos(\delta) + m_2g(l_1\cos(\delta) + l_2\cos(\beta)) \\
&+ P(l_1\cos(\delta) + l_2\cos(\beta))) \qquad (30)
\end{aligned}$$

The Euler-Lagrange equation is a fundamental equation of motion in the calculus of variations and it is used to obtain the equations of motion for systems that are described by the Lagrangian L. Given that L is a function of generalized coordinates, their derivatives, and possibly time, the Euler-Lagrange equation is given by:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = Q$$

Where:

- q is the generalized coordinate.
- $\dot{q}$ is the time derivative of the generalized coordinate.
- Q is the generalized force associated with the generalized coordinate q.

$\frac{\partial L}{\partial x}$: This involves differentiating the kinetic energy terms in L with respect to x

$$\frac{\partial L}{\partial x} = 0$$

$\frac{\partial L}{\partial \dot{x}}$: This involves differentiating the kinetic energy terms in L with respect to $\dot{x}$:

$$\begin{aligned}\frac{\partial L}{\partial \dot{x}} = {} & M\dot{x} \\ & + m_1(\dot{x} + l_1\dot{\delta}\cos(\delta)) \\ & + m_2(\dot{x} + l_1\dot{\delta}\cos(\delta) \\ & + l_2\dot{\beta}\cos(\beta))\end{aligned} \tag{31}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = M\ddot{x} + m_1(\ddot{x} - l_1\dot{\delta}^2\sin(\delta) + l_1\ddot{\delta}\cos(\delta))$$

$$+m_2(\ddot{x} - l_1\dot{\delta}^2\sin(\delta) + l_1\ddot{\delta}\cos(\delta) - l_2\dot{\beta}^2\sin(\beta) + l_2\ddot{\beta}\cos(\beta))$$

Using the Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = Q$$

Substituting in our results:

$$(M + m_1 + m_2)\ddot{x} + m_1 l_1\ddot{\delta}\cos(\delta) - m_1 l_1\dot{\delta}^2\sin(\delta) + m_2 l_1\ddot{\delta}\cos(\delta)$$

$$-m_2 l_1\dot{\delta}^2\sin(\delta) + m_2 l_2\ddot{\beta}\cos(\beta) - m_2 l_2\dot{\beta}^2\sin(\beta) = F$$

This equation describes the dynamics of the cart (with coordinate x) in the presence of the two pendulums and the external force P.

To determine the equation of motion for δ, we'll apply the Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\delta}}\right) - \frac{\partial L}{\partial \delta} = 0$$

For $\frac{\partial L}{\partial \delta}$:

$$\frac{\partial L}{\partial \delta} = m_1 g l_1 \sin(\delta) + m_2 g l_1 \sin(\delta) - P l_1 \sin(\delta)$$
$$+m_1 l_1 \dot{x}\dot{\delta}\sin(\delta) + m_2 l_1 \dot{x}\dot{\delta}\sin(\delta) + m_2 l_2 \dot{\beta}\dot{\delta}\sin(\beta)$$

For $\frac{\partial L}{\partial \dot{\delta}}$:

$$\frac{\partial L}{\partial \dot{\delta}} = m_1 l_1^2 \dot{\delta} + m_1 l_1 \dot{x}\cos(\delta) + m_2 l_1^2 \dot{\delta}$$
$$+m_2 l_1 \dot{x}\cos(\delta) + m_2 l_1 l_2 \dot{\beta}\cos(\delta - \beta)$$

Differentiate with respect to time:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\delta}}\right) = m_1 l_1^2 \ddot{\delta} - m_1 l_1 \dot{x}\dot{\delta}\sin(\delta) + m_1 l_1 \ddot{x}\cos(\delta)$$
$$+m_2 l_1^2 \ddot{\delta} - m_2 l_1 \dot{x}\dot{\delta}\sin(\delta) + m_2 l_1 \ddot{x}\cos(\delta)$$
$$+m_2 l_1 l_2 \ddot{\beta}\cos(\delta - \beta) - m_2 l_1 l_2 \dot{\beta}\sin(\delta - \beta)(\dot{\delta} - \dot{\beta})$$

Combining the above results using the Euler-Lagrange equation:

$$\begin{aligned}
&m_1 l_1^2 \ddot{\delta} + m_2 l_1^2 \ddot{\delta} + m_1 l_1 \ddot{x}\cos(\delta) \\
&+ m_2 l_1 \ddot{x}\cos(\delta) + m_2 l_1 l_2 \ddot{\beta}\cos(\delta - \beta) \\
&- m_2 l_1 l_2 \dot{\beta}^2 \sin(\delta - \beta)(\dot{\delta} - \dot{\beta}) \\
&= m_1 g l_1 \sin(\delta) + m_2 g l_1 \sin(\delta) \\
&- P l_1 \sin(\delta)
\end{aligned} \tag{32}$$

To determine the equation of motion for β, we'll apply the Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\beta}}\right) - \frac{\partial L}{\partial \beta}$$

$$\frac{\partial L}{\partial \dot{\beta}} = m_2 l_2^2 \dot{\beta} + m_2 l_1 l_2 \dot{\delta}\cos(\delta - \beta) + m_2 l_2 \dot{x}\cos(\beta)$$

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\beta}}\right) = {} & m_2 l_2^2 \ddot{\beta} \\
& + m_2 l_1 l_2 \ddot{\delta} \cos(\delta - \beta) \\
& - m_2 l_1 l_2 \dot{\delta} \dot{\beta} \sin(\delta - \beta)(\dot{\delta} - \dot{\beta}) \\
& + m_2 l_2 \ddot{x} \cos(\beta) \\
& - m_2 l_2 \dot{x} \dot{\beta} \sin(\beta)
\end{aligned} \tag{33}
$$

From the given Lagrangian, differentiating the terms that involve β, we get:

$$
\begin{aligned}
\frac{\partial L}{\partial \beta} = {} & m_2 g l_2 \sin(\beta) \\
& + m_2 l_1 l_2 \dot{\delta} \dot{\beta} \sin(\delta - \beta) \\
& - m_2 l_2 \dot{x} \dot{\beta} \sin(\beta) \\
& + P l_2 \sin(\beta)
\end{aligned} \tag{34}
$$

Substituting into the equation:

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\beta}}\right) - \frac{\partial L}{\partial \beta} = {} & m_2 l_2^2 \ddot{\beta} + m_2 l_1 l_2 \ddot{\delta} \cos(\delta - \beta) \\
& - m_2 l_1 l_2 \dot{\delta} \dot{\beta} \sin(\delta - \beta)(\dot{\delta} - \dot{\beta}) + m_2 l_2 \ddot{x} \cos(\beta) \\
& - 2 m_2 l_2 \dot{x} \dot{\beta} \sin(\beta) - m_2 g l_2 \sin(\beta) \\
& - m_2 l_1 l_2 \dot{\delta} \dot{\beta} \sin(\delta - \beta) - P l_2 \sin(\beta) = 0
\end{aligned} \tag{35}
$$

To solve for $\ddot{x}$, $\ddot{\delta}$, and $\ddot{\beta}$, we'll start by isolating each of these terms in the given equations.

Given the equations:

$$
\begin{aligned}
& (M + m_1 + m_2)\ddot{x} + (m_1 l_1 + m_2 l_1)\ddot{\delta} \cos(\delta) \\
& + m_2 l_2 \ddot{\beta} \cos(\beta) = F \\
& + (m_1 l_1 + m_2 l_1)\dot{\delta}^2 \sin(\delta) + m_2 l_2 \dot{\beta}^2 \sin(\beta)
\end{aligned} \tag{36}
$$

$$
\begin{aligned}
& (m_1 l_1^2 + m_2 l_1^2)\ddot{\delta} + (m_1 l_1 + m_2 l_1)\ddot{x} \cos(\delta) \\
& + m_2 l_1 l_2 \ddot{\beta} \cos(\delta - \beta) = m_1 g l_1 \sin(\delta) \\
& + m_2 g l_1 \sin(\delta) + P l_1 \cos(\delta) \\
& + m_2 l_1 l_2 \dot{\beta}^2 \sin(\delta - \beta)(\dot{\delta} - \dot{\beta})
\end{aligned} \tag{37}
$$

$$
\begin{aligned}
&m_2 l_2^2 \ddot{\beta} + m_2 l_1 l_2 \ddot{\delta} \cos(\delta - \beta) \\
&+ m_2 l_2 \ddot{x} \cos(\beta) = m_2 g l_2 \sin(\beta) \\
&+ 2 m_2 l_2 \dot{x} \dot{\beta} \sin(\beta) \\
&+ m_2 l_1 l_2 \dot{\delta} \dot{\beta} \sin(\delta - \beta)(\dot{\delta} - \dot{\beta}) \\
&+ P l_2 \cos(\beta)
\end{aligned} \tag{38}
$$

$$
\text{Isolating } \ddot{x}, \ddot{\delta}, \text{ and } \ddot{\beta} \text{ respectively:} \tag{39}
$$

From (1):

$$
\begin{aligned}
\ddot{x} = &\frac{F + (m_1 l_1 + m_2 l_1)\dot{\delta}^2 \sin(\delta) + m_2 l_2 \dot{\beta}^2 \sin(\beta)}{M + m_1 + m_2} \\
&- \frac{(m_1 l_1 + m_2 l_1)\ddot{\delta} \cos(\delta) + m_2 l_2 \ddot{\beta} \cos(\beta)}{M + m_1 + m_2}
\end{aligned} \tag{40}
$$

From (2):

$$
\begin{aligned}
\ddot{\delta} = &\frac{1}{m_1 l_1^2 + m_2 l_1^2} \Big(m_1 g l_1 \sin(\delta) + m_2 g l_1 \sin(\delta) \\
&+ P l_1 \cos(\delta) + m_2 l_1 l_2 \dot{\beta}^2 \sin(\delta - \beta)(\dot{\delta} - \dot{\beta}) \\
&- (m_1 l_1 + m_2 l_1)\ddot{x} \cos(\delta) - m_2 l_1 l_2 \ddot{\beta} \cos(\delta - \beta) \Big)
\end{aligned} \tag{41}
$$

From (3):

$$
\begin{aligned}
\ddot{\delta} = &\frac{1}{m_1 l_1^2 + m_2 l_1^2} \Big(m_1 g l_1 \sin(\delta) + m_2 g l_1 \sin(\delta) \\
&+ P l_1 \cos(\delta) + m_2 l_1 l_2 \dot{\beta}^2 \sin(\delta - \beta)(\dot{\delta} - \dot{\beta}) \\
&- (m_1 l_1 + m_2 l_1)\ddot{x} \cos(\delta) - m_2 l_1 l_2 \ddot{\beta} \cos(\delta - \beta) \Big)
\end{aligned} \tag{42}
$$

Linearization of the equation of motion:

To linearize the expressions, we use small-angle approximations (where relevant) and assume that terms that are products of two or more small quantities can be neglected. Specifically, for small angles α, we have:

$$
\sin(\alpha) \approx \alpha \tag{43}
$$

$$
\cos(\alpha) \approx 1 \tag{44}
$$

Let's linearize the given equations for $\ddot{x}$, $\ddot{\delta}$, and $\ddot{\beta}$:

1) For the expression of $\ddot{x}$:

$$\ddot{x} \approx \frac{F - (m_1 l_1 + m_2 l_1)\ddot{\delta} - m_2 l_2 \ddot{\beta}}{M + m_1 + m_2} \tag{45}$$

2) For $\ddot{delta}$:

$$\ddot{\delta} \approx \frac{m_1 g l_1 \delta + m_2 g l_1 \delta + P l_1 - (m_1 l_1 + m_2 l_1)\ddot{x} - m_2 l_1 l_2 \ddot{\beta}}{m_1 l_1^2 + m_2 l_1^2} \tag{46}$$

Again, the $\dot{\beta}^2$ term is nonlinear and is dropped in the linearization.

1. For the expression of $\ddot{x}$:

$$\ddot{x} = \frac{F - (m_1 l_1 + m_2 l_1)\ddot{\delta} - m_2 l_2 \ddot{\beta}}{M + m_1 + m_2}$$

2. For $\ddot{\delta}$:

$$\ddot{\delta} = \frac{m_1 g l_1 \delta + m_2 g l_1 \delta + P l_1 - (m_1 l_1 + m_2 l_1)\ddot{x} - m_2 l_1 l_2 \ddot{\beta}}{m_1 l_1^2 + m_2 l_1^2}$$

3. For $\ddot{\beta}$:

$$\ddot{\beta} \approx \frac{m_2 g l_2 \beta + P l_2 - m_2 l_1 l_2 \ddot{\delta} - m_2 l_2 \ddot{x}}{m_2 l_2^2}$$

Terms like $\dot{\delta}\dot{\beta}$ are products of two small quantities and have been neglected in the linearization.

From the state-space representation perspective, we can put this system in the form:

$$\dot{x} = Ax + Bu$$

Where $x = [x, \dot{x}, \delta, \ddot{\delta}, \beta, \ddot{\beta}]^T$ is the state vector.

To fill the matrix A and B using the given differential equations:

We introduce the following substitutions to simplify the matrix representation:

$$
\begin{aligned}
a_1 &= \frac{-(m_1 g l_1 + m_2 g l_1)}{M + m_1 + m_2} \\
a_2 &= \frac{-m_2 g l_2}{M + m_1 + m_2} \\
a_3 &= \frac{m_1 g l_1 + m_2 g l_1}{m_1 l_1^2 + m_2 l_1^2} = \frac{g}{l_1} \\
a_4 &= \frac{m_1 l_1 + m_2 l_1}{m_1 l_1^2 + m_2 l_1^2} = \frac{1}{l_1} \\
a_5 &= \frac{-m_2 l_1 l_2}{m_1 l_1^2 + m_2 l_1^2} = \frac{-m_2 l_2}{(m_1 + m_2) l_1} \\
a_6 &= \frac{m_2 l_2}{m_2 l_2^2} = \frac{1}{l_2} \\
a_7 &= \frac{m_2 l_1 l_2}{m_2 l_2^2} = \frac{l_1}{l_2} \\
a_8 &= \frac{m_2 g l_2}{m_2 l_2^2} = \frac{g}{l_2} \\
b_1 &= \frac{1}{M + m_1 + m_2} \\
b_2 &= \frac{-l_1}{m_1 l_1^2 + m_2 l_1^2} = \frac{-1}{(m_1 + m2) l_1} \\
b_3 &= \frac{-l_2}{m_2 l_2^2} = \frac{-1}{m_2 l_2}
\end{aligned}
$$

Using these substitutions, the system dynamics can be represented as:

$$
\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\delta} \\ \ddot{\delta} \\ \dot{\beta} \\ \ddot{\beta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_1 & 0 & a_2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ a_3 & a_4 & 0 & 0 & a_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & a_6 & a_7 & 0 & a_8 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \delta \\ \ddot{\delta} \\ \beta \\ \ddot{\beta} \end{bmatrix} + \begin{bmatrix} 0 \\ b_1 \\ 0 \\ b_2 \\ 0 \\ b_3 \end{bmatrix} u
$$

Where $u = [F]^T$ is the input vector.

2 Control Design

Before Designing the controller to stabilize the double inverted pendulum let analyze the open loop system responses let assuming the initial state of the double inverted pendulum is

$$x0 = \begin{bmatrix} 2 & 0.1 & \pi/3 & 0 & -\pi/6 & 0 \end{bmatrix}^T$$

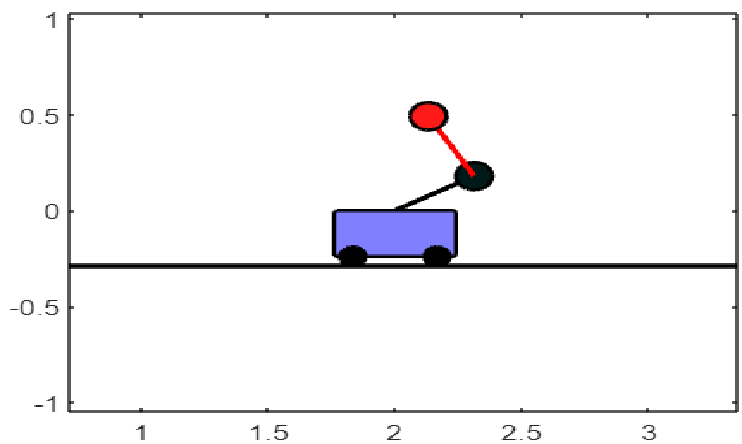

Figure 3: double inverted pendulum with initial state $x0$

with this initial condition we simulated the double inverted pendulum as shown in Figure 3 Since the system is totally unstable with this initial condition the system increasing of the state value with out limit which shown in Figure 4

The optimal control(LQR) is to determine control signal so that the system to be controlled can meet physical constraints and minimize/maximize performance function. the optimization problem is to bring the system's state $x(t)$ to the desired trajectory $x(t)^*$ by minimizing the state cost and use of the control inputs.

$$J = \int_{t_0}^{\infty} \left\{ \mathbf{u}(t)^T \cdot \mathbf{R} \cdot \mathbf{u}(t) + [\mathbf{x}(t) - \mathbf{x}(\mathbf{t})^*]^T \cdot \mathbf{Q} \cdot \left[\mathbf{x}(t) - \mathbf{x}(\mathbf{t})^*\right] \right\} dt$$

where Q and R are a positive semi definite state and input cost matrix respectively After some algebraic manipulation the optimization become the Riccati's algebraic equation which solving for the value of P where

$$PA + A^T P - PBR^{-1}B^T P + C^T QC = 0$$

The controller gain

$$K = R^{-1}B^T P$$

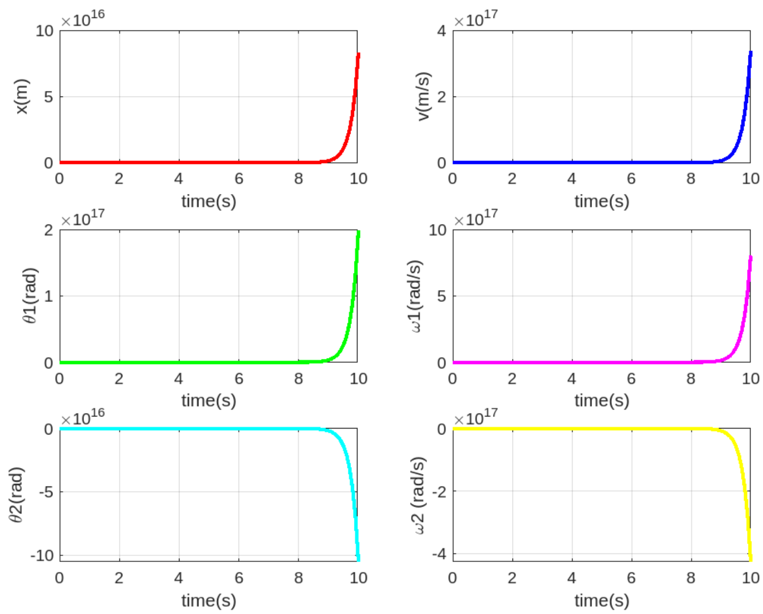

Figure 4: open loop responses of double inverted pendulum

The control policy become

$$u(t) = -K(x(t) - x(t)^*)$$

in such we use our linear version state space model of the double inverted pendulum in order to apply control feedback. By solving K for different parameter state in the Table 1 we get the result of the closed loop res ponces as show in the Figure 5 and Figure 6

Table 1: Different Q and R cost parameter for controller tuning

parameter	Q	R
1	$diag([\ 1\ 1\ 1\ 1\ 1\ 1\ ])$	0.01
2	$diag([\ 1000\ 1\ 10\ 1\ 10\ 1\ ])$	0.1
3	$diag([\ 1000\ 1\ 100\ 1\ 1\ 1\ ])$	0.1

we also apply pole placement control design with out considering the cost function which we achieved stabilization of the double inverted pendulum that settle with the final value in faster time but with a high control effort which the closed system responses shown in Figure 7

finally the simulation result show in figure 8 after an desired control input signal is provided for the system in which the controller able to stabilize the double inverted pendulum with a given references state/upright position $[0, 0, 0, 0, 0, 0]$

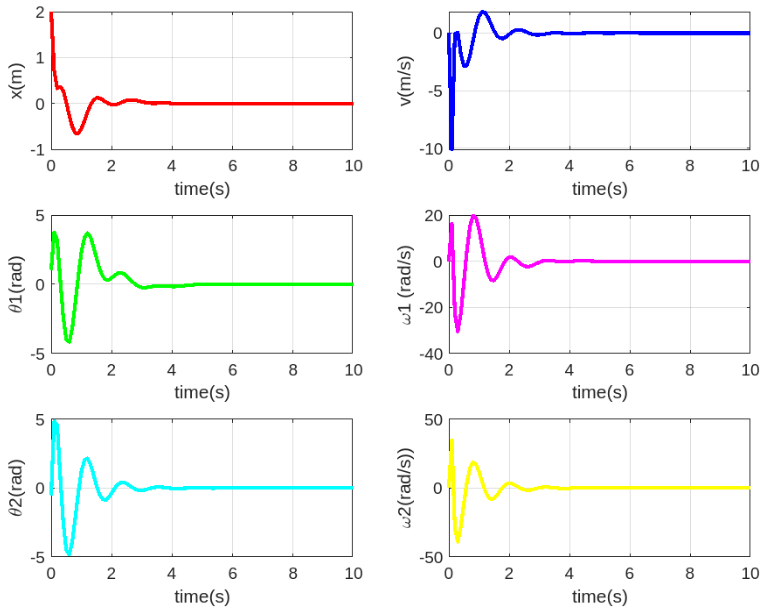

Figure 5: Responses of double inverted pendulum with parameter 2

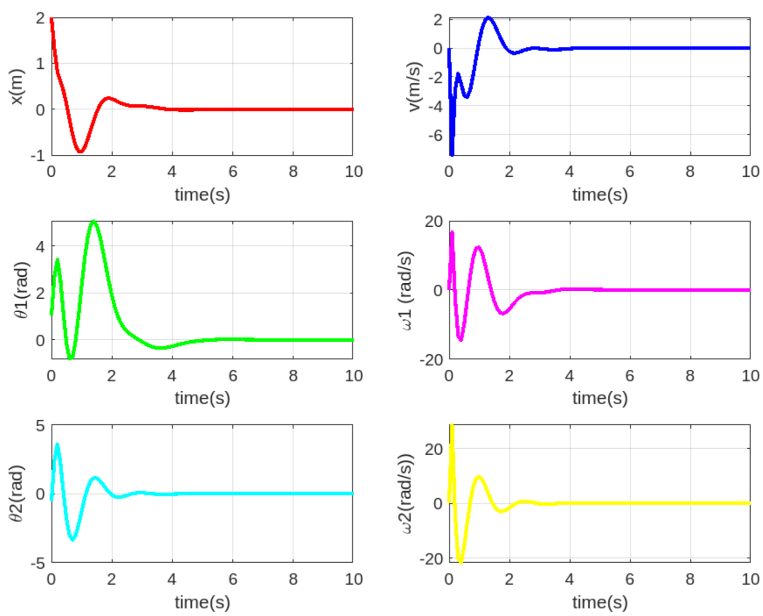

Figure 6: Responses of double inverted pendulum with parameter 3

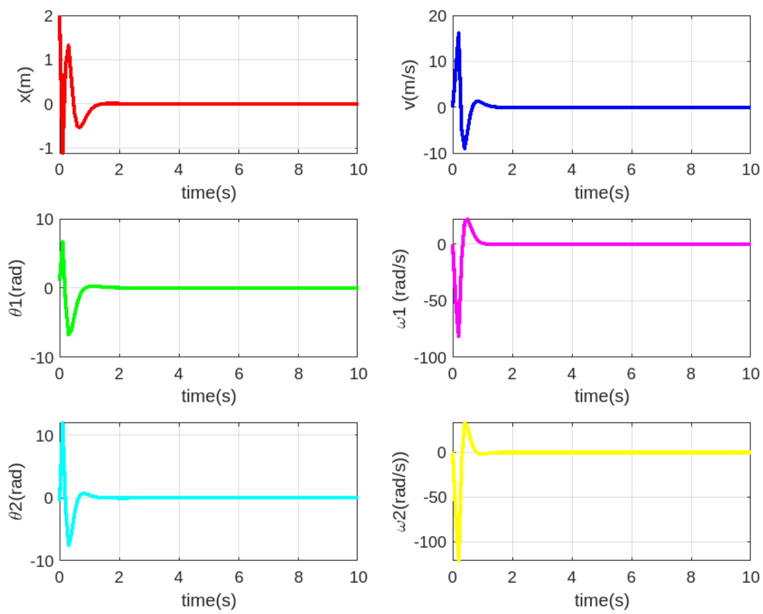

Figure 7: Responses of double inverted pendulum using pole placement

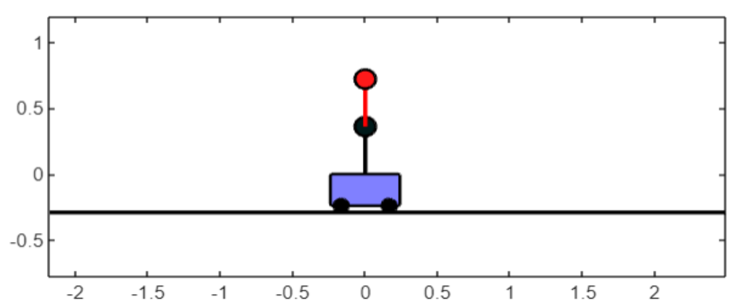

Figure 8: stabilize double inverted pendulum