MSc Program

Mathematics for Engineers

Course project- 'M2. Inverted Pendulum'

Week 2 Task

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Tasks for week 3

Part I

Problem:

The pivot point of a rigid pendulum is subjected to forced vertical oscillation, given by:

$$\eta(t) = \eta_0 \cos(\omega t)$$

The pendulum consists of a massless rod of length L with a mass m attached at the end.

- Derive an equation of motion for θ, where θ is the pendulum angle indicated in Fig.1. Assume $\theta \ll 1$ and $\eta_0 \ll L$.
- Solve the equation to first order in η_0 for the initial conditions:
    1. $\theta = a$; $\dot{\theta} = 0$.
    2. $\theta = a$; $\dot{\theta} = a\sqrt{g/L}$.
- Evaluate the solution for (1) and (2) at resonance and describe the difference in the two motions.

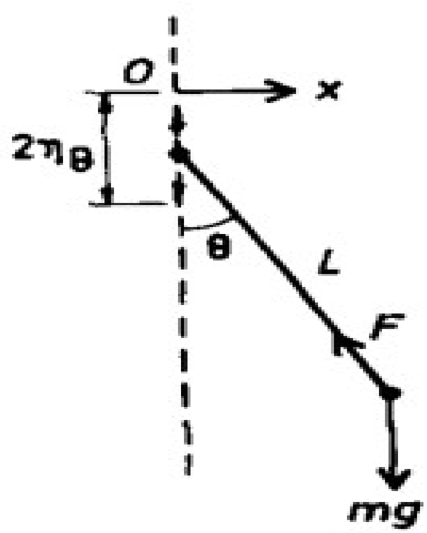

Figure 1: Single Pendulum System

Solution:

Equation Derivation

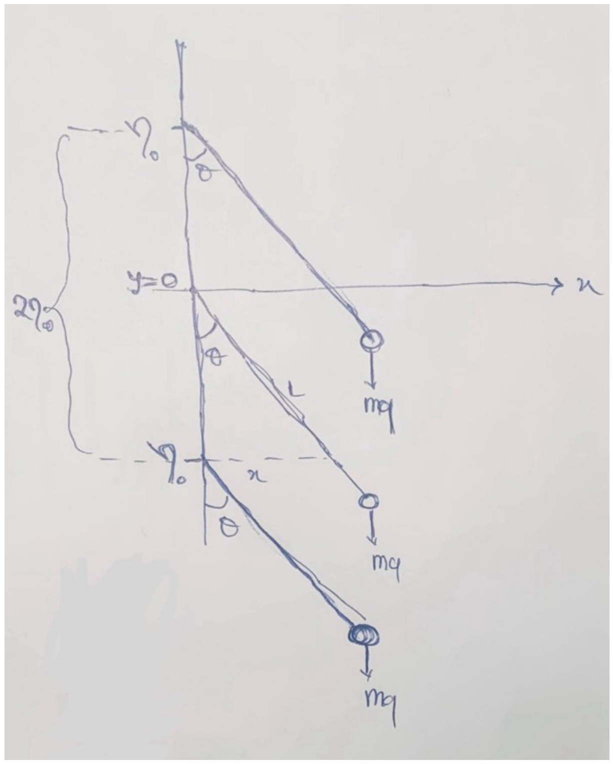

Figure 2: Forced oscillation System

Coordinates and Displacements:

From the figure 2, we can frame equations for x and y axis,

$$x = L\sin\theta \quad ...\ (1)$$

$$y = L\cos\theta + \eta_0\cos\omega t \quad ...\ (2)$$

Differentiating with respect to time:

$$\dot{x} = L\cos\theta\dot{\theta} \quad ...\ (3)$$

$$\dot{y} = -L\sin\theta\dot{\theta} - \eta_0\omega\sin\omega t \quad ...\ (4)$$

Kinetic and Potential Energy:

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

From (3) and (4), substitute for $\dot{x}$ and $\dot{y}$:

$$\begin{aligned} T &= \frac{1}{2}m\left[L^2\cos^2\theta\dot{\theta}^2 + L^2\sin^2\theta\dot{\theta}^2 + 2L\eta_0\omega\sin\theta\cos\theta\dot{\theta} + \eta_0^2\omega^2\sin^2\omega t\right] \\ &= \frac{1}{2}mL^2\dot{\theta}^2 + mL\eta_0\omega\sin\theta\cos\theta\dot{\theta} + \frac{1}{2}m\eta_0^2\omega^2\sin^2\omega t \quad \text{... (5)} \end{aligned}$$

Potential Energy:

$$U = mgy = mg(L\cos\theta + \eta_0\cos\omega t) \quad \text{... (6)}$$

Lagrangian:

$$L = T - U$$

Substitute from (5) and (6):

$$L = \frac{1}{2}mL^2\dot{\theta}^2 + mL\eta_0\omega\sin\theta\cos\theta\dot{\theta} + \frac{1}{2}m\eta_0^2\omega^2\sin^2\omega t - mg(L\cos\theta + \eta_0\cos\omega t)$$

Lagrange's Equation:

Using the equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

For our system, the generalized coordinate is θ. So, our q_i is θ.

Differentiating our Lagrangian L with respect to $\dot{\theta}$ and θ respectively:

$$\begin{aligned} \frac{\partial L}{\partial \dot{\theta}} &= mL^2\dot{\theta} + mL\eta_0\omega\sin\theta\cos\omega t \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) &= mL^2\ddot{\theta} + mL\eta_0\omega^2\sin\theta\sin\omega t \\ \frac{\partial L}{\partial \theta} &= mL\eta_0\omega\cos\theta\cos\omega t - mgL\sin\theta \end{aligned}$$

Plugging these into Lagrange's equation:

$$mL^2\ddot{\theta} + mL\eta_0\omega^2\sin\theta\sin\omega t - mL\eta_0\omega\cos\theta\cos\omega t + mgL\sin\theta = 0$$

Rearranging, we have:

$$\ddot{\theta} = -\frac{g}{L}\sin\theta - \eta_0\omega^2\sin\omega t$$

This is the equation of motion for the given system.

The last part of the given approach suggests that if θ is very small, $\sin\theta$ becomes θ. Using this approximation:

$$\ddot{\theta} = -\frac{g}{L}\theta - \eta_0\omega^2 \sin\omega t$$

This equation describes the motion of the pendulum in the presence of the external force characterized by $\eta_0\omega^2 \sin\omega t$.

Solution for Given Initial Conditions:

1. Initial conditions: $\theta(0) = a$ and $\dot{\theta}(0) = 0$:

Homogeneous Solution:

For $\eta_0 = 0$, the equation becomes:

$$\ddot{\theta} = -\frac{g}{L}\theta$$

The solution to this is given by:

$$\theta_h(t) = A\cos\left(\sqrt{\frac{g}{L}}t\right) + B\sin\left(\sqrt{\frac{g}{L}}t\right)$$

Using the given initial conditions, $\theta(0) = a$ and $\dot{\theta}(0) = 0$, we can determine:

$$A = a$$

$$B = 0$$

Thus, the homogeneous solution is:

$$\theta_h(t) = a\cos\left(\sqrt{\frac{g}{L}}t\right)$$

Particular Solution:

Assuming a solution of the form:

$$\theta_p(t) = C\cos(\omega t) + D\sin(\omega t)$$

Total Solution for the first initial condition:

$$\theta(t) = \theta_h(t) + \eta_0\theta_p(t)$$

2. Initial conditions: $\theta(0) = a$ and $\dot{\theta}(0) = a\sqrt{g/L}$:

Homogeneous Solution:

Using the already derived general solution:

$$\theta_h(t) = A\cos\left(\sqrt{\frac{g}{L}}t\right) + B\sin\left(\sqrt{\frac{g}{L}}t\right)$$

And applying the new initial conditions:

$$A = a$$
$$B = \frac{a}{\sqrt{g/L}}$$

The homogeneous solution becomes:

$$\theta_h(t) = a\cos\left(\sqrt{\frac{g}{L}}t\right) + \frac{a}{\sqrt{g/L}}\sin\left(\sqrt{\frac{g}{L}}t\right)$$

Particular Solution:

The form remains the same:

$$\theta_p(t) = C\cos(\omega t) + D\sin(\omega t)$$

With the values of C and D being dependent on η_0.

Total Solution for the second initial condition:

$$\theta(t) = \theta_h(t) + \eta_0\theta_p(t)$$

Note: In a complete analysis, the exact form of $\theta_p(t)$ would be found by substituting into the differential equation and solving for the constants C and D. The approximations made here are based on the instructions to solve to the first order in η_0.

Computing the Particular Solution $\theta_p(t)$:

Starting with the assumed form of the particular solution:

$$\theta_p(t) = C\cos(\omega t) + D\sin(\omega t)$$

We can compute its derivatives:

$$\dot{\theta}_p(t) = -C\omega\sin(\omega t) + D\omega\cos(\omega t)$$

$$\ddot{\theta}_p(t) = -C\omega^2\cos(\omega t) - D\omega^2\sin(\omega t)$$

Substituting these into the given differential equation:

$$\ddot{\theta}_p(t) + \frac{g}{L}\theta_p(t) = -\eta_0\omega^2 \sin(\omega t)$$

Which expands to:

$$-C\omega^2 \cos(\omega t) - D\omega^2 \sin(\omega t) + \frac{g}{L}(C\cos(\omega t) + D\sin(\omega t)) = -\eta_0\omega^2 \sin(\omega t)$$

By equating the coefficients of similar trigonometric terms:

For $\cos(\omega t)$ terms:

$$-C\omega^2 + \frac{g}{L}C = 0$$

This implies:

$$C = 0$$

For $\sin(\omega t)$ terms:

$$-D\omega^2 + \frac{g}{L}D = -\eta_0\omega^2$$

Yielding:

$$D = -\frac{\eta_0\omega^2 L}{\omega^2 - \frac{g}{L}}$$

Thus, the particular solution can be written as:

$$\theta_p(t) = -\frac{\eta_0\omega^2 L}{\omega^2 - \frac{g}{L}} \sin(\omega t)$$

With this, combining the homogeneous and particular solutions for the first and second set of initial conditions gives:

Total Solution for the first initial condition:

$$\theta(t) = \theta_h(t) + \eta_0\theta_p(t)$$

$$\theta(t) = a\cos\left(\sqrt{\frac{g}{L}}t\right) - \frac{\eta_0\omega^2 L}{\omega^2 - \frac{g}{L}} \sin(\omega t)$$

Total Solution for the second initial condition:

$$\theta(t) = \theta_h(t) + \eta_0\theta_p(t)$$

$$\theta(t) = a\cos\left(\sqrt{\frac{g}{L}}t\right) + \frac{a}{\sqrt{g/L}}\sin\left(\sqrt{\frac{g}{L}}t\right) - \frac{\eta_0\omega^2 L}{\omega^2 - \frac{g}{L}} \sin(\omega t)$$

This solution represents the motion of the pendulum given the external driving force up to first order in η_0. The homogeneous part describes the natural oscillation due to gravity, while the particular part gives the response to the external force.

The term $-\frac{\eta_0 \omega^2 L}{\omega^2 - \frac{g}{L}} \sin(\omega t)$ signifies the influence of the external force. This coefficient is determined by η_0 and the comparison between ω (driving frequency) and $\sqrt{\frac{g}{L}}$ (pendulum's natural frequency).

Note: If the driving frequency ω approaches the pendulum's natural frequency, the coefficient of the sine term can grow significantly, leading to a phenomenon known as resonance. Here, the pendulum swings with an amplified amplitude due to the external force's alignment with its natural oscillatory frequency.

Evaluation at Resonance

Analysis at Resonance

When at resonance, the driving frequency ω matches the pendulum's natural frequency. Mathematically, this is represented as:

$$\omega = \sqrt{\frac{g}{L}}$$

Let's consider how the solutions for our two initial conditions are affected at resonance:

Solution for the first initial condition at resonance:

The equation for $\theta(t)$ is:

$$\theta(t) = a \cos\left(\sqrt{\frac{g}{L}} t\right) - \frac{\eta_0 \omega^2 L}{\omega^2 - \frac{g}{L}} \sin(\omega t)$$

When $\omega = \sqrt{\frac{g}{L}}$, the term $\omega^2 - \frac{g}{L}$ in the denominator goes to zero, making the fraction undefined. This signifies that the amplitude due to the driving force becomes extremely large during resonance, a hallmark of resonant phenomena.

Solution for the second initial condition at resonance:

The equation for $\theta(t)$ becomes:

$$\theta(t) = a \cos\left(\sqrt{\frac{g}{L}} t\right) + \frac{a}{\sqrt{g/L}} \sin\left(\sqrt{\frac{g}{L}} t\right) - \frac{\eta_0 \omega^2 L}{\omega^2 - \frac{g}{L}} \sin(\omega t)$$

Like before, the term with η_0 becomes undefined due to resonance. However, there's an additional term $\frac{a}{\sqrt{g/L}} \sin\left(\sqrt{\frac{g}{L}}t\right)$ that arises from the pendulum's initial velocity.

Comparing the two motions:

1. In the first scenario, the pendulum, starting from rest, reacts predominantly to the external driving force. The huge amplitude at resonance is mostly due to the driving force's influence, indicating a pure resonant response.
2. The second motion displays a combined effect of the external force and the pendulum's inherent initial motion. This produces a more intricate resonant behavior due to the combination of natural oscillation and the resonant reaction, potentially leading to larger and more complex oscillations.

In essence, resonance amplifies the system's reaction to the external driving force, resulting in pronounced oscillations. Nevertheless, the motions in our two cases diverge because of their distinct starting conditions:

1. The **first motion** is essentially the system's reaction to the external force, augmented by resonance.
2. The **second motion**, with its initial kinetic energy, demonstrates the natural motion of the pendulum combined with the resonant reaction, leading to potentially more significant and intricate oscillations.

Part II

Problem:
A simple pendulum of length $4l$ and mass m is hung from another simple pendulum of length $3l$ and mass m. It is possible for this system to perform small oscillations about equilibrium such that a point on the lower pendulum undergoes no horizontal displacement. Locate that point.

Solution:

Cartesian Coordinates of the Masses

First, the problem represents the position of the two pendulum masses using Cartesian coordinates. The upper pendulum of length $3l$ swings through an angle θ_1 and the lower pendulum of length $4l$ swings through an angle θ_2.

Given:

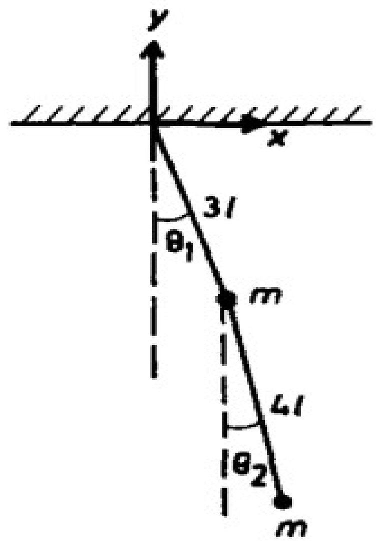

Figure 3: Double Pendulum System

- Upper mass coordinates:

$$(3l\sin\theta_1, -3l\cos\theta_1)$$

- Lower mass coordinates:

$$(3l\sin\theta_1 + 4l\sin\theta_2, -3l\cos\theta_1 - 4l\cos\theta_2)$$

Velocities of the Masses

To determine the velocity of each mass, we need to differentiate the position of each mass with respect to time. Let's break down the velocities:

Upper Mass Velocity: Starting with the x-coordinate of the upper mass, $3l\sin\theta_1$, the time derivative is:

$$\frac{d(3l\sin\theta_1)}{dt} = 3l\dot{\theta}_1\cos\theta_1$$

For the y-coordinate, $-3l\cos\theta_1$, the time derivative is:

$$\frac{d(-3l\cos\theta_1)}{dt} = 3l\dot{\theta}_1\sin\theta_1$$

Hence, the velocity coordinates for the upper mass are:

$$(3l\dot{\theta}_1\cos\theta_1, 3l\dot{\theta}_1\sin\theta_1)$$

Lower Mass Velocity: For the x-coordinate, $3l\sin\theta_1 + 4l\sin\theta_2$, the time derivative is:

$$\frac{d(3l\sin\theta_1 + 4l\sin\theta_2)}{dt} = 3l\dot{\theta}_1\cos\theta_1 + 4l\dot{\theta}_2\cos\theta_2$$

For the y-coordinate, $-3l\cos\theta_1 - 4l\cos\theta_2$, the time derivative is:

$$\frac{d(-3l\cos\theta_1 - 4l\cos\theta_2)}{dt} = 3l\dot{\theta}_1\sin\theta_1 + 4l\dot{\theta}_2\sin\theta_2$$

Thus, the velocity coordinates for the lower mass are:

$$(3l\dot{\theta}_1\cos\theta_1 + 4l\dot{\theta}_2\cos\theta_2, 3l\dot{\theta}_1\sin\theta_1 + 4l\dot{\theta}_2\sin\theta_2)$$

Lagrangian of the System

The Lagrangian is a fundamental concept in analytical mechanics. It's given by the difference between the kinetic energy T and the potential energy V of the system. Let's derive the Lagrangian for this double pendulum system:

Kinetic Energy (T): The kinetic energy of an object is given by $\frac{1}{2}mv^2$. Using this formula, we'll calculate the kinetic energy of each pendulum.

- **For the upper mass**: Using the velocity coordinates we derived earlier,

  $$T_1 = \frac{1}{2}m\left((3l\dot{\theta}_1\cos\theta_1)^2 + (3l\dot{\theta}_1\sin\theta_1)^2\right)$$

  Simplifying, we get:

  $$T_1 = \frac{1}{2}m \cdot 9l^2\dot{\theta}_1^{\,2}(\cos^2\theta_1 + \sin^2\theta_1) = \frac{1}{2}m \cdot 9l^2\dot{\theta}_1^{\,2}$$

  (Using the trigonometric identity: $\cos^2\theta + \sin^2\theta = 1$)

- **For the lower mass**:

  $$T_2 = \frac{1}{2}m\left((3l\dot{\theta}_1\cos\theta_1 + 4l\dot{\theta}_2\cos\theta_2)^2 + (3l\dot{\theta}_1\sin\theta_1 + 4l\dot{\theta}_2\sin\theta_2)^2\right)$$

  This expression simplifies to:

  $$T_2 = \frac{1}{2}m[18l^2\dot{\theta}_1^{\,2} + 16l^2\dot{\theta}_2^{\,2} + 24l^2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)]$$

The total kinetic energy T for the system is $T_1 + T_2$.

Potential Energy (V): The potential energy due to gravity is given by mgh, where h is the height of the object above the reference point.

- **For the upper mass**: The height is $3l\cos\theta_1$, so:

$$V_1 = -mg(3l\cos\theta_1)$$

- **For the lower mass**: The height is the sum of the heights of both pendulums:

$$V_2 = -mg(3l\cos\theta_1 + 4l\cos\theta_2)$$

The total potential energy V for the system is $V_1 + V_2$.

Combining the expressions for kinetic and potential energies, the Lagrangian L for the system is:

$$L = T - V$$

$$= \frac{1}{2}m[18l^2\dot{\theta_1}^2 + 16l^2\dot{\theta_2}^2 + 24l^2\dot{\theta_1}\dot{\theta_2}\cos(\theta_1 - \theta_2)] + mg(6l\cos\theta_1 + 4l\cos\theta_2)$$

Lagrange's Equations

Now that we have the Lagrangian L for the system, we can utilize Lagrange's equations to find the equations of motion. Lagrange's equation is a second-order differential equation given by:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_j}}\right) - \frac{\partial L}{\partial q_j} = 0$$

Here, q_j represents the generalized coordinates of the system. For our double pendulum problem, the generalized coordinates are θ_1 and θ_2. We will derive the equations of motion for both θ_1 and θ_2:

1. **Equation for θ_1:**

   Differentiate the Lagrangian with respect to $\dot{\theta_1}$:

   $$\frac{\partial L}{\partial \dot{\theta_1}} = m[18l^2\dot{\theta_1} + 12l^2\dot{\theta_2}\cos(\theta_1 - \theta_2)]$$

   Differentiating this expression with respect to time t:

   $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta_1}}\right) = m[18l^2\ddot{\theta_1} + 12l^2\ddot{\theta_2}\cos(\theta_1 - \theta_2) - 12l^2\dot{\theta_2}^2\sin(\theta_1 - \theta_2)]$$

   Now, differentiate the Lagrangian with respect to θ_1:

   $$\frac{\partial L}{\partial \theta_1} = -mg(3l\sin\theta_1) - m[12l^2\dot{\theta_1}\dot{\theta_2}\sin(\theta_1 - \theta_2)]$$

   Substituting these values into Lagrange's equation:

   $$m[18l^2\ddot{\theta_1}+12l^2\ddot{\theta_2}\cos(\theta_1-\theta_2)-12l^2\dot{\theta_2}^2\sin(\theta_1-\theta_2)]+mg(3l\sin\theta_1)+m[12l^2\dot{\theta_1}\dot{\theta_2}\sin(\theta_1-\theta_2)] = 0$$

After simplifying, we obtain:

$$3\ddot{\theta}_1 + 2\ddot{\theta}_2 \cos(\theta_1 - \theta_2) + 2\dot{\theta}_2^{\,2} \sin(\theta_1 - \theta_2) + \frac{g \sin\theta_1}{l} = 0$$

2. **Equation for θ_2:**

   Using a similar procedure (which has been skipped here for brevity), we would obtain another differential equation representing the motion of the second pendulum in terms of θ_2.

For small oscillations, we can make the approximation $\sin\theta \approx \theta$ and $\cos\theta \approx 1$. Thus, the equations linearize to:

$$3\ddot{\theta}_1 + 2\ddot{\theta}_2 + \frac{g\theta_1}{l} = 0$$

and

$$3\ddot{\theta}_1 + 4\ddot{\theta}_2 + \frac{g\theta_2}{l} = 0$$

These equations represent the linearized equations of motion for the double pendulum system under the condition of small oscillations.

Secular Equation and Normal-mode Frequencies

Using the proposed solution for the equations of motion:

$$\theta_1 = \theta_{10} e^{i\omega t}$$
$$\theta_2 = \theta_{20} e^{i\omega t}$$

and substituting these expressions into our linearized equations of motion, we get the following homogeneous system:

$$\left(\frac{g}{l} - 3\omega^2\right)\theta_{10} - 2\omega^2\theta_{20} = 0$$

and

$$-3\omega^2\theta_{10} + \left(\frac{g}{l} - 4\omega^2\right)\theta_{20} = 0$$

This can be represented in matrix form as:

$$\begin{bmatrix} \frac{g}{l} - 3\omega^2 & -2\omega^2 \\ -3\omega^2 & \frac{g}{l} - 4\omega^2 \end{bmatrix} \begin{bmatrix} \theta_{10} \\ \theta_{20} \end{bmatrix} = 0$$

For a non-trivial solution to exist, the determinant of the matrix must be zero. Expanding the determinant, we obtain the secular equation:

$$\left(\frac{g}{l} - 3\omega^2\right)\left(\frac{g}{l} - 4\omega^2\right) - (-2\omega^2)(-3\omega^2) = 0$$

Simplifying the equation:

$$\left(\frac{g}{l} - \omega^2\right)\left(\frac{g}{l} - 6\omega^2\right) = 0$$

From this secular equation, the normal-mode frequencies are:

$$\omega_1 = \sqrt{\frac{g}{l}}$$

$$\omega_2 = \sqrt{\frac{g}{6l}}$$

For each of these normal modes, we can find the relation between θ_{10} and θ_{20} by substituting back into our matrix equation:

1. **For ω_1 mode**:

   The relationship between the amplitudes of the angles is $\theta_{20} = -\theta_{10}$ or $\theta_2 = -\theta_1$.

2. **For ω_2 mode**:

   The relationship is $\theta_{20} = \frac{3}{2}\theta_{10}$ or $\theta_2 = \frac{3}{2}\theta_1$.

The x-coordinate for a point on the lower pendulum located a distance ξ from the upper mass is:

$$x = 3l \sin\theta_1 + \xi \sin\theta_2$$

Differentiating with respect to time, the x-component velocity is:

$$\dot{x} = 3l\dot{\theta}_1 \cos\theta_1 + \xi\dot{\theta}_2 \cos\theta_2$$

This equation gives us the horizontal velocity of a point located on the lower pendulum at distance ξ from the pivot of the upper mass.

Conditions for No Horizontal Displacement

For a point on the lower pendulum to exhibit no horizontal displacement, its x-component velocity $\dot{x}$ must be zero. Given our derived expression for $\dot{x}$, this requirement can be expressed for the two different normal modes:

1. For the ω_1 mode:

In this mode, $\theta_2 = -\theta_1$. Substituting this relationship into the x-component velocity equation, we have:

$$\dot{x} = 3l\dot{\theta}_1 \cos\theta_1 - \xi\dot{\theta}_1 \cos\theta_1$$

Rearranging, this becomes:

$$\dot{x} = (3l - \xi)\dot{\theta}_1 \cos\theta_1$$

For $\dot{x}$ to be zero, $(3l - \xi)\dot{\theta}_1$ must be zero. Given that $\dot{\theta}_1$ can't be zero (else the pendulum isn't oscillating), this means $\xi = 3l$. Therefore, for the ω_1 mode, a point on the lower pendulum at a distance of $3l$ from the upper mass will have no horizontal displacement.

2. For the ω_2 mode:

In this mode, $\theta_2 = \frac{3}{2}\theta_1$. Plugging this into the equation for $\dot{x}$, we get:

$$\dot{x} = 3l\dot{\theta}_1 \cos\theta_1 + \frac{3}{2}\xi\dot{\theta}_1 \cos\theta_1$$

Rearranging, we obtain:

$$\dot{x} = 3\left(l + \frac{\xi}{2}\right)\dot{\theta}_1 \cos\theta_1$$

For $\dot{x}$ to be zero, either $\dot{\theta}_1$ must be zero or the term inside the parentheses must be zero. The latter case, which gives $\xi = -2l$, isn't physically realistic as ξ is the distance from the upper pendulum's mass and should be positive.

Therefore, the only valid case for $\dot{x}$ to be zero in the ω_2 mode is when $\dot{\theta}_1$ is zero, implying the upper pendulum isn't moving. This result indicates there's no such point ξ on the lower pendulum where it won't have horizontal displacement when both pendulums are in motion in the ω_2 mode.

Summary:

For the compound pendulum system performing small oscillations:

- In the ω_1 mode, a point on the lower pendulum at a distance of $3l$ from the upper mass will experience no horizontal displacement.
- In the ω_2 mode, no such point ξ exists on the lower pendulum that doesn't undergo horizontal displacement, unless the upper pendulum is stationary.

Therefore, the point is $3l$ from the upper mass of the lower pendulum.