MSc Program

Mathematics for Engineers

Course project- 'M2. Inverted Pendulum'

Week 1 Task

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Tasks for week 1

1. Describe the mathematical model of the system from Fig.1.
2. Draw the free body/kinematic diagram of the system from Fig.1.
3. Write the equations of motion for the system from Fig.1.
4. Write down the asymptotic solution of the system from Fig.1.

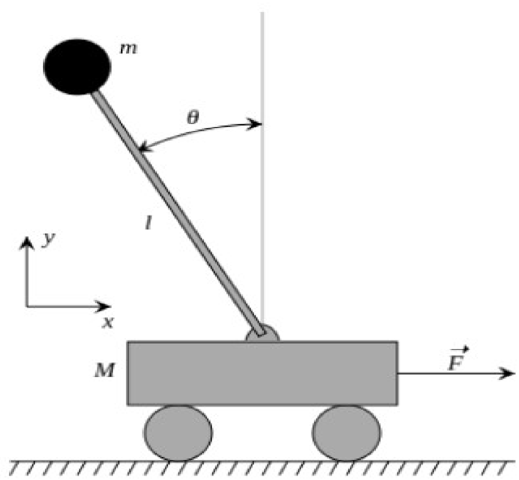

Figure 1: Inverted Pendulum System

The diagram in Figure 3 shows an inverted pendulum system. A pendulum with mass m at the end is attached to a rod of length l. This pendulum makes an angle θ with the vertical axis. The rod is attached to a cart of mass M that moves horizontally on a surface. The horizontal position of the cart is represented by x. A force F is applied to the cart, pushing it in the right direction.

1 Mathematical model of the system

1.1 Variables:

- m: mass of the pendulum (ball).
- M: mass of the cart.

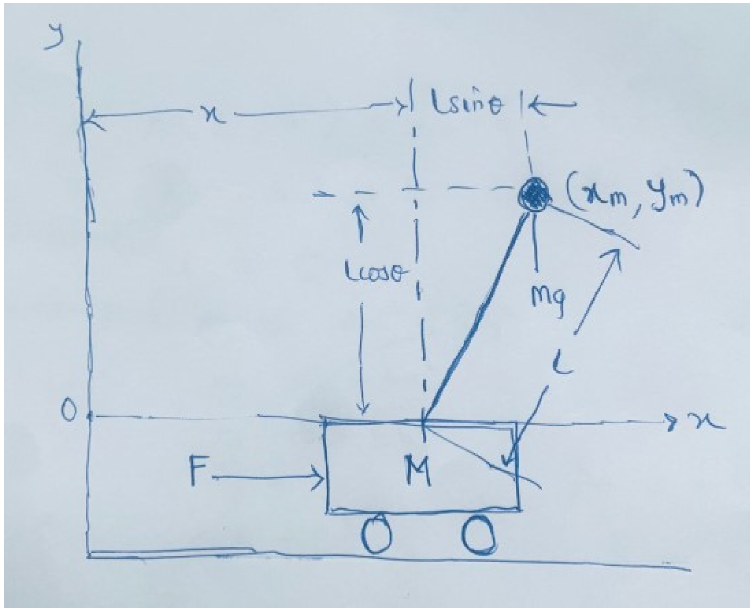

Figure 2: Inverted Pendulum System

- l: length of the rod (distance from the pivot point to the center of mass of the ball).
- θ: angle of the pendulum with the vertical.
- F: horizontal force applied to the cart.
- x: horizontal position of the cart.

1.2 Equations of motion:

Using the Lagrangian method, we derive the equations of motion for the system. This results in two differential equations, one for the cart and one for the pendulum.

1. For the cart:

$$(M + m)\ddot{x} + ml\ddot{\theta}\cos(\theta) - ml\dot{\theta}^2\sin(\theta) = F$$

2. For the pendulum:

$$l\ddot{\theta} + g\sin(\theta) = \ddot{x}\cos(\theta)$$

Where:

- g is the acceleration due to gravity.

- $\ddot{x}$ and $\ddot{\theta}$ are the second time derivatives of x and θ, respectively, i.e., the accelerations.
- $\dot{\theta}$ is the first time derivative of θ, i.e., the angular velocity.

1.3 Assumptions:

1. The pendulum rod is massless and rigid.
2. The system operates in a plane.
3. All movements are frictionless.

2. Kinematics:

Position:

$$x_m = x + L\sin\theta \tag{1}$$
$$y_m = L\cos\theta \tag{2}$$

Velocity:

$$\dot{x} = \dot{x} + L\dot{\theta}\cos\theta \tag{3}$$
$$\dot{y_m} = -L\dot{\theta}\sin\theta \tag{4}$$

Potential Energy:

Using $y_m = L\cos\theta$:
$$V = mgy_m = mgL\cos\theta \tag{5}$$

Kinetic Energy:

Starting with:
$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x_m}^2 + \dot{y_m}^2)$$

Substitute from (3) and (4):

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left[(\dot{x} + L\dot{\theta}\cos\theta)^2 + (-L\dot{\theta}\sin\theta)^2\right]$$

Expanding:

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left[\dot{x}^2 + 2\dot{x}L\dot{\theta}\cos\theta + L^2\dot{\theta}^2\cos^2\theta + L^2\dot{\theta}^2\sin^2\theta\right]$$

Grouping like terms:

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left[\dot{x}^2 + 2\dot{x}L\dot{\theta}\cos\theta + L^2\dot{\theta}^2(\cos^2\theta + \sin^2\theta)\right]$$

Using the trigonometric identity $\cos^2\theta + \sin^2\theta = 1$:

$$T = \frac{1}{2}(M+m)\dot{x}^2 + mL\dot{x}\dot{\theta}\cos\theta + \frac{1}{2}mL^2\dot{\theta}^2 \tag{6}$$

Lagrange's Equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i \tag{7}$$

$$L = T - V \tag{8}$$

Substituting the expressions for T and V gives:

$$L = \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}ml^2\dot{\theta}^2 + ml\dot{\theta}\dot{x}\cos\theta - mgl\cos\theta \tag{9}$$

Substituting (9) in (7):
With respect to $\dot{x}$:

$$(M+m)\ddot{x} + (ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta) = F(t) \tag{10}$$

$$(M+m)\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta = F(t) \tag{11}$$

With respect to $\dot{\theta}$:

$$ml^2\ddot{\theta} + ml\ddot{x}\cos\theta - ml\dot{\theta}\dot{x}\sin\theta + ml\dot{\theta}\dot{x}\sin\theta - mgl\sin\theta = 0$$

Dividing through by ml:

$$l\ddot{\theta} + \ddot{x}\cos\theta - g\sin\theta = 0 \tag{12}$$

Rearranging:

$$\ddot{\theta} = \frac{g\sin\theta - \ddot{x}\cos\theta}{l} \tag{13}$$

$$\ddot{x} = \frac{g\sin\theta - l\ddot{\theta}}{\cos\theta} \tag{14}$$

Substituting $\ddot{\theta}$ from (13) into the expression from (11):

$$(M+m)\ddot{x} + ml\cos\theta\left(\frac{g\sin\theta - \ddot{x}\cos\theta}{l}\right) - ml\dot{\theta}^2\sin\theta = F(t)$$

Now, we can simplify the expression further:

$$(M+m)\ddot{x} + m\cos\theta(g\sin\theta - \ddot{x}\cos\theta) - ml\dot{\theta}^2\sin\theta = F(t)$$
$$(M+m)\ddot{x} + mg\sin\theta\cos\theta - m\ddot{x}\cos^2\theta - ml\dot{\theta}^2\sin\theta = F(t)$$

Rearranging we get:

$$(M + m + m\cos^2\theta)\ddot{x} = F(t) + ml\dot{\theta}^2\sin\theta - mg\sin\theta\cos\theta$$

$$\ddot{x}[M + m - m\cos^2\theta] = F(t) + m\ell\dot{\theta}^2\sin\theta - mg\sin\theta\cos\theta \tag{15}$$

$$\ddot{x}[M + m(1 - \cos^2\theta)] = F(t) + m\ell\dot{\theta}^2\sin\theta - mg\sin\theta\cos\theta$$

$$\ddot{x}(M + m\sin^2\theta) = F(t) + m\ell\dot{\theta}^2\sin\theta - mg\sin\theta\cos\theta$$

Rearranging for $\ddot{x}$:

$$\ddot{x} = \frac{F(t) + m\ell\dot{\theta}^2\sin\theta - mg\sin\theta\cos\theta}{M + m\sin^2\theta} \tag{16}$$

Substituting (16) in the expression:

$$\ddot{\theta} = \frac{g\sin\theta - \cos\theta \times \frac{F(t)+m\ell\dot{\theta}^2\sin\theta - mg\sin\theta\cos\theta}{M+m\sin^2\theta}}{\ell}$$

Expanding and simplifying:

$$\ddot{\theta} = \frac{g\sin\theta(M + m\sin^2\theta) - F(t)\cos\theta - m\ell\dot{\theta}^2\sin\theta\cos\theta + mg\sin\theta\cos^2\theta}{\ell(M + m\sin^2\theta)}$$

Using the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$:

$$\ddot{\theta} = \frac{Mg\sin\theta + mg\sin\theta - F(t)\cos\theta - m\ell\dot{\theta}^2\sin\theta\cos\theta}{\ell(M + m\sin^2\theta)} \tag{17}$$

Equations (16) and (17) are the non-linear equations of the model.
To obtain the linear model of the system, the following conditions are set:

$$\begin{aligned} \theta &\to 0 \\ \sin\theta &\to \theta \\ \cos\theta &\to 1 \\ \dot{\theta} &\to 0 \end{aligned}$$

Using the non-linear equation (from previous discussions):

$$\ddot{x} = \frac{F(t) + m\ell\dot{\theta}^2\sin\theta - mg\sin\theta\cos\theta}{M + m\sin^2\theta}$$

Linearizing the above equation, we obtain:

$$\ddot{x} = \frac{F(t) - mg\theta}{M} \tag{18}$$

Similarly, for the other non-linear equation:

$$\ddot{\theta} = \frac{-F(t)\cos\theta + (M+m)g\sin\theta - m\ell\dot{\theta}^2\sin\theta\cos\theta}{\ell(M + m\sin^2\theta)}$$

Linearizing this equation gives:

$$\ddot{\theta} = \frac{-F(t) + (M+m)g\theta}{\ell M} \tag{19}$$

Equations (18) and (19) are to be used to develop the state-space model. The state variables are defined as:

$$\begin{aligned}
x_1 &= \theta \\
\dot{x}_1 &= \dot{\theta} = x_2 \\
\ddot{x}_1 &= \ddot{\theta} = \dot{x}_2 \\
x_3 &= x \\
\dot{x}_3 &= \dot{x} - x_4 \\
x_5 &= \ddot{x} - x_4
\end{aligned}$$

Equations (18) and (19) can be reformulated as:

$$\ddot{x} = \dot{x}_4 = -\frac{mg}{M}x_1 + \frac{F}{M} \tag{20}$$

$$\ddot{\theta} = \dot{x}_2 = \frac{(M+m)g}{\ell M}x_1 - \frac{F}{\ell M} \tag{21}$$

State space representation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{(M+m)g}{M\ell} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{mg}{M} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{\ell M} \\ 0 \\ \frac{1}{M} \end{bmatrix} F \tag{22}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} F \tag{23}$$

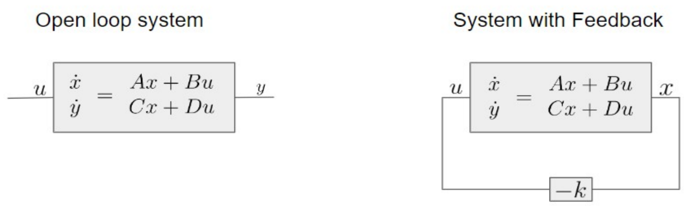

Figure 3: open loop and feedback system

Stabilization Task/Controller Design

Variables and Parameters:

- u – Control input
- x – state of the system
- y – observation/output
- A – Transition matrix
- B – Input Matrix
- C – Output Matrix
- D – Feedforward term
- k – Controller gain

A, B, C, and D were found from the system model after linearization.
Assumption: All the state is Measurable/observable.

$$y = x$$
$$\text{New control input:} \quad u = -kx$$

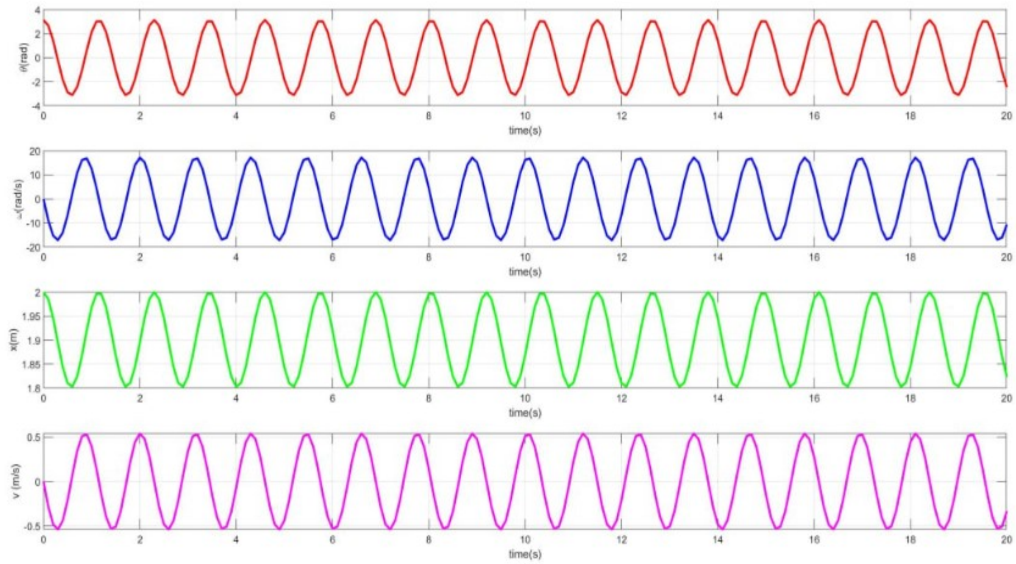

Figure 4: System Responses /Before Stabilization

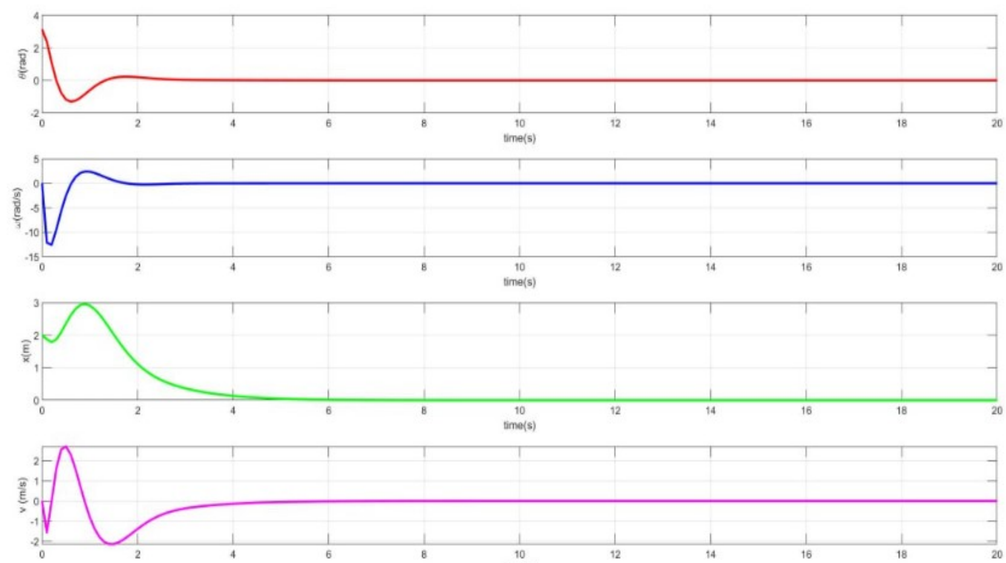

Figure 5: System Responses with Controller