lec06

February 5, 2026

1 Lecture 6: Decision Theory — Loss, Risk, Admissibility, and Optimality

Data 145, Spring 2026: Evidence and Uncertainty
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Please run the setup cell below before reading.

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[1]:
```
import numpy as np
import matplotlib.pyplot as plt
from scipy import stats

plt.style.use('fivethirtyeight')
%matplotlib inline

# Color scheme for this lecture (colorblind-safe palette)
# Using IBM's colorblind-safe palette
COLOR_MLE = '#648FFF'       # Blue - MLE (unbiased)
COLOR_LAPLACE1 = '#785EF0'  # Purple - Laplace +1/+1
COLOR_LAPLACE2 = '#DC267F'  # Magenta - Laplace +2/+2
COLOR_BAD = '#FE6100'       # Orange - Inadmissible
COLOR_TRUE = '#000000'      # Black - True value
```

1.1 Introduction: Beyond Asymptotic Efficiency

In Lectures 3–5, we developed the theory of maximum likelihood estimation. We showed that the MLE is: - **Consistent**: $\hat{\theta}_n \xrightarrow{P} \theta_0$ - **Asymptotically normal**: $\sqrt{n}(\hat{\theta}_n - \theta_0) \xrightarrow{d} N(0, 1/I(\theta_0))$ - **Efficient**: Among unbiased estimators, it achieves the Cramér-Rao lower bound

This is a powerful result: if you want an unbiased estimator with small variance, the MLE is (asymptotically) the best you can do.

But who said we have to be unbiased?

Today we'll see that: 1. Biased estimators can sometimes have *lower* mean squared error than the MLE 2. There's no single "best" estimator — different estimators are optimal under different criteria 3. This leads naturally to **Bayesian statistics** as one principled way to choose among estimators

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1.2 1. Decision Theory in Statistical Estimation

1.2.1 The Setup

Recall our standard setup: - **Data**: X (often $X_1, \ldots, X_n$ i.i.d.) - **Model**: f_θ indexed by parameter $\theta \in \Theta$ - **Estimand**: Some function $g(\theta)$ we want to estimate (often just θ itself) - **Estimator**: A function $T(X)$ that produces our estimate

When estimating θ itself, we typically write $\hat{\theta}(X)$ or just $\hat{\theta}$.

1.2.2 Loss Functions

A **loss function** $L(\theta, a)$ measures how bad it is to report the estimate a when the true parameter is θ.

Squared error loss is the most common choice:

$$L(\theta, a) = (\theta - a)^2$$

Other loss functions exist (absolute error, 0-1 loss, etc.), but we'll focus on squared error.

1.2.3 Risk Functions

The loss for a single dataset doesn't tell us much — one estimator might beat another by luck. We want to average over the sampling distribution.

The **risk function** is the expected loss:

$$R(\theta, T) = E_\theta[L(\theta, T(X))]$$

For squared error loss, this is the **mean squared error (MSE)**:

$$R(\theta, T) = \mathrm{MSE}_\theta(T) = E_\theta[(T(X) - \theta)^2]$$

1.2.4 The Fundamental Problem

The risk function depends on θ, which we don't know! Different estimators may be better for different values of θ.

This creates a fundamental tension: **how do we compare estimators when their relative performance depends on the unknown parameter?**

1.3 2. The Binomial Example

1.3.1 Setup

Let's work with a concrete example where we can visualize everything.

Example (Coin flipping): Flip a coin n times. Let X = number of heads. We want to estimate the probability p of heads.

Equivalently: We have n patients in a drug trial, X respond positively, and we want to estimate the success rate p.

Mathematically: $X \sim \text{Binomial}(n, p)$, and we want to estimate $p \in [0, 1]$.

This example is nice because: - The parameter space $[0, 1]$ is bounded, so we can plot risk functions over the entire space - X is the sum of n i.i.d. Bernoulli(p) random variables, so results from Lectures 4–5 apply

We'll use $n = 16$ throughout (you'll see why this choice is nice later).

1.3.2 The MLE

The MLE for p is the sample proportion:

$$\hat{p} = \frac{X}{n}$$

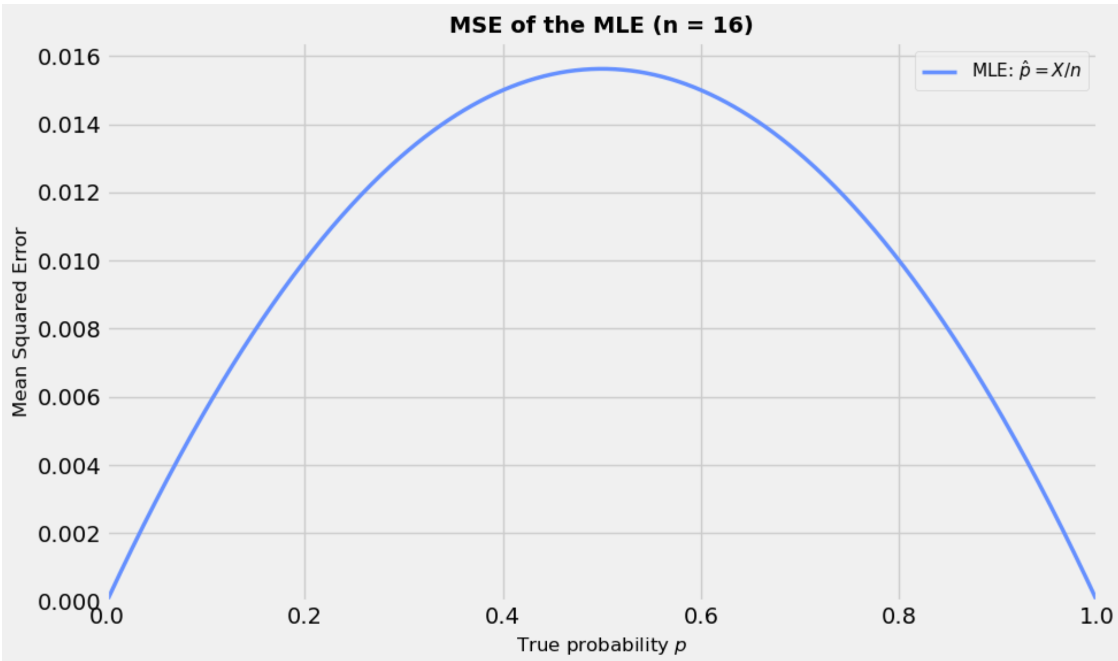
From Lectures 4–5 (applied to Bernoulli observations), we know: - **Unbiased**: $E_p[\hat{p}] = p$ - **Variance**: $\text{Var}_p(\hat{p}) = \frac{p(1-p)}{n}$ - **Fisher information** (per observation): $I(p) = \frac{1}{p(1-p)}$ - **Efficient**: $\text{Var}_p(\hat{p}) = \frac{1}{nI(p)}$ achieves the Cramér-Rao bound

Since the MLE is unbiased, its MSE equals its variance:

$$\text{MSE}_p(\hat{p}) = \frac{p(1-p)}{n}$$

[2]:
```
# Plot the MSE of the MLE
n = 16
p_grid = np.linspace(0.001, 0.999, 500)

# MSE of MLE
mse_mle = p_grid * (1 - p_grid) / n

fig, ax = plt.subplots(figsize=(10, 6))
ax.plot(p_grid, mse_mle, color=COLOR_MLE, linewidth=2.5, label=r'MLE: $\hat{p}␣
 ↪= X/n$')

ax.set_xlabel('True probability $p$', fontsize=12)
ax.set_ylabel('Mean Squared Error', fontsize=12)
ax.set_title(f'MSE of the MLE (n = {n})', fontsize=14, fontweight='bold')
ax.set_xlim(0, 1)
ax.set_ylim(0, None)
```

```
ax.legend(fontsize=11)

plt.tight_layout()
plt.show()

print(f"Maximum MSE occurs at p = 0.5: MSE = {0.5 * 0.5 / n:.4f}")
```




```
Maximum MSE occurs at p = 0.5: MSE = 0.0156
```

1.3.3 Is Unbiased + Efficient the End of the Story?

The MLE is unbiased and efficient. Is it the "best" estimator?

Consider a concrete scenario: $n = 16$ flips, and we observe $X = 12$ heads.

The MLE says $\hat{p} = 12/16 = 0.75$.

Question: Do we really believe the coin has a 75% chance of heads? Or might that be an overreaction to a small sample?

Most real coins are close to fair. If we had some prior belief that p is likely near 0.5, we might want to **shrink** our estimate toward 0.5.

The MLE is the best *unbiased* estimator. **But who said we have to be unbiased?**

1.4 3. Shrinkage Estimators

1.4.1 Laplace's Estimator: Adding Pseudodata

Intuition: If we think extreme values of p (near 0 or 1) are unlikely, maybe we should shrink our estimate toward 1/2.

Laplace's idea (1774): Pretend we observed two extra flips before seeing the data — one head and one tail. This gives:

$$\tilde{p}_1 = \frac{X+1}{n+2}$$

For our example with $n = 16$, $X = 12$: - MLE: $\hat{p} = 12/16 = 0.75$ - Laplace: $\tilde{p}_1 = 13/18 \approx 0.722$

The Laplace estimator shrinks toward 0.5.

1.4.2 More Aggressive Shrinkage

What if we add *two* heads and *two* tails of pseudodata?

$$\tilde{p}_2 = \frac{X+2}{n+4}$$

For $n = 16$, $X = 12$: - Laplace +2/+2: $\tilde{p}_2 = 14/20 = 0.70$

This shrinks even more toward 0.5.

1.4.3 An Inadmissible Estimator

What about this estimator?

$$\hat{p}_{\text{bad}} = \frac{X+1}{n}$$

This adds 1 to the numerator but nothing to the denominator.

Note: This does NOT correspond to adding pseudodata (that would be $(X+1)/(n+1)$). It's just inflating the estimate.

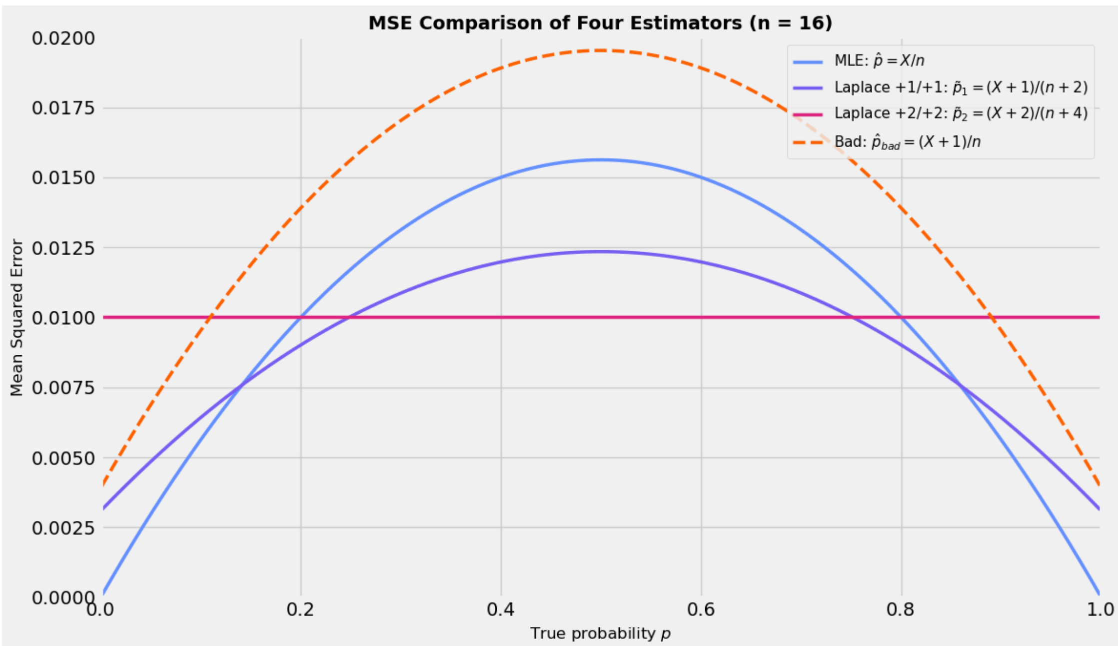
For $n = 16$, $X = 12$: - Bad estimator: $\hat{p}_{\text{bad}} = 13/16 = 0.8125$ — even higher than the MLE!

Let's see how all four estimators compare.

[3]:
```
# Define MSE functions for all four estimators
def mse_mle(p, n):
    """MSE of MLE: X/n"""
    return p * (1 - p) / n

def mse_laplace1(p, n):
    """MSE of Laplace +1/+1: (X+1)/(n+2)"""
    # E[(X+1)/(n+2)] = (np + 1)/(n+2)
    # Bias = (np + 1)/(n+2) - p = (1 - 2p)/(n+2)
```

```python
    # Var = n * p(1-p) / (n+2)^2
    bias = (1 - 2*p) / (n + 2)
    var = n * p * (1 - p) / (n + 2)**2
    return bias**2 + var

def mse_laplace2(p, n):
    """MSE of Laplace +2/+2: (X+2)/(n+4)"""
    # E[(X+2)/(n+4)] = (np + 2)/(n+4)
    # Bias = (np + 2)/(n+4) - p = (2 - 4p)/(n+4) = 2(1 - 2p)/(n+4)
    # Var = n * p(1-p) / (n+4)^2
    bias = 2 * (1 - 2*p) / (n + 4)
    var = n * p * (1 - p) / (n + 4)**2
    return bias**2 + var

def mse_bad(p, n):
    """MSE of bad estimator: (X+1)/n"""
    # E[(X+1)/n] = p + 1/n
    # Bias = 1/n (constant!)
    # Var = p(1-p)/n (same as MLE)
    bias = 1 / n
    var = p * (1 - p) / n
    return bias**2 + var

# Compute MSE for all estimators
n = 16
p_grid = np.linspace(0.001, 0.999, 500)

mse_vals = {
    'MLE': mse_mle(p_grid, n),
    'Laplace +1/+1': mse_laplace1(p_grid, n),
    'Laplace +2/+2': mse_laplace2(p_grid, n),
    'Bad': mse_bad(p_grid, n)
}

# Plot all MSE curves
fig, ax = plt.subplots(figsize=(12, 7))

ax.plot(p_grid, mse_vals['MLE'], color=COLOR_MLE, linewidth=2.5,
        label=r'MLE: $\hat{p} = X/n$')
ax.plot(p_grid, mse_vals['Laplace +1/+1'], color=COLOR_LAPLACE1, linewidth=2.5,
        label=r'Laplace +1/+1: $\tilde{p}_1 = (X+1)/(n+2)$')
ax.plot(p_grid, mse_vals['Laplace +2/+2'], color=COLOR_LAPLACE2, linewidth=2.5,
        label=r'Laplace +2/+2: $\tilde{p}_2 = (X+2)/(n+4)$')
ax.plot(p_grid, mse_vals['Bad'], color=COLOR_BAD, linewidth=2.5, linestyle='--',
        label=r'Bad: $\hat{p}_{bad} = (X+1)/n$')

ax.set_xlabel('True probability $p$', fontsize=12)
```

```
ax.set_ylabel('Mean Squared Error', fontsize=12)
ax.set_title(f'MSE Comparison of Four Estimators (n = {n})', fontsize=14, fontweight='bold')
ax.set_xlim(0, 1)
ax.set_ylim(0, 0.02)
ax.legend(fontsize=11, loc='upper right')

plt.tight_layout()
plt.show()
```



1.4.4 Observations from the Plot

1. **The bad estimator is strictly worse than the MLE everywhere!** Its MSE curve lies entirely above the MLE curve.
2. **The other curves cross.** No single estimator (among MLE, Laplace +1/+1, Laplace +2/+2) is best for all values of p.
3. **Near** $p = 0.5$, the Laplace estimators beat the MLE. Their shrinkage toward 0.5 helps when p really is near 0.5.
4. **Near** $p = 0$ **or** $p = 1$, the MLE beats the Laplace estimators. Shrinking toward 0.5 hurts when the truth is extreme.

To understand *why* these patterns occur, we need to examine the bias-variance tradeoff.

1.5 4. Bias-Variance Calculations and Admissibility

1.5.1 The Bias-Variance Decomposition

Recall that for any estimator T:

$$\text{MSE}_\theta(T) = \text{Bias}_\theta(T)^2 + \text{Var}_\theta(T)$$

where $\text{Bias}_\theta(T) = E_\theta[T] - \theta$.

Let's calculate the bias, variance, and MSE for each of our four estimators.

1.5.2 MLE: $\hat{p} = X/n$

Mean: $E_p[\hat{p}] = E_p[X/n] = p$

Bias: $\text{Bias}_p(\hat{p}) = p - p = 0$

Variance: $\text{Var}_p(\hat{p}) = \frac{\text{Var}_p(X)}{n^2} = \frac{np(1-p)}{n^2} = \frac{p(1-p)}{n}$

MSE: $\text{MSE}_p(\hat{p}) = 0 + \frac{p(1-p)}{n} = \frac{p(1-p)}{n}$

1.5.3 Laplace +1/+1: $\tilde{p}_1 = (X+1)/(n+2)$

Mean: $E_p[\tilde{p}_1] = \frac{E_p[X]+1}{n+2} = \frac{np+1}{n+2}$

Bias:

$$\text{Bias}_p(\tilde{p}_1) = \frac{np+1}{n+2} - p = \frac{np+1-p(n+2)}{n+2} = \frac{np+1-np-2p}{n+2} = \frac{1-2p}{n+2}$$

Notice: the bias is **positive when** $p < 0.5$ (pulls estimate up toward 0.5) and **negative when** $p > 0.5$ (pulls estimate down toward 0.5). The estimator shrinks toward 0.5!

Variance: $\text{Var}_p(\tilde{p}_1) = \frac{\text{Var}_p(X)}{(n+2)^2} = \frac{np(1-p)}{(n+2)^2}$

Note: the variance is **smaller** than the MLE's variance $p(1-p)/n$ because $(n+2)^2 > n \cdot n$ and the numerator only has n (not $n+2$).

MSE:

$$\text{MSE}_p(\tilde{p}_1) = \frac{(1-2p)^2}{(n+2)^2} + \frac{np(1-p)}{(n+2)^2} = \frac{(1-2p)^2 + np(1-p)}{(n+2)^2}$$

1.5.4 Laplace +2/+2: $\tilde{p}_2 = (X+2)/(n+4)$

Mean: $E_p[\tilde{p}_2] = \frac{np+2}{n+4}$

Bias:

$$\text{Bias}_p(\tilde{p}_2) = \frac{np+2}{n+4} - p = \frac{np+2-p(n+4)}{n+4} = \frac{2-4p}{n+4} = \frac{2(1-2p)}{n+4}$$

Same pattern as before, but with more shrinkage toward 0.5.

Variance: $\text{Var}_p(\tilde{p}_2) = \frac{np(1-p)}{(n+4)^2}$

Even smaller variance than Laplace +1/+1.

MSE:

$$\mathrm{MSE}_p(\tilde{p}_2) = \frac{4(1-2p)^2}{(n+4)^2} + \frac{np(1-p)}{(n+4)^2} = \frac{4(1-2p)^2 + np(1-p)}{(n+4)^2}$$

1.5.5 Bad Estimator: $\hat{p}_{\text{bad}} = (X+1)/n$

Mean: $E_p[\hat{p}_{\text{bad}}] = \frac{E_p[X]+1}{n} = \frac{np+1}{n} = p + \frac{1}{n}$

Bias: $\mathrm{Bias}_p(\hat{p}_{\text{bad}}) = \frac{1}{n}$ — **positive for all p!**

Variance: $\mathrm{Var}_p(\hat{p}_{\text{bad}}) = \frac{\mathrm{Var}_p(X)}{n^2} = \frac{p(1-p)}{n}$ — **same as the MLE!**

MSE:

$$\mathrm{MSE}_p(\hat{p}_{\text{bad}}) = \frac{1}{n^2} + \frac{p(1-p)}{n} = \mathrm{MSE}_p(\hat{p}) + \frac{1}{n^2}$$

The bad estimator has the same variance as the MLE but strictly positive bias. Its MSE is strictly higher than the MLE's MSE for all p!

[4]:
```
# Summary table of bias, variance, MSE formulas
print("Summary of Estimator Properties")
print("=" * 80)
print(f"{'Estimator':<25} {'Bias':^20} {'Variance':^20} {'MSE':^20}")
print("-" * 80)
print(f"{'MLE: X/n':<25} {'0':^20} {'p(1-p)/n':^20} {'p(1-p)/n':^20}")
print(f"{'Laplace +1: (X+1)/(n+2)':<25} {'(1-2p)/(n+2)':^20} {'np(1-p)/(n+2)²':
↪^20} {'[formula]':^20}")
print(f"{'Laplace +2: (X+2)/(n+4)':<25} {'2(1-2p)/(n+4)':^20} {'np(1-p)/(n+4)²':
↪^20} {'[formula]':^20}")
print(f"{'Bad: (X+1)/n':<25} {'1/n':^20} {'p(1-p)/n':^20} {'p(1-p)/n + 1/n²':
↪^20}")
print("=" * 80)
```

```
Summary of Estimator Properties
================================================================================
Estimator                         Bias                Variance                MSE
--------------------------------------------------------------------------------
MLE: X/n                           0                  p(1-p)/n
p(1-p)/n
Laplace +1: (X+1)/(n+2)       (1-2p)/(n+2)        np(1-p)/(n+2)²
[formula]
Laplace +2: (X+2)/(n+4)      2(1-2p)/(n+4)        np(1-p)/(n+4)²
[formula]
Bad: (X+1)/n                      1/n                 p(1-p)/n          p(1-p)/n +
1/n²
================================================================================
```

1.5.6 What Went Wrong with the Bad Estimator?

The bad estimator has: - **Same variance** as the MLE - **Strictly positive bias** for all p

It adds bias without any compensating variance reduction. That's a bad trade!

The Laplace estimators, by contrast, make a **smart trade**: - They add bias (toward 0.5) - But they reduce variance (by inflating the denominator) - When p is near 0.5, the bias is small and the variance reduction helps - When p is near 0 or 1, the bias penalty outweighs the variance benefit

The bad estimator just shifts everything up — pure bias, no variance benefit.

1.5.7 Admissibility

Definition: An estimator T_1 is **inadmissible** if there exists another estimator T_2 such that: - $R(\theta, T_2) \leq R(\theta, T_1)$ for all $\theta \in \Theta$ - $R(\theta, T_2) < R(\theta, T_1)$ for at least one θ

In this case, we say T_2 **dominates** T_1. An estimator that is not inadmissible is called **admissible**.

Our estimators: - The **bad estimator is inadmissible** — it is dominated by the MLE - The **MLE, Laplace +1/+1, and Laplace +2/+2 are all admissible** — none dominates another

Admissibility is a minimal requirement: we should never use an inadmissible estimator. But among admissible estimators, we still need a way to choose.

1.6 5. Optimality Criteria

1.6.1 The Fundamental Problem

We have multiple admissible estimators. How do we choose among them?

Question: Is it possible to find an estimator that's best for *all* values of p?

Answer: No! (At least not for MSE.)

Why not? Think about it: if $p = 0.75$ is really the truth, then the constant estimator $T(X) \equiv 0.75$ has MSE $= 0$ at that point. No other estimator can beat it there. But that constant estimator would be terrible if $p = 0.25$.

No estimator can beat every other estimator at every p.

1.6.2 Two Approaches to Choosing Among Admissible Estimators

Approach 1: Restrict to a class of estimators

For example, among **unbiased** estimators, the MLE is optimal.

Why? For unbiased estimators, MSE = Variance. So minimizing MSE is the same as minimizing variance. By the Cramér-Rao bound, the MLE achieves the minimum variance among unbiased estimators.

Another example: among **equivariant** estimators (you'll see this in Worksheet 3, Problem 5), there's often a unique best choice.

Note: We won't pursue optimality among restricted classes further in this course. Instead, we'll focus on the second approach.

Approach 2: Summarize the risk function by a single number

Instead of comparing entire risk curves, we can reduce each estimator to a single number and compare those.

1.6.3 Worst-Case Risk (Minimax)

Define the **worst-case risk**:

$$R_{\max}(T) = \max_{p \in [0,1]} \mathrm{MSE}_p(T)$$

A **minimax estimator** minimizes the worst-case risk.

This is a conservative approach: we guard against the worst possible scenario.

[5]:
```python
# Calculate worst-case risk for each estimator
n = 16
p_grid = np.linspace(0.001, 0.999, 1000)

worst_case = {
    'MLE': np.max(mse_mle(p_grid, n)),
    'Laplace +1/+1': np.max(mse_laplace1(p_grid, n)),
    'Laplace +2/+2': np.max(mse_laplace2(p_grid, n)),
    'Bad': np.max(mse_bad(p_grid, n))
}

print("Worst-Case Risk (Maximum MSE over all p)")
print("=" * 50)
for name, wc in worst_case.items():
    print(f"{name:<20}: {wc:.6f}")
print("=" * 50)
print(f"\nMinimax estimator: Laplace +2/+2 (smallest worst-case risk)")
```

```
Worst-Case Risk (Maximum MSE over all p)
==================================================
MLE                 : 0.015625
Laplace +1/+1       : 0.012346
Laplace +2/+2       : 0.010000
Bad                 : 0.019531
==================================================

Minimax estimator: Laplace +2/+2 (smallest worst-case risk)
```

For $n = 16$, the **Laplace +2/+2 estimator** $\tilde{p}_2 = (X+2)/(n+4)$ is the minimax estimator!

Its MSE curve is relatively flat, with its maximum at $p = 0.5$. By shrinking toward the center, it avoids extremely bad performance anywhere.

1.6.4 Average-Case Risk

Instead of worst-case, we could consider **average-case** risk:

$$R_{\mathrm{avg}}(T) = \int_0^1 \mathrm{MSE}_p(T)\, dp$$

This averages the MSE uniformly over all possible values of p.

Question: We've considered worst-case and average-case risk. Why not **best-case** risk?

Answer: Every estimator can achieve arbitrarily small best-case risk — just find some p where it happens to do well (e.g., the constant estimator $T \equiv c$ has MSE $= 0$ at $p = c$). Best-case risk doesn't discriminate between estimators, so it's not useful.

Which estimator minimizes average-case risk? Let's find out.

1.7 6. Minimizing Average-Case Risk: A Surprising Answer

1.7.1 The Problem

We want to find the estimator $T(X)$ that minimizes:

$$R_{\text{avg}}(T) = \int_0^1 \text{MSE}_p(T)\,dp = \int_0^1 E_p[(T(X)-p)^2]\,dp$$

1.7.2 Rewriting the Objective

Let's interchange the integral and expectation (justified by Fubini's theorem):

$$R_{\text{avg}}(T) = \int_0^1 E_p[(T(X)-p)^2]\,dp = E\left[(T(X)-p)^2\right]$$

where the expectation is over the **joint distribution** of (X,p) with: - $p \sim \text{Uniform}(0,1)$ - $X \mid p \sim \text{Binomial}(n,p)$

1.7.3 Finding the Optimal $T(X)$

For any fixed value of $X = x$, what choice of $T(x)$ minimizes $E[(T(x)-p)^2 \mid X = x]$?

This is a familiar problem from probability: the value that minimizes expected squared error is the **conditional mean**:

$$T^*(x) = E[p \mid X = x]$$

So the optimal estimator is $T^*(X) = E[p \mid X]$.

1.7.4 Computing $E[p \mid X]$

We need the conditional distribution of p given X.

Click to expand: Full derivation of the conditional distribution

We compute the conditional distribution using Bayes' rule.

Joint distribution:

For $p \in [0,1]$ and $x \in \{0,1,\dots,n\}$:

$$f(x,p) = P(X = x \mid p) \cdot f(p) = \binom{n}{x} p^x (1-p)^{n-x} \cdot 1$$

Marginal distribution of X:

$$P(X = x) = \int_0^1 \binom{n}{x} p^x (1-p)^{n-x} dp$$

This integral involves the **Beta function**. Recall that:

$$B(a,b) = \int_0^1 p^{a-1}(1-p)^{b-1} dp = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \frac{(a-1)!(b-1)!}{(a+b-1)!}$$

for positive integers a, b.

So:

$$P(X = x) = \binom{n}{x} \int_0^1 p^x (1-p)^{n-x} dp = \binom{n}{x} B(x+1, n-x+1)$$

$$= \binom{n}{x} \frac{x!(n-x)!}{(n+1)!} = \frac{n!}{x!(n-x)!} \cdot \frac{x!(n-x)!}{(n+1)!} = \frac{1}{n+1}$$

Interesting! Under a uniform prior on p, the marginal distribution of X is uniform on $\{0, 1, \dots, n\}$.

Conditional distribution of p given X:

$$f(p \mid x) = \frac{f(x,p)}{P(X = x)} = \frac{\binom{n}{x} p^x (1-p)^{n-x}}{1/(n+1)} = (n+1)\binom{n}{x} p^x (1-p)^{n-x}$$

This is proportional to $p^x(1-p)^{n-x}$, which is the kernel of a **Beta distribution**:

$$p \mid X = x \sim \text{Beta}(x+1, n-x+1)$$

Conditional mean:

The mean of a $\text{Beta}(\alpha, \beta)$ distribution is $\frac{\alpha}{\alpha+\beta}$.

So:

$$E[p \mid X] = \frac{X+1}{(X+1)+(n-X+1)} = \frac{X+1}{n+2}$$

1.7.5 The Punchline

The estimator that minimizes average-case risk $\int_0^1 \text{MSE}_p(T)\, dp$ is:

$$T^*(X) = E[p \mid X] = \frac{X+1}{n+2}$$

This is exactly Laplace's estimator $\tilde{p}_1$!

Laplace's "add one head and one tail" rule, which seemed like an intuitive hack, is actually the *optimal* estimator if we measure quality by average MSE over $p \in [0,1]$.

1.7.6 The Bayesian Interpretation

What we just computed has a name: it's the **posterior mean** of p given X.

- The distribution $p \sim \text{Uniform}(0,1)$ is called the **prior distribution** — it represents our beliefs about p before seeing data
- The distribution $p \mid X \sim \text{Beta}(X+1, n-X+1)$ is called the **posterior distribution** — it represents our updated beliefs after seeing data
- The posterior mean $E[p \mid X]$ is the **Bayes estimator** for squared error loss

You've already seen Beta-Binomial conjugacy in your probability class — this is the same calculation! The uniform distribution $\text{Uniform}(0,1)$ is the same as $\text{Beta}(1,1)$, and after observing X successes in n trials, the posterior is $\text{Beta}(1+X, 1+n-X) = \text{Beta}(X+1, n-X+1)$.

An estimator that minimizes average risk (with respect to some prior distribution) is called a Bayes estimator.

1.7.7 What About $\tilde{p}_2$?

The Laplace +2/+2 estimator $\tilde{p}_2 = (X+2)/(n+4)$ is also a Bayes estimator — for the **Beta(2, 2) prior** instead of the uniform prior.

The Beta(2,2) prior puts more weight near $p = 0.5$ and less weight near the extremes. If we believe p is likely to be moderate, this prior (and its corresponding Bayes estimator) makes sense.

Different priors lead to different Bayes estimators, each optimal for its own average-case criterion.

[6]:
```python
# Calculate average-case risk for each estimator
from scipy import integrate

n = 16

# Average risk = integral of MSE over p from 0 to 1
avg_risk_mle, _ = integrate.quad(lambda p: mse_mle(p, n), 0, 1)
avg_risk_lap1, _ = integrate.quad(lambda p: mse_laplace1(p, n), 0, 1)
avg_risk_lap2, _ = integrate.quad(lambda p: mse_laplace2(p, n), 0, 1)
avg_risk_bad, _ = integrate.quad(lambda p: mse_bad(p, n), 0, 1)

print("Average-Case Risk (Mean MSE over uniform p)")
print("=" * 50)
print(f"{'MLE':<20}: {avg_risk_mle:.6f}")
print(f"{'Laplace +1/+1':<20}: {avg_risk_lap1:.6f}  <-- Minimum!")
print(f"{'Laplace +2/+2':<20}: {avg_risk_lap2:.6f}")
print(f"{'Bad':<20}: {avg_risk_bad:.6f}")
print("=" * 50)
print(f"\nLaplace +1/+1 minimizes average-case risk (it's the Bayes estimator␣
↪for uniform prior)")
```

```
Average-Case Risk (Mean MSE over uniform p)
==================================================
```

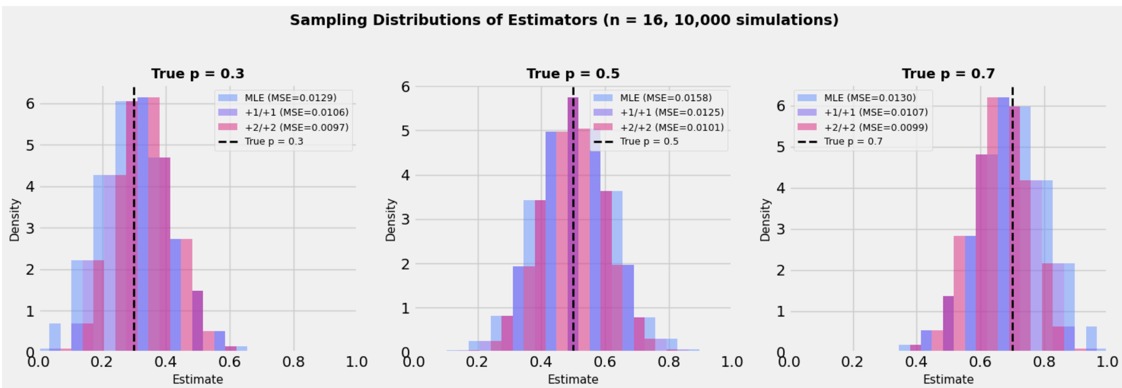
```
MLE                    : 0.010417
Laplace +1/+1          : 0.009259  <-- Minimum!
Laplace +2/+2          : 0.010000
Bad                    : 0.014323
==================================================

Laplace +1/+1 minimizes average-case risk (it's the Bayes estimator for uniform
prior)
```

1.8 Simulation: Sampling Distributions of the Estimators

Let's visualize how the different estimators behave by simulating their sampling distributions for a specific true value of p.

[7]:
```python
# Simulate sampling distributions for different true values of p
np.random.seed(42)

n = 16
n_sims = 10000
true_p_values = [0.3, 0.5, 0.7]

fig, axes = plt.subplots(1, 3, figsize=(15, 5))

for ax, true_p in zip(axes, true_p_values):
    # Simulate X ~ Binomial(n, true_p)
    X = np.random.binomial(n, true_p, size=n_sims)

    # Compute estimates
    est_mle = X / n
    est_lap1 = (X + 1) / (n + 2)
    est_lap2 = (X + 2) / (n + 4)

    # Plot histograms
    bins = np.linspace(0, 1, 30)
    ax.hist(est_mle, bins=bins, alpha=0.5, color=COLOR_MLE,
            label=f'MLE (MSE={np.mean((est_mle - true_p)**2):.4f})',␣
 ↪density=True)
    ax.hist(est_lap1, bins=bins, alpha=0.5, color=COLOR_LAPLACE1,
            label=f'+1/+1 (MSE={np.mean((est_lap1 - true_p)**2):.4f})',␣
 ↪density=True)
    ax.hist(est_lap2, bins=bins, alpha=0.5, color=COLOR_LAPLACE2,
            label=f'+2/+2 (MSE={np.mean((est_lap2 - true_p)**2):.4f})',␣
 ↪density=True)

    # Mark true value
```

```
    ax.axvline(true_p, color=COLOR_TRUE, linestyle='--', linewidth=2,␣
↪label=f'True p = {true_p}')

    ax.set_xlabel('Estimate', fontsize=11)
    ax.set_ylabel('Density', fontsize=11)
    ax.set_title(f'True p = {true_p}', fontsize=13, fontweight='bold')
    ax.legend(fontsize=9)
    ax.set_xlim(0, 1)

plt.suptitle(f'Sampling Distributions of Estimators (n = {n}, {n_sims:,}␣
↪simulations)',
             fontsize=14, fontweight='bold', y=1.02)
plt.tight_layout()
plt.show()
```



1.8.1 Observations from the Simulation

- **At** $p = 0.5$: The Laplace estimators have lower MSE than the MLE. Their shrinkage toward 0.5 helps because 0.5 is the truth!
- **At** $p = 0.3$ **and** $p = 0.7$: The MLE and Laplace estimators are more comparable. The Laplace estimators are biased toward 0.5, which slightly hurts them here.
- **All three admissible estimators** concentrate around the true value, but with different bias-variance tradeoffs.

1.9 7. Summary

1.9.1 Key Concepts

Concept	Definition
Loss function	$L(\theta, a)$ — measures how bad estimate a is when truth is θ
Risk function	$R(\theta, T) = E_\theta[L(\theta, T(X))]$ — expected loss
MSE	$E_\theta[(T(X) - \theta)^2] = \text{Bias}^2 + \text{Variance}$
Admissible	Not dominated by any other estimator
Minimax	Minimizes worst-case risk $\max_\theta R(\theta, T)$
Bayes estimator	Minimizes average risk $\int R(\theta, T)\pi(\theta)d\theta$

1.9.2 What We Learned

1. **The MLE is not always the best estimator.** It's the best *unbiased* estimator, but biased estimators can have lower MSE.
2. **Shrinkage can help.** The Laplace estimators shrink toward 0.5, trading bias for reduced variance. This helps when the true p is near 0.5.
3. **Admissibility is a minimal requirement.** The bad estimator $(X + 1)/n$ is inadmissible — strictly dominated by the MLE. But many admissible estimators exist.
4. **No uniformly best estimator exists.** Different estimators win for different values of p.
5. **Optimality criteria help us choose:**
   - **Minimax**: Laplace +2/+2 minimizes worst-case risk
   - **Bayes (uniform prior)**: Laplace +1/+1 minimizes average-case risk
6. **The Bayes estimator is the posterior mean.** This is our first glimpse of Bayesian statistics — arrived at through a purely frequentist argument!

1.9.3 Next Time

In Lecture 7, we'll develop the **Bayesian perspective** more fully: - Treat θ as a random variable with a **prior distribution** - Update to a **posterior distribution** after seeing data - Understand why the **likelihood** is all that matters (given the prior) - Explore the asymptotic behavior of the posterior

1.9.4 Worksheet 3 Preview

Problem 5 explores **scale-equivariant estimation** and proves that unbiased equivariant estimators are always inadmissible — you can always improve by shrinking! This complements today's theme: unbiasedness is not always a virtue.

```
[ ]:
```