* Problem 1

$\boxed{1\text{-(b)}}$ $f_X(x) = \lambda e^{-\lambda x}\mathbb{1}(x \ge 0)$

$$
\begin{aligned}
E(X) &= \int_0^\infty \lambda x e^{-\lambda x}dx \\
&= \left[\lambda x \cdot \left(-\tfrac{1}{\lambda}e^{-\lambda x}\right)\right]_0^\infty - \int_0^\infty \lambda \cdot \left(-\tfrac{1}{\lambda}e^{-\lambda x}\right)dx \\
&= \cancel{\left[-x e^{-\lambda x}\right]_0^\infty}^{\,0} + \int_0^\infty e^{-\lambda x}dx \\
&= \left[-\tfrac{1}{\lambda}e^{-\lambda x}\right]_0^\infty \\
&= \frac{1}{\lambda} \qquad \therefore E(X) = \frac{1}{\lambda}
\end{aligned}
$$

$$
\begin{aligned}
E(X^2) &= \int_0^\infty \lambda x^2 e^{-\lambda x}dx \\
&= \left[\lambda x^2 \cdot \left(-\tfrac{1}{\lambda}e^{-\lambda x}\right)\right]_0^\infty - \int_0^\infty 2\lambda x \cdot \left(-\tfrac{1}{\lambda}e^{-\lambda x}\right)dx \\
&= \cancel{\left[-x^2 e^{-\lambda x}\right]_0^\infty}^{\,0} + \int_0^\infty 2x \cdot e^{-\lambda x}dx \\
&= \frac{2}{\lambda}\int_0^\infty \lambda x e^{-\lambda x}dx \\
&= \frac{2}{\lambda}E(X) = \frac{2}{\lambda^2}
\end{aligned}
$$

$$Var(X) = E(X^2) - E(X)^2 = \frac{1}{\lambda^2}$$

$\boxed{1\text{-(d)}}$ $Y_1 \sim Gamma(a_1, b)$

$$f_{Y_1}(y) = \frac{b^{a_1} y^{a_1-1} e^{-by}}{\Gamma(a_1)}\mathbb{1}\{y \ge 0\}$$

& $Y_2 \sim Gamma(a_2, b)$

$M_{Y_1}(t)$

$$
\begin{aligned}
&= E_{Y_1}(e^{tY_1}) = \int_0^\infty \frac{b^{a_1} y_1^{a_1-1} e^{-by_1}\, e^{tY_1}}{\Gamma(a_1)}dy_1 \\
&= \int_0^\infty \frac{b^{a_1}}{\Gamma(a_1)} y_1^{a_1-1} \cdot e^{-(b-t)y_1}dy_1 \\
&= \left(\frac{b}{b-t}\right)^{a_1} \underbrace{\int_0^\infty \frac{(b-t)^{a_1}}{\Gamma(a_1)} y_1^{a_1-1} e^{-(b-t)y_1}dy_1}_{=1}
\end{aligned}
$$

$$M_{Y_1}(t) = \left(\frac{b}{b-t}\right)^{a_1}$$

$$M_{Y_2}(t) = \left(\frac{b}{b-t}\right)^{a_2}$$

Let $Z = Y_1 + Y_2$ $(Y_1 \perp\!\!\!\perp Y_2)$

$$
\begin{aligned}
M_Z(t) &= E(e^{tZ}) = E(e^{t(Y_1+Y_2)}) \\
&= \prod_{i=1}^{2} E[\exp(tY_i)] \quad \because Y_1 \perp\!\!\!\perp Y_2 \\
&= \prod_{i=1}^{2}\left(\frac{b}{b-t}\right)^{a_i} \\
&= \left(\frac{b}{b-t}\right)^{a_1+a_2} \sim \underline{\Gamma(a_1+a_2, b)}
\end{aligned}
$$

$\boxed{1\text{-(e)}}$ Gamma distribution with shape parameter 1 is nothing but exponential distribution.

So, I can rewrite $Gamma(1,1) := exp(1)$

As I proved in 1-(d), $Y \sim Gamma(m,1)$ can be rewritten

as: $X \sim Gamma(1,1) \Rightarrow \underline{Y = mX}\ (= \overbrace{X+X+\cdots+X}^{m})$

It means $Gamma(m,1)$ is identical with $m\,exp(1)$

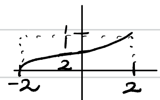
This is the theoretical reason why I could code corresponding to sampling from $Y \sim Gamma(m,1)$ in a two short lines.

$\boxed{2\text{-(a)}}$ $g(x) = \frac{3}{16}x^2\mathbb{1}[-2 \le x \le 2]$

$$
\begin{aligned}
\text{cdf of } g(x) : F_g(x) &= \int_{-2}^{x}\frac{3}{16}t^2 dt \\
&= \left[\tfrac{1}{16}t^3\right]_{-2}^{x} \\
&= \frac{1}{16}x^3 + \frac{1}{2}
\end{aligned}
$$

Let $F_g(x) \coloneqq y$

$$y = \frac{1}{16}x^3 + \frac{1}{2} \Rightarrow$$



$0 \le y \le 1$

get F_g^{-1}

$$\left(x - \frac{1}{2}\right) \times 16 = y^3$$

$$y^3 = 16x - 8$$

$$\boxed{y = \sqrt[3]{16x-8} \qquad \underline{0 \le x \le 1}}$$

sample from np.rand & put the value into $\sqrt[3]{16x-8}$.

Then, might get the sample from $q(x)$

Mean : $E_q(X) = \int_{-2}^{2} x \cdot \frac{3}{16}x^2 \, dx$

$$= \int_{-2}^{2} \frac{3}{16}x^3 \, dx$$

$$= \left[\frac{3}{64}x^4 \right]_{-2}^{2} = 0$$

$$E_q(X^2) = \int_{-2}^{2} x^2 \cdot \frac{3}{16}x^2 \, dx$$

$$= \int_{-2}^{2} \frac{3}{16}x^4 \, dx$$

$$= \left[\frac{3}{80}x^5 \right]_{-2}^{2} = \frac{3}{80} \times (32+32)$$

$$= \frac{3}{40} \times 32 = \frac{12}{5} = 2.4$$

Mean : $E(X) = 0$ as proven on above

Var : $Var(X) = E(X^2) - E(X)^2 = 2.4$