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Assignment 1

1. (a) $P(X=x; a,b) = \int_0^1 \mathbb{P}(X=x|\theta) f(\theta; a,b)\, d\theta$

$$= \int_0^1 \binom{n}{x} \theta^x (1-\theta)^{n-x} \cdot \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \theta^{a-1} (1-\theta)^{b-1} \mathbb{1}_{\{0\le\theta\le1\}}\, d\theta$$

$$= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \binom{n}{x} \int_0^1 \theta^{x+a-1} (1-\theta)^{n+b-x-1} \mathbb{1}_{\{0\le\theta\le1\}}\, d\theta$$

$$= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \binom{n}{x} \frac{\Gamma(x+a)\Gamma(n+b-x)}{\Gamma(n+a+b)} \underbrace{\int_0^1 \frac{\Gamma(n+a+b)}{\Gamma(x+a)\Gamma(n+b-x)} \theta^{x+a-1} (1-\theta)^{n+b-x-1}\, d\theta}_{=1}$$

$$= \binom{n}{x} \cdot \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \cdot \frac{\Gamma(x+a)\Gamma(n+b-x)}{\Gamma(n+a+b)}$$

$$= \frac{n!}{x!(n-x)!} \cdot \frac{(a+b-1)!}{(a-1)!(b-1)!} \cdot \frac{(x+a-1)!(n+b-x-1)!}{(n+a+b-1)!}$$

$$= \frac{\binom{x+a-1}{x}\binom{n+b-x-1}{n-x}}{\binom{n+a+b-1}{n}} = \underbrace{\frac{{}_{x+a-1}C_x \cdot {}_{n+b-x-1}C_{n-x}}{{}_{n+a+b-1}C_n}}$$

(b) Calculate Bayes Factor $\dfrac{\mathbb{P}(X=x|M_1)}{\mathbb{P}(X=x|M_2)}$

$\rightarrow P(X=x|M_1) = \int_0^1 \binom{n}{x} \theta^x (1-\theta)^{n-x} \cdot \delta_{\frac{1}{2}}(\theta) \mathbb{1}_{\{0\le\theta\le1\}}\, d\theta$

$$= \binom{n}{x}\left(\frac{1}{2}\right)^n$$

$\rightarrow P(X=x|M_2) = \dfrac{1}{n+1}$ ← When $a=b=1$

$$BF = \frac{\binom{10}{9}\left(\frac{1}{2}\right)^{10}}{\frac{1}{11}} = \frac{110}{1024} = \frac{55}{512}$$

2. (a) $E = \{1,2,3,4\}$, $A = \{\{1\},\{2\}\}$ Find $\sigma(A)$.

$\sigma(A) = \{\emptyset, \{1\}, \{2\}, \{2,3,4\}, \{1,3,4\}, \{1,2,3,4\}\}$

(b) 1. $A \subset C \Rightarrow \sigma(A) \subset \sigma(C)$

Let B an another collection of subsets of set E which belongs to C but not to A.

Then, I can express C as below

$\Rightarrow C = \{A, B\}$

Then, $\sigma(C)$ is $\{\sigma(A), \sigma(B)\}$.

$\sigma(A) \subset \{\sigma(A), \sigma(B)\}$
$= \sigma(C)$

2. $A \subset \sigma(C)$, $C \subset \sigma(A) \Rightarrow \sigma(A) = \sigma(C)$

- $A \subset \sigma(C) \Rightarrow \sigma(A) \subset \sigma(\sigma(C)) = \sigma(C)$
  $\Rightarrow \sigma(A) \subset \sigma(C)$
- $C \subset \sigma(A) \Rightarrow \sigma(C) \subset \sigma(\sigma(A)) = \sigma(A)$
  $\Rightarrow \sigma(C) \subset \sigma(A)$

$\sigma(A) \subset \sigma(C)$, $\sigma(C) \subset \sigma(A)$
$\Rightarrow \therefore \sigma(A) = \sigma(C)$

3. $A \subset C \subset \sigma(A)$

$A \subset C \Rightarrow \sigma(A) \subset \sigma(C)$ according to #1.

$C \subset \sigma(A) \Rightarrow \sigma(C) \subset \sigma(A)$

$\Rightarrow \sigma(A) \subset \sigma(C) \subset \sigma(A)$

$\Rightarrow \therefore \sigma(A) = \sigma(C)$

3. $M_X(t) = \mathbb{E}[e^{tX}]$

(a) $$\mathbb{E}[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x^2 - 2tx)} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-t)^2 + \frac{1}{2}t^2} dx$$

$$= e^{\frac{1}{2}t^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-t)^2} dx \qquad x - t \Rightarrow z, \quad dx = dz$$

$$= e^{\frac{1}{2}t^2} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz}_{=1} = e^{\frac{1}{2}t^2}$$

(b) $Y_n := c(X_1 + X_2 + \cdots + X_n)$

$$\mathbb{E}_Y(e^{tY}) = \mathbb{E}(e^{ct(X_1 + \cdots + X_n)})$$

$$= \mathbb{E}(e^{ctX_1 + ctX_2 + \cdots + ctX_n})$$

$$= \mathbb{E}(e^{ctX_1} \cdot e^{ctX_2} \cdots e^{ctX_n}) \qquad X's \text{ all i.i.d.}$$

$$= \underbrace{\mathbb{E}(e^{ctX_1})}_{M_{X_1}(ct)} \underbrace{\mathbb{E}(e^{ctX_2})}_{M_{X_2}(ct)} \cdots \underbrace{\mathbb{E}(e^{ctX_n})}_{M_{X_n}(ct)}$$

$$= \{M_X(ct)\}^n \quad \text{because } X_1, \cdots, X_n \text{ are i.i.d}$$

(c) Additionally need to assume that all X's are $\perp\!\!\!\perp$ as stated in (b). ☆

$X \sim (\mu, \sigma^2)$

$Y_i := \frac{X_i - \mu}{\sigma}$, $Z_n := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Y_i$

$$M_{Z_n}(t) = \mathbb{E}(e^{tZ_n}) = \mathbb{E}(e^{\frac{t}{\sqrt{n}} \sum^n Y_i})$$

$$= \{\mathbb{E}(e^{\frac{t}{\sqrt{n}} Y})\}^n$$

Let it as $o(\frac{t}{\sqrt{n}})$

$$= \{\mathbb{E}(1 + \frac{t}{\sqrt{n}} Y + \frac{1}{2!} \cdot \frac{t^2}{n} Y^2 + \underbrace{\frac{1}{3!} (\frac{t}{\sqrt{n}})^3 Y^3 + \cdots}\)\}^n$$

$$\mathbb{E}(Y) = \mathbb{E}\left(\frac{X_i - \mu}{\sigma}\right) = 0$$

$$\mathbb{E}(Y^2) = \mathbb{E}\left(\left(\frac{X_i - \mu}{\sigma}\right)^2\right) = \mathbb{E}\left(\frac{X_i^2 - 2\mu X_i + \mu^2}{\sigma^2}\right)$$

$$= E\left(\frac{\sigma^2+\mu^2-2\mu^2+\mu^2}{\sigma^2}\right)$$

$$= 1$$

$$= \left[1+\frac{t^2}{2n}+o\left(\frac{t}{\sqrt{n}}\right)\right]^n$$

In statistics, O notation is used to represent really tiny terms, which are so small that they will not influence on the result of computation.

Using the fact that

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$$

$$\lim_{n\to\infty}\left(1+\frac{k}{n}\right)^n = e^k.$$

$$M_{Z_n}(t) \to e^{\frac{1}{2}t^2} \text{ as } n\to\infty$$

This is same as the Mgf of standard normal dist'n.

$$\therefore Z_n \xrightarrow{d} N(0,1)$$

4. (a) $\lim_{n\to\infty} (|\hat{\sigma}_n^2 - \sigma^2| > \varepsilon) = 0$

$$\hat{\sigma}_n^2 = \frac{1}{n}\Sigma(X_i - \bar{X}_n)^2$$
$$= \frac{1}{n}\Sigma\{(X_i-\mu)-(\bar{X}-\mu)\}^2$$
$$= \frac{1}{n}\Sigma\{(X_i-\mu)^2 - 2(X_i-\mu)(\bar{X}-\mu)+(\bar{X}-\mu)^2\}$$
$$= \frac{1}{n}\Sigma(X_i-\mu)^2 - 2(\bar{X}-\mu)\frac{1}{n}\Sigma(X_i-\mu) + (\bar{X}-\mu)^2$$
$$= \sigma^2 - (\bar{X}-\mu)^2$$

$$|\hat{\sigma}_n^2 - \sigma^2| = |-(\bar{X}-\mu)^2|$$
$$= (\bar{X}_n - \mu)^2$$

Then, what I need to show is that $\lim_{n\to\infty}((\bar{X}_n-\mu)^2 \geq \varepsilon) = 0$

By Chevyshev's inequality,

$$\lim_{n\to\infty}(|\bar{X}_n-\mu|^2 > \varepsilon) \leq \frac{\sigma^2}{n\varepsilon}$$
$$= 0 \text{ as } n\to\infty$$

$\therefore \hat{\sigma}_n^2 \xrightarrow{P} \sigma^2$ as $n\to\infty$

(b) $E(\hat{\sigma}_n^2)$

$$= E\{\sigma^2 - (\bar{X}-\mu)^2\} = \sigma^2 - E\{(\bar{X}-\mu)^2\}$$
$$= \sigma^2 - \frac{\sigma^2}{n} = \frac{n-1}{n}\sigma^2$$

Bias $= E(\hat{\sigma}_n^2) - \sigma^2 = -\frac{\sigma^2}{n} \neq 0 \Rightarrow$ It is biased!

(c) Letting $\hat{\sigma}_n^2 = \frac{1}{n-1}\Sigma(X_i-\bar{X})^2$ is the solution.

$$E(\hat{\sigma}_n^2) = E\left(\frac{n}{n-1}\cdot\frac{1}{n}\Sigma(X_i-\bar{X}_n)^2\right)$$
$$= \frac{n}{n-1}\cdot E\left(\frac{1}{n}\Sigma(X_i-\bar{X}_n)^2\right)$$
$$= \frac{n}{n-1}\cdot\frac{n-1}{n}\sigma^2 = \sigma^2$$

Bias $= \sigma^2 - \sigma^2 = 0 \Rightarrow$ It is unbiased now.

5. What I need to show is $X_n \xrightarrow{d} c$

For all $\varepsilon > 0$,

- 1) $F_{X_n}(c) = P(X_n \le c, X \le c+\varepsilon) + P(X_n \le c, \overbrace{X > c+\varepsilon}^{c < X-\varepsilon})$

$$\le P(X \le c+\varepsilon) + P(X_n < X-\varepsilon)$$
$$= P(X \le c+\varepsilon) + P(X - X_n > \varepsilon)$$
$$\le \underline{P(X \le c+\varepsilon)} + P(|X_n - X| > \varepsilon)$$
$$= F_X(c+\varepsilon) + P(|X_n - X| > \varepsilon)$$

idea

- 2) $P(X \le c-\varepsilon) = P(X \le c-\varepsilon, \overbrace{|X_n - X| < \varepsilon}^{X-\varepsilon < X_n < X+\varepsilon}) + P(X \le c-\varepsilon, |X_n - X| \ge \varepsilon)$

$$= P(X-\varepsilon < X_n < X+\varepsilon \le c) + P(X \le c-\varepsilon, |X_n - X| \ge \varepsilon)$$
$$\le P(X_n < c) + P(|X_n - X| \ge \varepsilon)$$
$$= F_{X_n}(c) + P(|X_n - X| \ge \varepsilon)$$

$\Rightarrow$ 1) : $F_{X_n}(c) \le F_X(c+\varepsilon) + P(|X_n - X| > \varepsilon)$

2) : $P(X \le c-\varepsilon) - P(|X_n - X| \ge \varepsilon) \le F_{X_n}(c)$

1) $\cup$ 2) $\Rightarrow$ $F_X(c-\varepsilon) - P(|X_n - X| \ge \varepsilon) \le F_{X_n}(c) \le F_X(c+\varepsilon) + P(|X_n - X| > \varepsilon)$

as $n \to \infty$: lim lim lim

$$\Rightarrow \lim_{n\to\infty}\{F_X(c-\varepsilon) - \underbrace{P(|X_n - X| \ge \varepsilon)}_{\xrightarrow{P} \text{ by definition}}\} \le \lim_{n\to\infty} F_{X_n}(c) \le \lim_{n\to\infty}\{F_X(c+\varepsilon) + \underbrace{P(|X_n - X| > \varepsilon)}_{\xrightarrow{P} \text{ by definition}}\}$$

0 0

By Sandwitch Theorem $\Rightarrow$ $\lim_{n\to\infty} F_{X_n}(c) = F_X(c)$

$\therefore$ $\underline{X_n \xrightarrow{d} c}$