1st Help Session for Mathematical Finance

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How to Solve 1st Order Linear ODEs

- A typical first order linear ODE:

$$y' + a(t)y = h(t).$$

- Its general solution is $y(t) = y_h(t) + y_p(t)$.
- $y_h(t)$ is the general solution to $y' + a(t)y = 0$:

$$y_h(t) = Ck(t),$$

  where $k(t) = e^{-\int^t a(x)dx}$.
- $y_p(t) = k(t)\int^t h(x)/k(x)dx$ is a particular solution.

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- Exercise: solve $y' + 2y = t^2 + 1$ with $y(1) = 1/2$.

Quadratic Function Optimization

- A quadratic function can be written as

$$f(x) = ax^2 + bx + c,$$

  where $a \neq 0$.
- A stationary point of $f(x)$ is a solution to $f'(x) = 0$.
- The unique stationary point of $f(x)$ is $x^* = -\frac{b}{2a}$.
- First year calculus: $f(x)$ is maximized (minimized) at x^* if $a < (>)0$.

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- Exercise: maximize $f(x) = -x^2 + 4x + 5$.

Some Concepts in Probability Theory

Definition: $\sigma-$algebra

Let $\Omega \neq \emptyset$ and $\mathcal{F} \subset 2^{\Omega}$. $\mathcal{F}$ is a $\sigma-$algebra if (i) $\emptyset \in \mathcal{F}$; (ii) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$; (iii) $A_1, A_2, \ldots$, belongs to $\mathcal{F} \Rightarrow \cup_{n=1}^{\infty} A_n \in \mathcal{F}$.

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Definition: Probability Measure

A mapping $P : \mathcal{F} \to [0, 1]$ is a probability measure if (i) $P(\Omega) = 1$; (ii) $P(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n)$ where $A_i \cap A_j = \emptyset$ whenever $i \neq j$.

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Definition: Probability Space

The triple $(\Omega, \mathcal{F}, P)$ is a probability space.

Filtration and Martingale

Definition: Filtration

A filtration is an incerasing sequence of σ-algebras $(\mathcal{F}_t)_{t\geq 0}$ with $\mathcal{F}_m \subset \mathcal{F}_n \subset \mathcal{F}$ for all $m \leq n$.

Definition: Martingale

Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration. A stochastic process $(X_t, \mathcal{F}_t)_{t\geq 0}$ is a martingale if (i) $\mathbb{E}[|X_t|] < \infty, \forall t \geq 0$; (ii) X_t is $\mathcal{F}_t$-measurable $\forall t \geq 0$; (iii) For any $m \leq t$, $\mathbb{E}[X_t|\mathcal{F}_m] = X_m$ a.s.

Exercise: prove a standard Brownian motion is a martingale w.r.t. to its natural filtration.

Itô's Lemma

- A typical Itô process is

$$dX_t = \mu(t)dt + \sigma(t)dW_t.$$

- Itô's Lemma: If $f \in \mathcal{C}^2$ (i.e., f is twice continuously differentiable), then

$$df(t, X_t) = \left[\mu(t)f_x + f_t + \frac{1}{2}\sigma^2(t)f_{xx}\right]dt + \sigma(t)f_x dW_t.$$

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- Exercise: assume $dX_t = \mu(t)dt + \sigma(t)dW_t$; $g(\cdot)$ is smooth enough; $\alpha \geq 0$; find $df(t, X_t)$ if (i) $f(t,x) = g(t)\frac{x^{1-\alpha}}{1-\alpha}$; (ii) $f(t,x) = g(t)\ln(x)$.