4th Help Session for Mathematical Finance

Leifu Zhang

February 13, 2019

The Fundamental Theorem of Asset Pricing

Definition

A stochastic process $(\bar{Z}_t, t \geq 0)$ is a state price density (SPD) if (i) $\bar{Z}_0 = 1$, (ii) $\bar{Z}_t > 0, \forall t \geq 0$, and (iii) $\forall s < t, \mathbb{E}_s[\bar{Z}_t P_t] = \bar{Z}_s P_s$, where $(P_t, t \geq 0)$ is the price process of an arbitrary asset in this economy.

Theorem (FTAP)

If the market is complete and there is no arbitrage, then there exists a unique SPD $(\bar{Z}_t, t \geq 0)$.

Deriving the SPD

Proposition

In the environment of Chapter 2, the unique SPD is

$$\bar{Z}_t = e^{-(r+\theta^2/2)t-\theta W_t}.$$

Proof.

The FTAP implies there exists a unique SPD $(\bar{Z}_t, t \geq 0)$. By the martingale representation theorem, we can assume

$$\frac{d\bar{Z}_t}{\bar{Z}_t} = \mu(t)dt + \sigma(t)dW_t. \tag{1}$$

Let B_t be the price of the risk-free bond at time t and define $X_t := \bar{Z}_t B_t$. Note that $dB_t/B_t = rdt$, then by Itô's lemma, $dX_t/X_t = [\mu(t) + r]dt + \sigma(t)dW_t$. By the definition of SPD, $(X_t, t \geq 0)$ is a local martingale. Hence, $\mu(t) = -r$. Define $Y_t := \bar{Z}_t S_t$, then use the same trick (left as an exercise), we have $\sigma(t) = -(\mu - r)/\sigma := -\theta$. With the boundary condition $\bar{Z}_0 = 1$, we can solve (1), which yields $\bar{Z}_t = e^{-(r+\theta^2/2)t-\theta W_t}$ (left as another exercise). □

Deriving the Static Budget Constraint

- Itô's lemma implies

$$d\left[\int_0^t \bar{Z}_s c_s ds + Z_t X_t\right] = \bar{Z}_t(\pi_t\sigma - \theta X_t)dW_t.$$

- $\left(\int_0^t \bar{Z}_s c_s ds + \bar{Z}_t X_t, t \geq 0\right)$ is a martingale means that

$$\int_0^t \bar{Z}_s c_s ds + \bar{Z}_t X_t = \mathbb{E}_t\left[\int_0^T \bar{Z}_s c_s ds + \bar{Z}_T X_T\right].$$

- Set $t = 0$, then we get the static budget constraint

$$\mathbb{E}\left[\int_0^T \bar{Z}_s c_s ds + \bar{Z}_T X_T\right] = X_0.$$

The Envelope Theorem

Theorem (for Unconstrained Problems)

Consider the following differentiable maximization problem:

$$\max_x f(x, q),$$

where q is a parameter. Let $x^*(q)$ be a solution to this problem. We define the value function

$$f^*(q) := f(x^*(q), q).$$

We fix $q = q_0$. If $x^*(q)$ is differentiable at q_0, then

$$\left.\frac{\partial f^*}{\partial q}\right|_{q=q_0} = \left.\frac{\partial f}{\partial q}\right|_{(x,q)=(x^*(q_0),q_0)}.$$

Proof.

Left as an exercise. (Hint: Use the chain rule.) □

The Envelope Theorem

An example: $f(x, q) = q - 2(x - q)^2$.

- First year calculus: $x^*(q) = q$.
- The value function $f^*(q) = q$.
  - $\frac{\partial f^*}{\partial q} = 1$.
  - $\frac{\partial f}{\partial q}\big|_{(x,q)=(x^*(q),q)} = 1 + 4(x^*(q) - q) = 1$.

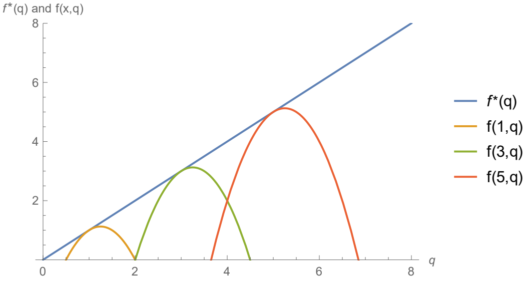
Why is it called the **Envelope** Theorem?

The Envelope Theorem

An example: $f(x,q) = q - 2(x-q)^2$.

- First year calculus: $x^*(q) = q$.
- The value function $f^*(q) = q$.
  - $\frac{\partial f^*}{\partial q} = 1$.
  - $\frac{\partial f}{\partial q}\big|_{(x,q)=(x^*(q),q)} = 1 + 4(x^*(q) - q) = 1$.

Why is it called the **Envelope** Theorem?



Lagrange Duality for Constrained Optimization

- Consider a standard optimization problem (P):

$$\max_{x \in \mathbb{R}^n} f(x)$$

such that

$$h(x) = 0.$$

- Suppose $X := \{x | h(x) = 0\} \neq \emptyset$.
- Let x^* be a solution to P and define $P^* := f(x^*)$.
- The Lagrangian to P is

$$\mathfrak{L}(x, \lambda) := f(x) + \lambda h(x),$$

and we call λ the Lagrange multiplier.

Lagrange Duality for Constrained Optimization

- Then we can define the Lagrangian dual function

$$g(\lambda) := \sup_x \mathfrak{L}(x, \lambda).$$

- I claim $g(\lambda) \geq P^*$, i.e., $g(\lambda)$ is a upper bound on optimal value.
  - To see this, consider an arbitrary $x' \in X$. Then

$$\mathfrak{L}(x', \lambda) = f(x') + \lambda h(x') = f(x').$$

  - Hence, $\forall x' \in X$,

$$g(\lambda) = \sup_x \mathfrak{L}(x, \lambda) \geq \mathfrak{L}(x', \lambda) = f(x').$$

  - Then we have $g(\lambda) \geq P^*$.

Lagrange Duality for Constrained Optimization

- The Lagrange dual problem (D) to P is

$$\min_{\lambda} g(\lambda).$$

- D gives us the best upper bound.
- Let $D^* := \min_\lambda g(\lambda)$.
- By the upper bound property in the previous slide, we have

$$D^* = \min_{\lambda} g(\lambda) \geq P^*,$$

which is the so-called weak duality.

Lagrange Duality for Constrained Optimization

- If

$$D^* = P^*,$$

  we say the strong duality holds.
- Then we can solve primal by solving dual.
- The strong duality does not always hold...
- But in this class we are fine...
- Ref: Luenberger, D.G., 1997. *Optimization by vector space methods*. John Wiley & Sons.
- You will learn optimization theory as well as functional analysis...

Finding the Optimal Portfolio by the Dual Approach

- The investor's problem at time t:

$$\max_{(c_s,\pi_s)_{s\geq t}} \mathbb{E}_t\left[\int_t^T e^{-\beta(s-t)}u_1(c_s)ds + e^{-\beta(T-t)}u_2(X_T)\right]$$

  such that

$$\mathbb{E}_t\left[\int_t^T \bar{Z}_s c_s ds + \bar{Z}_T X_T\right] = \bar{Z}_t x_t. \tag{2}$$

- A fact (I will use it without proof) is that instead of choosing $(c_s,\pi_s)_{s\geq t}$, the investor can directly choose $(c_s)_{s\geq t}$ and X_T subject to (2) to maximize the objective function.

Finding the Optimal Portfolio by the Dual Approach

- Set up the Lagrangian

$$\begin{aligned}\mathfrak{L}(c_s, X_T, \lambda) = \mathbb{E}_t &\left[\int_t^T e^{-\beta(s-t)} u_1(c_s) ds + e^{-\beta(T-t)} u_2(X_T)\right] \\ &+ \lambda \left\{ \bar{Z}_t x_t - \mathbb{E}_t \left[\int_t^T \bar{Z}_s c_s ds + \bar{Z}_T X_T\right]\right\}.\end{aligned}$$

- The first-order conditions (FOCs) are

$$\begin{aligned}\frac{\partial \mathfrak{L}}{\partial c_s} = 0 \Rightarrow u_1'(\hat{c}_s) = \lambda e^{\beta(s-t)} \bar{Z}_s, \\ \frac{\partial \mathfrak{L}}{\partial X_T} = 0 \Rightarrow u_2'(\hat{X}_T) = \lambda e^{\beta(T-t)} \bar{Z}_T.\end{aligned}$$

- Then

$$\begin{aligned}\hat{c}_s = (u_1')^{-1}(\lambda e^{\beta(s-t)} \bar{Z}_s), \\ \hat{X}_T = (u_2')^{-1}(\lambda e^{\beta(T-t)} \bar{Z}_T).\end{aligned}$$

- Then we can use (2) to solve $\lambda, (\hat{c}_s)_{s \geq t}$ and $\hat{X}_T$. Done?! But...

Finding the Optimal Portfolio by the Dual Approach

- The Lagrange dual function is $\max_{c_s, X_T} \mathfrak{L}(c_s, X_T, \lambda) = \mathfrak{L}(\hat{c}_s, \hat{X}_T, \lambda)$.
- The Lagrange dual problem is

$$\min_{\lambda} \mathfrak{L}(\hat{c}_s, \hat{X}_T, \lambda).$$

- If the strong duality holds, then

$$\min_{\lambda} \mathfrak{L}(\hat{c}_s, \hat{X}_T, \lambda) = V(t, x).$$

- Note that

$$\begin{aligned}\mathfrak{L}(\hat{c}_s, \hat{X}_T, \lambda) = \mathbb{E}_t \Big\{ \int_t^T & e^{-\beta(s-t)}[u_1(\hat{c}_s) - \lambda e^{\beta(s-t)} \bar{Z}_s \hat{c}_s] ds \\ & + e^{-\beta(T-t)}[u_2(\hat{X}_T) - \lambda e^{\beta(T-t)} \bar{Z}_T \hat{X}_T] \Big\} + \lambda \bar{Z}_t X_t.\end{aligned}$$

Finding the Optimal Portfolio by the Dual Approach

- We define

$$\hat{Z}_s := \lambda e^{\beta(s-t)} \bar{Z}_s,$$
$$\tilde{u}_1(z) := \max_c \{u_1(c) - zc\},$$
$$\tilde{u}_2(z) := \max_x \{u_2(c) - zx\}.$$

- Then

$$\begin{aligned}\mathfrak{L}(\hat{c}_s, \hat{X}_T, \lambda) &= \mathbb{E}_t \left[\int_t^T e^{-\beta(s-t)} \tilde{u}_1(\hat{Z}_s) ds + e^{-\beta(T-t)} \tilde{u}_2(\hat{Z}_T) \right] + \lambda \bar{Z}_t X_t \\ &= \tilde{V}(t, z) + \lambda \bar{Z}_t X_t,\end{aligned}$$

where

$$\tilde{V}(t, z) := \mathbb{E}_t \left[\int_t^T e^{-\beta(s-t)} \tilde{u}_1(\hat{Z}_s) ds + e^{-\beta(T-t)} \tilde{u}_2(\hat{Z}_T) \middle| \hat{Z}_t = z \right].$$

Finding the Optimal Portfolio by the Dual Approach

- Again $e^{-\beta t}\tilde{V}(t,z)+\int_0^t e^{-\beta s}\tilde{u}_1(\hat{Z}_s)ds$ is a martingale.
- Itô's lemma implies $d\hat{Z}_t = -(r-\beta)\hat{Z}_t dt - \theta \hat{Z}_t dW_t$.
- Then (see HW4)

$$\begin{aligned} d\left[e^{-\beta t}\tilde{V}(t,z)+\int_0^t e^{-\beta s}\tilde{u}_1(\hat{Z}_s)ds\right] =& e^{-\beta t}[\tilde{V}_t - \beta\tilde{V} + \tilde{u}_1(z) + \theta^2 z^2 \tilde{V}_{zz}/2 \\ & - (r-\beta)z\tilde{V}_z]dt - e^{-\beta t}\theta z \tilde{V}_z dW_t. \end{aligned}$$

- The dual HJB equation is

$$\tilde{V}_t + \frac{1}{2}\theta^2 z^2 \tilde{V}_{zz} - (r-\beta)z\tilde{V}_z - \beta\tilde{V} + \tilde{u}_1(z) = 0,$$

with the terminal condition $\tilde{V}(T,z) = \tilde{u}_2(z)$.

Finding the Optimal Portfolio by the Dual Approach

- Recall the strong duality, then

$$\begin{aligned} V(t,x) &= \min_{\lambda} \mathfrak{L}(\hat{c}_s, \hat{X}_T, \lambda) \\ &= \min_{\lambda} [\tilde{V}(t,z) + \lambda \bar{Z}_t X_t] \\ &= \min_{z} [\tilde{V}(t,z) + zx]. \end{aligned}$$

- The FOC of $\min_z[\tilde{V}(t,z) + zx]$ is $\tilde{V}_z + x = 0 \Rightarrow$

$$\hat{X}_t = -\tilde{V}_z(t, \hat{Z}_t).$$

- By Itô's lemma,

$$d\hat{X}_t = \left[(r-\beta)\hat{Z}_t \tilde{V}_{zz} - \tilde{V}_{zt} - \frac{1}{2}\theta^2 \hat{Z}_t^2 \tilde{V}_{zzz} \right] dt + \theta \hat{Z}_t \tilde{V}_{zz} dW_t.$$

Finding the Optimal Portfolio by the Dual Approach

- Recall the dynamic budget constraint

$$dX_t = [rX_t + \pi_t(\mu - r) - c_t]dt + \pi_t \sigma dW_t.$$

- Compare the diffusion terms, then we have

$$\hat{\pi}_t = \frac{\theta \hat{Z}_t \tilde{V}_{zz}}{\sigma}.$$