2nd Help Session for Mathematical Finance

Leifu Zhang

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Some Results in the Theory of Expected Utility

Definition: Affine Transformation

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Definition: Positive Affine Transformation

If $a > 0$, we say $w(u)$ is a positive affine transformation of u.

Some Results in the Theory of Expected Utility

Proposition: The Uniqueness of Expected Utilities

If preference $\succeq$ is represented by $U(c) = \mathbb{E}[u(c)]$, then $W(c) = \mathbb{E}[w(c)]$ represents the same preference if $w(u)$ is a positive affine transformation of u.

Proof.

Show that for any two consumption plans c_1 and c_2,
$U(c_1) \geq U(c_2) \Leftrightarrow W(c_1) \geq W(c_2)$. □

Some Results in the Theory of Expected Utility

- The logarithmic utility is just the power utility with $\gamma = 0$.
  - $w(c) = \frac{c^\gamma - 1}{\gamma}$ represents the same preference as $u(c) = \frac{c^\gamma}{\gamma}$.
  - L'Hôspital's rule implies

$$\lim_{\gamma \to 0} \frac{c^\gamma - 1}{\gamma} = \lim_{\gamma \to 0} \frac{c^\gamma \ln(c)}{1} = \ln(c).$$

Some Results in the Theory of Expected Utility

Exercise

For a risk averse investor, $CE(c) \leq \mathbb{E}[c]$.

Proof.

By the definition of certainty equivalent, $u(CE(c)) = \mathbb{E}[u(c)]$. Since the investor is risk averse, we have $\mathbb{E}[u(c)] \leq u(\mathbb{E}[c])$. Hence, $u(CE(c)) = \mathbb{E}[u(c)] \leq u(\mathbb{E}[c]) \Rightarrow CE(c) \leq \mathbb{E}[c]$ since u is increasing. $\square$

The Dynamic Budget Constraint

- Let X_t be the wealth level.
- Two assets: a risk-free asset with return rate r and a risky asset whose price S_t follows
$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t.$$
- π_t is the money amount invested in the risky asset.
- The dynamic budget constraint is
$$dX_t = rX_t dt + \pi_t(\mu - r)dt + \pi_t \sigma dW_t - c_t dt.$$

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$$dX_t = rX_t dt + \pi_t(\mu - r)dt + \pi_t \sigma dW_t - c_t dt.$$
  - Consider a short time period dt, then
$$dX_t = \frac{\pi_t}{S_t} dS_t + (X_t - \pi_t) r dt - c_t dt.$$
  - We plug the price process into the above equation.

Conditional Expectation and Martingale

Theorem: The Law of Iterated Expectations

If $\mathcal{G} \subset \mathcal{F}$ are $\sigma-$algebras, and Y is a random variable adapted to them, then

$$\mathbb{E}[\mathbb{E}[Y|\mathcal{F}]|\mathcal{G}] = \mathbb{E}[Y|\mathcal{G}].$$

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Proposition

Let Y be a r.v. and $(\mathcal{F}_t, t \geq 0)$ be a filtration on $(\Omega, \mathcal{F}, P)$. Let $Z_t := \mathbb{E}[Y|\mathcal{F}_t]$, then $(Z_t, t \geq 0)$ is a martingale w.r.t. $(\mathcal{F}_t, t \geq 0)$.

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Proof.

By the definition of conditional expectation, we have $\sigma(Z_t) \subset \mathcal{F}_t$, i.e., Z_t is adapted to $\mathcal{F}_t$.
Since $(\mathcal{F}_t : t \geq 0)$ is a filtration, for all $0 \leq s < t$, we have $\mathcal{F}_s \subset \mathcal{F}_t$.
Also, $\mathbb{E}[Z_t|\mathcal{F}_s] = \mathbb{E}[\mathbb{E}[Y|\mathcal{F}_t]|\mathcal{F}_s] = \mathbb{E}[Y|\mathcal{F}_s] = Z_s$. The first and third equalities are directly from the definition of Z_t. The second equality holds because of the law of iterated expectations. $\square$

Value Function and Martingale

Question

Is $J(t, X_t; c, \pi) = \mathbb{E}_t \left[\int_t^T e^{-\beta(s-t)} u_1(c_s) ds + e^{-\beta(T-t)} u_2(X_T) \right]$ a martingale?

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Answer

No. But why?

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Answer

No. But why?

Quesion

Is $e^{-\beta t} J(t, X_t; c, \pi) + \int_0^t e^{-\beta s} u_1(c_s) ds = \mathbb{E}_t \left[\int_0^T e^{-\beta s} u_1(c_s) ds + e^{-\beta T} u_2(X_T) \right]$ a martingale?

Value Function and Martingale

Answer

Yes. Let $\mathcal{F}_t$ be the information set ($\sigma-$algebra) at time t, then $\left(Z_t := \mathbb{E}_t\left[\int_0^T e^{-\beta s}u_1(c_s)ds + e^{-\beta T}u_2(X_T)\right], t \geq 0\right)$ is a stochastic process adapted to the filtration $(\mathcal{F}_t : t \geq 0)$. Then apply the proposition in the previous slide.

You can also verify that by the law of iterated expectations, $\forall m < t$,

$$\begin{aligned}
\mathbb{E}_m[Z_t] &= \mathbb{E}\left[\mathbb{E}\left[\int_0^T e^{-\beta s}u_1(c_s)ds + e^{-\beta T}u_2(X_T)\middle|\mathcal{F}_t\right]\middle|\mathcal{F}_m\right] \\
&= \mathbb{E}\left[\int_0^T e^{-\beta s}u_1(c_s)ds + e^{-\beta T}u_2(X_T)\middle|\mathcal{F}_m\right] \\
&= Z_m.
\end{aligned}$$

Deriving the HJB Equation

- Recall Itô's Lemma: If $f \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R})$, then

$$df(t, X_t) = \left[\mu(t) f_x + f_t + \frac{1}{2}\sigma^2(t) f_{xx}\right] dt + \sigma(t) f_x dW_t.$$

- Recall the dynamic constraint:

$$dX_t = \underbrace{[rX_t + \pi_t(\mu - r) - c_t]}_{:=\mu(t)} dt + \underbrace{\pi_t \sigma}_{:=\sigma(t)} dW_t.$$

- Let $f(t, X_t) := e^{-\beta t} J(t, X_t; c, \pi) + \int_0^t e^{-\beta s} u_1(c_s) ds$.

Deriving the HJB Equation

- First year calculus:

$$
\begin{aligned}
f_x &= e^{-\beta t} J_x, \\
f_t &= -\beta e^{-\beta t} J + e^{-\beta t} J_t + e^{-\beta t} u_1(c_t), \\
f_{xx} &= e^{-\beta t} J_{xx}.
\end{aligned}
$$

- By Itô's Lemma,

$$
\begin{aligned}
df(t, X_t) &= \left[\mu(t) f_x + f_t + \frac{1}{2}\sigma^2(t) f_{xx}\right] dt + \sigma(t) f_x dW_t \\
&= e^{-\beta t}\Bigg[J_t - \beta J + \frac{1}{2}\sigma^2 \pi_t^2 J_{xx} + rxJ_x \\
&\quad + \pi_t(\mu - r) J_x - c_t J_x + u_1(c_t)\Bigg] dt + e^{-\beta t} \pi_t \sigma J_x dW_t.
\end{aligned}
$$

- We have proved that $f(t, X_t)$ is a martingale, hence

$$
J_t - \beta J + \frac{1}{2}\sigma^2 \pi^2 J_{xx} + rxJ_x + \pi(\mu - r) J_x - cJ_x + u_1(c) = 0.
$$

Deriving the HJB Equation

- Let $V(t,x) := \max_{c,\pi} J(t,x;c,\pi)$. Then

$$\begin{aligned}\beta V &= \beta \max_{c,\pi} J \\ &= \max_{c,\pi} \left[V_t + \frac{1}{2}\sigma^2\pi^2 V_{xx} + rxV_x + \pi(\mu - r)V_x - cV_x + u_1(c) \right].\end{aligned}$$

- Then we have the so called Hamilton-Jacobi-Bellman (HJB) equation:

$$\beta V = \max_{c,\pi} \left\{ V_t + [rx + \pi(\mu - r) - c]V_x + \frac{1}{2}\sigma^2\pi^2 V_{xx} + u_1(c) \right\}.$$

- $J(T, X_T; c, \pi) = \mathbb{E}_T \left[\int_T^T e^{-\beta(s-t)} u_1(c_s) ds + e^{-\beta(T-T)} u_2(X_T) \right] = u_2(X_T) \Rightarrow$ $V(T,x) = u_2(x)$, which is the terminal condition.

Pricing a Derivative

- Assume the underlying asset (stock) price follows

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t.$$

- Assume the risk-free rate is a constant r, then the bond price follows

$$dB_t = rB_t dt.$$

- Let $V(t, S_t)$ be the price of a contingent claim on the stock maturing at T with payoff $P(T, S)$ (i.e. it is a derivative).

Pricing a Derivative

- Since the market is complete, we can use this derivative and the stock to mimic the risk-free bond.
- Suppose at $t \in [0, T]$ we hold a unit of this contingent claim and $-\Delta_t$ units of the stock.
- The value of this portfolio is $\Xi(t, S_t) = V(t, S_t) - \Delta_t S_t$.
- By Itô's Lemma,

$$d\Xi(t, S_t) = \left[\mu S_t(V_S - \Delta_t) + V_t + \frac{1}{2} S_t^2 \sigma^2 V_{SS}\right] dt + \sigma S_t (V_S - \Delta_t) dW_t.$$

- Since we want to use this portfolio to mimic the risk-free bond, the diffusion term should be zero and then

$$\Delta_t = V_S.$$

- Then $d\Xi(t, S_t) = \left[V_t + \frac{1}{2} S_t^2 \sigma^2 V_{SS}\right] dt$.

Pricing a Derivative

- If there is no arbitrage, our portfolio should be identical to the risk-free bond:

$$d\Xi(t, S_t) = r\Xi(t, S_t)dt \Rightarrow r(V(t, S_t) - V_S S_t) = V_t + \frac{1}{2}S_t^2\sigma^2 V_{SS}.$$

- Rearrange the above equation, we get the fundamental partial differential equation (PDE):

$$-rV + V_t + rSV_S + \frac{1}{2}\sigma^2 S^2 V_{SS} = 0,$$

- with the terminal (boundary) condition

$$V(T, S) = P(T, S).$$