ALY-6010 Probability & Statistics
Northeastern University
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Part 1: Introduction Data Analysis, and Statistics

An Overview of the Course Coverage:

1. Descriptive Statistics – Theory, Applications, Microsoft Excel / R Programming (optional)
2. Concepts in Probability Theory – Theory, Applications
3. Discrete Probability Distributions – Theory, Applications, Microsoft Excel / R Programming (optional)
4. Continuous Probability Distributions – Theory, Applications, Microsoft Excel / R Programming (optional)
5. Simulation of Probability Distribution – Theory, Applications, Microsoft Excel / R Programming (optional)
6. Inferential Statistics – Confidence Intervals – Theory, Applications, Microsoft Excel / R Programming (optional)
7. Inferential Statistics – Hypothesis Testing– Theory[1]
8. Inferential Statistics – Hypothesis Testing – Applications, Microsoft Excel / R Programming (optional)
9. Simple & Multiple Regression Analysis – Theory, Applications, Microsoft Excel / R Programming (optional)
10. Topics in Statistical Data Analysis – Theory[2]
11. Topics in Statistical Data Analysis – Applications

(i) Meanings of the word "*Statistics*":

There are two parallel meanings of statistics:

1. It is the science of data; that is, collecting, organizing, and analyzing data
2. It is the plural of the word "*Statistic*"

(ii) Related Terminologies:

- ***Parameter***: A characteristic of a population (often, a numerical characteristic such as a population mean, a population variance, a population standard deviation, . . . , etc.)
- ***Statistic***: A characteristic of a sample (such as a sample mean or a standard deviation, . . . , etc.)
- ***Data***: A collection of information (literally, data is the plural of datum; meaning: what is given)

[1] . Inferential Statistics will consist of the *z* test, the *t* test, the *Chi-squared* test, and the *F* test

[2] . Topics will be selected from *Tests of Normality*, *Chi-squared Goodness of Fit test, Chi-squared Test of Normality, and ANOVA.*

(iii) Data Types:

1. *Categorical* (Qualitative)
   (a) *Nominal*: According to Name / Examples: Data containing names, genders, races, …
   (b) *Ordinal*: According to Order / Example: Data containing ranks, data that has been organized according to the alphabetic order, …, etc.

2. *Numerical* (Quantitative)
   (a) According to the *ratio scale* (a possible value of zero in the data is an inherent zero) / Examples: Data containing heights, weights, time durations, grades, …, etc.
   (b) According to the *interval scale* (a zero is not inherently zero) / Example: Data containing temperatures.

   Statistically, a numerical set of data may be *discrete* or *continuous*.
   (a) A ***discrete data*** is one in which the measurements take a countable set of isolated values. For example, the number of chairs, the number of patients, the number of accidents, …, are all examples of discrete data
   (b) A continuous set of data is one in which the measurements can take any real value within a certain range. For example, the amount of rainfall in Charlotte in January during the last 30 years or the amount of customer waiting times at a local bank are examples of continuous data sets.

(iv) Types of Statistics

There are two main types of statistics: *Descriptive* and *Inferential*

1. ***Descriptive statistics***: is to describe a set of data; *graphically* or *numerically*

(a) Graphical Descriptive Statistics
Describing a set of data graphically by creating *bar graphs*, *pie charts*, *histograms*, *line plots*, *scatter plots*, . . ., etc.

(b) Numerical Descriptive Statistics
There are a number of particular characteristics of data that are often the focus of interest to the data analyst. These are:

1. Measures describing the center of data
   Examples of such measures are: ***mean*** (arithmetic average), ***median***, ***mode***, and the ***weighted mean***
2. Measures describing the variability (spread or dispersion) of data
   Examples of these types of measures are: the ***range***, the ***variance***, and the ***standard deviation*** of data
3. Measures of location
   Examples of such measures are the ***percentile ranking*** and the ***z-score***. These measures describe where a particular measurement stands compared to the rest of the data.

4. Measures describing the shape of the distribution of data
   Skewness and ***Kurtosis*** are two measures that describe the shape of the distribution of a data set

2. ***Inferential statistics***: is to utilize one or more random samples in order to gain insight about the population from where those samples were randomly selected. Often a data analyst may be interested in obtaining information about one or more particular parameters of a given population. However since the entire population is not accessible in the majority of situations, the data analyst must select one or more samples from the population of interest and perform statistical analysis on these samples. Once sample characteristics have been verified or revealed, the analyst will then use the methods of inferential statistics to transform the sample information into population information.

   There are three main methods of inferential statistics:

   (a) Constructing ***Confidence Intervals***: This is to estimate a population parameter to within two limits: a lower limit and an upper limit
   (b) Performing ***Hypothesis Testing***: This is to verify or to reject hypotheses or claims
   (c) ***Modeling or Testing Relationships*** between Data sets

Loading the Data Analysis ToolPak of Microsoft Excel[3]

One of the tools of technology that will be used in this program is Microsoft Excel and its data analytic capabilities. To activate the Data Analysis Tool-Pak of Excel, follow the instructions below:

1. Click the **File** tab, and then click **Options**.
2. Click **Add-Ins**, and then in the **Manage** box, select **Excel Add-ins**.
3. Click **Go**.
4. In the **Add-Ins available** box, select the "**Analysis ToolPak"** and the "**Excel** Solver" check boxes, and then click **OK**.
   (a) If any of the **Analysis ToolPak** or **Solver** is not listed in the **Add-Ins available** box, click **Browse** to locate it.
   (b) If you get prompted that any of the above two add-ins is not currently installed on your computer, click **Yes** to install it.

[3] . The instructions given in here for loading the Data Analysis ToolPak apply to the 2013 and the 2016 version of Excel for Windows. The loading procedure for older versions of Excel for Windows is very similar to the procedure described in here. For Mac users, the data Analysis ToolPak is included in the Excel 2016 for Mac version, and can be loaded by: 1. Click **Tools,** and then Click **Add-ins**; 2. Click the **Data Analysis ToolPak** and the **Solver** options to enable them. Then click **OK**; 3. The Data Analysis ToolPak and the Solver can now be located on the **Data** tab. Note that older versions of Excel for Mac do not come with the Analysis ToolPak.

5. After you load the Analysis ToolPak and the Solver, the**ir** commands will become available in the **Analysis** group on the **Data** tab.

Part 2: Descriptive Statistics

Related Excel Formulas:

mean: **=AVERAGE(*data range*)**

median: **=MEDIAN(*data range*)**

mode: **=MODE(*data range*)**

largest measurement: **=MAX(*data range*)**

smallest measurement: **=MIN(*data range*)**

range: **=MAX(*data range*) – MIN(*data range*)**

number of measurements: **=COUNT(*data range*)**

sample variance: **=VAR.S(*data range*)**

population variance: **=VAR.P(*data range*)**

sample standard deviation: **=STDEV.S(*data range*)**

population standard deviation: **=STDEV.P(*data range*)**

skewness: **=SKEW(*data range*)**

quartile 1: **=QUARTILE(*data range , 1*)**

quartile 2: **=QUARTILE(*data range , 2*)** (Note: *quartile 2 = median*)

quartile 3: **=QUARTILE(*data range , 3*)**

(i) **Bar Graphs and Pie Charts**

Bar graphs, pie charts, and Pareto charts are used to display a categorical data.

(ii) **Frequency Histograms, Relative Frequency Histograms, Cumulative Frequency Line Plots, Relative Cumulative Frequency Line Plots, Box & Whisker Plots**

The above chart types are used to display a numerical (quantitative) data set.

(iii) Numerical Descriptive Statistics

- **(a)** Measures describing the center of data / Examples of such measures are: ***mean*** (arithmetic average), ***median***, ***mode***, the ***mean of a distribution***, and the ***weighted mean***
- **(b)** Measures describing the variability (spread or dispersion) of data / Examples of these measures are: the ***range***, the ***variance***, and the ***standard deviation*** of data
- **(c)** Measures of location / Examples of such measures are the ***percentile ranking*** and the ***z-score***. These measures describe where a particular measurement stands compared to the rest of the data.
- **(d)** Measures describing the shape of the distribution of data / ***Skewness*** and ***Kurtosis*** are two measures that describe the shape of the distribution of a data set

In the sequel, the following notations will be used. The mathematical formula for calculating each of the quantities is written next to the characteristic:

$\boldsymbol{x}$ **:** a measurement**,**

$\boldsymbol{n}$ **:** sample size

N **:** population size

w **: measurement** weight

m**:** class midpoint

$\boldsymbol{f}$**:** Class frequency

$\sum$ **:** The summation Notation

$\bar{x}$ **(x-bar)**: sample mean $\quad \boldsymbol{\bar{x}} = \frac{\sum \boldsymbol{x}}{\boldsymbol{n}}$

$\boldsymbol{\mu}$ **(mu)**: population mean $\quad \boldsymbol{\mu} = \frac{\sum \boldsymbol{x}}{\boldsymbol{N}}$

$\bar{x}_w$: weighted mean $\quad \boldsymbol{\bar{x}} = \frac{\sum \boldsymbol{w}\,\boldsymbol{x}}{\sum \boldsymbol{w}}$

$\bar{x}_f$: mean of a distribution $\quad \boldsymbol{\bar{x}_f} = \frac{\sum \boldsymbol{f}\,\boldsymbol{m}}{\sum \boldsymbol{f}}$

s^2 : sample variance $\quad \boldsymbol{s^2} = \frac{\sum(\boldsymbol{x}-\bar{\boldsymbol{x}})^2}{\boldsymbol{n}-\mathbf{1}} = \frac{\sum \boldsymbol{x}^2 - \frac{(\sum \boldsymbol{x})^2}{\boldsymbol{n}}}{\boldsymbol{n}-\mathbf{1}}$

s : sample standard deviation $\quad s = \sqrt{s^2}$

σ^2 **(sigma-squared)**: population variance $\quad s^2 = \frac{\sum(x-\bar{x})^2}{n-1} = \frac{\sum x^2 - \frac{(\sum x)^2}{n}}{n-1}$

σ **(sigma)**: population standard deviation $\quad \sigma = \sqrt{\sigma^2}$

b : sample skewness $\quad b = \frac{\frac{1}{n}\sum(x-\bar{x})^3}{\left(\frac{1}{n-1}\sum(x-\bar{x})^2\right)^{3/2}} = \frac{\frac{1}{n}\sum(x-\bar{x})^3}{s^{3/2}}$

Part 3 : Concepts of Probability & Counting Rules

Keywords: Probability, Experiment, Observation, Outcome (Sample Point), Sample Space, Event, Union, Intersection, Mutually Exclusive, Contingency Tables, Addition Rule, Multiplication Rule, Independent Events, the Total Law of Probability, Bayes' Theorem.

We begin the study of probability with a straightforward example. Suppose a coin is tossed and the up face is recorded. The result is called an ***observation***, and the process of making an observation is called an ***experiment***. The two possible outcomes of this experiment are:

Observe a tail *(T)*, Observe a head *(H)*.

Each one of the above possible outcomes is called an ***outcome***, or a ***simple event***, or a ***sample point***. A sample point is the most basic outcome of the experiment. The ***sample space*** of an experiment is the collection of all its sample points. In our example, the sample space, denoted by ***S***, is: $S = \{T, H\}$.

Example1

A coin is tossed twice. Write the sample space of this experiment.

Solution

Even for a seemingly trivial experiment, we must be careful when listing the sample points. There are four possible outcomes, and the sample space is the collection of all above sample points:

Sample space ***S*** = { *TT, TH, HT, HH* }.

We are now ready to discuss probabilities of sample points. We have some intuitive idea about the meaning of probability. Probability is generally used synonymously with "chance", "odds", and similar concepts. For example when a fair coin is tossed, we might reason that both sample points, observing a head and observing a tail, have the same chance of occurring. Thus we might state that "the chance of observing a head is 50%" or "the odds of seeing a head is 50:50". Both these statements are based on informal knowledge of probability.

The ***probability*** of a sample point is a **number between 0 and 1 (inclusive)** that measures the likelihood that the outcome will occur when the experiment is performed. This number is usually taken to be the relative frequency of the occurrence of a sample point when the experiment is repeated a very large number of times.

$$\textbf{Probability of an Outcome} = \frac{\textbf{The number of times the outcome is observed}}{\textbf{The number of times the experiment is repeated}} \qquad \textbf{(1)}$$

For example, if we are assigning probabilities to the sample points in the coin tossing experiment (observing a head and observing a tail), we might reason that if we toss a coin a very large number of times, the outcomes of observing a head and observing a tail will have the same relative frequency or probability of 0.5.

The probabilities assigned must always obey two rules:

Probability Rules for basic outcomes (sample points):

1. All sample point probabilities must lie between 0 and 1.
2. The probabilities of all the sample points within a sample space must sum to 1.

When a coin is tossed twice, the sample space ***S*** consists of four possible outcomes: ***S*** = *{TT, TH, HT, HH}.* Assuming a fair coin, the sample points are equally likely to happen, i.e., the probability of each sample point is $\frac{1}{4}$ and the probabilities of all sample points sum to 1: $\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}=1$.

An ***event*** is a specific collection of sample points. The probability of an event ***A*** is denoted by ***P(A)***.

Example 3

Consider the experiment of tossing a coin twice. Suppose the coin is not balanced and the probabilities of sample points are given in the table below

Outcome	Probability
HH	0.16
HT	0.24
TH	0.24
TT	0.36

Consider the events

A: {Observe exactly one tail}

B: {Observe at least one tail}

Calculate the probability of ***A***, and the probability of ***B***.

Solution

Event ***A*** contains the sample points *HT* and *TH*. We can calculate the probability of event *A* by summing the probabilities of its two sample points:

$$P(A) = P(HT) + P(TH) = 0.24 + 0.24 = 0.48$$

Similarly, since ***B*** contains the sample points *HT, TH,* and *TT,* and

$$P(B) = P(HT) + P(TH) + P(TT) = 0.24 + 0.24 + 0.36 = 0.84.$$

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The above example leads us to a general procedure for finding the probability of an event:

The probability of an event A is calculated by summing the probabilities of sample points (outcomes) in A.

The following is a summary of steps for calculating the probability of an event:

1. ***Define the experiment.***
2. ***List the sample points in the sample space.***
3. ***Assign probabilities to sample points.***
4. ***Determine the collection of sample points contained in the event of interest.***
5. ***Sum the sample point probabilities to get the event probability.***

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Example 4

The following table describes the income of the adult population of a small suburb of a southern city:

	INCOME		
	< $25,000	*$25,000-$50,000*	*> $50,000*
AGE			
< 25	*200*	*400*	*300*
25-45	*300*	*100*	*100*
> 45	*100*	*500*	*600*

Consider the following events:

A*:* The person is between 25 and 45.

B*:* The person has income less than $25,000.

Find ***P(A)*** and ***P(B)****.*

Solution

The suburb has a total adult population of **2600**. There are **500** adults who are between 25 and 45. Therefore ***P(A)* = 500/2600 = 0.192**. Similarly, there are **600** adults whose income is less than $25000. Therefore, ***P(B)* = 600/2600 = 0.231**.

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Exercises

1. Refer to example II.5 and consider the following events:

   C*:* The person is 25 or older.

   D*:* The person has an income of $50,000 or less.

   Find ***P(C)****,* and ***P (D)****.*

2. A box contains 3 blue marbles and two red marbles. If two marbles are

drawn at random and without replacement, find the probability that (a) at least one of the marbles is blue, (b) at least one of the marbles is red, (c) two blue marbles are drawn, (d) two red marbles are drawn.

3. The genes obtained from each parent determine an individual's genetic

   makeup. For every genetic trait, each parent possesses a gene pair; and each contributes half of this gene pair, with equal probability, to their offspring, forming a new gene pair. The offspring's traits (eye color, baldness, etc.) come from the new gene pair, where each gene in this pair possesses some characteristic. For the gene pair that determines eye color, each gene trait may be one of the two types: dominant brown (*B*), or recessive blue (*b*). A person possessing the gene pair *BB* or *Bb* has brown eyes, whereas the gene pair *bb* produces blue eyes.

(a) Suppose both parents of an individual are brown-eyed each with a gene pair of the type *Bb*. What is the probability that a randomly selected child of this couple has blue eyes?
(b) If one parent has brown eyes of type *Bb* and the other parent has blue eyes, what is the probability that a randomly selected child of this couple has blue eyes?

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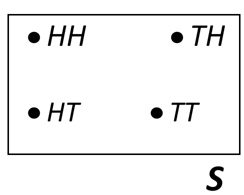
Complements, Unions, Intersections, and Basic Rules

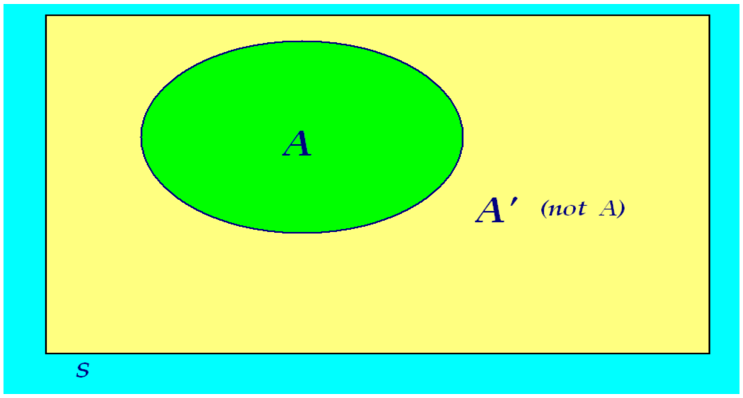
A pictorial method for presenting the sample space and events can often be helpful. The figure below shows such a representation for the experiment of tossing a coin twice:



The sample space ***S*** is shown as a closed figure, labeled ***S***, and containing all possible sample points. Such graphical representation is called a ***Venn diagram***. An event ***A*** belonging to a sample space ***S*** is shown as a round closed figure inside ***S***.

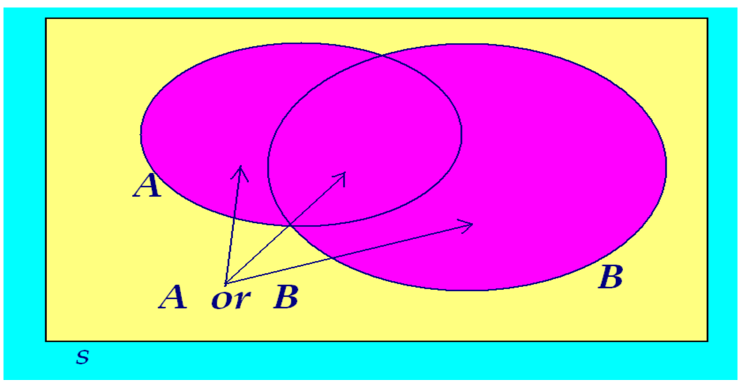
The ***complement*** of an event ***A***, denoted by $\mathbf{A}^c$ or ***(not A)***, or $\mathbf{A}'$, is the event that occurs if ***A*** does not occur; that is, the event consisting of all sample points that are not in ***A***.

The figure below is the Venn diagram indicating an event ***A*** and its complement ***A'***.



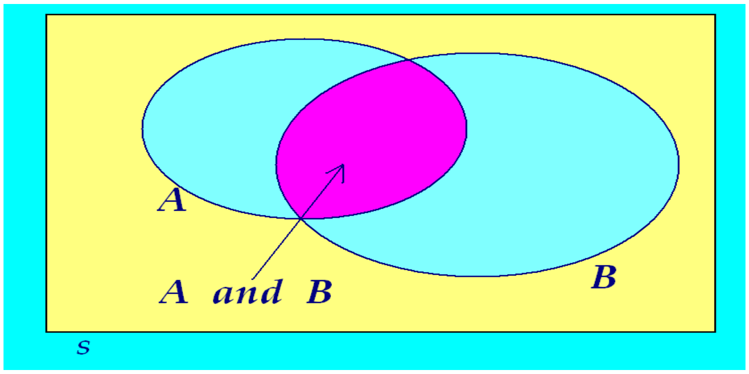
A Venn diagram showing the sample space *S*, an event *A,* and its complement *A′*

The ***union*** of two events ***A*** and ***B***, is the event that occurs if either ***A*** or ***B*** or both occur on a single performance of the experiment. The union of two events is usually denoted by $(A \cup B)$ or by (**A or B**), and consists of sample points that belong to ***A*** or to ***B*** or to both.



The event $(A \cup B)$ is indicated by the shaded region

The ***intersection*** of two events ***A*** and ***B***, denoted by $(A \cap B)$ or ***(A and B)***, is the event that occurs if both ***A*** and ***B*** occur on a single performance of the experiment. ***(A and B)*** consists of all sample points belonging to both ***A*** and ***B.***



The event $(A \cap B)$ is indicated by the shaded region

Example 5

Consider the die-toss experiment. Define the following events:

***A**:* Toss an odd number

***B**:* Toss a number less than or equal to 3.

Describe $(A \cup B)$, $(A \cap B)$, A', and B', and find the probability of each one.

Solution

S*={1, 2, 3, 4, 5, 6}**, ***A*={1, 3, 5}**, and ***B*={1, 2, 3}**. The union of ***A and ***B*** is the event that occurs if we observe either an odd number or a number less than or equal to **3** or both on a single throw of the die. We find

$$(A \cup B) = \{1,2,3,5\}$$

$$\Rightarrow$$

$$P(A \cup B) = P(1) + P(2) + P(3) + P(5)$$

$$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{2}{3}.$$

The intersection of ***A*** and ***B*** is the event that occurs if we observe both an odd number and a number less than or equal to **3** on a single throw of the die:

$$(A \cap B) = \{1,3\} \quad \Rightarrow$$

$$P(A \cap B) = P(1) + P(3) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}.$$

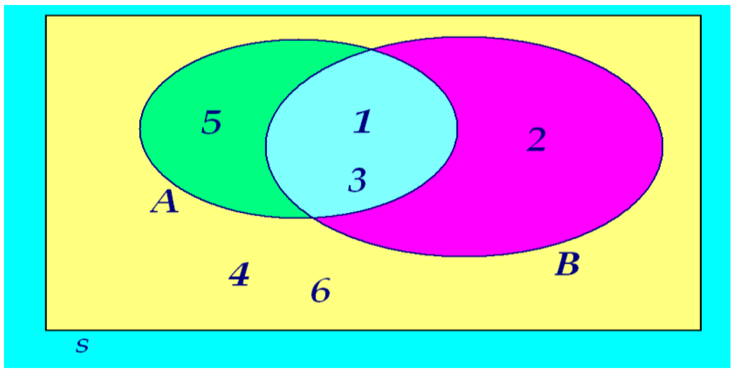
The complements of ***A*** and ***B*** are given below:

$$A' = \{2,4,6\} \quad \Rightarrow$$

$$P(A') = P(2) + P(4) + P(6) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}.$$

$$B' = \{4,5,6\} \quad \Rightarrow$$

$$P(B') = P(4) + P(5) + P(6) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}.$$



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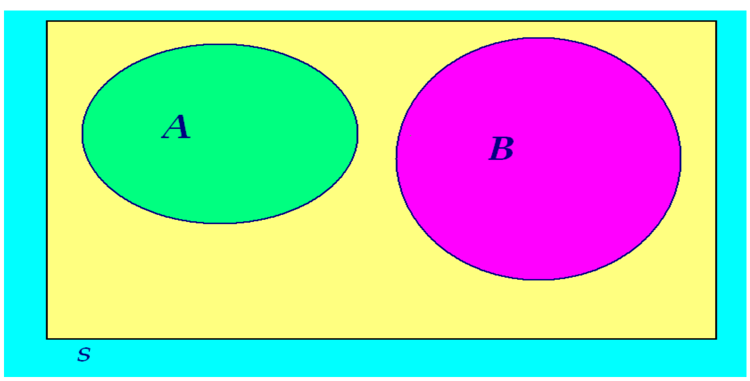
Two events ***A*** and ***B*** are said to be ***mutually exclusive*** or ***disjoint*** if their intersection, $(A \cap B)$, contains no sample points, that is, if ***A*** and ***B*** have no sample points in common.



Two mutually exclusive events

The Complementary Rule of Probability:

The sum of the probabilities of complementary events equals 1: that is:

$$P(A)+P(A')=1 \qquad \textbf{(2)}$$

The Addition Rule of Probability:

The probability of the union of events ***A*** and ***B*** is the sum of the probability of events ***A*** and ***B*** minus the probability of the intersection of events ***A*** and ***B***, that is,

$$P(A\cup B)=P(A)+P(B)-P(A\cap B) \qquad \textbf{(3)}$$

If two events are mutually exclusive, the probability of their union equals the sum of their respective probabilities:

$$P(A\cup B)=P(A)+P(B) \qquad \textbf{(4)}$$

Example 6
Consider the experiment of tossing a fair coin twice. Use the complementary rule to calculate the probability of event ***A*:{observing at least one tail}**.

Solution
The complement of ***A*** is defined as the event that occurs when ***A*** does not occur. Therefore,

$$A'=\{observing\ no\ tails\}=\{HH\} \Rightarrow P(A')=P(HH)=\frac{1}{4}.$$

$$P(A)=1-P(A')=1-\frac{1}{4}=\frac{3}{4}.$$

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Example 7
Hospital records show that 15% of all female patients are admitted for surgical treatment, 25% are admitted for obstetrics, and 5% receive both obstetrics and surgical treatments. If a new female patient is admitted to the hospital, what is the probability that the patient will be admitted either for surgery, obstetrics, or both?

Solution

Define the following events:

A:{ A female patient admitted to the hospital receives surgical treatment},
B:{ A female patient admitted to the hospital receives obstetrics treatment}.

Then, from the given information, ***P(A)*=** **0.15**, ***P(B)*=** **0.25**, and $P\left(A \cap B\right) = 0.05$. Therefore,

$$P\left(A \cup B\right) = P\left(A\right) + P\left(B\right) - P\left(A \cap B\right) = 0.15 + 0.25 - 0.05 = 0.35.$$

Thus, 35% of all female patients admitted to the hospital receive either surgical treatment, obstetrics treatment, or both.

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Exercises

1. Refer to example 4, and find the probabilities of the following events:

   A:{ A resident is younger than 25 *and* makes between 25K and 50K}
   B:{ a resident is younger than 25 *or* makes between 25K and 50K}

2. Refer to example 4, and find the following probabilities.
   a. The probability that a resident is older than 45 and makes less than 25K.
   b. The probability that a resident is 25 or older.
   c. The probability that a resident is older than 45 or makes more than 50K.
   d. The probability that a resident is 25 or older and whose income is between 25K and 50K.
   e. The probability that a resident's income is not more than 50K.

3. A fair coin is tossed 10 times. What is the probability of observing at least one tail?

Conditional Probability and Independent Events

The event probabilities we have been discussing so far are often called ***unconditional probabilities*** since no special conditions other than those that define the experiment are assumed. Sometimes, on the other hand, we may have additional knowledge that might alter the probability of an event. A probability that reflects such additional knowledge is called the ***conditional probability*** of the event.

We represent the probability of event ***A***, given that event ***B*** occurs by the symbol $P\left(A \mid B\right)$ (it reads: the probability of ***A*** condition ***B***). For the above experiment, and is given by:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \qquad \textbf{(5)}$$

Note that $P(A \mid B) \neq P(B \mid A)$, since

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} \qquad \textbf{(6)}$$

The Multiplication Rule of Probability

Formulas (5) and (6), after cross multiplication, can be written as

$$P(A \cap B) = P(A \mid B)\, P(B) \qquad \textbf{(7)}$$

$$P(A \cap B) = P(B \mid A)\, P(A) \qquad \textbf{(8)}$$

Example 8

The human resources director of a company constructed the following table, which describes the talent and motivation levels of company employees. The number in each cell is the number of employees that fall in that category.

		TALENT	
	high	**medium**	**low**
MOTIVATION			
high	20	40	30
medium	30	100	22
low	7	5	16

(a) Suppose that an employee has a low level of motivation. What is the probability that he/she has a medium level of talent?
(b) Suppose that an employee has a low level of talent. What is the probability that he/she is highly motivated?

Solution

Define the following events:

A = {an employee of the company is highly motivated.}

B = {an employee of the company has a medium talent level.}

C = {an employee of the company has a low talent level.}

D = {an employee of the company has a low motivation level.}

(a) We must find $P\left(B \mid D\right)$:

***P(D)*= 28/270**, and $P\left(B \cap D\right) = \frac{5}{270}$. Therefore,

$$P\left(B \mid D\right) = \frac{P\left(B \cap D\right)}{P\left(D\right)} = \frac{5/270}{28/270} = \frac{5}{28} = 0.179.$$

This result indicates that 17.9% of those who have a low level of motivation also have medium talent levels.

(b) In this part, we must find $P\left(A \mid C\right)$:

***P(C)* = 68/270**, and $P\left(A \cap C\right) = \frac{30}{270}$. Therefore,

$$P\left(A \mid C\right) = \frac{P\left(A \cap C\right)}{P\left(C\right)} = \frac{30/270}{68/270} = \frac{30}{68} = 0.441.$$

This result indicates that 44.1% of the employees who have low levels of talent also are highly motivated.

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Independent Events

Two events ***A*** and ***B*** are said to be ***independent*** if the outcome of one does not influence the outcome of the other. Mathematically, events ***A*** and ***B*** are independent if and only if

$$P\left(A \mid B\right) = P\left(A\right). \qquad \textbf{(9)}$$

Criterion (9) means that the occurrence of event ***B*** does not affect the occurrence of event ***A***. The "if and only if" statement implies that if events ***A*** and ***B*** are independent then $P\left(A \mid B\right) = P\left(A\right)$; and conversely, if $P\left(A \mid B\right) = P\left(A\right)$, then events ***A*** and ***B*** are independent. Events that are not independent are said to be ***dependent***.

When the Multiplicative Rule of Probability (formulas (7) or (8)) and the independency criterion (9) are combined, we obtain the following alternative criterion for establishing the dependency or independency of two events:

Events A and B are independent if and only if

$$P\left(A \cap B\right) = P\left(A\right)P\left(B\right) \qquad \textbf{(10)}$$

In practice, formula (10) is easier to use than formula (9) because it does not include conditional probabilities. Hence the above equation may serve as a criterion to determine whether two events are dependent or independent.

Example 10

A manufacturer of an electromechanical kitchen utensil conducted an analysis of a large number of consumer complaints and found that they fell into six categories shown in the table below:

		Reason of Complaint	
Time of Complaint	**electrical**	**mechanical**	**appearance**
During Warranty period	20%	15%	30%
After warranty period	10%	15%	10%

Define the following events:

***A**:* {Cause of complaint is product appearance},

***B**:* {Complaint occurred during the warranty term}.

Are ***A*** and ***B*** independent events?

Solution

Events ***A*** and ***B*** will be independent when $P(A \cap B) = P(A)\,P(B)$. Otherwise, they will be dependent events.

$$P(A \cap B) = 0.30,$$

$$\begin{cases} P(A) = 0.30 + 0.10 = 0.40 \\ P(B) = 0.20 + 0.15 + 0.30 = 0.65 \end{cases} \quad \Rightarrow \quad P(A)P(B) = (0.40)(0.65) = 0.26$$

Since $P(A \cap B) \neq P(A)P(B)$, we conclude that ***A*** and ***B*** are <u>dependent</u> events.

■

Exercises

1. Refer to example 4. Are ***A*** and ***B*** independent events?
2. Refer to example 4. (a) Given that a resident is younger than 25, what is the probability that his/her salary is more than 50K? (b) Given that a resident's salary is less than 25k, what is the probability that his/her age is between 25 and 45?

3. Explain, both in words and mathematically, whether two mutually exclusive events are dependent or independent.

4. Two events ***A*** and ***B*** are independent with ***P(A) = 0.5*** and ***P(B)= 0.4***. Find $P(A \cup B)$.

5. Two events **A** and **B** are dependent with ***P(A) = 0.6, P(B) = 0.7***, and $P(A \mid B) = 0.8$. Find $P(A \cup B)$.

6. Refer to example 8. (a) what is the probability that the cause of complaint was mechanical given that the complaint originated during the guarantee period? (b) What is the probability that the complaint originated after the guarantee period given that the cause of the complaint was appearance? (c) What percentage of complaints whose reason was electrical originated during the guarantee period?

7. Many medical researchers have conducted experiments to examine the relationship between cigarette smoking and cancer. Consider an individual randomly selected from an adult male population. Let ***A*** represent the event that the individual smokes, and let ***B*** represent the event that the individual develops cancer. Therefore, $A \cap B$ is the event that an individual adult male smokes and develops cancer; $A \cap B'$ is the event that an adult male smokes and does not develop cancer, etc. The probabilities associated with the four possible sample points are shown in the following table.

Sample Points	*Probabilities*
$A \cap B$	0.04
$A \cap B'$	0.21
$A' \cap B$	0.02
$A' \cap B'$	0.73

(a) What percentage of smokers (i) develop cancer? (ii) do not develop cancer?
(b) What percentage of non-smokers (i) develop cancer? (ii) do not develop cancer?
(c) What percentage of cancer patients (i) are smokers? (ii) are non-smokers?
(d) What is the probability that a person who does not have cancer (i) is a smoker? (ii) is not a smoker?
(e) Explain how can the above information be used to examine the relationship between smoking and cancer?

The Law of Total Probability and the Bayes' Theorem

A group of events $A_1, A_2, \ldots, A_n$ is said to be ***exhaustive*** if they satisfy the following two conditions:

(i) $A_1 \cup A_2 \cup \ldots \cup A_n = S \;\; ; \quad or, \;\; \bigcup_{i=1}^{n} A_i = S$

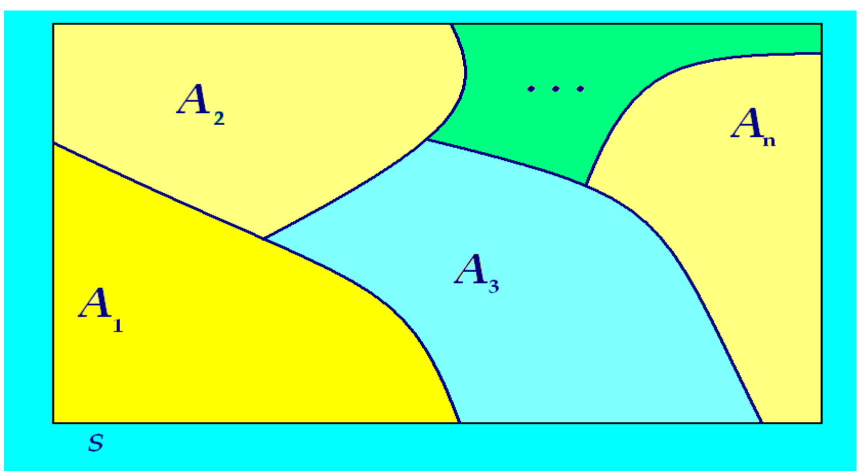
(ii) $For\ any\ pair\ A_i\ and\ A_j,\ with\ i \neq j,\ A_i \cap A_j = \varnothing$, where $\varnothing$ denotes the ***empty*** set.

The first condition states that the union of exhaustive events is the entire sample space. More clearly, this means that at least one of them must occur. The second property states that any two pairs of exhaustive events are mutually exclusive (disjoint). This means that it is impossible for any two of them to occur at the same time. The above two properties can also be expressed in terms of the probabilities:

(i) $P(A_1) + P(A_2) + \ldots + P(A_n) = 1 \;\; ; \quad or, \;\; \sum_{i=1}^{n} P(A_i) = 1$

(ii) $P(A_i \cap A_j) = 0 \;\; for \;\; i \neq j$

A Venn diagram of exhaustive events is shown below:



A collection of exhaustive events

The Law of Total Probability

Suppose $A_1, A_2, \ldots, A_n$ is a collection of exhaustive events, and suppose ***B*** is any nonempty event. Then ***B*** can be expressed as the union of its individual intersections $B \cap A_i$, $i=1,2,\ldots,n$, with those exhaustive events. That is,

$$B = \left(B \cap A_1\right) \cup \left(B \cap A_2\right) \cup \ \ldots \ \cup \left(B \cap A_n\right).$$

In other words, the intersections $B \cap A_i$ serve as building blocks for constructing the event ***B***. In terms of probabilities, since each pair $B \cap A_i$ and $B \cap A_j$, $i \neq j$, are disjoint, we obtain

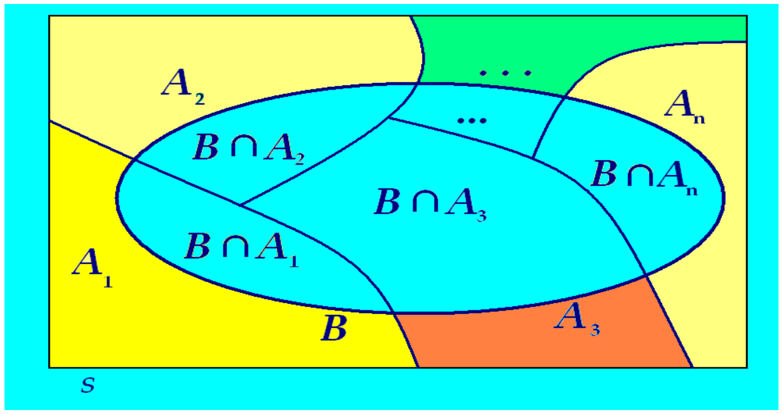
$$P\left(B\right) = P\left(B \cap A_1\right) + P\left(B \cap A_2\right) + \ldots + P\left(B \cap A_n\right) = \sum_{i=1}^{n} P\left(B \cap A_i\right).$$

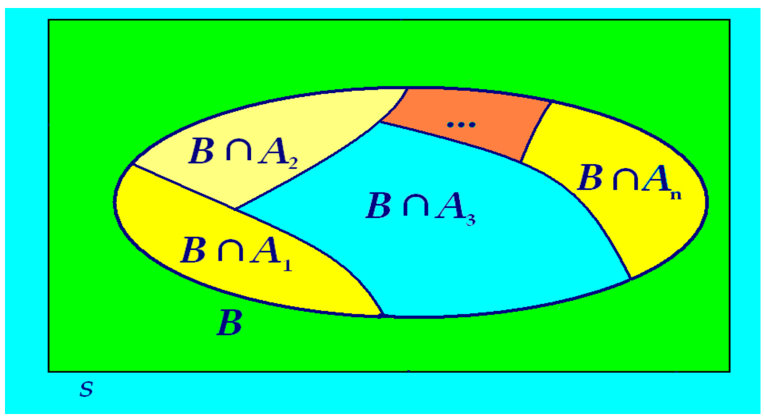
(11)

Equivalently, replacing $P\left(B \cap A_i\right)$ by $P\left(B \mid A_i\right)P\left(A_i\right)$ from the multiplication rule (formula (8)), we obtain the ***Law of Total Probability***:

$$P\left(B\right) = \sum_{i=1}^{n} P\left(B \mid A_i\right)P\left(A_i\right) \qquad \textbf{(12)}$$

The following two Venn diagrams demonstrate a pictorial description of the Law of Total Probability.





The Law of Total Probability: An event *B* is expressed as the union of its individual intersections with a collection of exhaustive events $A_1, A_2, \ldots,$ and A_n

Bayes' Theorem

Let $A_1, A_2, \ldots, A_n$ be a collection of exhaustive events, and suppose ***B*** is any nonempty event. Then for any j, $j = 1, 2, \ldots, n$, we have

$$P\left(A_j \mid B\right) = \frac{P\left(B \mid A_j\right) P\left(A_j\right)}{\sum_{i=1}^{n} P\left(B \mid A_i\right) P\left(A_i\right)} \quad \textbf{(13)}$$

Example 11

A gas station has three types of fuel: regular unleaded, mid-grade unleaded, and premium unleaded. Of the customers who buy fuel at this station, 50% purchase regular unleaded, 30% purchase mid-grade unleaded, and the rest purchase premium unleaded. 30% of those who purchase regular unleaded, buy a full tank of gas, whereas the percentages of fill-ups for the other two groups are 40% and 60% respectively.

(a) What percentage of the customers purchase a full tank of gas?

(b) Given that a customer has purchased a full tank of gas, what is the probability

the customer has purchased (i) regular unleaded, (ii) premium unleaded?

Solution

Define the events A_1, A_2, A_3, and B as follows:

A_1 : The customer has purchases regular unleaded.

A_2 : The customer has purchases mid-grade unleaded.

A_3 : The customer has purchases premium unleaded.

B : The customer has purchases a full tank of gas.

Then events A_1, A_2, and A_3 form a collection of exhaustive events since a customer who purchases gas, would purchase at least one of the above gas types. This fact satisfies the first condition for exhaustiveness:

$$A_1 \cup A_2 \cup A_3 = S.$$

Also, a customer would purchase exactly one type of gas (here, we omit the rare possibility that a customer may purchase two or three different types of gas at the same time). This fact satisfies the second condition for exhaustiveness:

$$A_1 \cap A_2 = \varnothing, \quad A_1 \cap A_3 = \varnothing \quad , \quad A_2 \cap A_3 = \varnothing.$$

Furthermore, the following probabilities are given:

$$P(A_1) = 0.50, \qquad P(A_2) = 0.30, \qquad P(A_3) = 1 - (0.5 + 0.3) = 0.20, \qquad .$$

$$P(B \mid A_1) = 0.30, \quad P(B \mid A_2) = 0.40, \quad P(B \mid A_3) = 0.60.$$

(a) By the law of total probability, we have:

$$\begin{aligned} P(B) &= P(B \mid A_1)P(A_1) + P(B \mid A_2)P(A_2) + P(B \mid A_3)P(A_3) \\ &= (0.30)\,(0.50) + (0.40)\,(0.30) + (0.60)\,(0.20) \\ &= 0.15 + 0.12 + 0.12 \\ &= 0.39. \end{aligned}$$

Therefore, **39%** of the customers who purchase gas at this gas station, purchase a full tank of gas.

(b) In part (i), we must find $P(A_1 \mid B)$. By the Bayes' Law,

$$P(A_1 \mid B) = \frac{P(B \mid A_1)P(A_1)}{P(B)} = \frac{(0.30)(0.50)}{0.39} = \frac{0.15}{0.39} = 0.384.$$

For part (ii), applying the Bayes' Law again, we get

$$P(A_3 \mid B) = \frac{P(B \mid A_3)P(A_3)}{P(B)} = \frac{(0.60)(0.20)}{0.39} = \frac{0.12}{0.39} = 0.307.$$

■

Example 12

A factory has 4 machines that produce the same item. Machines 1 and 2 each produce 20% of the total output, while machines 3 and 4 each produce 30% of the total output. It's known that 6% of machine 1's output is defective, while machine 2 produces 5% defective items, and machines 3 and 4 each produce 8% defective items. An item is chosen at random from the output of the factory by the quality control manager.

(a) What is the probability that this item is defective?

(b) Given that the selected item is defective, what is the probability that it was produced by machine 2?

Solution

(a) Define the events A_1, A_2, A_3, A_4, and B as follows:

A_1 : The item was produced by machine 1.

A_2 : The item was produced by machine 2.

A_3 : The item was produced by machine 3.

A_4 : The item was produced by machine 4.

B : The selected item is defective.

Note that events A_1, A_2, A_3, and A_4 are exhaustive since a chosen item must have been produced by <u>exactly</u> one of the four machines. Also, from the given information we have:

$$P(A_1) = 0.20, \quad P(A_2) = 0.20, \quad P(A_3) = 0.30, \quad P(A_4) = 0.30.$$

$$P(B \mid A_1) = 0.06, \quad P(B \mid A_2) = 0.05, \quad P(B \mid A_3) = 0.08, \quad P(B \mid A_4) = 0.08.$$

By the Law of Total probability (12),

$$P(B)=\sum_{i=1}^{4}P(B \mid A_i)P(A_i)=(0.06)(0.2)+(0.05)(0.2)+(0.08)(0.3)+(0.08)(0.3)=0.07$$

This result indicates that 7% of total production of this factory is defective.

(b) Using Bayes' Law,

$$P(A_2 \mid B)=\frac{P(B \mid A_2)P(A_2)}{P(B)}=\frac{(0.05)(0.2)}{0.07}=\frac{1}{7}=0.143\,.$$

■

Exercises

1. There are three car rental agencies in a town (agencies 1, 2, and 3). Suppose 4% of the cars of agency 1 are unsafe, whereas the percentages of unsafe cars for agencies 2 and 3 are 8% and 5% respectively. An agency is chosen at random and a car rented from it was found to be unsafe. Find the probability that the car came from agency 1.
2. Of the 100 senators in US Senate, 51 are Republicans and 49 are Democrats. In a recent voting on an important financial bill, 41 of the Republicans voted "Yes" and 45 of the Democrats votes "Yes".
   (a) What percentage of the senators voted "No" to this bill?
   (b) Given that a senator voted "No" to the bill, what is the probability the he/she is a Democrat? A Republican?
   (c) The senate has declared that the voting has been conducted in a bi-partisan manner. Explain why you agree or disagree with this statement.
3. A gas station sell four types of fuel, regular unleaded, mid-grade unleaded, premium unleaded, and diesel. Of the customers who buy fuel at this station, 40% purchase regular unleaded, 30% buy mid-grade unleaded, 20% buy premium unleaded, and 10% buy diesel. 40% of the customers who buy regular unleaded buy a full tank of gas, 50% of those who buy mid-grade fill up their tanks, 80% who buy premium buy a full tank of gas, and all diesel customers fill up their tanks. A customer is randomly chosen.

   (a) What percentage of the customers buy a full tank of gas?

   (b) Of those customers who purchased a full tank of gas, what purchase

   premium gas? Diesel gas?

Solutions to Exercises:

Exercise 1:

Define the events A_1, A_2, A_3, and B as follows:

A_1 : The car was rented from agency 1

A_2 : The car was rented from agency 2

A_3 : The car was rented from agency 3

B : The car rented is unsafe.

We must calculate $P(A_1 \mid B)$.

Assuming that the three agencies have an equal share of the market,

$$P(A_1) = \frac{1}{3},\ P(A_2) = \frac{1}{3},\ P(A_3) = \frac{1}{3}$$

Furthermore,

$$P(B \mid A_1) = 0.04$$

$$P(B \mid A_2) = 0.08$$

$$P(B \mid A_3) = 0.05$$

By Bayes' law:

$$P(A_1 \mid B) = \frac{P(B \mid A_1)\,P(A_1)}{P(B)} = \frac{P(B \mid A_1)\,P(A_1)}{\sum P(B \mid A_i)\,P(A_i)}$$

$$P(A_1 \mid B) = \frac{(0.04)(\frac{1}{3})}{(0.04)\left(\frac{1}{3}\right)+(0.08)\left(\frac{1}{3}\right)+(0.05)(\frac{1}{3})} = \mathbf{0.2353}$$

Exercise 2:

Define the events A_1, A_2, and B as follows:

A_1 : The senator is a Republican

A_2 : The senator is a Democrat

B : Senator voted Yes

$$P(A_1) = \frac{51}{100} = 0.51, \;\; P(A_2) = \frac{49}{100} = 0.49$$

$$P(B \mid A_1) = \frac{41}{51}, \;\; P(B \mid A_2) = \frac{45}{49}$$

(a) We must calculate $P(\bar{B})$, where $\bar{B}$ is the complement of B. By the Law of Total probability,

$$P(B) = \sum P(B \mid A_i)\, P(A_i) = \left(\frac{41}{51}\right)\left(\frac{51}{100}\right) + \left(\frac{45}{49}\right)\left(\frac{49}{100}\right) = 0.86$$

Therefore,

$$P(\bar{B}) = 1 - P(B) = 1 - 0.86 = 0.14$$

(b) Of 14 senators who voted "No", 10 (=51-41) were Republican and 4 (=49-45) were Democrats. Therefore,

$$P(A_1 \mid \bar{B}\,) = \frac{10}{14} = \mathbf{0.7143}$$
$$P(A_2 \mid \bar{B}\,) = \frac{4}{14} = \mathbf{0.2857}$$

(c) According to the results obtained in part (b), 71% of those who voted "No" were Republican senators; whereas the percentage of Democrats who voted "No" was about 29%. This voting manner does not seem to have been a "bi-partisan" manner.

Exercise 3:

Define the events A_1, A_2, A_3, A_4, and B as follows:

A_1 : Customer purchases regular unleaded gas

A_2 : Customer purchases mid-grade unleaded gas

A_3 : Customer purchases premium unleaded gas

A_4 : Customer purchases diesel gas

B : Customer purchases a full tank of gas

Then, $P(A_1) = 0.40$, $P(A_2) = 0.30$, $P(A_3) = 0.20$, and $P(A_4) = 0.10$

Furthermore, $P(B \mid A_1) = 0.40$, $P(B \mid A_2) = 0.50$, $P(B \mid A_3) = 0.80$, and $P(B \mid A_4) = 1$

(a) What percentage of the customers buy a full tank of gas?

According to the *Law of Total Probability*,

$$P(B) = \sum P(B \mid A_i)\, P(A_i) = (0.4)(0.4) + (0.5)(0.3) + (0.8)(0.2) + (1)(0.1)$$

$$P(B) = 0.57$$

(b) For this part, we need to calculate two values: $P(\,A_3 \mid B)$ and $P(\,A_4 \mid B)$

By the Bayes' Law,

$$P(\,A_3 \mid B) = \frac{P(B \mid A_3)\, P(A_3)}{P(B)} = \frac{(0.8)(0.2)}{0.57} = 0.2807$$

Answer: **28%**

$$P(\,A_4 \mid B) = \frac{P(B \mid A_4)\, P(A_4)}{P(B)} = \frac{(1)(0.1)}{0.57} = 0.1754$$

Answer: **18%**

Counting Principles

Counting rules are often used to calculate probabilities of events. There are three such principles:

1. The Fundamental Counting Principal
2. The Permutation Principal
3. The combinations Principal

1. Fundamental Counting Principal:

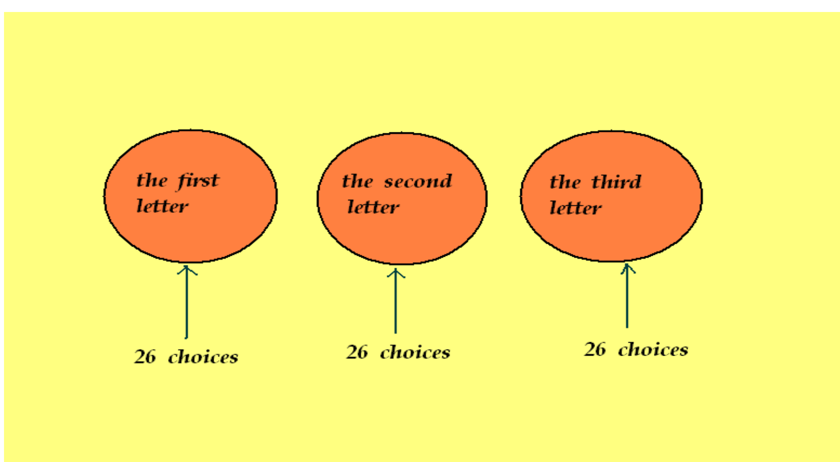
Suppose an event ***A*** can take place in ***m*** different ways, a second event ***B*** can take place in ***n*** different ways, and so on…. Then the total number of ways the above events can occur in sequence is given by:

$$Total\ number\ of\ ways\ of\ occurance = m\ n\ \ldots. \qquad \textbf{(1)}$$

Example 1

How many different 3 letter words can be constructed from the 26 letters of English alphabet?

Solution



There are three letters in a 3-letter word. Each one of these three letters describes an event. There are 26 choices for the first letter, 26 choices for the second letter, and 26 choices for the third letter. Therefore the total number of combinations is:

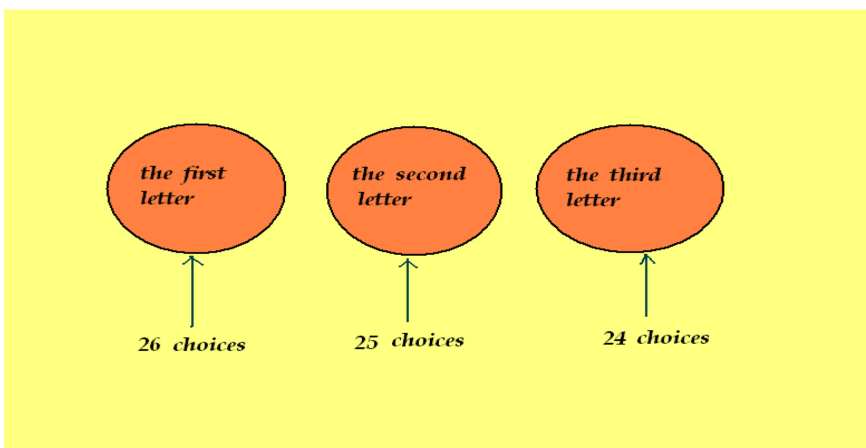
$$Total\ number\ of\ ways = 26 \times 26 \times 26 = 17576$$

Therefore a total of 17576 three –letter words can be constructed.

Example 2

How many different 3 letter words can be constructed from the 26 letters of English alphabet such that no one letter is repeated?

Solution



There are 26 choices for the first letter. Since repetition is not allowed, there remain 25 choices for the second letter. Finally, since two letters have already been chosen as the first and the second letters of the word, there are 24 letters left for the third place. Therefore,

$$Total\ number\ of\ ways = 26 \times 25 \times 24 = 15600$$

Therefore a total of $\mathbf{15600}$ three –letter words can be constructed in which all three letters are distinct.

Example 3

Suppose a three-letter code is to be randomly constructed. What is the probability that no two letters of this word are the same?

Solution:

Define the event of interest to be:

$$A: \textit{in a random selection of } \mathbf{3} \textit{ letters, no two letters are the same.}$$

To find the probability of **A**, we must divide the number three-letter words, in which no two letters are repeated, by the total number of three-letter words.

$$P(A) = \frac{\textit{number three} - \textit{letter words in which no letter is repeated}}{\textit{total number of three} - \textit{letter words}}$$

Using the results from examples 1 and 2, we obtain:

$$P(A) = \frac{15600}{17576} = 0.89$$

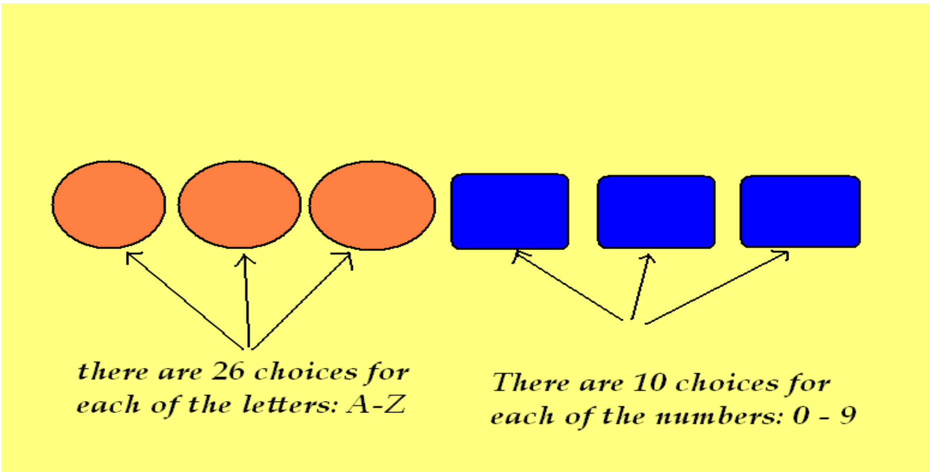
Example 4

Vehicle license tags contain three letters followed by three numbers.

(a) How many different such license tag numbers can be constructed?
(b) If a license tag is randomly chosen, what is the probability that it contains no “O” and no “0”.

Solution

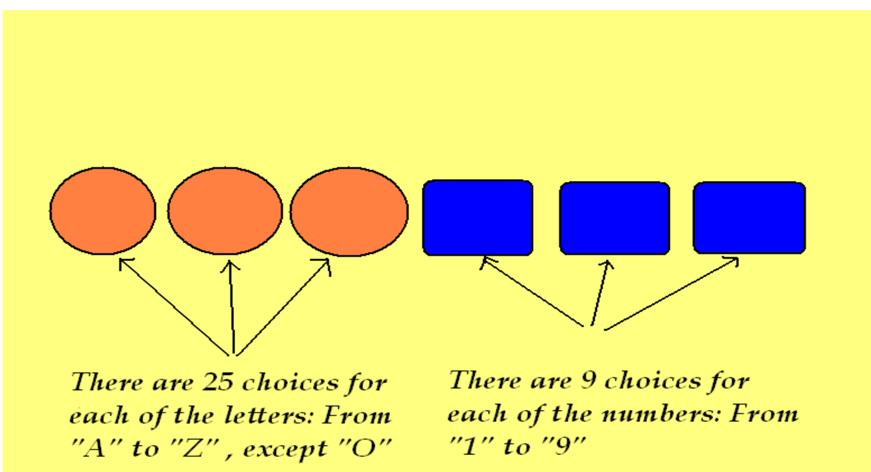
(a)



$$Total\ number\ of\ license\ tags = 26 \times 26 \times 26 \times 10 \times 10 \times 10$$
$$= 17{,}576{,}000$$

(b)

First, we must find the total number of license tags which do not contain an "O" or a zero.



$$\mathit{Total\ number\ of\ license\ tags} = \mathbf{25 \times 25 \times 25 \times 9 \times 9 \times 9}$$
$$= \mathbf{11,390,625}$$

Therefore, the probability that a randomly selected license tag does not contain an "O" or a "0", is:

$$\frac{\mathbf{11,390,625}}{\mathbf{17,576,000}} = \mathbf{0.65}.$$

■

2. Permutation:

There are <u>3 types of permutations</u>:

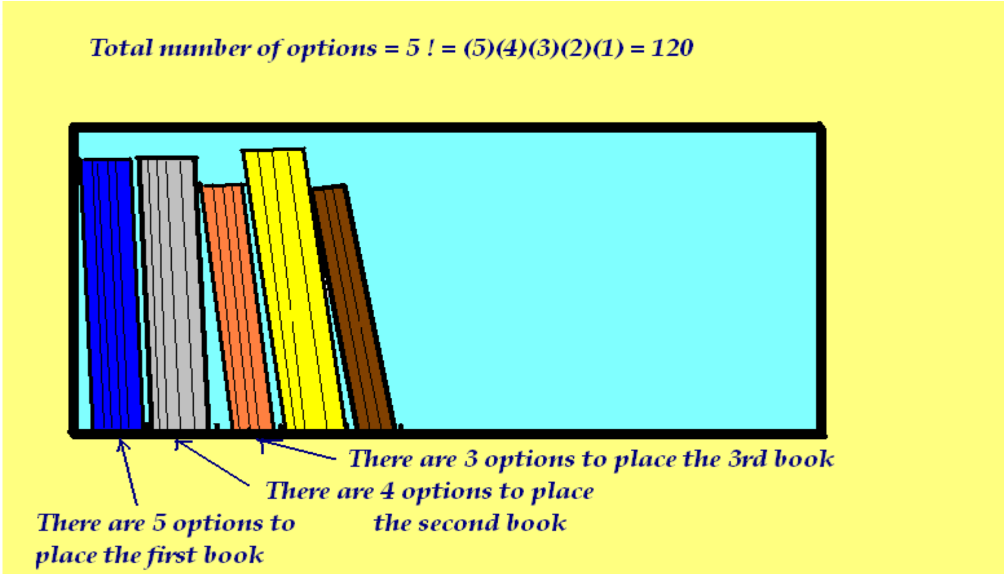
Type (i): <u>An ordered arrangement of *n* distinct objects</u>

The number of different ways to arrange n distinct objects according to a certain order is called a permutation of n elements. There are $n!$ different ordered arrangements of n elements.

Example 5

In how many different ways can 5 books be placed in a bookshelf?

Solution



This is a permutation of 5 elements. There are

$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

different ways to place (or to order) 5 books in a bookshelf.

Example 6

In how many different ways can 5 books be placed in a bookshelf so that the first and the second volume of the same book are placed next to each other?

Solution

Since the two volumes of the same book are to be placed next to each other, then they can be considered to form one unit (element). Therefore, together with the other 2 books, they form 4 units (elements).

There are $4! = 4 \times 3 \times 2 \times 1 = 24$ different ways to arrange the 4 units on a bookshelf. Now for each of the 24 way above, there are 2 options of placing the volumes 1 & 2 next to each other: either volume 1 is placed on the left of volume 2, or the other way around. Therefore, the total number of ways one can place 5 books on the bookshelf so that volumes 1 & 2 are placed together is:

$$(2)(4\,!) = 48$$

Example 7

Suppose we wish to randomly arrange 5 books in a bookshelf. What is the probability that volumes 1 and 2 of the same book will be placed next to each other?

Solution

Define the event of interest to be:

$$A : volumes\ 1\ \&\ 2\ are\ together.$$

To find the probability of **A**, we must divide the number of ways of arrangement with volumes 1 and 2 being together by the total number of arrangements.

$$P(A) = \frac{number\ of\ arrangement\ of\ 5\ books\ so\ that\ volumes\ 1\ \&\ 2\ are\ together}{total\ number\ of\ arrangement\ of\ 5\ books}$$

Using the results from examples 5 and 6, we obtain:

$$\boldsymbol{P(A) = \frac{48}{120} = \frac{2}{5} = 0.4}$$

Type (ii): <u>Permutation of *n* elements taken *r* at a time</u>

Choosing r objects from a group of n objects ($r \leq n$), and putting them in order, is another type of ***permutation***. This type of permutation is called: ***a permutation of n elements taken r at a time***. The number of permutations of n elements taken r at a time is denoted by ${}_nP_r$, and is given by:

$$ {}_{n}P_{r} = \frac{n\,!}{(n-r)\,!} \qquad \textbf{(2)} $$

Example 8

In how many different ways can we chose a committee of 3 people from a group of 10 people such that one of them becomes the president, the other becomes the vice president, and the third becomes a financial officer.

Solution:

Note that since a president, a vice president, and a financial officer are to be chosen from 10 elements, this is an ordered selection of 3 elements from a group of 10 elements; that is, a permutation of 10 elements taken 3 at a time. The total number of such a permutation is given by:

$$ {}_{10}P_{3} = \frac{10\,!}{(10-3)\,!} = \frac{10\,!}{7\,!} = \frac{10\times 9\times 8\times 7!}{7!} = 10\times 9\times 8 = 720 $$

Type (iii): Distinguishable <u>permutation of n elements, where one group of elements is of one kind, a second group is of another kind, … and so on.</u>

The number of distinguishable permutations of n elements where n_1 are of one kind, n_2 are of another kind, and so on, is given by:

$$ \frac{n\,!}{(n_1!)(n_2!)(n_3!)\ldots} \qquad \textbf{(3)} $$

Example 9

In how many ways can the letters in the word "CALCULUS" be arranged?

Solution

The word "CALCULUS" consists of 8 elements ($n = 8$), where

$n_1 = 2$ Number of "C"'s.

$n_2 = 1$ Number of "A"'s.

$n_3 = 2$ Number of "L"'s.

$n_4 = 2$ Number of "U"'s.

$n_5 = 1$ Number of "S"'s.

Therefore,

$$\text{number of distinguishable permutations} = \frac{8\,!}{2\,!\,1\,!\,2!\,2\,!\,1\,!}$$
$$= \frac{40320}{8} = 5040$$

■

3. Combinations:

Choosing r objects from a group of n objects ($r \leq n$) without any regard to order, is called a combination. More precisely, ***a combination of n elements taken r at a time***. The number of combinations of n elements taken r at a time is denoted by ${}_nC_r$, and is given by:

$${}_nC_r = \frac{n\,!}{r\,!\,\left(n-r\right)!} \qquad \textbf{(4)}$$

Example 10:

In how many different ways can a committee of 3 people be selected from a group of 7 people?

Solution:

This is a combination of 7 elements taken 3 at a time; i.e., $n = 7$, and $r = 3$. Therefore,

$$_{7}C_{3}=\frac{7\,!}{3\,!\,\left(7-3\right)!}=\frac{7\,!}{3\,!\,\left(4\,!\right)}=\frac{7\times6\times5\times\left(4!\right)}{6\times\left(4\,!\right)}=35\,.$$

Example 11:

In how many different ways can a committee of 2 secretaries and 3 receptionists be selected from a group of 8 secretaries and 12 receptionists?

Solution:

There are $_{8}C_{2}=28$ different ways to choose 2 secretaries from a group of 8 secretaries, and there are $_{12}C_{3}=220$ different ways to choose 3 receptionists from a group of 12 receptionists. Therefore, by the Fundamental Counting Principal, there are $_{8}C_{2}\times{}_{12}C_{3}=28\times220=6160$ different ways to choose 2 secretaries from 8, and 3 receptionists from12.

$$_{8}C_{2}=\frac{8!}{2!\,\left(8-2\right)!}=\frac{8\times7\times\left(6\,!\right)}{2\times\left(6\,!\right)}=28\,,$$

$$_{12}C_{3}=\frac{12!}{3!\,\left(12-3\right)!}=\frac{12\times11\times10\times\left(9\,!\right)}{6\times\left(9\,!\right)}=220\,,$$

$$_{8}C_{2}\times{}_{12}C_{3}=28\times220=6160\,.$$

Excel Functions:

$n!: \; = \textbf{FACT}\,(\boldsymbol{n})$

${}_{n}P_{r}: \; = \textbf{PERMUT}\,(\boldsymbol{n}\,, \boldsymbol{r})$

${}_{n}C_{r}: \; = \textbf{COMBIN}\,(\boldsymbol{n}\,, \boldsymbol{r})$

EXERCISES:

1. A three-letter word is randomly constructed. Find the probability that the word has the following characteristics:
   (a) There are no vowels in the word.
   (b) The middle letter is a vowel.
   (c) Only the middle letter is a vowel; but the other two letters are not vowels.
   (d) There is at least one vowel in the word.
   (e) The first letter is a "C", and the second letter is a vowel.
   (f) The first and the last letters are the same.
2. Evaluate ${}_{12}P_{2}$, and explain what it means.
3. In how many different ways can 8 people stand in front of a camera to have their picture taken?
4. In how many different ways can 8 people stand in front of a camera to have their picture taken, so that a husband and wife would be together?
5. 8 people are to randomly stand in front of the camera to have their picture taken. What is the probability that a husband and wife would be standing next to each other?
6. In how many different ways can 8 people stand in front of a camera to have their picture taken, so that three members of the same family would be together?
7. 8 people are to randomly stand in front of the camera to have their picture taken. What is the probability that three members of the same family would next to each other?
8. In how many different ways can 2 people be selected from a group of 11? In how many different ways can a president and a vice president be chosen from a committee of 11 people?
9. Suppose you are given a letter "B", a letter "K", and two of the letter "O". You are asked to randomly arrange the four letters. What is the probability that your random arrangement results in the word "BOOK"?
10. In how many different ways can 4 social and 5 medical volunteers be selected from a group of 12 social and 10 medical volunteers?