Chapter 13: Linear Optimization

Statistics, Data Analysis, and Decision Modeling, Fifth Edition

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Optimization

- The process of selecting values of decision variables that maximize or minimize an objective function.
- Optimal solution – the best set of decision variables

Constrained Optimization

- Constraints - limitations or requirements that decision variables must satisfy.
  - The amount of material used to produce a set of products cannot exceed the available amount of 850 square feet.
  - The amount of money spent on research and development projects cannot exceed the assigned budget of $300,000.
  - Contractual requirements specify that at least 500 units of product must be produced.
  - A mixture of fertilizer must contain exactly 30 percent nitrogen.
  - We cannot produce a negative amount of product *(nonnegativity).*

Constraint Functions

- Amount of material used ≤ 850 square feet
- Amount spent on research and development ≤ \$300,000
- Number of units of product produced ≥ 500
- Amount of nitrogen in mixture/total amount in mixture = 0.30
- Amount of product produced ≥ 0

The left hand sides are called constraint functions.

Mathematical Representation

- Suppose that the material requirements for three products are 3.0, 3.5, and 2.3 square feet per unit.
- Let A, B, and C represent the number of units of each product to produce.
  - The amount of material used to produce A units of product A = $3.0A$
  - The amount of material used to produce B units of product B = $3.5B$
  - The amount of material used to produce C units of product C = $2.3C$
- Constraint: $3.0A + 3.5B + 2.3C \leq 850$

Definitions

- Feasible solution: any solution that satisfies all constraints
- A problem that has no feasible solutions is called infeasible.

Developing an Optimization Model

1. Define the decision variables.
2. Identify the objective function.
3. Identify all appropriate constraints.
4. Write the objective function and constraints as mathematical expressions.

Example: Sklenka Ski Company

- Sklenka Ski Company (SSC) is a small manufacturer of two types of popular all-terrain snow skis, the Jordanelle and Deercrest models. The manufacturing process consists of two principal departments: fabrication and finishing. The fabrication department has 12 skilled workers, each of whom works 7 hours per day. The finishing department has three workers, who also work a 7-hour shift. Each pair of Jordanelle skis requires 3.5 labor hours in the fabricating department and one labor hour in finishing. The Deercrest model requires four labor hours in fabricating and 1.5 labor hours in finishing. The company operates five days per week. SSC makes a net profit of $50 on the Jordanelle model, and $65 on the Deercrest model. In anticipation of the next ski sale season, SSC must plan its production of these two models. Because of the popularity of its products and limited production capacity, its products are in high demand and SSC can sell all it can produce each season. The company anticipates selling at least twice as many Deercrest models as Jordanelle models. The company wants to determine how many of each model should be produced on a daily basis to maximize net profit.

Optimization Model

- Maximize Total Profit = 50 Jordanelle + 65 Deercrest
- Total labor used ≤ the amount of labor available
  - 3.5 Jordanelle + 4 Deercrest ≤ 84
  - 1 Jordanelle + 1.5 Deercrest ≤ 21
- Number of pairs of Deercrest skis must be at least twice the number of Jordanelle skis
  - Deercrest - 2 Jordanelle ≥ 0
- Nonnegativity
  - Deercrest ≥ 0, Jordanelle ≥ 0

Characteristics of Linear Optimization Models

- The objective function and all constraints are linear functions of the decision variables
- All variables are continuous (fractional values are allowed)

Transformation to Linear Functions

- What appears to be nonlinear can often be transformed to a linear function.
  - Example: If two ingredients contain 20 percent and 33 percent nitrogen, respectively, then the fraction of nitrogen in a mixture of x pounds of the first ingredient and y pounds of the second ingredient is expressed by the constraint function $(0.20x + 0.33y)/(x + y)$
  - If a constraint requires that the fraction to be 0.3, this can be rewritten as a linear function

$$(0.20x + 0.33y) = 0.3(x + y)$$

or

$$-0.1x + 0.03y = 0$$

Spreadsheet Modeling

- Set up a logical format
- Define cells for the decision variables
- Define separate cells for the objective function and each constraint function
- Avoid Excel functions ABS, MIN, MAX, INT, ROUND, IF, COUNT

Sklenka Skis

	A	B	C	D
1	**Sklenka Skis**			
2				
3	**Data**			
4		**Product**		
5	**Department**	Jordanelle	Deercrest	Limitation (min.)
6	Fabrication	3.5	4	84
7	Finishing	1	1.5	21
8				
9	Profit/unit	$ 50.00	$ 65.00	
10				
11				
12	**Model**			
13		Jordanelle	Deercrest	
14	Quantity Produced	0	0	Hours Used
15	Fabrication	0	0	0
16	Finishing	0	0	0
17				
18				Excess Deercrest
19	Market mixture			0
20				
21				Total Profit
22	Profit Contribution	$ -	$ -	$ -

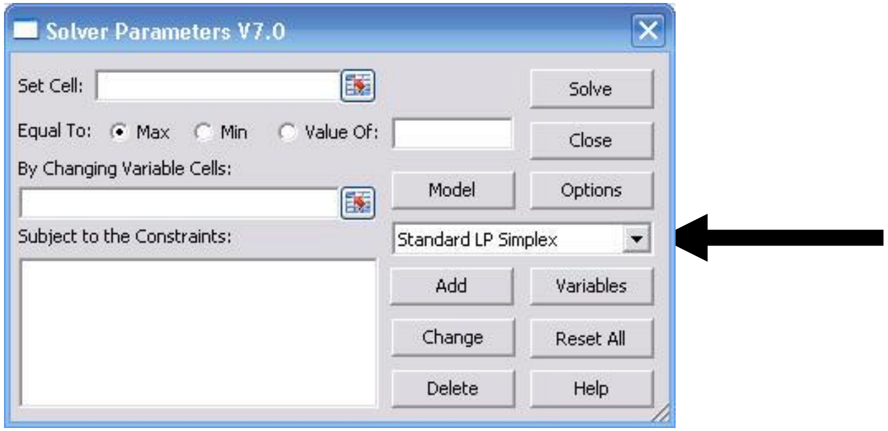
	A	B	C	D
1	**Sklenka Skis**			
2				
3	**Data**			
4		**Product**		
5	**Department**	Jordanelle	Deercrest	Limitation (min.)
6	Fabrication	3.5	4	84
7	Finishing	1	1.5	21
8				
9	Profit/unit	50	65	
10				
11				
12	**Model**			
13		Jordanelle	Deercrest	
14	Quantity Produced	0	0	Hours Used
15	Fabrication	=B6*$B$14	=C6*$C$14	=B15+C15
16	Finishing	=B7*$B$14	=C7*$C$14	=B16+C16
17				
18				Excess Deercrest
19	Market mixture			=C14-2*B14
20				
21				Total Profit
22	Profit Contribution	=B9*$B$14	=C9*$C$14	=B22+C22

Excel *Solver*

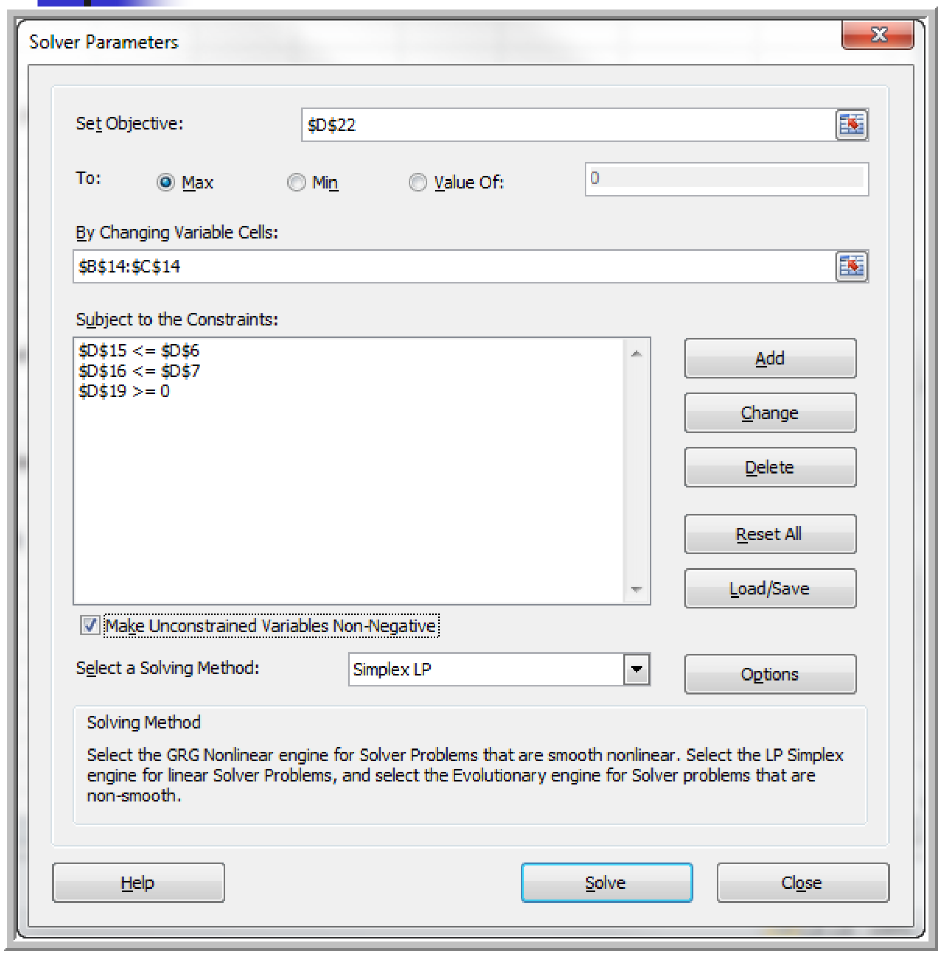
- Standard Solver (packaged with Excel)
- Premium Solver (available as a download with the book)

Premium Solver

- Solution procedures
  - *Standard GRG Nonlinear* – used for solving nonlinear optimization problems
  - *Standard Simplex LP* – used for solving linear and linear integer optimization problems
  - *Standard Evolutionary* – used for solving complex nonlinear and nonlinear integer problems

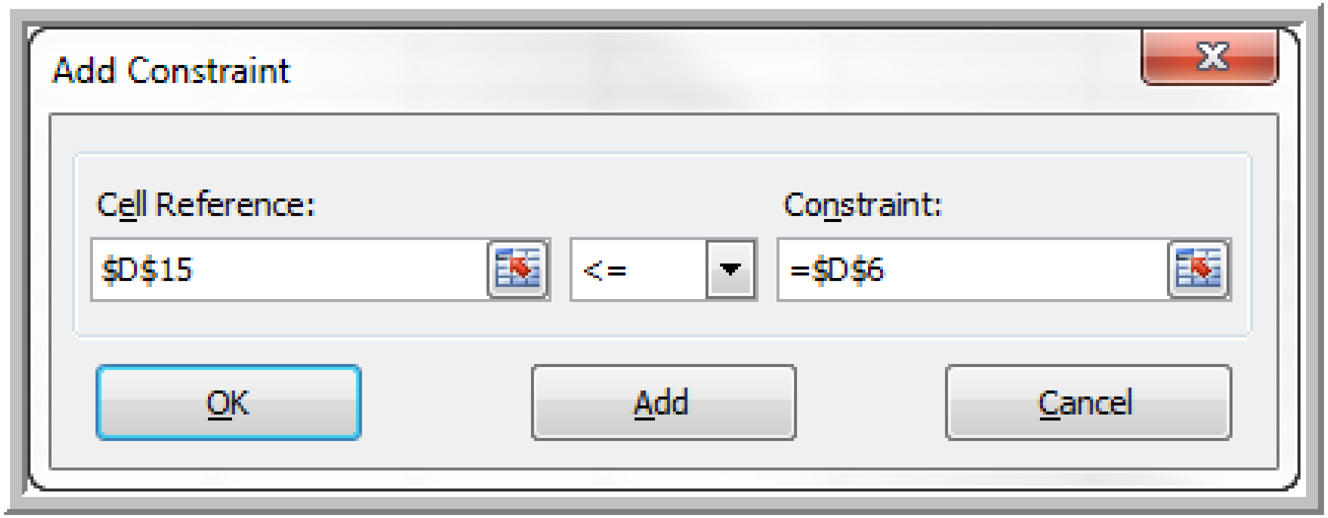


Using Standard Solver

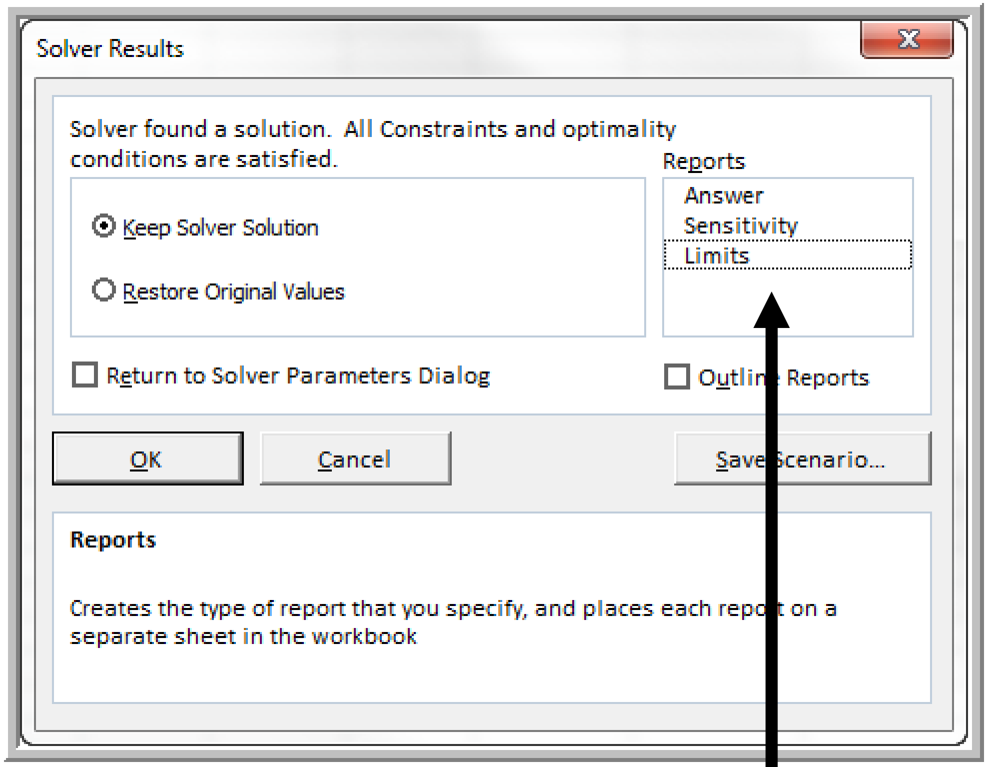


	A	B	C	D
1	**Sklenka Skis**			
2				
3	**Data**			
4		**Product**		
5	**Department**	Jordanelle	Deercrest	Limitation (min.)
6	Fabrication	3.5	4	84
7	Finishing	1	1.5	21
8				
9	Profit/unit	$ 50.00	$ 65.00	
10				
11				
12	**Model**			
13		Jordanelle	Deercrest	
14	Quantity Produced	0	0	Hours Used
15	Fabrication	0	0	0
16	Finishing	0	0	0
17				
18				Excess Deercrest
19	Market mixture			0
20				
21				Total Profit
22	Profit Contribution	$ -	$ -	$ -

Add Constraint Dialog



Solver Results Dialog

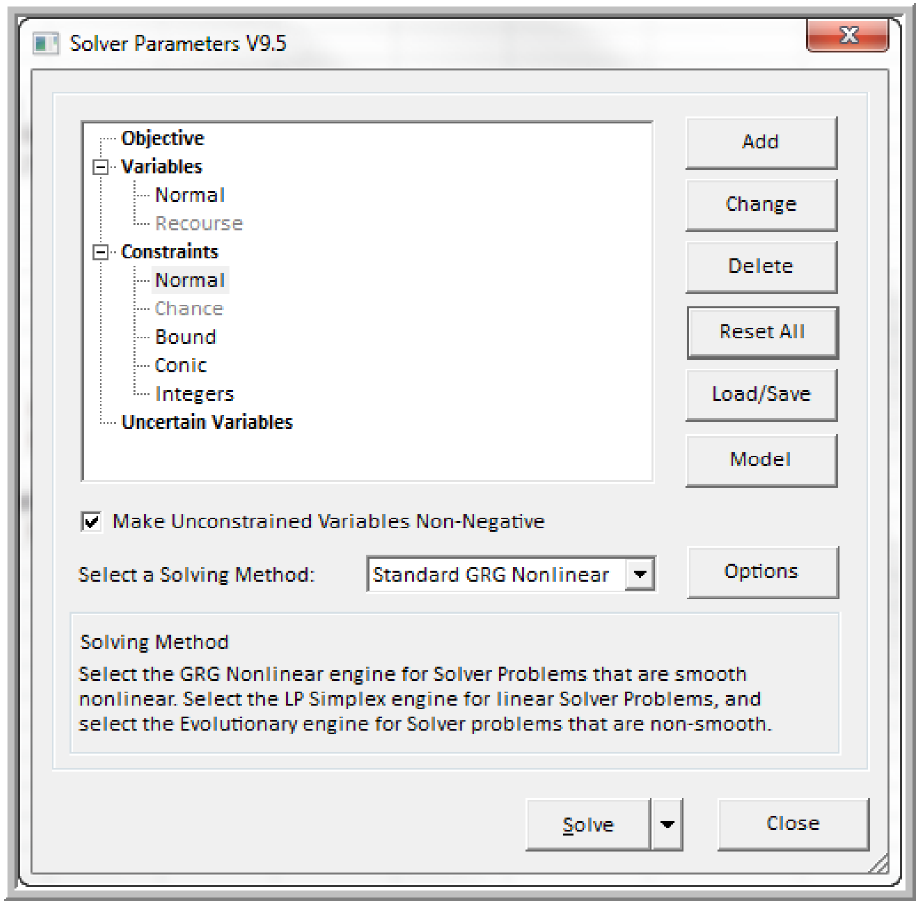


Select reports to save

SSC *Solver* Solution

	A	B	C	D
1	**Sklenka Skis**			
2				
3	**Data**			
4		**Product**		
5	**Department**	Jordanelle	Deercrest	Limitation (min.)
6	Fabrication	3.5	4	84
7	Finishing	1	1.5	21
8				
9	Profit/unit	$ 50.00	$ 65.00	
10				
11				
12	**Model**			
13		Jordanelle	Deercrest	
14	Quantity Produced	5.25	10.5	Hours Used
15	Fabrication	18.375	42	60.375
16	Finishing	5.25	15.75	21
17				
18				Excess Deercrest
19	Market mixture			0
20				
21				Total Profit
22	Profit Contribution	$ 262.50	$ 682.50	$ 945.00

Premium Solver Parameters Dialog



Possible Outcomes

- Unique optimal solution
- Alternate optimal solution
- Unbounded problem
  - “The Set Cell values do not converge”
- Infeasible problem
  - “Solver could not find a feasible solution”

Answer Report

Objective Cell (Max)

Cell	Name	Original Value	Final Value
$D$22	Profit Contribution Total Profit	0	945

Decision Variable Cells

Cell	Name	Original Value	Final Value
$B$14	Quantity Produced Jordanelle	0	5.25
$C$14	Quantity Produced Deercrest	0	10.5

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$D$15	Fabrication Hours Used	60.375	$D$15<=$D$6	Not Binding	23.625
$D$16	Finishing Hours Used	21	$D$16<=$D$7	Binding	0
$D$19	Market mixture Excess Deercrest	0	$D$19>=0	Binding	0

Slack and Binding Constraints

- Slack is the difference between the left- and right-hand sides of a constraint when the optimal solution is substituted for the variables.
- A constraint is binding if the slack is zero.
- Example:
  - Amount of Resource Used ≤ Amount Available
  - Amount of Resource Used + Amount of Resource Unused = Amount Available
  - Slack = Amount of Resource Unused

    = Amount Available - Amount of Resource Used

Sensitivity Report

Objective Cell (Max)

Cell	Name	Final Value
$D$22	Profit Contribution Total Profit	945

Decision Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$14	Quantity Produced Jordanelle	5.25	0	50	1E+30	6.6666668
$C$14	Quantity Produced Deercrest	10.5	0	65	10.0000002	90.00000013

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$D$15	Fabrication Hours Used	60.375	0	84	1E+30	23.625
$D$16	Finishing Hours Used	21	45	21	8.217391304	21
$D$19	Market mixture Excess Deercrest	0	-2.5	0	14	42

Interpreting the Sensitivity Report

- Reduced cost – how much the objective coefficient needs to change for a variable to become positive in the optimal solution
- Allowable Increase/Decrease – how much an individual objective coefficient can change before the optimal values of the decision variables will change
- Shadow price – how much the value of the objective function will change as the right-hand-side of a constraint is increased by 1.

Solver Limits Report

- Shows the lower limit and upper limit that each variable can assume while satisfying all constraints and holding all of the other variables constant.

	Objective	
Cell	**Name**	**Value**
$D$22	Profit Contribution Total Profit	$ 945.00

	Decision Variable		Lower	Objective	Upper	Objective
Cell	**Name**	**Value**	**Limit**	**Result**	**Limit**	**Result**
$B$14	Quantity Produced Jordanelle	5.25	0	$ 682.50	5.25	$ 945.00
$C$14	Quantity Produced Deercrest	10.5	10.5	$ 945.00	10.5	$ 945.00

How Solver Creates Names in Reports

- How you design your spreadsheet model will affect on how Solver creates the names used in the output reports.
- Poor spreadsheet design can make it difficult or confusing to interpret the Answer and Sensitivity reports.
- Solver assigns names to target cells, changing cells, and constraint function cells by concatenating the text in the first cell containing text to the left of the cell with the first cell containing text above it.

Example

	A	B	C	D
1	**Sklenka Skis**			
2				
3	**Data**			
4		**Product**		
5	**Department**	Jordanelle	Deercrest	Limitation (min.)
6	Fabrication	3.5	4	84
7	Finishing	1	1.5	21
8				
9	Profit/unit	$ 50.00	$ 65.00	
10				
11				
12	**Model**			
13		Jordanelle	Deercrest	
14	Quantity Produced	5.25	10.5	Hours Used
15	Fabrication	18.375	42	60.375
16	Finishing	5.25	15.75	21
17				
18				Excess Deercrest
19	Market mixture			0
20				
21				Total Profit
22	Profit Contribution	$ 262.50	$ 682.50	$ 945.00

	A	B	C	D	E	F	G
11							
12	Objective Cell (Max)						
13		**Cell**	**Name**	**Original Value**	**Final Value**		
14		$D$22	Profit Contribution Total Profit	0	945		
15							
16							
17	Decision Variable Cells						
18		**Cell**	**Name**	**Original Value**	**Final Value**		
19		$B$14	Quantity Produced Jordanelle	0	5.25		
20		$C$14	Quantity Produced Deercrest	0	10.5		
21							
22	Constraints						
23		**Cell**	**Name**	**Cell Value**	**Formula**	**Status**	**Slack**
24		$D$15	Fabrication Hours Used	60.375	$D$15<=$D$6	Not Binding	23.625
25		$D$16	Finishing Hours Used	21	$D$16<=$D$7	Binding	0
26		$D$19	Market mixture Excess Deercrest	0	$D$19>=0	Binding	0

Difficulties With Solver

- A poorly scaled model—one in which the parameters of the objective and constraint functions differ by several orders of magnitude — may cause round-off errors in internal computations or error messages such as "The conditions for Assume Linear Model are not satisfied." The values of the coefficients in the objective function and constraints, as well as the right hand sides, should not differ from each other by a factor of more than 1,000 or 10,000.
- Remedies
  - Keep the solution that *Solver* found and run *Solver* again starting from that solution.
  - *Solver* also has a checkbox for *Use Automatic Scaling* that can be used, especially if solver gives an error message that linearity is not satisfied.

Applications of Linear Optimization

- Product mix
- Process selection
- Blending
- Portfolio selection
- Transportation
- Multiperiod production planning
- Financial management
- Production/Marketing

TABLE 13.1 Generic Examples of Linear Optimization Models

Type of Model	Decision Variables	Objective Function	Typical Constraints
Product mix	Quantities of product to produce and sell	Maximize contribution to profit	Resource limitations (e.g., production time, labor, material); minimum sales requirements; maximum sales potential
Process selection	Quantities of product to make using alternative processes	Minimize cost	Demand requirements; resource limitations
Blending	Quantity of materials to mix to produce one unit of output	Minimize cost	Specifications on acceptable mixture
Portfolio selection	Proportions to invest in different financial instruments	Maximize future return or minimize risk exposure	Limit on available funds; sector requirements or restrictions; proportional relationships on investment mix
Transportation	Amount to ship between sources of supply and destinations	Minimize total transportation cost	Limited availability at sources; required demands met at destinations
Multiperiod production planning	Quantities of product to produce in each of several time periods; amount of inventory to hold between periods	Minimize total production and inventory costs	Limited production rates; material balance equations
Multiperiod financial management	Amounts to invest in short-term instruments	Maximize cash on hand	Cash balance equations; required cash obligations
Production/ marketing	Allocation of advertising expenditures; production quantities	Maximize profit	Budget limitation; production limitations; demand requirements

Constraint Categories

- Simple bounds
- Limitations
- Requirements
- Proportional relationships
- Balance constraints

Example: Process Selection

- A textile mill produces three types of fabrics. The decision facing the plant manager is on what type of loom to process each fabric during the next 13 weeks. The mill has 15 regular looms and 3 dobbie looms. Dobbie looms can be used to make all fabrics and are the only looms that can weave certain fabrics. After weaving, fabrics are sent to the finishing department and then sold. Any fabrics that cannot be woven in the mill because of limited capacity will be purchased from an external supplier, finished at the mill, and sold at the selling price. In addition to determining which looms to process the fabrics, the manager also needs to determine which fabrics to buy externally.

TABLE 13.2 Textile Production Data

Fabric	Demand (Yards)	Dobbie Loom Capacity (Yards/Hour)	Regular Loom Capacity (Yards/Hour)	Mill Cost ($/Yard)	Outsourcing Cost ($/Yard)
1	45,000	4.7	0.0	$0.65	$0.85
2	76,500	5.2	5.2	$0.61	$0.75
3	10,000	4.4	4.4	$0.50	$0.65

LP Model

- D_i = number of yards of fabric i to produce on dobbie looms, $i = 1, \ldots, 3$
- R_i = number of yards of fabric i to produce on regular looms, $i = 1, \ldots, 3$
- P_i = number of yards of fabric i to purchase from an outside supplier, $i = 1, \ldots, 3$

Min $0.65D_1 + 0.61D_2 + 0.50D_3 ++ 0.61R_2 + 0.50R_3 + 0.85P_1 + 0.75P_2 + 0.65P_3$

$D_1 + P_1 = 45{,}000$ (Demand, fabric 1)

$D_2 + R_2 + P_2 = 76{,}500$ (Demand, fabric 2)

$D_3 + R_3 + P_3 = 10{,}000$ (Demand, fabric 3)

$0.213D_1 + 0.192D_2 + 0.227D_3 \leq 6552$ (Dobbie loom production time)

$0.192R_2 + 0.227R_3 \leq 32{,}760$ (Regular loom production time)

Example: Blending

TABLE 13.3 Birdseed Nutrition Data

Ingredient	Protein %	Fat %	Fiber %	Cost/lb
Sunflower seeds	16.9	26	29	$0.22
White millet	12	4.1	8.3	$0.19
Kibble corn	8.5	3.8	2.7	$0.10
Oats	15.4	6.3	2.4	$0.10
Cracked corn	8.5	3.8	2.7	$0.07
Wheat	12	1.7	2.3	$0.05
Safflower	18	17.9	28.8	$0.26
Canary grass seed	11.9	4	10.9	$0.11

LP Model

Minimize $0.22X_1 + 0.19X_2 + 0.10X_3 + 0.10X_4 + 0.07X_5 + 0.05X_6 + 0.26X_7 + 0.11X_8$

$X_1 + X_2 + X_3 + X_4 + X_5 + X_6 + X_7 + X_8 = 1$ (proportion)

$0.169X_1 + 0.12X_2 + 0.085X_3 + 0.154X_4 + 0.085X_5 + 0.12X_6 + 0.18X_7 + 0.119X_8 \geq 0.13$ (protein)

$0.26X_1 + 0.041X_2 + 0.038X_3 + 0.063X_4 + 0.038X_5 + 0.017X_6 + 0.179X_7 + 0.04X_8 \geq 0.15$ (fat)

$0.29X_1 + 0.083X_2 + 0.027X_3 + 0.024X_4 + 0.027X_5 + 0.023X_6 + 0.288X_7 + 0.109X_8 \leq 0.14$ (fiber)

$X_i \geq 0$, for $i = 1,2,\ldots 8$

Portfolio Investment

- $500,000 to invest
- No more than $200,000 in any fund
- At least $50,000 in each of multinational and balanced funds
- At least 40% in Income Equity and Balanced funds
- Required average return of at least 5%
- Minimize risk

	Fund	Expected Annual Return	Risk Measure
1.	Innis Low-Priced Stock Fund	8.13%	10.57
2.	Innis Multinational Fund	9.02%	13.22
3.	Innis Mid-Cap Stock Fund	7.56%	14.02
4.	Innis Mortgage Fund	3.62%	2.39
5.	Innis Income Equity Fund	7.79%	9.30
6.	Innis Balanced Fund	4.40%	7.61

LP Model

Minimize Total Risk = $110.57X1 + 13.22X2 + 14.02X3 + 2.39X4 + 9.30X5 + 7.61X6/500{,}000$

$X1 + X2 + X3 + X4 + X5 + X6 = 500{,}000$

$(8.13X1 + 9.02X2 + 7.56X3 + 3.62X4 + 7.79X5 + 4.40X62)/500{,}000 \geq 5.00$

$X5 + X6 \geq .4(500{,}000)$

$X2 \geq 50{,}000$

$X6 \geq 50{,}000$

$Xj \leq 200{,}000$ for $j = 1, \ldots, 6$

$Xj \geq 0$ for $j = 1, \ldots, 6$

Transportation Problem

TABLE 13.4 Cost, Capacity, and Demand Data

	Distribution Center				
Plant	**Cleveland**	**Baltimore**	**Chicago**	**Phoenix**	**Capacity**
Marietta	$12.60	$14.35	$11.52	$17.58	1,200
Minneapolis	$9.75	$16.26	$8.11	$17.92	800
Demand	150	350	500	1,000	

LP Model

Minimize $12.60X_{11} + 14.35X_{12} + 11.52X_{13} + 17.58X_{14} + 9.75X_{21} + 16.26X_{22} + 8.11X_{23} + 17.92X_{24}$

$X_{11} + X_{12} + X_{13} + X_{14} \leq 1200$
$X_{21} + X_{22} + X_{23} + X_{24} \leq 800$

Supply constraints

$X_{11} + X_{21} = 150$
$X_{12} + X_{22} = 350$
$X_{13} + X_{23} = 500$
$X_{14} + X_{24} = 1000$

Demand constraints

$X_{ij} \geq 0$, for all i and j

Spreadsheet Implementation

	A	B	C	D	E	F
1	**Transportation Model**					
2						
3	**Data**					
4		**Distribution Center**				
5	**Plant**	Cleveland	Baltimore	Chicago	Phoenix	**Capacity**
6	Marietta	$ 12.60	$ 14.35	$ 11.52	$ 17.58	1200
7	Minneapolis	$ 9.75	$ 16.26	$ 8.11	$ 17.92	800
8	**Demand**	150	350	500	1000	
9						
10	**Model**					
11		**Distribution Center**				
12	**Plant**	Cleveland	Baltimore	Chicago	Phoenix	**Total shipped**
13	Marietta	0	0	0	0	0
14	Minneapolis	0	0	0	0	0
15	**Demand met**	0	0	0	0	
16						
17	**Total cost**					
18	$ -					

Sensitivity Report Formatting Issues

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$13	Marietta Cleveland	0	3	12.6	1E+30	3.19
$C$13	Marietta Baltimore	350	0	14.35	1.57	1E+30
$D$13	Marietta Chicago	0	4	11.52	1E+30	3.75
$E$13	Marietta Phoenix	850	0	17.58	0.34	1.57
$B$14	Minneapolis Cleveland	150	0	9.75	3.19	1E+30
$C$14	Minneapolis Baltimore	0	2	16.26	1E+30	1.57
$D$14	Minneapolis Chicago	500	0	8.11	3.75	1E+30
$E$14	Minneapolis Phoenix	150	0	17.92	1.57	0.34

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$F$13	Marietta Total shipped	1200	0	1200	150	0
$F$14	Minneapolis Total shipped	800	0	800	1E+30	0
$B$15	Demand met Cleveland	150	10	150	0	150
$C$15	Demand met Baltimore	350	15	350	0	150
$D$15	Demand met Chicago	500	8	500	0	500
$E$15	Demand met Phoenix	1000	18	1000	0	150

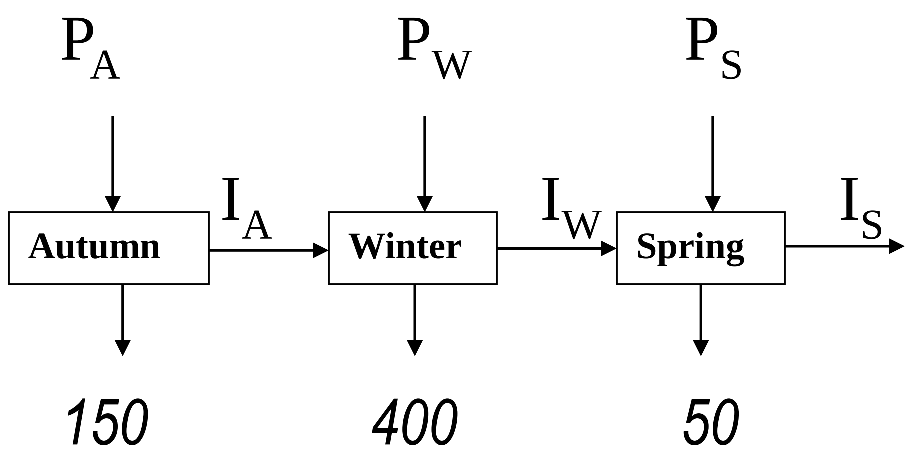
Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$13	Marietta Cleveland	0	3.19	12.6	1E+30	3.19
$C$13	Marietta Baltimore	350	0.00	14.35	1.57	1E+30
$D$13	Marietta Chicago	0	3.75	11.52	1E+30	3.75
$E$13	Marietta Phoenix	850	0.00	17.58	0.34	1.57
$B$14	Minneapolis Cleveland	150	0.00	9.75	3.19	1E+30
$C$14	Minneapolis Baltimore	0	1.57	16.26	1E+30	1.57
$D$14	Minneapolis Chicago	500	0.00	8.11	3.75	1E+30
$E$14	Minneapolis Phoenix	150	0.00	17.92	1.57	0.34

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$F$13	Marietta Total shipped	1200	-0.34	1200	150	0
$F$14	Minneapolis Total shipped	800	0.00	800	1E+30	0
$B$15	Demand met Cleveland	150	9.75	150	0	150
$C$15	Demand met Baltimore	350	14.69	350	0	150
$D$15	Demand met Chicago	500	8.11	500	0	500
$E$15	Demand met Phoenix	1000	17.92	1000	0	150

Multiperiod Planning



Minimize $11P_A + 14P_W + 12.50P_S + 1.20I_A + 1.20I_W + 1.20I_S$

$P_A - I_A = 150$

$P_W + I_A - I_W = 400$

$P_S + I_W - I_S = 50$

$P_i \geq 0$, for all i

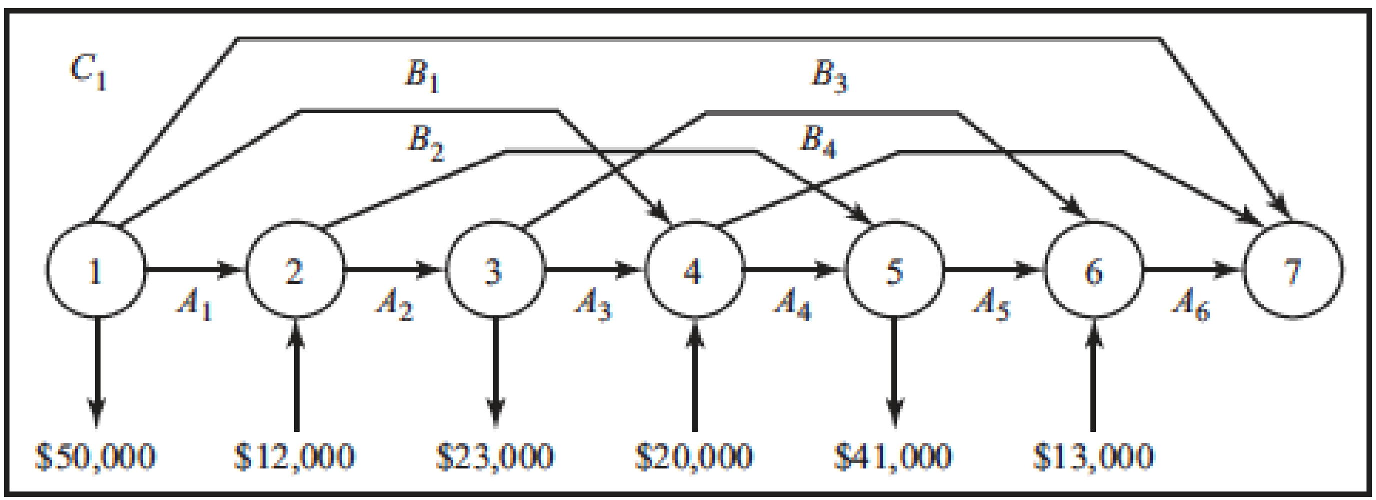
$I_i \geq 0$, for all i

Multiperiod Financial Planning

- A financial manager at D.A. Branch & Sons must ensure that funds are available to pay company expenditures in the future, but would also like to maximize investment income. Three short-term investment options are available over the next six months: *A*, a one-month CD that pays 0.25%, available each month; *B*, a three-month CD that pays 1.00%, available at the beginning of the first four months; and *C*, a six-month CD that pays 2.3%, available in the first month. The net expenditures for the next six months are forecast as $50,000, ($12,000), $23,000, ($20,000), $41,000, ($13,000). Amounts in parentheses indicate a net inflow of cash. The company must maintain a cash balance of at least $10,000 at the end of each month. The company currently has $200,000 in cash.

Model Development

- A_i = amount ($) to invest in a one-month CD at the start of month i
- B_i = amount ($) to invest in a three-month CD at the start of month i
- C_i = amount ($) to invest in a six-month CD at the start of month i



LP Model

$$\text{Maximize } 1.0025A_6 + 1.00B_4 + 1.023C_1$$

Subject to

$$
\begin{aligned}
200{,}000 - (A_1 + B_1 + C_1 + 50{,}000) &\geq 10{,}000 && \text{(Month 1)} \\
1.0025A_1 + 12{,}000 - (A_2 + B_2) &\geq 10{,}000 && \text{(Month 2)} \\
1.0025A_2 - (A_3 + B_3 + 23{,}000) &\geq 10{,}000 && \text{(Month 3)} \\
1.0025A_3 + 1.00B_1 + 20{,}000 - (A_4 + B_4) &\geq 10{,}000 && \text{(Month 4)} \\
1.0025A_4 + 1.00B_2 - (A_5 + 41{,}000) &\geq 10{,}000 && \text{(Month 5)} \\
1.0025A_5 + 1.00B_3 + 13{,}000 - A_6 &\geq 10{,}000 && \text{(Month 6)} \\
A_i, B_i, C_i &\geq 0, && \text{for all } i
\end{aligned}
$$

Models With Bounded Variables

- Bounded variables are listed in the Adjustable Cells section.
- Reduced costs may be interpreted as shadow prices.

Example: J&M Manufacturing

TABLE 13.5 J&M Manufacturing Data

Grill Model	Selling Price/Unit	Variable Cost/Unit	Minimum Monthly Sales Requirements	Maximum Monthly Sales Potential
A	\$250	\$210	0	4,000
B	\$300	\$240	0	3,000
C	\$400	\$300	500	2,000
D	\$650	\$520	500	1,000

Department	*A*	*B*	*C*	*D*	Hours Available
Stamping	40	30	10	10	320
Painting		20	10	10	320
Assembly	25	15	15	12	320
Inspection	20	20	25	15	320
Packaging	50	40	40	30	320

Maximize Total profit=(250 – 210)A + (300 – 240)B + (400 – 300)C + (650 – 520)D = 40A + 60B + 100C + 130D

A/40 + B/30 + C/10 + D/10 ≤ 320 (Stamping)

B/20 + C/10 + D/10 ≤ 320 (Painting)

A/25 + B/15 + C/15 + D/12 ≤ 320 (Assembly)

A/20 + B/20 + C/25 + D/15 ≤ 320 (Inspection)

A/50 + B/40 + C/40 + D/30 ≤ 320 (Packaging)

Spreadsheet and Solver Models

	A	B	C	D	E	F
1	**J&M Manufacturing**					
2						
3	**Data**					
4	Grill model	Selling price	Variable cost	Min Sales	Max Sales	
5	A	$ 250.00	$ 210.00	0	4000	
6	B	$ 300.00	$ 240.00	0	3000	
7	C	$ 400.00	$ 300.00	500	2000	
8	D	$ 650.00	$ 520.00	500	1000	
9						
10	Production rates (hours/unit)	A	B	C	D	Hours Available
11	Stamping	40	30	10	10	320
12	Painting		20	10	10	320
13	Assembly	25	15	15	12	320
14	Inspection	20	20	25	15	320
15	Packaging	50	40	40	30	320
16						
17	**Model**					
18	Department	A	B	C	D	Hours Used
19	Stamping	96.429	0.000	123.571	100.000	320.000
20	Painting		0.000	123.571	100.000	223.571
21	Assembly	154.286	0.000	82.381	83.333	320.000
22	Inspection	192.857	0.000	49.429	66.667	308.952
23	Packaging	77.143	0.000	30.893	33.333	141.369
24						
25	Number produced	3857.14	0.00	1235.71	1000.00	
26	Net profit/unit	$ 40.00	$ 60.00	$ 100.00	$ 130.00	Total Profit
27	Profit contribution	$ 154,285.71	$ -	$ 123,571.43	$ 130,000.00	$ 407,857.14

- **Objective**
  - $F$27 (Max)
- **Variables**
  - Normal
    - $B$25:$E$25
  - Recourse
- **Constraints**
  - Normal
    - $F$19:$F$23 <= $F$11:$F$15
  - Chance
  - Bound
    - $B$25:$E$25 <= $E$5:$E$8
    - $B$25:$E$25 >= $D$5:$D$8
  - Conic
  - Integers
- **Uncertain Variables**

J&M Answer Report

Objective Cell (Max)

Cell	Name	Original Value	Final Value
$F$27	Profit contribution Total Profit	0	407857.1429

Decision Variable Cells

Cell	Name	Original Value	Final Value
$B$25	Number produced A	0	3857.142857
$C$25	Number produced B	0	0
$D$25	Number produced C	0	1235.714286
$E$25	Number produced D	0	1000

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$F$19	Stamping Hours Used	320.000	$F$19<=$F$11	Binding	0
$F$20	Painting Hours Used	223.571	$F$20<=$F$12	Not Binding	96.42857143
$F$21	Assembly Hours Used	320.000	$F$21<=$F$13	Binding	0
$F$22	Inspection Hours Used	308.952	$F$22<=$F$14	Not Binding	11.04761905
$F$23	Packaging Hours Used	141.369	$F$23<=$F$15	Not Binding	178.6309524
$B$25	Number produced A	3857.142857	$B$25<=$E$5	Not Binding	142.8571429
$C$25	Number produced B	0	$C$25<=$E$6	Not Binding	3000
$D$25	Number produced C	1235.714286	$D$25<=$E$7	Not Binding	764.2857143
$E$25	Number produced D	1000	$E$25<=$E$8	Binding	0
$B$25	Number produced A	3857.142857	$B$25>=$D$5	Not Binding	3857.142857
$C$25	Number produced B	0	$C$25>=$D$6	Binding	0
$D$25	Number produced C	1235.714286	$D$25>=$D$7	Not Binding	735.7142857
$E$25	Number produced D	1000	$E$25>=$D$8	Not Binding	500

J&M Sensitivity Report

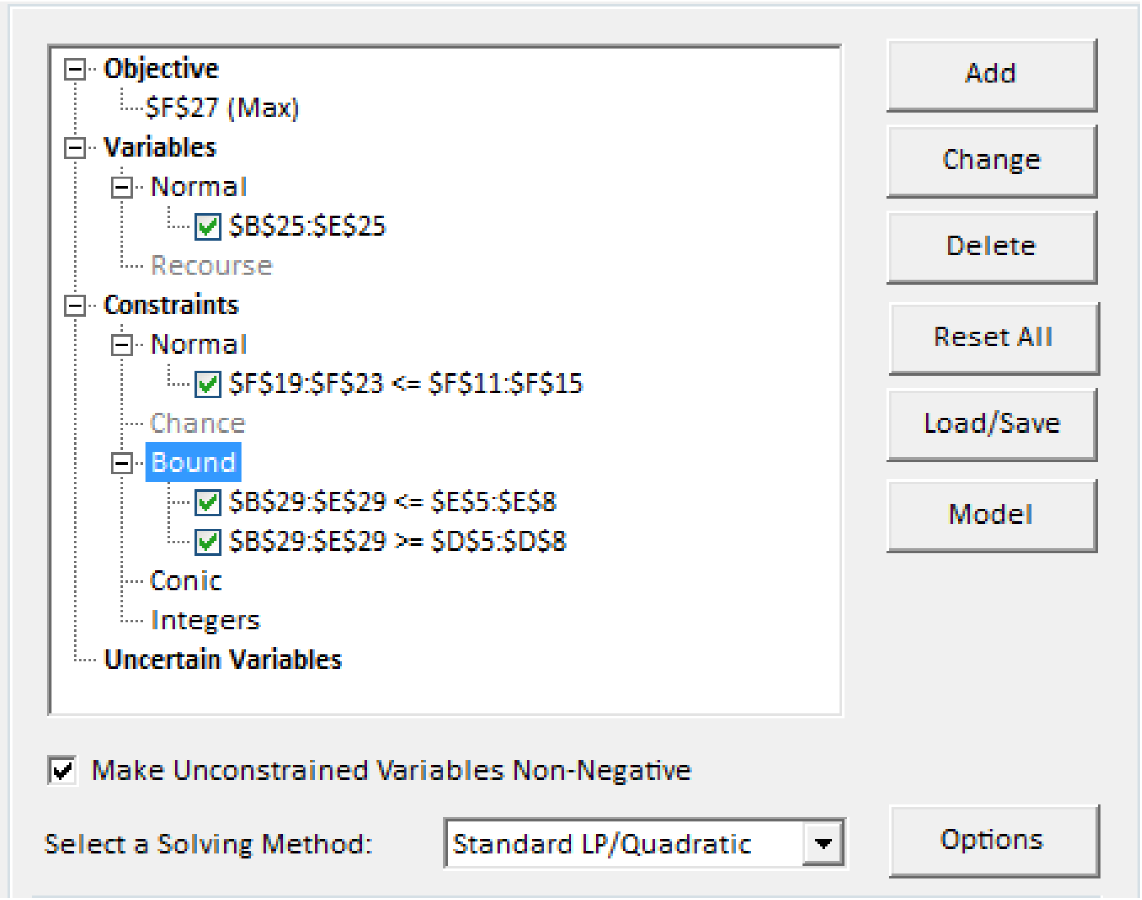
	A	B	C	D	E	F	G	H
4								
5	Objective Cell (Max)							
6		Cell	Name	Final Value				
7		$F$27	Profit contribution Total Profit	407857.1429				
8								
9	Decision Variable Cells							
10				Final	Reduced	Objective	Allowable	Allowable
11		Cell	Name	Value	Cost	Coefficient	Increase	Decrease
12		$B$25	Number produced A	3857.142857	0	40	20.00000004	1.000000042
13		$C$25	Number produced B	0	-1.904761905	60	1.904761905	1E+30
14		$D$25	Number produced C	1235.714286	0	100	13.33333389	33.33333339
15		$E$25	Number produced D	1000	19.28571429	130	1E+30	19.28571429
16								
17	Constraints							
18				Final	Shadow	Constraint	Allowable	Allowable
19		Cell	Name	Value	Price	R.H. Side	Increase	Decrease
20		$F$19	Stamping Hours Used	320.000	571.429	320	44.58333333	5
21		$F$20	Painting Hours Used	223.571	0.000	320	1E+30	96.42857143
22		$F$21	Assembly Hours Used	320.000	642.857	320	3.333333333	71.33333333
23		$F$22	Inspection Hours Used	308.952	0.000	320	1E+30	11.04761905
24		$F$23	Packaging Hours Used	141.369	0.000	320	1E+30	178.6309524

Modeling Trick

- Using reduced costs as shadow prices can be a bit confusing.
- In your spreadsheet model, *define a new set of cells for any decision variables that have upper or lower bound constraints by referencing (not copying) the original changing cells*.
- In the Solver model, use this auxiliary variable cell to define the bound constraint.

	A	B	C	D	E	F
24						
25	Number produced	0	0	0	0	
26	Net profit/unit	=B5-C5	=B6-C6	=B7-C7	=B8-C8	Total Profit
27	Profit contribution	=B25*B26	=C25*C26	=D25*D26	=E25*E26	=SUM(B27:E27)
28						
29	Auxiliary variable	=B25	=C25	=D25	=E25	

Solver Model with Auxiliary Variables



Solver Results Using Auxiliary Variable Approach

Objective Cell (Max)

Cell	Name	Final Value
$F$27	Profit contribution Total Profit	407857.1429

Decision Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$25	Number produced A	3857.1	0.0	40	20.00000004	1.000000053
$C$25	Number produced B	0.0	-1.9	60	1.904761905	1E+30
$D$25	Number produced C	1235.7	0.0	100	13.33333403	33.33333339
$E$25	Number produced D	1000.0	0.0	130	1E+30	19.28571439

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$B$29	Auxiliary variable A	3857.14	0.00	4000	1E+30	142.8571429
$C$29	Auxiliary variable B	0.00	0.00	3000	1E+30	3000
$D$29	Auxiliary variable C	1235.71	0.00	2000	1E+30	764.2857143
$E$29	Auxiliary variable D	1000.00	19.29	1000	895.6521739	200
$B$29	Auxiliary variable A	3857.14	0.00	0	3857.142857	1E+30
$C$29	Auxiliary variable B	0.00	0.00	0	0	1E+30
$D$29	Auxiliary variable C	1235.71	0.00	500	735.7142857	1E+30
$E$29	Auxiliary variable D	1000.00	0.00	500	500	1E+30
$F$19	Stamping Hours Used	320.000	571.429	320	44.58333333	5
$F$20	Painting Hours Used	223.571	0.000	320	1E+30	96.42857143
$F$21	Assembly Hours Used	320.000	642.857	320	3.333333333	71.33333333
$F$22	Inspection Hours Used	308.952	0.000	320	1E+30	11.04761905
$F$23	Packaging Hours Used	141.369	0.000	320	1E+30	178.6309524

Production/Marketing Allocation Model

maximize profit = (\$6.25S + \$5.25M) – (\$1.60S + \$1.40M + As + Am)

= 4.65S + 4.85M - As – Am

Budget cannot be exceeded:

$\$1.60S + \$1.40M + As + Am \le \$50{,}000$

Contractual requirements must be met:

$0.4 \ldots S/(S + M) \le 0.7$

or, expressed in linear form:

$0.6S - 0.4M \ge 0$ and $0.3S - 0.7M \le 0$

Production must not exceed demand:

$S \le 1{,}000 + 5As$

$M \le 2{,}000 + 8Am$

Nonnegativity

Spreadsheet Model with Solver Results

	A	B	C	D	E	F
1	**Walker Wines Product Mix**					
2						
3	**Data**					
4			Shiraz	Merlot		
5		Cost/bottle	$ 1.60	$ 1.40		
6		Price/bottle	$ 6.25	$ 5.25		
7						
8		Base demand	1,000.00	2,000.00		
9		Increase/$1 Adv.	5	8		
10		Min. percent requirement	40%			
11		Max. percent limitation	70%			
12						
13		Total Budget	$ 50,000.00			
14						
15	**Model**					
16						
17		Total profit				
18		$ 124,775.84				
19			Shiraz	Merlot	Total	
20		Unit profit	$ 4.65	$ 3.85		
21		Advertising dollars	$ 3,912.37	$ 851.53	$ 4,763.90	
22		Demand	20,561.86	8,812.23	29,374.09	
23		Quantity produced	20,561.86	8,812.23	29,374.09	
24						
25		Min. percent requirement	8812.227074	>=	0	
26		Max. percent limitation	0	<=	0	
27						
28					Used	Unused
29		Budget	$ 36,811.35	$ 13,188.65	$ 50,000.00	$ -

	A	B	C	D	E	F
1	**Walker Wines Product Mix**					
2						
3	**Data**					
4			Shiraz	Merlot		
5		Cost/bottle	1.6	1.4		
6		Price/bottle	6.25	5.25		
7						
8		Base demand	1000	2000		
9		Increase/$1 Adv.	5	8		
10		Min. percent requirement	0.4			
11		Max. percent limitation	0.7			
12						
13		Total Budget	50000			
14						
15	**Model**					
16						
17		Total profit				
18		=(C20*C23)+(D20*D23)-C21-D21				
19			Shiraz	Merlot	Total	
20		Unit profit	=C6-C5	=D6-D5		
21		Advertising dollars	0	0	=SUM(C21:D21)	
22		Demand	=C8+(C9*C21)	=D8+(D9*D21)	=SUM(C22:D22)	
23		Quantity produced	0	0	=SUM(C23:D23)	
24						
25		Min. percent requirement	=(1-C10)*C23-C10*D23	>=	0	
26		Max. percent limitation	=(1-C11)*C23-C11*D23	<=	0	
27						
28					Used	Unused
29		Budget	=C21+(C23*C5)	=D21+(D23*D5)	=SUM(C29:D29)	=C13-E29

Sensitivity Report

Objective Cell (Max)

Cell	Name	Final Value
$B$18	Total profit	124775.837

Decision Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$C$21	Advertising dollars Shiraz	$ 3,912.37	$ -	-1	3.771791052	0.266394356
$D$21	Advertising dollars Merlot	$ 851.53	$ -	1	0.36111235	112.8666705
$C$23	Quantity produced Shiraz	20,561.86	0.00	4.65	1E+30	0.053278871
$D$23	Quantity produced Merlot	8,812.23	0.00	3.85	0.045139044	14.10833381

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$C$23	Quantity produced Shiraz	20,561.86	0.69	1000	1E+30	252250
$C$25	Min. percent requirement Shiraz	8812.227074	0	0	8812.227074	1E+30
$C$26	Max. percent limitation Shiraz	0	0.047307132	0	1E+30	9256.880734
$D$23	Quantity produced Merlot	8,812.23	0.43	2000	1E+30	403600
$E$29	Budget Used	$ 50,000.00	$ 2.46	50000	1E+30	50450

Changing Shiraz Grape Cost to $1.65

	A	B	C	D	E	F
1	**Walker Wines Product Mix**					
2						
3	**Data**					
4			Shiraz	Merlot		
5		Cost/bottle	$ 1.65	$ 1.40		
6		Price/bottle	$ 6.25	$ 5.25		
7						
8		Base demand	1,000.00	2,000.00		
9		Increase/$1 Adv.	5	8		
10		Min. percent requirement	40%			
11		Max. percent limitation	70%			
12						
13		Total Budget	$ 50,000.00			
14						
15	**Model**					
16						
17		Total profit				
18		$ 122,231.12				
19			Shiraz	Merlot	Total	
20		Unit profit	$ 4.60	$ 3.85		
21		Advertising dollars	$ 2,238.67	$ 2,036.25	$ 4,274.92	
22		Demand	12,193.35	18,290.03	30,483.38	
23		Quantity produced	12,193.35	18,290.03	30,483.38	
24						
25		Min. percent requirement	0	>=	0	
26		Max. percent limitation	-9145.01511	<=	0	
27						
28					Used	Unused
29		Budget	$ 22,357.70	$ 27,642.30	$ 50,000.00	$ -

Interpreting Sensitivity Analysis

- Sensitivity report assumes that *all other model parameters are held constant*!
- In this case, the parameter that changed is also a part of the budget constraint, violating this assumption.