UNIT 5
Confidence Interval & Hypothesis Testing of Differences – Two Sample Tests:

There are many instances when one wishes to compare two sample means, variances, or proportions to determine whether the null hypothesis should be rejected. Yet in other situations, the objective may be to estimate such differences by using confidence intervals. For example, the average lifetimes of two different brands of televisions might be compared to see whether there is any difference between those average lifetimes. Two different over-the counter headache medicines may be tested to see whether one brand is more effective than the other. Or the popularity rates of two different politicians may be compared to see who is more popular.

1. Confidence Interval and Hypothesis Testing for the Difference of Two Population Means $\boldsymbol{\mu_1 - \mu_2}$:

(i) Using the Z Distribution:
The Z distribution can be used to create a confidence interval for the difference of two population means, or a Z-test can be performed to test the difference of two population means if the following conditions are satisfied:

(a) Both samples selected from the two groups are large; i.e., $\boldsymbol{n_1 \geq 30}$ and $\boldsymbol{n_2 \geq 30}$. If any of the two samples is small, then both populations must be normally distributed.

(b) The two selected Samples are independent; meaning that the selection in one group in no way influences the selection in the other group.

(c) The standard deviations of both populations ($\boldsymbol{\sigma_1}$ and $\boldsymbol{\sigma_2}$) are known.

In this case, $\boldsymbol{\overline{x}_1 - \overline{x}_2}$ will be normally distributed, and the standard deviation of this distribution (the ***standard error***) will be:

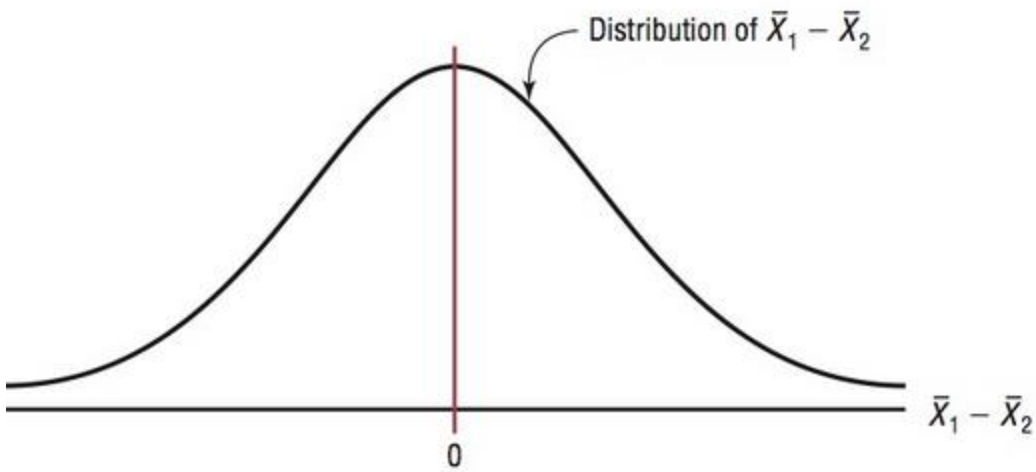
$$\boldsymbol{\sigma_{\overline{x}_1 - \overline{x}_2} = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

For creating a confidence interval for $\boldsymbol{\mu_1 - \mu_2}$ and given a confidence level $\boldsymbol{c}$, the standard normal value $\boldsymbol{z_c}$ is then calculated by using the following Excel formula:

$$= \mathbf{NORM.S.INV}\left(\frac{\mathbf{1} + \boldsymbol{c}}{\mathbf{2}}\right)$$

The confidence interval can now be constructed according to the following inequality:

$$\boldsymbol{(\overline{x}_1 - \overline{x}_2) - z_c \sigma_{\overline{x}_1 - \overline{x}_2} \leq \mu_1 - \mu_2 \leq (\overline{x}_1 - \overline{x}_2) + z_c \sigma_{\overline{x}_1 - \overline{x}_2}}$$



The procedure for constructing a confidence interval is summarized in the following template:

Confidence Interval for $\boldsymbol{\mu_1 - \mu_2}$:

Large Independent Samples ($\boldsymbol{n_1 \geq 30}$ and $\boldsymbol{n_2 \geq 30}$)
$\boldsymbol{\sigma_1}$ and $\boldsymbol{\sigma_2}$ are known
Sample Mean1= $\boldsymbol{\bar{x}_1}$, Sample Mean2 = $\boldsymbol{\bar{x}_2}$
Standard Error:

$$\boldsymbol{\sigma_{\bar{x}_1 - \bar{x}_2} = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

Confidence Level = $\boldsymbol{c}$

$$\boldsymbol{z_c}: \quad = \mathbf{NORM.S.INV}\left(\frac{\mathbf{1+c}}{\mathbf{2}}\right)$$

Margin of Error $\boldsymbol{E = z_c \sigma_{\bar{x}_1 - \bar{x}_2}}$
CI Lower Limit = $\boldsymbol{(\bar{x}_1 - \bar{x}_2) - E}$
CI Upper Limit = $\boldsymbol{(\bar{x}_1 - \bar{x}_2) + E}$

Similarly, the procedures for performing a hypothesis testing about the difference $\boldsymbol{\mu_1 - \mu_2}$ for right-tailed, left-tailed, and two-tailed problems are summarized in the following three templates:

Z-Test Hypothesis Testing for $\mu_1 - \mu_2$:

Right Tailed problem

Large Independent Samples ($n_1 \geq 30$ and $n_2 \geq 30$)

σ_1 and σ_2 are known

Null Hypothesis: H_0: $\mu_1 - \mu_2 \leq \mu_0$

Alternative Hypothesis: H_A: $\mu_1 - \mu_2 > \mu_0$

μ_0 = ***Hypothesized Difference***

Sample Mean₁= $\bar{x}_1$, Sample Mean₂ = $\bar{x}_2$

Standard Error:

$$\sigma_{\bar{x}_1 - \bar{x}_2} = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

Test Statistic $z = \frac{(\bar{x}_1 - \bar{x}_2) - \mu_0}{\sigma_{\bar{x}_1 - \bar{x}_2}}$

Level of significance $= \alpha$

Critical Z value (z^*) = z with an area of α on its right.

Use Excel for ***critical z^* value***:

$$= \text{NORM.S.INV}(1 - \alpha)$$

P value = $P(Z \geq z)$

Use Excel for ***P value***:

$$= 1 - \text{NORM.S.DIST}(z, 1)$$

Decision:

Method 1: ***Reject Ho if*** $sample\ z \geq Critical\ z^*$

Method 2: ***Reject Ho if*** $Pvalue \leq \alpha$

Z-Test Hypothesis Testing for $\mu_1 - \mu_2$:

Left Tailed problem

Large Independent Samples ($n_1 \geq 30$ and $n_2 \geq 30$)

σ_1 and σ_2 are known

Null Hypothesis: H_0: $\mu_1 - \mu_2 \geq \mu_0$

Alternative Hypothesis: H_A: $\mu_1 - \mu_2 < \mu_0$

μ_0 = ***Hypothesized Difference***

Sample Mean1= $\bar{x}_1$, Sample Mean2 = $\bar{x}_2$

Standard Error:

$$\sigma_{\bar{x}_1-\bar{x}_2} = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

Test Statistic $z = \frac{(\bar{x}_1-\bar{x}_2)-\mu_0}{\sigma_{\bar{x}_1-\bar{x}_2}}$

Level of significance $= \alpha$

Critical *Z* value (z^*) = *z* with an area of α on its left.

Use Excel for ***critical*** z^* ***value***:

$$= \text{NORM.S.INV}(\alpha)$$

P value = $P(Z \leq z)$

Use Excel for ***P value***:

$$= \text{NORM.S.DIST}(z, 1)$$

Decision:

Method 1: ***Reject Ho if*** $sample\ z \leq Critical\ z^*$

Method 2: ***Reject Ho if*** $Pvalue \leq \alpha$

Z-Test Hypothesis Testing for $\mu_1 - \mu_2$:

Two Tailed problem

Large Independent Samples ($n_1 \geq 30$ and $n_2 \geq 30$)

σ_1 and σ_2 are known

Null Hypothesis: H_0: $\mu_1 - \mu_2 = \mu_0$

Alternative Hypothesis: H_A: $\mu_1 - \mu_2 \neq \mu_0$

μ_0 = ***Hypothesized Difference***

Sample Mean1= $\bar{x}_1$, Sample Mean2 = $\bar{x}_2$

Standard Error:

$$\sigma_{\bar{x}_1 - \bar{x}_2} = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

Test Statistic $z = \frac{(\bar{x}_1 - \bar{x}_2) - \mu_0}{\sigma_{\bar{x}_1 - \bar{x}_2}}$

Level of significance $= \alpha$

Critical Z value 1 (z_1^*) **=** ***z*** **with an area of $\alpha/2$ on its right.**

Critical Z value 2 (z_2^*) **=** ***z*** **with an area of $\alpha/2$ on its left.**

Use Excel for critical z_1^* and z_2^* values:

z_1^*: $= \mathrm{NORM.S.INV}(1 - \alpha/2)$

z_2^*: $= \mathrm{NORM.S.INV}(\alpha/2)$

P value **=** $2\ P(Z \geq z)$ **if the test statistic** z **is positive**

P value **=** $2\ P(Z \leq z)$ **if the test statistic** z **is negative**

Use Excel for ***P values***:

If $z > 0$, $= 2 * \left(1 - \mathrm{NORM.S.DIST}(z\,, 1)\right)$

If $z < 0$, $= 2 * \ \mathrm{NORM.S.DIST}(z\,, 1)$

Decision:

Method 1: Reject Ho if:

test statistic $z \geq z_1^*$ **for** $z > 0$

test statistic $z \leq z_2^*$ **for** $z < 0$

Method 2: ***Reject Ho if*** $Pvalue \leq \alpha$

Example:

A study using two random samples of 40 people each found that the average amount of time those in the age group of 16–22 years spent per week on watching TV was 27.4 hours, and those in the age group of 26–35 years spent 26.2 hours. Assume that the population standard deviation for those in the first age group found by previous studies is 2.3 hours, and the population standard deviation of those in the second group found by previous studies was 1.8 hours. At $\boldsymbol{\alpha = 0.05}$, can it be concluded that there is a significant difference in the average times each group spends on watching TV?

solution:

Step 1: State the problem hypotheses:

Let $\boldsymbol{\mu_1}$ denote the average weekly time the population of the age group 16–22 years watches TV, and let $\boldsymbol{\mu_2}$ denote the average weekly time the population of the age group 26–35 years watches TV. Then,

$\boldsymbol{H_0}$: $\boldsymbol{\mu_1 = \mu_2}$, or $\boldsymbol{H_0}$: $\boldsymbol{\mu_1 - \mu_2 = 0}$
$\boldsymbol{H_A}$: $\boldsymbol{\mu_1 \neq \mu_2}$, or $\boldsymbol{H_A}$: $\boldsymbol{\mu_1 - \mu_2 \neq 0}$ (claim)
Therefore, this hypothesis problem is a two-tailed problem.

Step 2: Calculate the standard error $\boldsymbol{\sigma_{\bar{x}_1 - \bar{x}_2}}$, and the test statistic $\boldsymbol{Z}$:

$$\boldsymbol{\sigma_{\bar{x}_1 - \bar{x}_2} = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} = \sqrt{\frac{2.3^2}{40} + \frac{1.8^2}{40}} = 0.4618}$$

Test Statistic $\boldsymbol{z = \frac{(27.4-26.2)-0}{0.4618} = 2.599 \approx 2.6}$

Step 3:

Method 1: Compute the critical values, and compare the test statistic with the appropriate critical value:

For two-tailed problems, there are two critical values: one with an area of $\boldsymbol{\alpha/2}$ on its right ($\boldsymbol{z_1^*}$), and the other with an area of $\boldsymbol{\alpha/2}$ on its left ($\boldsymbol{z_2^*}$).

For $\boldsymbol{z_1^*}$, enter: $= \mathbf{NORM.S.INV(1 - 0.05/2)}$ to obtain: $\boldsymbol{z_1^* = 1.96.}$
For $\boldsymbol{z_2^*}$, enter: $= \mathbf{NORM.S.INV(0.05/2)}$ to obtain: $\boldsymbol{z_2^* = -1.96.}$

However, since the test statistic $\boldsymbol{z}$ is positive, then it should be compared with $\boldsymbol{z_1^*}$. According to the decision rule, we should reject the null hypothesis if $\boldsymbol{z \geq z_1^*}$. This is exactly the case in here since $\boldsymbol{2.6 > 1.96}$.

Method 2: Compute the Pvalue and compare it with the level of significance $\boldsymbol{\alpha}$:

Since the test statistic $\boldsymbol{z}$ is positive, then the ***P value*** will be twice the area on its right; that is:
P value = $\boldsymbol{2\ P(Z \geq z)}$
This can be calculated in Excel by entering:

$$= \mathbf{2 * (1 - NORM.S.DIST(2.6\,, 1))}$$

Thus, we obtain:

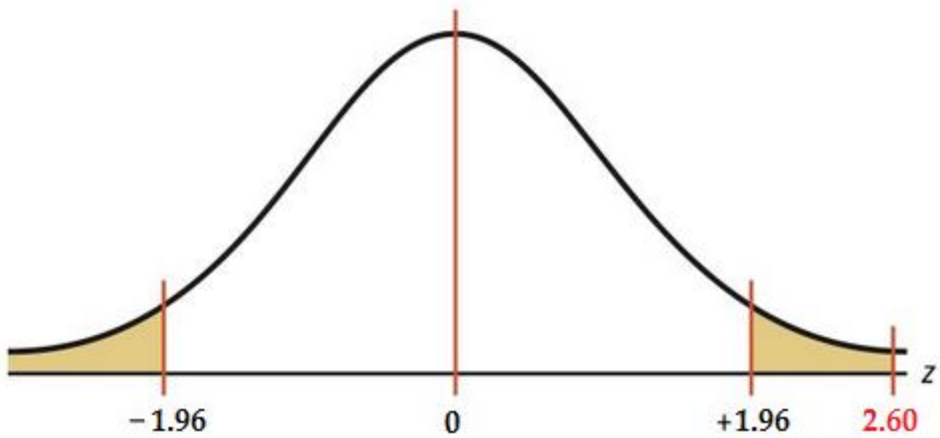
$$\boldsymbol{Pvalue = 0.009}$$

According to the decision rule, we should reject the null hypothesis if $\boldsymbol{Pvalue} \leq \boldsymbol{\alpha}$. This is exactly the case in here since $\mathbf{0.009 < 0.05}$.

Step 4: Make a decision, and describe that decision in the context of the problem:
Both methods 1 and 2 of the previous step have resulted in rejecting the Null hypothesis. This decision is then described as follows:

At 0.05 level of significance, there exists sufficient evidence to conclude that average weekly hours of watching TV for the age group 16–22 years is different than that of the age group 26–35.

The chart below shows the location of the critical values and the test statistic and the P value of the test.



Technology Note:

Excel has a two-sample z test included in the Data Analysis Add-in. To perform a z test for the difference between the means of two populations, given two independent samples, do this:

1. Enter the first sample data set into column **A** and the second sample data set into column **B**.
2. If the population variances are not known but $\boldsymbol{n_1 \geq 30}$ and $\boldsymbol{n_2 \geq 30}$ for the two samples, use the formulas **=VAR.S(data range)** to calculate the variances of the sample data sets.
3. Select the **Data** tab from the toolbar. Then select **Data Analysis**.
4. In the Analysis Tools box, select **z test: Two sample for Means**.
5. Type the ranges for the data in columns **A** and **B** and type a value (usually 0) for the Hypothesized Difference $\boldsymbol{\mu_0}$.
6. If the population variances are known, type them for **Variable 1** and **Variable 2**. Otherwise, use the sample variances obtained in step 2.
7. Specify the level of significance $\boldsymbol{\alpha}$.
8. Specify a location for the output, and click **OK**.

Note: In situations when the two samples are not given explicitly but the sample means $\overline{x}_1$ and $\overline{x}_2$ are known, we can first create a total of n_1 of values $\overline{x}_1$ and a total of n_2 of values $\overline{x}_2$ in respectively columns **A** and **B** before using the two-sample z test of Data Analysis.

(ii) Using the T Distribution (Independent Samples): The T distribution should be used to construct confidence intervals or to perform hypothesis testing when the following conditions are satisfied:

(a) The two samples are independent.
(b) When any of the two samples sizes is smaller than 30, then samples are assumed to be taken from two normally or approximately normally distributed populations.
(c) The population variances are not known
(d) The population variances are not equal.

In this case, $\overline{x}_1 - \overline{x}_2$ will have a T distribution, and the standard deviation of this distribution (the ***standard error***) will be:

$$\sigma_{\overline{x}_1-\overline{x}_2} = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

For creating a confidence interval for $\mu_1 - \mu_2$ and given a confidence level c, the t_c value from the T distribution is calculated by using the following Excel formula:

$$= \mathbf{T.INV}\left(\frac{1+c}{2}, df\right),$$

where the degrees of freedom ($\boldsymbol{df}$) of the T distribution equals:

$$\boldsymbol{df = min\{(n_1 - 1)\,, (n_2 - 1)\}}$$

The confidence interval can now be constructed according to the following inequality:

$$\boldsymbol{(\bar{x}_1 - \bar{x}_2) - t_c\sigma_{\bar{x}_1-\bar{x}_2} \le \mu_1 - \mu_2 \le (\bar{x}_1 - \bar{x}_2) + t_c\sigma_{\bar{x}_1-\bar{x}_2}}$$

The procedure for constructing a confidence interval for this case is summarized in the following template:

Confidence Interval for $\boldsymbol{\mu_1 - \mu_2}$:

Samples are independent
$\boldsymbol{\sigma_1}$ and $\boldsymbol{\sigma_2}$ are unknown, but are assumed to be unequal
If any of the two samples is small, then the two populations are normally or approximately normally distributed
Sample Mean1= $\boldsymbol{\bar{x}_1}$, Sample Mean2 = $\boldsymbol{\bar{x}_2}$
Sample Variance1= $\boldsymbol{s_1^2}$, Sample Variance2 = $\boldsymbol{s_2^2}$

Standard Error:

$$\boldsymbol{\sigma_{\bar{x}_1-\bar{x}_2} = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

Degrees of Freedom $\boldsymbol{df = min\{(n_1 - 1)\,, (n_2 - 1)\}}$
Confidence Level = $\boldsymbol{c}$
$\boldsymbol{t_c:\quad = \mathrm{T.INV}\left(\frac{1+c}{2}\,, df\right)}$
Margin of Error $\boldsymbol{E = t_c\sigma_{\bar{x}_1-\bar{x}_2}}$
CI Lower Limit = $\boldsymbol{(\bar{x}_1 - \bar{x}_2) - E}$
CI Upper Limit = $\boldsymbol{(\bar{x}_1 - \bar{x}_2) + E}$

The procedures for performing a ***T test*** hypothesis testing about the difference $\boldsymbol{\mu_1 - \mu_2}$ for right-tailed, left-tailed, and two-tailed problems are summarized in the following three templates:

T-Test Hypothesis Testing for $\mu_1 - \mu_2$:

Right Tailed problem

Samples are independent

σ_1 and σ_2 are unknown, but are assumed to be unequal

If any of the two samples is small, then the two populations are normally or approximately normally distributed

Null Hypothesis: H_0: $\mu_1 - \mu_2 \leq \mu_0$

Alternative Hypothesis: H_A: $\mu_1 - \mu_2 > \mu_0$

μ_0 = ***Hypothesized Difference***

Sample Mean$_1$= $\bar{x}_1$, Sample Mean$_2$ = $\bar{x}_2$

Sample Variance$_1$= s_1^2, Sample Variance$_2$ = s_2^2

Degrees of Freedom $df = min\{(n_1 - 1), (n_2 - 1)\}$

Standard Error:

$$\sigma_{\bar{x}_1 - \bar{x}_2} = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

Test Statistic $t = \frac{(\bar{x}_1 - \bar{x}_2) - \mu_0}{\sigma_{\bar{x}_1 - \bar{x}_2}}$

Level of significance $= \alpha$

Critical *T* value (t^*) = t with an area of α on its right.

Use Excel for ***critical t^* value***:

$$= \mathrm{T.INV}(1 - \alpha, df)$$

***P value* = $P(T \geq t)$**

Use Excel for ***P value***:

$$= 1 - \mathrm{T.DIST}(t, df, 1)$$

Decision:

Method 1: ***Reject Ho if*** $sample\ t \geq Critical\ t^*$

Method 2: ***Reject Ho if*** $Pvalue \leq \alpha$

T-Test Hypothesis Testing for $\mu_1 - \mu_2$:

Left Tailed problem

Samples are independent

σ_1 and σ_2 are unknown, but are assumed to be unequal

If any of the two samples is small, then the two populations are normally or approximately normally distributed

Null Hypothesis: H_0: $\mu_1 - \mu_2 \geq \mu_0$

Alternative Hypothesis: H_A: $\mu_1 - \mu_2 < \mu_0$

μ_0 = ***Hypothesized Difference***

Sample Mean$_1$= $\bar{x}_1$, Sample Mean$_2$ = $\bar{x}_2$

Sample Variance$_1$= s_1^2, Sample Variance$_2$ = s_2^2

Degrees of Freedom $df = min\{(n_1 - 1), (n_2 - 1)\}$

Standard Error:

$$\sigma_{\bar{x}_1 - \bar{x}_2} = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

Test Statistic $t = \frac{(\bar{x}_1 - \bar{x}_2) - \mu_0}{\sigma_{\bar{x}_1 - \bar{x}_2}}$

Level of significance $= \alpha$

Critical *T* value (t^*) = *t* with an area of α on its left.

Use Excel for ***critical t^* value***:

$$= \mathrm{T.INV}(\alpha, df)$$

***P value* = $P(T \leq t)$**

Use Excel for ***P value***:

$$= \mathrm{T.DIST}(t, df, 1)$$

Decision:

Method 1: ***Reject Ho if*** $sample\ t \leq Critical\ t^*$

Method 2: ***Reject Ho if*** $Pvalue \leq \alpha$

T-Test Hypothesis Testing for $\mu_1 - \mu_2$:

Two Tailed problem

Samples are independent

σ_1 and σ_2 are unknown, but are assumed to be unequal

If any of the two samples is small, then the two populations are normally or approximately normally distributed

Null Hypothesis: H_0: $\mu_1 - \mu_2 = \mu_0$

Alternative Hypothesis: H_A: $\mu_1 - \mu_2 \neq \mu_0$

μ_0 = ***Hypothesized Difference***

Sample Mean$_1$= $\bar{x}_1$, Sample Mean$_2$ = $\bar{x}_2$

Sample Variance$_1$= s_1^2, Sample Variance$_2$ = s_2^2

Degrees of Freedom $df = min\{(n_1 - 1), (n_2 - 1)\}$

Standard Error:

$$\sigma_{\bar{x}_1 - \bar{x}_2} = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

Test Statistic $t = \frac{(\bar{x}_1 - \bar{x}_2) - \mu_0}{\sigma_{\bar{x}_1 - \bar{x}_2}}$

Level of significance $= \alpha$

Critical T value 1 **(t_1^*) = t with an area of $\alpha/2$ on its right.**

Critical T value 2 **(t_2^*) = t with an area of $\alpha/2$ on its left.**

Use Excel for critical t_1^* and t_2^* values:

t_1^*: $= \mathrm{T.INV}\left(1 - \frac{\alpha}{2}, df\right)$

t_2^*: $= \mathrm{T.INV}\left(\frac{\alpha}{2}, df\right)$

P value **= $2\ P(T \geq t)$ if the test statistic t is positive**

P value **= $2\ P(T \leq t)$ if the test statistic t is negative**

Use Excel for ***P values***:

If $t > 0$, $= 2 * \left(1 - \mathrm{T.DIST}(t, df, 1)\right)$

If $t < 0$, $= 2 * \ \mathrm{T.DIST}(t, df, 1)$

Decision:

Method 1: Reject Ho if:

test statistic $t \geq t_1^*$ for $t > 0$

test statistic $t \leq t_2^*$ for $t < 0$

Method 2: ***Reject Ho if*** $Pvalue \leq \alpha$

Example:

A researcher wishes to see if the average weights of newborn male infants are higher than the average weights of newborn female infants. The researcher selects a random sample of 12 male infants and finds the mean weight is 7 pounds 8 ounces (i.e., 120 Ounces) and the standard deviation of the sample is 7 ounces. The researcher also selects a random sample of 14 female infants and finds that the mean weight is 7 pounds 1 ounces (i.e., 113 ounces) and the standard deviation of the sample is 4 ounces. Can it be concluded at the 0.01 level of significance that the mean weight of the males is higher than the mean weight of the females? Assume that the variables are normally distributed.

solution:

Step 1: State the problem hypotheses:

Let $\boldsymbol{\mu_1}$ denote the average weights of newborn male infants, and let $\boldsymbol{\mu_2}$ denote the average weights of newborn female infants. Then,

$\boldsymbol{H_0}$: $\boldsymbol{\mu_1 \leq \mu_2}$, or $\boldsymbol{H_0}$: $\boldsymbol{\mu_1 - \mu_2 \leq 0}$
$\boldsymbol{H_A}$: $\boldsymbol{\mu_1 > \mu_2}$, or $\boldsymbol{H_A}$: $\boldsymbol{\mu_1 - \mu_2 > 0}$ (claim)
Therefore, this hypothesis problem is a right-tailed problem. Furthermore,
Degrees of Freedom $\boldsymbol{df = min\{(12-1), (14-1)\} = 11}$

Step 2: Calculate the standard error $\boldsymbol{\sigma_{\bar{x}_1-\bar{x}_2}}$, and the test statistic $\boldsymbol{t}$:

$$\boldsymbol{\sigma_{\bar{x}_1-\bar{x}_2} = \sqrt{\frac{s_1^2}{n_1}+\frac{s_2^2}{n_2}} = \sqrt{\frac{7^{\,2}}{12}+\frac{4^{\,2}}{14}} = 2.286}$$

Test Statistic $\quad \boldsymbol{t = \frac{(120-113)-0}{2.286} = 3.062}$

Step 3:

Method 1: Compute the critical values, and compare the test statistic with the appropriate critical value:

For a right-tailed problem, the critical $\boldsymbol{t^*}$ value is a $\boldsymbol{t}$ value with an area of $\boldsymbol{\alpha}$ on its right.
To calculate this value in Excel, we enter:

$= \mathbf{T.INV(1-0.01\,,\ 11)}$ to obtain: $\boldsymbol{t^* = 2.718.}$

According to the decision rule, we should reject the null hypothesis if $\boldsymbol{t \geq t^*}$; and this is exactly the case in here since $\mathbf{3.062 > 2.718}$.

Method 2: Compute the Pvalue and compare it with the level of significance $\boldsymbol{\alpha}$:

In a right-tailed problem, the ***P value*** is the area on the right of the test statistic $\boldsymbol{t}$.
P value = $\boldsymbol{P(T \geq t)}$
This can be calculated in Excel by entering: $= \mathbf{1 - T.DIST(3.062\,, 11\,, 1).}$ Thus, we obtain:
$\boldsymbol{Pvalue = 0.0054}$

According to the decision rule, we should reject the null hypothesis if $\boldsymbol{Pvalue \leq \alpha}$. This is exactly the case in here since $\mathbf{0.0054 < 0.01}$.

Step 4: Make a decision, and describe that decision in the context of the problem:
Both methods 1 and 2 of the previous step have resulted in rejecting the Null hypothesis. This decision is then described as follows:

At 0.01 level of significance, there exists sufficient evidence to conclude that average weights of newborn male infants are higher than the average weights of newborn female infants.

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Note:
There are often two different options for the use of T tests. One option is used when the *variances of the populations are not equal*. This is the option that we have thus far used in our formulation of the T test. The other option is used when the *variances are equal*. In this case, most statistical packages (such as Excel) use a different formula to determine the degrees of freedom of the problem and the standard error of the sampling. Those formulas are shown below:

$$df = \frac{\left(s_1^2/n_1 + s_2^2/n_2\right)^2}{\frac{\left(s_1^2/n_1\right)^2}{n_1-1} + \frac{\left(s_2^2/n_2\right)^2}{n_2-1}}$$

$$\sigma_{\bar{x}_1-\bar{x}_2} = \sqrt{\frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1+n_2-2}}\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}$$

To determine whether two sample variances are equal, we can use an **F test**, as shown in Section 3 of these notes.

Technology Note:
The Data Analysis ToolPak of Excel has two options for performing a two-sample T test: One for the case when the population variances are assumed to be equal, and one for unequal population variances. To perform a T test for the difference between the means of two populations, given two independent samples, do this:

1. Enter the first sample data set into column **A** and the second sample data set into column **B**.
2. Select the **Data** tab from the toolbar. Then select **Data Analysis**.
3. In the Analysis Tools box, select one of the following: **t test: Two Sample Assuming Unequal Variances** or **t test: Two Sample Assuming Equal Variances**. Click **OK**.
4. Type the ranges for the data in columns **A** and **B** and type a value (usually 0) for the Hypothesized Difference $\boldsymbol{\mu_0}$.
5. Specify the level of significance $\boldsymbol{\alpha}$.
6. Specify a location for the output, and click **OK**.

Note: In situations when the two samples are not given explicitly but the sample means $\boldsymbol{\overline{x}_1}$ and $\boldsymbol{\overline{x}_2}$ are known, we can first create a total of $\boldsymbol{n_1}$ of values $\boldsymbol{\overline{x}_1}$ and a total of $\boldsymbol{n_2}$ of values $\boldsymbol{\overline{x}_2}$ in respectively columns **A** and **B** before using the two-sample T test of Data Analysis.

(iii) Using the T Distribution (Dependent Paired Samples): For matched paired samples (that are inherently dependent), a different version of the T test is used to perform a hypothesis testing of the population means differences. For example, a nutritionist may want to test whether a certain diet program is effective after it has been administered to a group of subjects for a period of one week. In this case, the subjects' weights before and after the diet period will constitute two dependent samples.

The three hypothesis testing problem types in this case are summarized in the following table:

	Right-tailed	Left-Tailed	Two-Tailed
Null Hypothesis	H_0: $\mu_D \leq \mu_0$	H_0: $\mu_D \geq \mu_0$	H_0: $\mu_D = \mu_0$
Alternative Hypothesis	H_A: $\mu_D > \mu_0$	H_A: $\mu_D < \mu_0$	H_A: $\mu_D \neq \mu_0$

Here, $\boldsymbol{\mu_D}$ is the symbol for the expected mean of the difference of the matched pairs. The general procedure for finding the test value involves several steps as shown below:

Let the data in the two samples be denoted by $\boldsymbol{X_1}$ and $\boldsymbol{X_2}$ respectively, and let $\boldsymbol{n}$ denote the number of pairs. We first calculate the differences $\boldsymbol{D}$ for each sampled pair:

$$\boldsymbol{D = X_1 - X_2}$$

We then calculate the mean $\boldsymbol{\bar{D}}$ and the standard deviation $\boldsymbol{s_D}$ of the differences:

$$\boldsymbol{\bar{D} = \frac{\sum D}{n}}, \qquad \boldsymbol{s_D = \sqrt{\frac{n\sum D^2 - (\sum D)^2}{n(n-1)}}}$$

Next, the standard error of the differences is given by:

$$\boldsymbol{s_{\bar{D}} = \frac{s_D}{\sqrt{n}}}$$

The test statistic $\boldsymbol{t}$ is based on $\boldsymbol{df = n - 1}$; and is calculated by the following expression:

$$\boldsymbol{t = \frac{\bar{D} - \mu_0}{s_{\bar{D}}}}$$

The rest of the procedure of hypothesis testing in this case is similar to other T test types that we have thus far discussed. It is important to note that for small samples, an important

assumption is that the population or populations must be normally or approximately normally distributed for the T test to apply.
The three hypothesis testing problems for dependent samples are summarized in the following templates:

T-Test Hypothesis Testing for $\mu_1 - \mu_2$:
Right Tailed problem **Dependent (paired Samples)** **Sample size (Number of pairs) =** n **Degrees of Freedom** $df = n - 1$ Sample 1: X_1; Sample 2: X_2 Differences $D = X_1 - X_2$ **Null Hypothesis:** H_0: $\mu_D \leq \mu_0$ **Alternative Hypothesis:** H_D: $\mu_D > \mu_0$ Mean of differences = $\bar{D}$ Standard deviation of differences = s_D **Standard Error** $s_{\bar{D}} = \frac{s_D}{\sqrt{n}}$ **Test Statistic** $t = \frac{\bar{D} - \mu_0}{s_{\bar{D}}}$ **Level of significance** $= \alpha$ **Critical *T* value (t^*) = t with an area of α on its right.** Use Excel for ***critical t^* value***: $= \mathrm{T.INV}(1 - \alpha, df)$ ***P value*** **=** $P(T \geq t)$ Use Excel for ***P value***: $= 1 - \mathrm{T.DIST}(t, df, 1)$ **Decision:** **Method 1:** ***Reject Ho if*** $sample\ t \geq Critical\ t^*$ **Method 2:** ***Reject Ho if*** $Pvalue \leq \alpha$

T-Test Hypothesis Testing for $\mu_1 - \mu_2$:

Left Tailed problem

Dependent (paired Samples)

Sample size (Number of pairs) = n

Degrees of Freedom $df = n - 1$

Sample 1: X_1; Sample 2: X_2

Differences $D = X_1 - X_2$

Null Hypothesis: H_0: $\mu_D \geq \mu_0$

Alternative Hypothesis: H_D: $\mu_D < \mu_0$

Mean of differences = $\bar{D}$

Standard deviation of differences = s_D

Standard Error $s_{\bar{D}} = \dfrac{s_D}{\sqrt{n}}$

Test Statistic $t = \dfrac{\bar{D} - \mu_0}{s_{\bar{D}}}$

Level of significance $= \alpha$

Critical *T* value (t^*) = t with an area of α on its left.

Use Excel for ***critical t^* value***:

$= \mathrm{T.INV}(\alpha\,, df)$

***P value* = $P(T \leq t)$**

Use Excel for ***P value***:

$= \mathrm{T.DIST}(t\,,\ df,\ 1)$

Decision:

Method 1: ***Reject Ho if*** $sample\ t \leq Critical\ t^*$

Method 2: ***Reject Ho if*** $Pvalue \leq \alpha$

T-Test Hypothesis Testing for $\mu_1 - \mu_2$:

Two Tailed problem

Dependent (paired Samples)

Sample size (Number of pairs) = n

Degrees of Freedom $df = n - 1$

Sample 1: X_1; Sample 2: X_2

Differences $D = X_1 - X_2$

Null Hypothesis: H_0: $\mu_D = \mu_0$

Alternative Hypothesis: H_A: $\mu_D \neq \mu_0$

Mean of differences = $\overline{D}$

Standard deviation of differences = s_D

Standard Error $s_{\overline{D}} = \dfrac{s_D}{\sqrt{n}}$

Test Statistic $t = \dfrac{\overline{D} - \mu_0}{s_{\overline{D}}}$

Level of significance $= \alpha$

Critical T value 1 **(t_1^*) = t with an area of $\alpha/2$ on its right.**

Critical Tvalue 2 **(t_2^*) = t with an area of $\alpha/2$ on its left.**

Use Excel for critical t_1^* and t_2^* values:

t_1^*: $= \mathrm{T.INV}\left(1 - \frac{\alpha}{2}, df\right)$

t_2^*: $= \mathrm{T.INV}\left(\frac{\alpha}{2}, df\right)$

P value **=** $2\ P(T \geq t)$ **if the test statistic t is positive**

P value **=** $2\ P(T \leq t)$ **if the test statistic t is negative**

Use Excel for ***P value***:

If $t > 0$, $= 2 * \left(1 - \mathrm{T.DIST}(t\,, df\,, 1)\right)$

If $t < 0$, $= 2 * \ \mathrm{T.DIST}(t\,, df\,, 1)$

Decision:

Method 1: Reject Ho if:

test statistic $t \geq t_1^*$ for $t > 0$

test statistic $t \leq t_2^*$ for $t < 0$

Method 2: ***Reject Ho if*** $Pvalue \leq \alpha$

Example:

A nutritionist wishes to see if a person's cholesterol level will change if the diet is supplemented by a certain combination of minerals and vitamins. Eight randomly selected subjects were pretested and their cholesterol levels were measured. The subjects then took the supplement for a period of two months, and their cholesterol levels were measured again. The results are shown in the table below. (Cholesterol level is measured in milligrams per deciliter.) Can it be concluded that the cholesterol level has been changed at a significance level of 0.10? Assume the variable is approximately normally distributed.

Subject	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
Before (X_1)	198	212	285	174	196	178	264	172
After (X_2)	172	218	223	176	184	182	222	173

solution:

Since we wish to test whether there has been a change in the cholesterol levels, then this is a two-tailed hypothesis testing problem. The two hypotheses are:

$\boldsymbol{H_0}$: $\boldsymbol{\mu_D = 0}$

$\boldsymbol{H_A}$: $\boldsymbol{\mu_D \neq 0}$ **(claim)**

Furthermore, we note that $\boldsymbol{df = n - 1 = 8 - 1 = 7}$.

For each subject, the difference $\boldsymbol{D = X_1 - X_2}$ has been calculated and displayed in the table below:

Difference *D*	26	-6	62	-2	12	-4	42	-1

The mean and the standard deviation of $\boldsymbol{D}$ are then calculated:

$$\boldsymbol{\bar{D} = 16.125}, \qquad \boldsymbol{s_D = 25.085}$$

The next step is to calculate the standard error and the test statistic $\boldsymbol{t}$**:**

$$\boldsymbol{s_{\bar{D}} = \frac{s_D}{\sqrt{n}} = \frac{25.085}{\sqrt{8}} = 8.869}$$

$$\boldsymbol{t = \frac{\bar{D} - \mu_0}{s_{\bar{D}}} = \frac{16.125 - 0}{8.869} = 1.818}$$

Since this problem is a two-tailed problem and $\boldsymbol{t}$ is positive, then the *P-value* will be equal to twice the area on the right of the test statistic $\boldsymbol{t}$**.** This can be calculated in Excel by entering:

$$\boldsymbol{= 2 * \left(1 - \mathrm{T.DIST}(1.818\,, 7\,, 1)\right)}$$

Thus, we obtain:

$$\boldsymbol{Pvalue = 0.1119}$$

The next step is to compare this P-value with the level of significance (0.10), and reject the null hypothesis if $\boldsymbol{Pvalue \leq 0.10}$. However, we observe that this condition is not satisfied since $\boldsymbol{0.1119 > 0.10}$. Therefore, the decision is: *not to reject the null hypothesis*; i.e.,

At 0.10 level of significance, there does not exist sufficient evidence to support the claim that the above supplement will reduce the cholesterol levels.

•

Note:

A confidence interval for $\boldsymbol{\mu_D}$ can be constructed by calculating the following two limits:

Confidence interval lower limit for $\boldsymbol{\mu_D}$: $\boldsymbol{\bar{D} - t_c\, s_{\bar{D}}}$

Confidence interval upper limit for $\boldsymbol{\mu_D}$: $\boldsymbol{\bar{D} + t_c\, s_{\bar{D}}}$

Technology Note:

The Data Analysis ToolPak of Excel can be used to perform a two dependent sample T test. To perform this test,

1. Enter the first sample data set into column **A** and the second sample data set into column **B**.
2. Select the **Data** tab from the toolbar. Then select **Data Analysis**.
3. In the Analysis Tools box, select: **t-test: Paired Two Sample for Means**. Then click **OK**.
4. Type (or enter) the ranges for the data in columns **A** and **B** and type a value (usually 0) for the Hypothesized Mean Difference $\boldsymbol{\mu_0}$.
5. Specify the level of significance $\boldsymbol{\alpha}$.
6. Specify a location for the output, and click **OK**.

2. Confidence Interval and Z-test Hypothesis Testing for the Difference of Two Population Proportions $\boldsymbol{p_1 - p_2}$:

A Z-test with minor modifications can be used test a hypothesis involving the difference between two population proportions $\boldsymbol{p_1}$ and $\boldsymbol{p_2}$. Let $\boldsymbol{n_1}$ and $\boldsymbol{n_2}$ be the sizes of the two samples collected from the two populations and let $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ denote the respective number of successes in the two populations. Then $\boldsymbol{\hat{p}_1 = \frac{x_1}{n_1}}$ and $\boldsymbol{\hat{p}_2 = \frac{x_2}{n_2}}$ are the two sample proportions used to estimate $\boldsymbol{p_1}$ and $\boldsymbol{p_2}$ respectively. Furthermore, $\boldsymbol{\hat{q}_1 = 1 - \hat{p}_1}$ and $\boldsymbol{\hat{q}_2 = 1 - \hat{p}_2}$. A set of four primary conditions that must be satisfied for the following methods to apply are:

(i) $\boldsymbol{n_1\hat{p}_1 \geq 10}$, and $\boldsymbol{n_1\hat{q}_1 \geq 10}$

(ii) $\boldsymbol{n_2\hat{p}_2 \geq 10}$, and $\boldsymbol{n_2\hat{q}_2 \geq 10}$

The above four conditions are meant to ensure that the two sample sizes $\boldsymbol{n_1}$ and $\boldsymbol{n_2}$ are large -enough for the sampling distribution of $\boldsymbol{\hat{p}_1 - \hat{p}_2}$ to approximately become a normal distribution. For the case of confidence intervals, the standard deviation (the ***sampling error***) of the distribution of $\boldsymbol{\hat{p}_1 - \hat{p}_2}$ is given by:

$$\boldsymbol{\sigma_{\hat{p}_1-\hat{p}_2} = \sqrt{\frac{\hat{p}_1\,\hat{q}_1}{n_1} + \frac{\hat{p}_2\,\hat{q}_2}{n_2}}}$$

The margin of error $\boldsymbol{E}$ of the confidence interval is then equal to:

$$\boldsymbol{E = z_c\,\sigma_{\hat{p}_1-\hat{p}_2}},$$

And the two limits of the confidence interval are:

Confidence interval lower limit for $\boldsymbol{p_1 - p_2}$: $\boldsymbol{(\hat{p}_1 - \hat{p}_2) - E}$
Confidence interval upper limit for $\boldsymbol{p_1 - p_2}$: $\boldsymbol{(\hat{p}_1 - \hat{p}_2) + E}$

The procedure for constructing a confidence interval for $\boldsymbol{p_1 - p_2}$ is summarized in the following template:

Confidence Interval for $\boldsymbol{p_1 - p_2}$:

Sample sizes: $\boldsymbol{n_1}$ and $\boldsymbol{n_2}$
Sample Successes: $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$
Sample Proportions: $\boldsymbol{\hat{p}_1 = \frac{x_1}{n_1}}$ and $\boldsymbol{\hat{p}_2 = \frac{x_2}{n_2}}$
$\boldsymbol{\hat{q}_1 = 1 - \hat{p}_1}$ and $\boldsymbol{\hat{q}_2 = 1 - \hat{p}_2}$

Standard Error $\boldsymbol{\sigma_{\hat{p}_1-\hat{p}_2}}$:

$$\boldsymbol{\sigma_{\hat{p}_1-\hat{p}_2} = \sqrt{\frac{\hat{p}_1\,\hat{q}_1}{n_1} + \frac{\hat{p}_2\,\hat{q}_2}{n_2}}}$$

Confidence Level = $\boldsymbol{c}$

$\boldsymbol{z_c}$: $\quad \mathbf{= NORM.S.INV}\left(\boldsymbol{\frac{1+c}{2}}\right)$

Margin of Error $\boldsymbol{E = z_c\,\sigma_{\hat{p}_1-\hat{p}_2}}$

CI Lower Limit = $\boldsymbol{(\hat{p}_1 - \hat{p}_2) - E}$
CI Upper Limit =$\boldsymbol{(\hat{p}_1 - \hat{p}_2) + E}$

As for performing a hypothesis testing for the difference $\boldsymbol{p_1 - p_2}$, the three hypothesis testing problem types are summarized in the following table:

	Right-tailed	Left-Tailed	Two-Tailed
Null Hypothesis	$H_0\colon p_1 - p_2 \le p_0$	$H_0\colon p_1 - p_2 \ge p_0$	$H_0\colon p_1 - p_2 = p_0$
Alternative Hypothesis	$H_A\colon p_1 - p_2 > p_0$	$H_A\colon p_1 - p_2 < p_0$	$H_A\colon p_1 - p_2 \ne p_0$

The quantity p_0 is called the *hypothesized difference* between the two population proportions (often, $p_0 = 0$).
For hypothesis testing problems, the sampling error of the difference is different from the one described above for the problems that involve confidence intervals. In this case, a ***pooled estimate*** $\bar{p}$ is first calculated as follows:

$$\bar{p} = \frac{n_1 \hat{p}_1 + n_2 \hat{p}_2}{n_1 + n_2} = \frac{x_1 + x_2}{n_1 + n_2}$$
$$\bar{q} = 1 - \bar{p}$$

Next, the sampling error $\sigma_{\hat{p}_1 - \hat{p}_2}$ is given by:

$$\sigma_{\hat{p}_1 - \hat{p}_2} = \sqrt{\bar{p}\,\bar{q}\,\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}$$

The value of the test statistic z is then calculated as follows:

$$z = \frac{(\hat{p}_1 - \hat{p}_2) - p_0}{\sigma_{\hat{p}_1 - \hat{p}_2}}$$

The three hypothesis testing procedures are summarized in the following templates:

Z-Test Hypothesis Testing for $p_1 - p_2$:

Right Tailed problems

Sample sizes: n_1 and n_2

Samples are randomly selected and independent of each other.

Sample Proportions: $\hat{p}_1 = \frac{x_1}{n_1}$ and $\hat{p}_2 = \frac{x_2}{n_2}$

Requirements: $n_1\hat{p}_1 \geq 10$, $n_1\hat{q}_1 \geq 10$, $n_2\hat{p}_2 \geq 10$, and $n_2\hat{q}_2 \geq 10$

Pooled Estimate of proportions:

$$\bar{p} = \frac{n_1\,\hat{p}_1 + n_2\,\hat{p}_2}{n_1 + n_2} = \frac{x_1 + x_2}{n_1 + n_2}$$

$$\bar{q} = 1 - \bar{p}$$

Null Hypothesis: H_0: $p_1 - p_2 \leq p_0$

Alternative Hypothesis: H_A: $p_1 - p_2 > p_0$

Standard Error $\sigma_{\hat{p}_1 - \hat{p}_2} = \sqrt{\bar{p}\,\bar{q}\,\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}$

Test Statistic $z = \frac{(\hat{p}_1 - \hat{p}_2) - p_0}{\sigma_{\hat{p}_1 - \hat{p}_2}}$

Level of significance $= \alpha$

Critical *Z* value (z^*) = *z* with an area of α on its right.

Use Excel for ***critical z^* value***:

$$= \mathrm{NORM.S.INV}(1 - \alpha)$$

P value = $P(Z \geq z)$

Use Excel for ***P value***:

$$= 1 - \mathrm{NORM.S.DIST}(z\,, 1)$$

Decision:

Method 1: ***Reject Ho if*** $sample\ z\ \geq Critical\ z^*$

Method 2: ***Reject Ho if*** $\_Pvalue\ \leq \alpha$

Z-Test Hypothesis Testing for $p_1 - p_2$:

Left Tailed problems

Sample sizes: n_1 and n_2

Samples are randomly selected and independent of each other.

Sample Proportions: $\hat{p}_1 = \frac{x_1}{n_1}$ and $\hat{p}_2 = \frac{x_2}{n_2}$

Requirements: $n_1\hat{p}_1 \geq 10$, $n_1\hat{q}_1 \geq 10$, $n_2\hat{p}_2 \geq 10$, and $n_2\hat{q}_2 \geq 10$

Pooled Estimate of proportions:

$$\bar{p} = \frac{n_1\,\hat{p}_1 + n_2\,\hat{p}_2}{n_1 + n_2} = \frac{x_1 + x_2}{n_1 + n_2}$$

$$\bar{q} = 1 - \bar{p}$$

Null Hypothesis: H_0: $p_1 - p_2 \geq p_0$

Alternative Hypothesis: H_A: $p_1 - p_2 < p_0$

Standard Error $\sigma_{\hat{p}_1 - \hat{p}_2} = \sqrt{\bar{p}\,\bar{q}\,\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}$

Test Statistic $z = \frac{(\hat{p}_1 - \hat{p}_2) - p_0}{\sigma_{\hat{p}_1 - \hat{p}_2}}$

Level of significance $= \alpha$

Critical *Z* value (z^*) = *z* with an area of α on its left.

Use Excel for ***critical z^* value***:

$$= \mathrm{NORM.S.INV}(\alpha)$$

P value = $P(Z \leq z)$

Use Excel for ***P value***:

$$= \mathrm{NORM.S.DIST}(z\,,1)$$

Decision:

Method 1: ***Reject Ho if*** $sample\ z\ \leq Critical\ z^*$

Method 2: ***Reject Ho if*** $Pvalue\ \leq \alpha$

Z-Test Hypothesis Testing for $p_1 - p_2$:

Two Tailed problems

Sample sizes: n_1 and n_2

Samples are randomly selected and independent of each other.

Sample Proportions: $\hat{p}_1 = \frac{x_1}{n_1}$ and $\hat{p}_2 = \frac{x_2}{n_2}$

Requirements: $n_1\hat{p}_1 \geq 10$, $n_1\hat{q}_1 \geq 10$, $n_2\hat{p}_2 \geq 10$, and $n_2\hat{q}_2 \geq 10$

Pooled Estimate of proportions:

$$\bar{p} = \frac{n_1\,\hat{p}_1 + n_2\,\hat{p}_2}{n_1 + n_2} = \frac{x_1 + x_2}{n_1 + n_2}$$

$$\bar{q} = 1 - \bar{p}$$

Null Hypothesis: H_0: $p_1 - p_2 = p_0$

Alternative Hypothesis: H_A: $p_1 - p_2 \neq p_0$

Standard Error $\sigma_{\hat{p}_1 - \hat{p}_2} = \sqrt{\bar{p}\,\bar{q}\,\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}$

Test Statistic $z = \frac{(\hat{p}_1 - \hat{p}_2) - p_0}{\sigma_{\hat{p}_1 - \hat{p}_2}}$

Level of significance $= \alpha$

Critical Z value 1 **(z_1^*) = *z* with an area of $\alpha/2$ on its right.**

Critical Z value 2 **(z_2^*) = *z* with an area of $\alpha/2$ on its left.**

Use Excel for critical z_1^* and z_2^* values:

z_1^*: $= \text{NORM.S.INV}(1 - \alpha/2)$

z_2^*: $= \text{NORM.S.INV}(\alpha/2)$

P value **= $2\ P(Z \geq z)$ if the test statistic z is positive**

P value **= $2\ P(Z \leq z)$ if the test statistic z is negative**

Use Excel for ***P values***:

If $z > 0$, $= 2 * \left(1 - \text{NORM.S.DIST}(z\,,1)\right)$

If $z < 0$, $= 2 * \ \text{NORM.S.DIST}(z\,,1)$

Decision:

Method 1: Reject Ho if:

test statistic $z \geq z_1^*$ for $z > 0$

or test statistic $z \leq z_2^*$ for $z < 0$

Method 2: ***Reject Ho if*** $Pvalue \leq \alpha$

Example:

A study is conducted to determine if the percent of men who receive financial aid in graduate school is less than the percent of women who receive financial aid in graduate school. In this study a random sample of 100 men who were enrolled in a graduate degree program, yielded 72 who were using financial aid; whereas 160 out of a randomly selected sample of 200 of their female counterparts were using financial aid. Use $\boldsymbol{\alpha = 0.1}$ to test the claim that the proportion of male graduate student who receive financial aid is less than the proportion of female graduate student who receive financial aid.

Solution:

Step 1: State the problem hypotheses:

Let $\boldsymbol{p_1}$ denote the proportion of male graduate student who receive financial aid, and let $\boldsymbol{p_2}$ denote the proportion of female graduate student who receive financial aid. Then,

$\boldsymbol{H_0}$: $\boldsymbol{p_1 \geq p_2}$, or $\boldsymbol{H_0}$: $\boldsymbol{p_1 - p_2 \geq 0}$
$\boldsymbol{H_A}$: $\boldsymbol{p_1 < p_2}$, or $\boldsymbol{H_A}$: $\boldsymbol{p_1 - p_2 < 0}$ (claim)

Therefore, this hypothesis problem is a Left-tailed problem. We must first check to ensure that the conditions for applying the methods described are satisfied:

$$\boldsymbol{\hat{p}_1 = \frac{x_1}{n_1} = \frac{72}{100} = 0.72 \quad \rightarrow \quad \hat{q}_1 = 1 - \hat{p}_1 = 0.28}$$
$$\rightarrow \quad \boldsymbol{n_1\hat{p}_1 = 72 \geq 10}, \text{ and } \quad \boldsymbol{n_1\hat{q}_1 = 28 \geq 10}$$

Also,

$$\boldsymbol{\hat{p}_2 = \frac{x_2}{n_2} = \frac{160}{200} = 0.8 \quad \rightarrow \quad \hat{q}_2 = 1 - \hat{p}_2 = 0.2}$$
$$\rightarrow \quad \boldsymbol{n_2\hat{p}_2 = 160 \geq 10}, \text{ and } \quad \boldsymbol{n_2\hat{q}_2 = 40 \geq 10}$$

Next, the ***pooled estimate*** $\boldsymbol{\overline{p}}$ is calculated as follows

$$\boldsymbol{\overline{p} = \frac{x_1 + x_2}{n_1 + n_2} = \frac{72 + 160}{100 + 200} = 0.7733}$$
$$\boldsymbol{\overline{q} = 1 - \overline{p} = 1 - 0.7733 = 0.2267}$$

Step 2: Calculate the standard error $\boldsymbol{\sigma_{\hat{p}_1 - \hat{p}_2}}$, and the test statistic $\boldsymbol{Z}$:

$$\boldsymbol{\sigma_{\hat{p}_1 - \hat{p}_2} = \sqrt{\overline{p}\,\overline{q}\,\left(\frac{1}{n_1} + \frac{1}{n_2}\right)} = \sqrt{0.7733 \times 0.2267\,\left(\frac{1}{100} + \frac{1}{300}\right)} = 0.0513}$$

$$z = \frac{(\hat{p}_1 - \hat{p}_2) - p_0}{\sigma_{\hat{p}_1 - \hat{p}_2}} = \frac{(0.72 - 0.8) - 0}{0.0513} = -1.56$$

Step 3:
Method 1: Compute the critical value, and compare the test statistic with the critical value:
For a left-tailed problem, the critical z^* value is a z value with an area of α on its leftt.
To calculate this value in Excel, we enter:

$= \mathbf{NORM.S.INV(0.10)}$ to obtain: $z^* = -1.28.$

According to the decision rule, we should reject the null hypothesis if $z \leq z^*$; and this is exactly the case in here since $-1.56 < -1.28$.

Method 2: Compute the Pvalue and compare it with the level of significance α:
In a left-tailed problem, the ***P value*** is the area on the left of the test statistic z.

P value = $P(Z \leq z)$

This can be calculated in Excel by entering: $= \mathbf{NORM.S.DIST(-1.56\ ,1)}.$ Thus, we obtain:

$$Pvalue = 0.0594$$

According to the decision rule, we should reject the null hypothesis if $Pvalue \leq \alpha$. This is exactly the case in here since $0.0594 < 0.1$.

Step 4: Make a decision, and describe that decision in the context of the problem:
Both methods 1 and 2 of the previous step have resulted in rejecting the Null hypothesis. This decision is then described as follows:

At 0.1 level of significance, there exists sufficient evidence to conclude that the proportion of male graduate student who receive financial aid is less than the proportion of female graduate student who receive financial aid.

●

3. F-Test Hypothesis Testing for the Difference or Ratio of Two Population Variances σ_1^2 / σ_2^2:

Researchers, statisticians, engineers, scientists, financial analysts, risk analysts, and managers are often interested to compare the variances (or standard deviations) of two populations. For example, a scientist may be interested in comparing the variations in temperatures in two different cities or a financial analyst may be interested in comparing the volatilities of two different stocks.

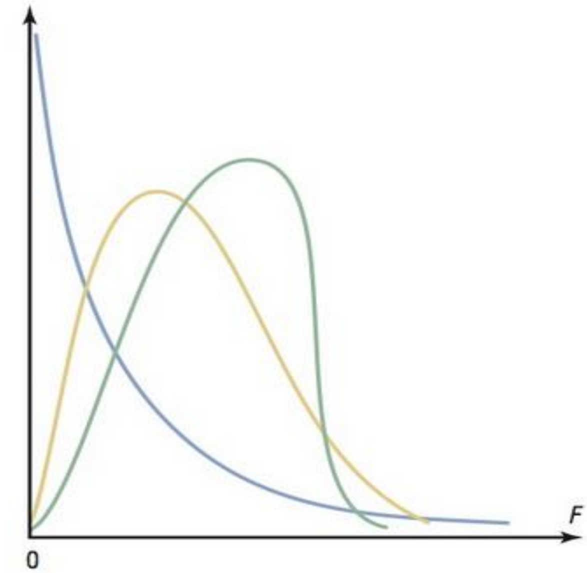
Recall from Unit 4 that the Chi-squared distribution was used when performing a hypothesis testing about the value of a single population variance. On the other hand and for comparing two variances, a different distribution called the ***F distribution*** should be used.

The use of the F distribution is based on the following fact:

If two independent samples are selected from two normally distributed populations in which the population variances are equal ($\boldsymbol{\sigma_1^2 = \sigma_2^2}$), and if the sample variances $\boldsymbol{s_1^2}$ and $\boldsymbol{s_2^2}$ are compared as $\frac{s_1^2}{s_2^2}$, then the sampling distribution of the variances' ratio is will be the *F distribution*. This distribution has the following characteristics:

1. The values of ***F*** are always non-negative, because variances are always positive or zero.
2. The ***F*** distribution is positively skewed.
3. The expected value (mean) of ***F*** is approximately equal to 1.
4. The ***F*** distribution is a family of curves based on the degrees of freedom of the variance of the numerator and the degrees of freedom of the variance of the denominator. In fact, the ***F*** distribution has two values for the degrees of freedom: that of the numerator, $\boldsymbol{df_1 = n_1 - 1}$, and that of the denominator, $\boldsymbol{df_2 = n_2 - 1}$, where $\boldsymbol{n_1}$ is the sample size from which the larger variance was obtained.

   The graph below shows several members of the F distribution family. Note that the shape of each curve depends on $\boldsymbol{df_1}$ and $\boldsymbol{df_2}$.



When performing a F test to compare two variances, the test statistic $\boldsymbol{f}$ is given by:

$$\boldsymbol{f} = \frac{s_1^2}{s_2^2}$$

An important characteristic of the F test is that, since s_1^2 can always be selected to be the larger sample variance, then the hypothesis testing problems about the ratio σ_1^2 / σ_2^2 are always of two types: right-tailed problems and two-tailed problems. These problem types are summarized in the following two templates:

Hypothesis Testing for $\sigma_1^2 - \sigma_2^2$: (or for σ_1^2 / σ_2^2)

Right Tailed problem

Use the *F distribution*

Sample sizes: n_1 and n_2

$df_1 = n_1 - 1,\ df_2 = n_2 - 1$

Sample variances s_1^2 (larger sample variance), and s_2^2 (smaller sample variance)

Null Hypothesis: H_0: $\sigma_1^2 \le \sigma_2^2$, or $\sigma_1^2/\sigma_2^2 \le 1$

Alternative Hypothesis: H_A: $\sigma_1^2 > \sigma_2^2$, or $\sigma_1^2/\sigma_2^2 > 1$

Test Statistic $f = \frac{s_1^2}{s_2^2}$

Level of significance $= \alpha$

Critical *F* value (F^*) = *F* with an area of α on its right.

Use Excel for ***critical F^* value***:

$= \mathrm{F.INV}(1-\alpha\,, df_1\,, df_2)$

P value = $P(F \ge F)$

Use Excel for ***P value***:

$= 1 - \mathrm{F.DIST}(f\ \,, df_1\,, df_2\,, 1)$

Decision:

Method 1: ***Reject Ho if*** $sample\ f\ \ge Critical\ F^*$

Method 2: ***Reject Ho if*** $\_Pvalue\ \le \alpha$

Hypothesis Testing for $\sigma_1^2 - \sigma_2^2$: (or for σ_1^2 / σ_2^2)

Two Tailed problem

Use the *F distribution*

Sample sizes: n_1 and n_2

$df_1 = n_1 - 1,\ df_2 = n_2 - 1$

Sample variances s_1^2 (larger sample variance), and s_2^2 (smaller sample variance)

Null Hypothesis: H_0: $\sigma_1^2 = \sigma_2^2$, *or* $\sigma_1^2/\sigma_2^2 = 1$

Alternative Hypothesis: H_A: $\sigma_1^2 \neq \sigma_2^2$, *or* $\sigma_1^2/\sigma_2^2 \neq 1$

Test Statistic $f = \dfrac{s_1^2}{s_2^2}$

Level of significance $= \alpha$

Critical *F* value (F^*) = *F* with an area of $\alpha/2$ on its right.

Use Excel for ***critical F^* value***:

$= \mathrm{F.INV}(1 - \alpha/2\ , df_1\ , df_2)$

P value **=** $2\ P(F \geq f)$

Use Excel for ***P value***:

$= 2 * (1 - \mathrm{F.DIST}(f\ \ , df_1\ , df_2\ , 1))$

Decision:

Method 1: ***Reject Ho if*** $sample\ f\ \geq Critical\ F^*$

Method 2: ***Reject Ho if*** $Pvalue\ \leq \alpha$

Example:

To compare the volatilities of two different types of stock, a financial analyst selected the 10 randomly selected values of stock A during on 10 randomly selected days and 8 randomly selected values of stock B during on 8 randomly selected days. The results are shown in the following two tables:

Date:	16-Jan	10-Feb	8-Mar	19-Apr	10-May	28-Jun	22-Jul	23-Aug	16-Sep	25-Oct
Stock A	110.8	114.5	99.87	112.5	101.29	109.4	116.9	129.34	110.3	118.6

Date:	18-Jan	10-Mar	18-May	19-Jun	10-Jul	13-Aug	16-Sep	19-Oct
Stock B	40.18	48.51	39.57	50.92	40.22	53.22	49.45	45.53

Is there sufficient evidence to conclude that stock A is more volatile than stock B?
Use $\boldsymbol{\alpha = 0.05}$.

Solution:

Testing whether stock A is more volatile than stock B is equivalent to testing whether Stock A has a larger variance in its values than stock B. let $\boldsymbol{\sigma_1^2}$ denote the variance of stock A and let $\boldsymbol{\sigma_1^2}$ denote that of stock B. We have the following information about the two stocks:

Sample sizes: $\boldsymbol{n_1 = 10}$ and $\boldsymbol{n_2 = 8}$
$\boldsymbol{df_1 = n_1 - 1 = 9,}$ and $\boldsymbol{df_2 = n_2 - 1 = 7}$

Sample variances: $\boldsymbol{s_1^2 = 71.72}$, and $\boldsymbol{s_2^2 = 29.05}$

Hypotheses:

$$\boldsymbol{H_0\colon\ \sigma_1^2 \le \sigma_2^2\,, \quad or \quad \sigma_1^2/\sigma_2^2 \le 1}$$

$$\boldsymbol{H_A\colon\ \sigma_1^2 > \sigma_2^2\,, \quad or \quad \sigma_1^2/\sigma_2^2 > 1}$$

The test statistic of this right-tailed problem is:

$$\boldsymbol{f = \frac{s_1^2}{s_2^2} = \frac{71.72}{29.05} = 2.468}$$

Next, we wish to compare this test statistic with the critical value of the ***F*** distribution. For a right tailed problem, the critical value for $\boldsymbol{\alpha = 0.05}$ is calculated by using the following Excel formula:

$$\boldsymbol{F^*:\quad = \mathrm{F.INV}(1 - 0.05\,, 9\,, 7) \quad \rightarrow \quad F^* = 3.677}$$

For us to reject the Null hypothesis, the test statistic $\boldsymbol{f}$ should be larger than the above critical value. However, since $\boldsymbol{2.468 < 3.677}$, we do not reject the null hypothesis.

Alternatively, we could have calculated the P-value to compare it with the level of significance. For a test statistic of $\boldsymbol{f} = \mathbf{2.468}$, the P-value is calculated by the following Excel formula:

$$\boldsymbol{Pvalue}: \quad = \mathbf{1 - F.DIST(2.468\ ,9\ ,7\ ,1)} \quad \rightarrow \quad \boldsymbol{Pvalue} = \mathbf{0.1234}$$

Since this P-value is larger than $\boldsymbol{\alpha} = \mathbf{0.05}$, we arrive at the same conclusion that we obtained when using the critical value. In summary,

At the 0.05 level of significance, there does not exist sufficient evidence to conclude that Stock A is more volatile than stock B.

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Technology Note:

Excel has a two-sample ***F*** test included in the Data Analysis Add-in. To perform an ***F*** test for the difference between the variances of two populations, given two independent samples, follow the following steps:

1. Enter the first sample data set into column **A** and the second sample data set into column **B**.
2. Select the **Data** tab from the toolbar. Then select **Data Analysis**.
3. In the Analysis Tools box, select: **F-test: Two Sample for Variances**. Then click **OK**.
4. Type (or enter) the ranges for the data in columns **A** and **B**.
5. Specify the level of significance $\boldsymbol{\alpha}$.
6. Specify a location for the output, and click **OK**.