Sharing and risk pooling in networks: a quick reference summary

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1 Model narrative and assumptions

Parameter List		
Name	Symbol	Range
Baseline payoff	b	$(1,\infty)$
Surplus payoff	B	(b,∞)
Per-connection cost	C	$(0,\infty)$
Rate of success	u	$(0,1)$
Risk	$1-u$	$(0,1)$
Sharing norm	s	$(0,1)$
Agent Degree	k_i	$N_{>0}^{\leq N-1}$
Agent's network mean degree	$\bar{k}_i$	$(0,N-1]$
Risk preference	δ	$(0,\infty)$

We start with a population of individuals who procure resources during their lifetimes. Lifetimes are separated into discrete steps, each representing a resource acquisition event (i.e. a foraging trip), with an initial resource level w_0 and individual payoff representing productivity across the lifetime. At every step, an individual attempts to get resource, sometimes getting a base rate of resource b with probability $1-u$ (risk) and a higher, surplus rate B with probability u (success rate). The payoff's growth is governed by multiplicative effects: every payoff gained at a particular step is multiplied by the payoff accumulated until that point. This assumes the payoff-accumulation process is not memory-less: my increase in payoff at a particular moment is not independent of the value of my payoffs up until that moment. This can be contrasted with an additive process, in which the increase in payoff at a time period is completely independent of the agent's payoff. We can set $b = 1$ for convenience, which assumes that, as far as the modeled individuals are concerned, the baseline is the same as getting nothing of value. The risk $1 - u$ then represents the risk of getting nothing out of your productive effort.

There are two strategies that we care about. The first is the **loner**, who individually procures resource without interacting with peers. The second one is the **sharer** strategy, which characterizes individuals that are part of a group employing a sharing norm. This means that individuals within the group agree on a proportion s of surplus B that should be shared with other group individuals that they have a reciprocity connection with. Essentially, whenever a sharer is successful in obtaining surplus B, the break off s of it and divide the resulting piece to give away to their connections within the group. In doing so, they can decrease the variance of their per-time-period resource acquisition rates. Furthermore, reciprocity connections are assumed to be costly, with sharers taking a per-connection loss C in units of resource in order to maintain each reciprocity connection.

Sharers establish connections with others, so they can be characterized by a number of connections k_i (where i is the sharer's index). A sharer's payoff will depend not only on their number of connections, but also on the number of connections that each of their connections has. This is because each one of a sharer's network peers is dividing their resource among their own peers. A sharer that is connected to individuals who have a higher number of connections can be seen as investing more into their network than what they are obtaining from it, while a an individual who is connected to others who have less connections enjoys a privileged position in the network. Therefore, an individual's position within their sharing network depends on how their number of connections k_i compares to the average number of connections of their network peers ($\bar{k}_i$). In fully regular networks, like lattices and cliques, k_i will equal $\bar{k}_i$, and sharing network hierarchy is not a thing. We call the parameter $\frac{k_i}{\bar{k}_i} = \beta_i$ an individual's *relative degree*, which tells us about their network position.

This simple setup allows us to explore different questions. I have concentrated on the following:

1. When do sharers enjoy higher payoffs than loners? When do sharing networks get "too crowded" to provide any benefit? What is the **maximum** $\bar{k}_i$?

2. What is the optimal network size? In other words, fixing an agent's relative degree, what is the **optimal value of** $\bar{k}_i$, the value which ensures the highest payoff growth rate for sharers?

3. What are the **optimal sharing norms** s? How much should individuals

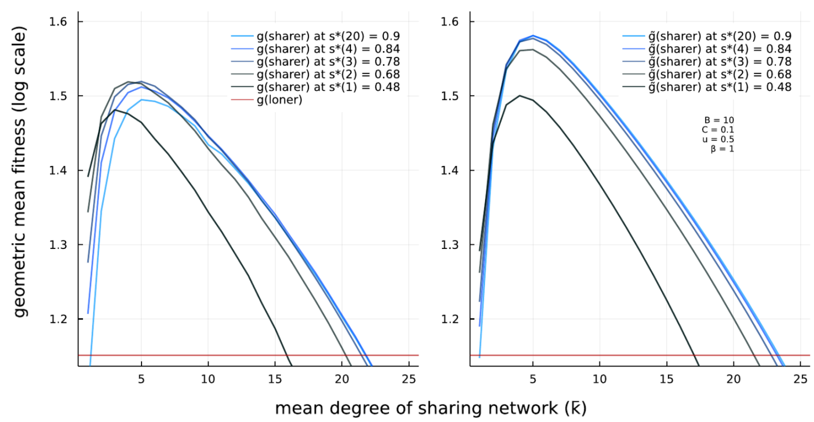

Figure 1: Numerically-simulated growth rate trajectories with increasing mean degree (left) versus trajectories from the mathematical geometric growth approximation (right). The approximation overestimates the growth rates as well as the optimal sharing norms, but remains an informative model.

in a sharing cluster share in order to optimize their growth rates?

4. How does risk, network position and per-connection costs to shared benefits ratio play into these relationships?

2 Results and equations

We simulate the production and sharing dynamic described above and construct mathematical approximations of it in order to get useful equations to describe it (see **Figure 1**). The simulations show us that:

1. The payoff of sharers exceeds that of loners already in small networks. This resonates with Winterhalder's classic risk-sharing model, where the increase in payoff due to variance reduction diminishes with every additional connection.

2. The payoff of sharers peaks ($\hat{k}$) with respect to their ego network's mean degree (fixing all other parameters). This means there is an optimal network density (network size, if the network is a clique). After this point, additional connection costs overtake the diminishing benefits of payoff variance reduction.

3. After the optimal point, sharers in networks can still enjoy higher payoffs than loners until they reach an overcrowding limit ($\bar{k}^*$), the point at which the network is so dense/large that the payoff of sharers intersects the payoff of loners. This is the maximum degree of an agent's sharing network.

4. Optimal sharing norms increase with higher surpluses, denser/larger networks, higher risk, and better position within the network hierarchy. If networks are large enough and networks are regular (so that hierarchy effects are muted), optimal sharing norms tend to full sharing ($s = 1$) as surplus size increases towards infinity.

I used two distinct mathematical approximations to get tractable estimates of the optimal sharing norms, optimal network density and maximum network density.

2.1 Optimal sharing norms

For the optimal sharing norm, I propose using a **long-term geometric growth rate model** (basically a geometric mean fitness, where sharing is a form of diversified bet-hedging), with a correction term penalizing the higher risk that smaller networks are exposed to (the smaller a network is, the higher the probability that no network peers are successful during a time period, yielding nothing but costs for the focal agent, and in a multiplicative dynamic the cases of heavy losses are very important).

To construct the approximation, we can consider the different components of a sharer's payoff. When successful during a time period, a sharer obtains

1. A **piece of surplus** $(1-s)B$ which is not shared among peers. The rest of this sB goes to the sharer's peers, so it is not present in the payoff.

2. The **shared surplus** from successful peers. Every peer of i gets surplus B with probability u, so on average there are uk_i successful peers per time period. Peers, on average, have $\bar{k}_i$ connections, and they share sB of their surplus among them. The average shared surplus received by a sharer i is then $uk_i \frac{sB}{\bar{k}_i} = u\beta_i sB$. I include a correction factor ρ_i into this average, meant to account for cases like correlated failures (see below). The whole thing is then $\rho_i u \beta_i sB$.

3. The **cumulative network costs** of connection maintenance. If every connection costs C to maintain, then the total cost of network maintenance can be written as $k_i C = \beta_i \bar{k}_i C$.

When a sharer fails, they get only get the shared surplus and pay the costs (remember, we are assuming here $b = 1$, which means the baseline is nothing). Given this, we can construct the approximate payoff. Or rather, its long-term average geometric growth rate, which is what we really care about:

$$G_i(\text{Sharer}) = u \log \{(1-s)B + S_i\} + (1-u) \log \{1 + S_i\} \tag{1}$$

where the term

$$S_i = \beta_i \left(\rho_i u s B - \bar{k}_i C\right) \tag{2}$$

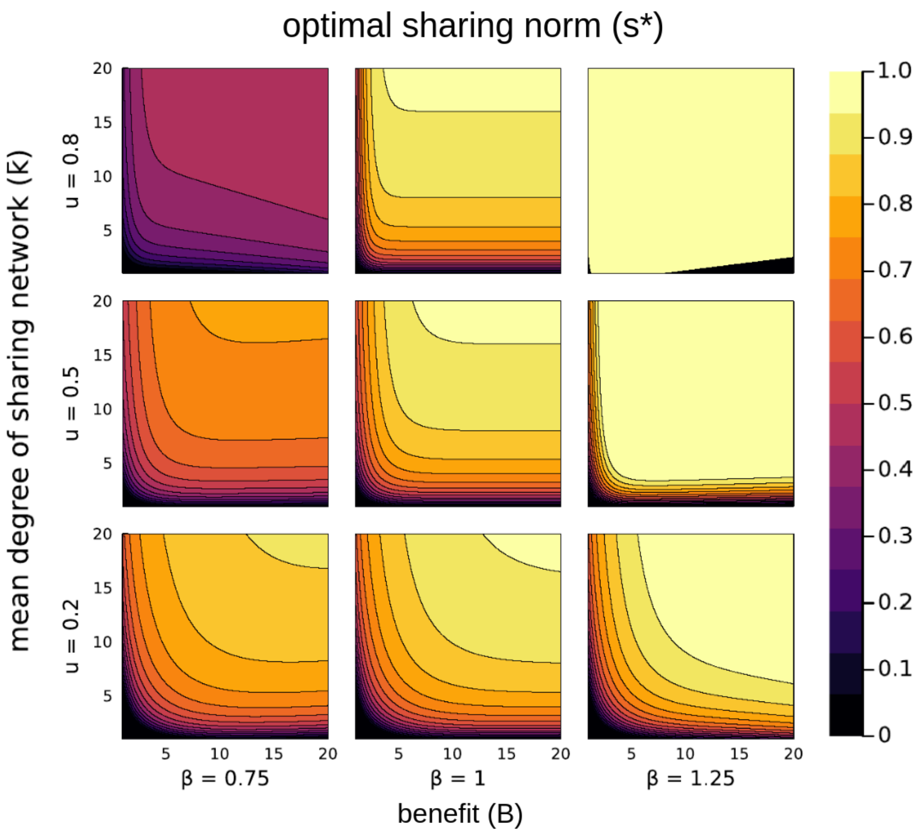

Figure 2: Optimal sharing norm as a function of $\bar{k}$ and B, for different values of u and β. Riskier environments allow for less divergence in optimal sharing norms across agents in a hierarchical cluster, where individuals differ in degree. More secure environments, on the other hand, exacerbate the difference in optimal sharing norms, and thus possible conflicts of interest between members of cluster with significant inequalities in sharing partners.

captures the network sharing components of the payoff. Here I set $\rho_i = 1-(1-u)^{k_i}$ as the above-mentioned correction factor, which is nothing more than the probability that at least one of the focal agent's network peers is successful and has something to share. As i's number of connections grows, this probability tends to 1. But having too little connections can put you in the situation of having a high chance of only experiencing costs in every time period, which is worse than just getting the baseline (worse than nothing, in this case). This happens because the chance of all of your peers failing is higher the less peers you have. In a multiplicative scenario like this one, this case is important, and this factor allows us to capture a bit of this importance, at least qualitatively. Here we assume that individuals' risks are uncorrelated, but we can also put the effect of risk correlation into this correction factor as well if we want to look at that.

By the same logic, the (geometric growth rate of the) payoff of a loner can be written as

$$G(\text{Loner}) = u \log B + (1-u)\log 1 = u \log B \tag{3}$$

Having these payoffs allows us to optimize for s (we only really need the sharer payoff for this):

$$s^* = \frac{\frac{(1-u)(B-\beta_i \bar{k} C)}{\beta_i u\left[(1-u)^{\beta_i k}-1\right]+1} + \frac{1-\beta_i \bar{k} C}{\beta_i\left[(1-u)^{\beta_i k}-1\right]}}{B} \tag{4}$$

with the limiting cases, first for large-enough networks

$$\lim_{\rho\to 1} s^* = \frac{\beta_i\left\{(\beta_i-1)\bar{k}C-(1-u)B-u\right\}+1}{\beta_i(\beta_i u-1)B} \tag{5}$$

and additionally regular networks

$$\lim_{\rho\to 1} s^* \mid_{\beta_i=1} = \frac{B-1}{B} \tag{6}$$

This approximation overshoots the optimal sharing norms in the simulations (since the expected value-based approach still throws away information on variance) but otherwise is an informative representation of the relationships between parameters and optimal sharing norms (see **Figure 1**). For a visual summary of the relationships, see **Figure 2**.

2.2 Optimal network density

For network density approximations, I propose a **rank-dependent utility model** which can be used to approximate geometric growth dynamics (see Price and Jones, 2020). This leads to payoffs that look like this:

$$V_i(\text{Sharer}) = u^\delta \left\{(1-s)B + S_i\right\} + (1-u^\delta)S_i \tag{7}$$

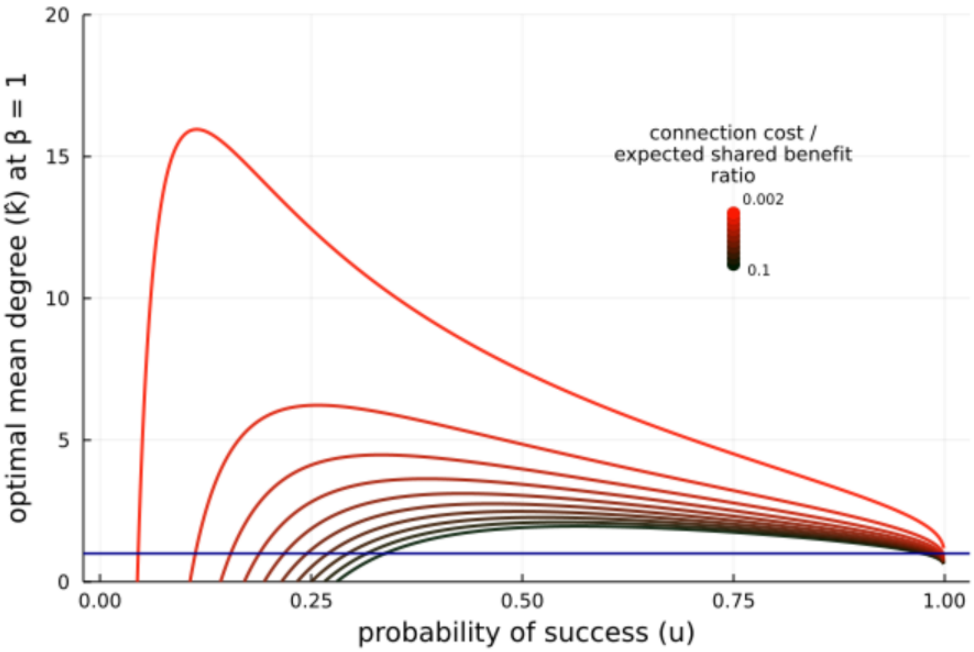

Figure 3: Optimal mean degree of the sharing cluster, for different values of connection cost to expected shared benefit ratio ($\frac{C}{sB}$) and $\beta = 1$ (an individual that sits at the mean degree of their network, or the degree of all individuals in a regular network). Riskier environments allow for larger optimal group sizes given that per-connection costs are low enough with respect to the expected shared benefits, but too much risk leads to sudden collapse. The size of optimal clusters decays as environments get more secure, with small ($\leq$ 5-agent) clusters dominating outside of the high-risk regions. The horizontal blue line represents a mean degree of 1 (purely dyadic sharing). Below this line, it is better to be a Loner.

$$V(\text{Loner}) = u^{\delta} B \tag{8}$$

We can think of this payoff structure as representing not an objective growth rate, but an individual decision-maker's perception of the payoffs in question. It includes an additional parameter δ that can be interpreted as an individual's attitude towards uncertainty. When $0 < \delta < 1$, individuals over-weight the success rate of acquiring surplus, leading to optimistic probability weighting. When $\delta = 1$, individuals take a standard arithmetic mean of the possible outcomes with respect to risk and success rates (neutral probability weighting). When $\delta > 1$, individuals increasingly over-weight risks, leading to pessimistic probability weighting. This allows for a tractable approximation of optimal network density, which we arrive by optimizing for $\bar{k}_i$, yielding the following equation:

$$\hat{k}_i = \frac{\log\left(\frac{-1}{\beta_i u \log(1-u)}\right) + \log\left(\frac{C}{sB}\right)}{\beta_i \log(1-u)} \tag{9}$$

From the equation it can already be seen that the effect of risk $(1-u)$ is mediated by network hierarchy effects (β), but can otherwise be separated from the effect of per-connection cost to shared benefit ration $(\frac{C}{sB})$. Because stuff on the log scale is not exactly easy to interpret, visual inspection helps (see **Figure 3**). We can see that, if the cost-benefit ratio is low enough, then optimal networks are at their densest when risk is high, with density decaying rapidly as risk decreases. At both ends of the risk spectrum (full risk and full security) optimal sharing completely collapses. Happily, this approximation is independent of δ, so we don't have to go down that rabbit hole when using this.

2.3 Maximum network density

By asking when $V_i(\text{Sharer}) > V(\text{Loner})$ and solving for $\bar{k}_i$, the rank-dependent model spits out this approximation for the maximum density:

$$\bar{k}_i^* = \frac{u\left(\beta_i - u^{\delta-1}\right)\frac{sB}{C} - \frac{W(\Sigma_i)}{\log(1-u)}}{\beta_i} \tag{10}$$

with

$$\Sigma_i = (1-u)^{u(\beta_i - u^{\delta-1})\frac{sB}{C}} \cdot \beta_i u \log(1-u) \cdot \frac{sB}{C} \tag{11}$$

the ugly $W(\Sigma)$ term basically serves to delimit the regions for which sharing is possible as a strategy at all (W is the Lambert W function, also known as the product-logarithm). This can be appreciated in **Figure 4**. If we assume that we are within the region where sharing is possible, then this term vanishes and we have

$$\bar{k}_i^* \approx u\left(\beta_i - u^{\delta-1}\right)\frac{sB}{\beta_i C} \tag{12}$$

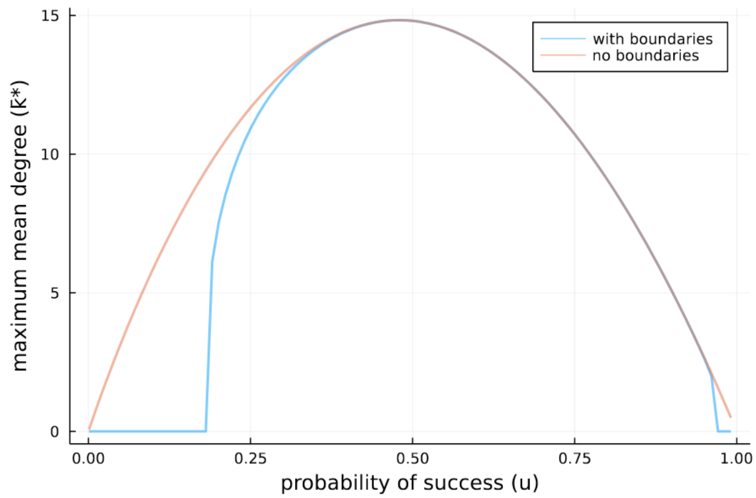

Figure 4: Difference between the derived expression for $\bar{k}^*$ when the "boundary" term is included versus when it is assumed away. Within the boundaries, the expression with no boundary approximates the expression that does account for boundaries, and it (predictably) overestimates sharing beyond the boundaries. Plotted for $B = 5$, $C = 0.05$, $s = 0.7$, $\beta = 1$ and $\delta = 0.2 + \log(B - 1)$.

which, for the case of regular networks, becomes

$$\bar{k}_i^* \mid_{\beta_i=1} \approx u\left(1 - u^{\delta-1}\right)\frac{sB}{C} \tag{13}$$

which tells us that we cannot ignore individuals' attitudes towards uncertainty when using this approximation. In particular, individuals who employ pessimistic probability weighting ($\delta > 1$) are the only ones that will choose to be sharers (in a regular network, this may change in an irregular one). As with the optimal density, the ratio of connection costs to shared benefits makes an appearance (although inverted), but in this case its effects are actually linear, making the equation easier to read. The effect of risk at the extremes of the spectrum is easier to see as well. The term involving risk looks like the variance of a Bernoulli random variable, but incorporating the effect of probability weighting. But at any level of probability weighting, sharing collapses when risk is too high or too low.

In order to get rid of δ, we can put assumptions on it and see how they fit the maximum density that we can get from simulating the original dynamics. I chose $\delta = \alpha + \log(B - b)$, where α represents a constant, baseline attitude towards uncertainty. Again, just assume $b = 1$. This expression assumes that pessimism grows with the (natural) logarithm of the difference between surplus and baseline payoffs. We still have α to reckon with, but the approximation is better. **Figure 5** plots both optimal and maximum density approximations

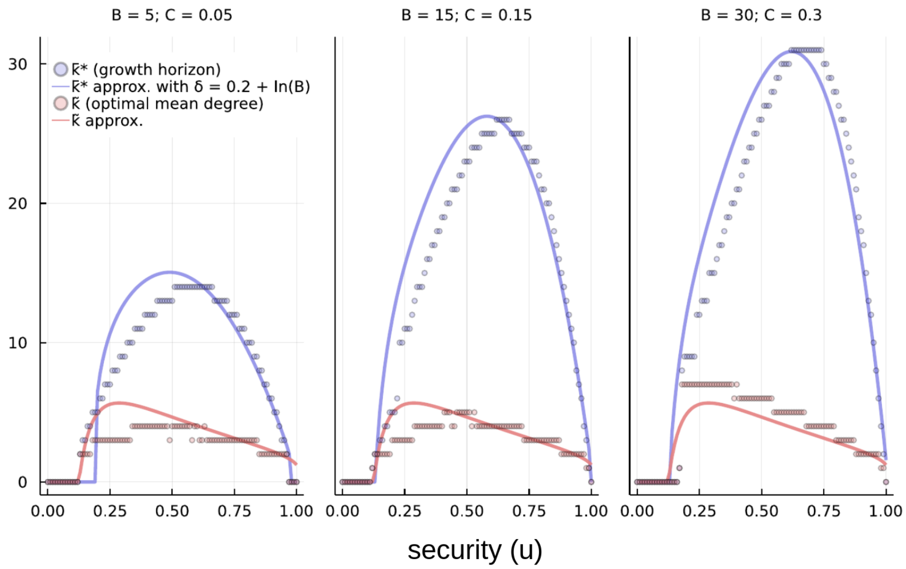

Figure 5: RDEUT approximations of $\hat{k}_i$ and $\bar{k}_i^*$ as functions of u (smooth) versus simulated values (dots) for increasing values of B and fixed surplus-to-connection-cost ratio. As B increases with respect to the baseline, so does the height and skew of the $\bar{k}_i^*$ curve, which in the RDEUT representation is due to surplus/stake dependence of the agent's risk attitudes. Also, with the increase of B the simulated $\hat{k}_i$ transitions from values below the RDEUT approximation to values above it. This is due to a valley in the payoff curve at network sizes that are not small enough to avoid high costs but not large enough to hedge for the risk of group failure. As B increases, the peak to the right of this valley overtakes the one to the left, leading to higher optimal network sizes and to the curve's final shape. Both shapes are well approximated by the optimal network density.

alongside simulated data, showing that they are not too bad.

Because of its dependence on psychological parameters, the maximum density approximation is perhaps better for experimental stuff, while the optimal density approximation could be useful for inference with naturalistic data.