

# Method of Partial Fractions

It is used to

integrate rational functions  $f(x) = \frac{P(x)}{Q(x)}$  where  $P(x)$  and  $Q(x)$  are polynomial functions and the power of  $P(x)$  is less than the power of  $Q(x)$ .

## Examples

1  $\int \frac{x}{x^3 - x} dx$

2  $\int \frac{x}{x^2 - x - 6} dx$

3  $\int \frac{1}{x^2 - 1} dx$

# Method of Partial Fractions

## Alert

If a rational function  $\frac{R(x)}{Q(x)}$  is such that the power of  $R(x)$  is greater than the power of  $Q(x)$ , then one must use long division and write the rational function in the form

$$\frac{R(x)}{Q(x)} = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n + \frac{P(x)}{Q(x)}$$

where now  $P(x)$  is a remainder term with the power of  $P(x)$  less than the power of  $Q(x)$  and our objective is to integrate each term of the above representation.

# Case 1: power of $P(x) \geq Q(x)$

Consider  $\int \frac{P(x)}{Q(x)} dx$

1. If power of  $P(x) \geq$  power of  $Q(x)$

Apply a long division.

$$\begin{array}{r|l} P(x) & Q(x) \\ \hline R(x) & q(x) \end{array}$$

And, then

$$\int \frac{P(x)}{Q(x)} dx = \underbrace{\int q(x) dx}_{\text{formula table}} + \int \frac{R(x)}{Q(x)} dx$$

## Case 2: power of $P(x) < Q(x)$

Consider  $\int \frac{P(x)}{Q(x)} dx$

1. The denominator  $Q(x)$  has only first power factors, none of which are repeated. Then,  $Q(x)$  has the form

$$Q(x) = (x - x_0)(x - x_1)(x - x_2) \cdots (x - x_n)$$

where  $x_0 \neq x_1 \neq x_2 \neq \cdots \neq x_n$ . One can then write

$$\frac{P(x)}{Q(x)} = \frac{A}{x - x_0} + \frac{B}{x - x_1} + \cdots + \frac{C}{x - x_n}$$

where  $A, B, \dots, C$  are constants to be determined.

## Case 2: power of $P(x) < Q(x)$

Consider  $\int \frac{P(x)}{Q(x)} dx$

2. The denominator  $Q(x)$  has only first power factors, but some of these factors may be repeated factors. For example, the denominator  $Q(x)$  might have a form such as

$$Q(x) = (x - x_0)^k (x - x_1)^p \cdots (x - x_n)^m$$

where  $k, p, \dots, m$  are integers.

Here the denominator has repeated factors of orders  $k, p, \dots, m$ .

In this case one can write the rational function in the form:

## Case 2: power of $P(x) < Q(x)$

### 2. (continue)

$$\begin{aligned} \frac{P(x)}{Q(x)} = & \frac{A_1}{x-x_0} + \frac{A_2}{(x-x_0)^2} + \cdots + \frac{A_k}{(x-x_0)^k} + \\ & + \frac{B_1}{x-x_1} + \frac{B_2}{(x-x_1)^2} + \cdots + \frac{B_p}{(x-x_1)^p} + \\ & + \cdots + \\ & + \frac{C_1}{x-x_n} + \frac{C_2}{(x-x_n)^2} + \cdots + \frac{C_m}{(x-x_n)^m} \end{aligned}$$

where  $A_1, \dots, A_k, B_1, \dots, B_p, \dots, C_1, \dots, C_m$  are constants to be determined.

## Case 2: power of $P(x) < Q(x)$

3. The denominator  $Q(x)$  has one or more quadratic factors of the form  $ax^2 + bx + c$  none of which are repeated. In this case, for each quadratic factor there corresponds a partial fraction of the form

$$\frac{P(x)}{Q(x)} = \frac{A_0x + B_0}{ax^2 + bx + c}$$

where  $A_0$  and  $B_0$  are constants to be determined.

## Case 2: power of $P(x) < Q(x)$

4. The denominator  $Q(x)$  has one or more quadratic factors, some of which are repeated quadratic factors. In this case, for each repeated quadratic factor  $(ax^2 + bx + c)^k$  there corresponds a sum of partial fractions of the form

$$\frac{P(x)}{Q(x)} = \frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_kx + B_k}{(ax^2 + bx + c)^k}$$

where  $A_1, B_1, \dots, A_k, B_k$  are constants to be determined.



# Integral of a Rational function — Summary

Let's consider the case 2 (power of denominator bigger than the power of numerator) and the following general form of partial fractions:

$$\begin{aligned}
 \frac{P(x)}{Q(x)} = & \frac{A_p}{(x-\alpha)^p} + \frac{A_{p-1}}{(x-\alpha)^{p-1}} + \cdots + \frac{A_2}{(x-\alpha)^2} + \frac{A_1}{x-\alpha} + \\
 & + \frac{B_q}{(x-\beta)^q} + \frac{B_{q-1}}{(x-\beta)^{q-1}} + \cdots + \frac{B_2}{(x-\beta)^2} + \frac{B_1}{x-\beta} + \cdots + \\
 & + \frac{C_r+D_rx}{((x-a)^2+b^2)^r} + \frac{C_{r-1}+D_{r-1}x}{((x-a)^2+b^2)^{r-1}} + \cdots + \frac{C_2+D_2x}{((x-a)^2+b^2)^2} + \frac{C_1+D_1x}{(x-a)^2+b^2}
 \end{aligned} \tag{1}$$

**Important:**

All the constants must be determined before applying the integral calculus.

# Techniques to find the constants

Consider Case 2.

Remark:

The 3 bellow sub-cases can appear in the integration. For each factor we should write the partial fractions as described above

## Techniques to find the constants

- Undetermined coefficients method
- The cover-up method
- Differentiation Rule

# Cover-up method

- If  $Q(x) = (x - \alpha)^p Q_1(x)$  and  $\alpha \in \mathbb{R}$

$$A_p = \left[ \frac{P(x)}{Q(x)} \right]_{x=\alpha}$$

- If  $Q(x) = ((x - a)^2 + b^2)^r Q_1(x)$  and  $x = a + bi \in \mathbb{C}$

$$\left[ C_r + D_r x = \frac{P(x)}{Q_1(x)} \right]_{x=a+bi}$$

# Cover-up method

## Example

$$\frac{6x - 1}{x^2 - 4x + 3} = \frac{6x - 1}{(x - 3)(x - 1)} = \frac{A}{x - 3} + \frac{B}{x - 1}$$

The constants  $A$  and  $B$  can be determined by cover-up method:

$$A = \left[ \frac{6x - 1}{x - 1} \right]_{x=3} = \frac{17}{2}$$

$$B = \left[ \frac{6x - 1}{x - 3} \right]_{x=1} = \frac{5}{-2} = -\frac{5}{2}$$

# Cover-up method

## Example

$$\frac{6x - 1}{(x - 3)^2(x - 1)} = \frac{A_1}{x - 3} + \frac{A_2}{(x - 3)^2} + \frac{B_1}{x - 1}$$

The constants  $A_2$  e  $B_1$  can be determined by cover-up method:

$$A_2 = \left[ \frac{6x - 1}{x - 1} \right]_{x=3} = \frac{26}{2} = 13$$

$$B_1 = \left[ \frac{6x - 1}{(x - 3)^2} \right]_{x=1} = \frac{6}{4} = -\frac{3}{2}$$

Be careful:

The constant  $A_1$  can not be determined using cover-up method

# Differentiation method

Useful when the denominator has roots (real or complex) with multiplicity greater than one.

- 1 Put the same denominator in the equation 1 (page 46).
- 2 Assign to  $x$  the roots values and find some coefficients.
- 3 Differentiate both sides of the equation and repeat step 2.
- 4 Differentiate again both sides of the equation until all the coefficients are found.

# Differentiation method

For real roots

If  $Q(x) = (x - \alpha)^p Q_1(x)$  and  $\alpha \in \mathbb{R}$

$$\left[ \frac{1}{r!} \cdot \frac{d^r}{dx^r} \left( \frac{P(x)}{Q_1(x)} \right) \right]_{x=\alpha} = A_{p-r}, \quad 0 \leq r \leq p-1$$

**Remark:** The derivative of order 0 is  $\frac{d^0}{dx^0} (f(x)) = f(x)$

# Differentiation method

## Examples

1 Consider  $\frac{x^2 + 1}{(x - 1)^3(x - 2)} = \frac{A_3}{(x - 1)^3} + \frac{A_2}{(x - 1)^2} + \frac{A_1}{x - 1} + \frac{B_1}{x - 2}$ .

Determine all the constants using the differentiation rule.

2 Consider  $\frac{2}{(x^2 + 1)^2x} = \frac{C_2 + D_2x}{(x^2 + 1)^2} + \frac{C_1 + D_1x}{x^2 + 1} + \frac{B_1}{x}$ . Determine all the constants using the undetermined coefficients method.



# Solution of First Example I

It is intended to find  $\int \frac{x^2 + 1}{(x - 1)^3(x - 2)} dx$

First, write the rational function as sums of partial fractions:

$$\frac{x^2 + 1}{(x - 1)^3(x - 2)} = \frac{A_3}{(x - 1)^3} + \frac{A_2}{(x - 1)^2} + \frac{A_1}{x - 1} + \frac{B_1}{x - 2}.$$

By comparison we obtain the following:

- $P(x) = x^2 + 1.$
- $(x - \alpha)^p = (x - 1)^3$ , then  $\alpha = 1 \wedge p = 3.$
- $Q_1(x) = x - 2.$

## Solution of First Example II

Since  $p = 3 \wedge 0 \leq r \leq p - 1 \implies 0 \leq r \leq 2$ .

$$A_{p-r} = \left[ \frac{1}{r!} \cdot \frac{d^r}{dx^r} \left( \frac{P(x)}{Q_1(x)} \right) \right]_{x=\alpha}, \quad 0 \leq r \leq 2.$$

Then, for:

- $r = 0, A_3 = \left[ \frac{1}{0!} \cdot \frac{d^0}{dx^0} \left( \frac{x^2+1}{x-2} \right) \right]_{x=1} = \left[ \left( \frac{x^2+1}{x-2} \right) \right]_{x=1} = -2.$
- $r = 1, A_2 = \left[ \frac{1}{1!} \cdot \left( \frac{x^2+1}{x-2} \right)' \right]_{x=1} = \left[ \left( \frac{x^2-4x-1}{(x-2)^2} \right) \right]_{x=1} = -4$
- $r = 2, A_1 = \left[ \frac{1}{2!} \cdot \left( \frac{x^2+1}{x-2} \right)'' \right]_{x=1} = \left[ \frac{1}{2!} \left( \frac{x^2-4x-1}{(x-2)^2} \right)' \right]_{x=1} = \dots = -5$

# Solution of First Example III

$B_1$  can be determined by hidden rule:

$$B_1 = \left[ \frac{x^2 + 1}{(x-1)^3} \right]_{x=2} = \frac{5}{1} = 5$$

Second, replace the constants and find the related integrals:

$$\int \frac{x^2 + 1}{(x-1)^3(x-2)} dx = \int \frac{A_3}{(x-1)^3} dx + \int \frac{A_2}{(x-1)^2} dx + \int \frac{A_1}{x-1} dx + \int \frac{B_1}{x-2} dx.$$

$$\begin{aligned} \int \frac{x^2 + 1}{(x-1)^3(x-2)} dx &= \int \frac{-2}{(x-1)^3} dx + \int \frac{-4}{(x-1)^2} dx + \int \frac{-5}{x-1} dx + \int \frac{5}{x-2} dx \\ &= \int -2(x-1)^{-3} dx + \int -4(x-1)^{-2} dx + \int \frac{-5}{x-1} dx + \int \frac{5}{x-2} dx \\ &= -2 \int (x-1)^{-3} dx - 4 \int (x-1)^{-2} dx - 5 \int \frac{1}{x-1} dx + 5 \int \frac{1}{x-2} dx \\ &= \dots \end{aligned}$$