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- **Spectrum**

  The spectrum of a matrix A, denoted by $S(A)$, is the set of all eigenvalues of A. In other words, it consists of all scalar values λ such that there exists a non-zero vector $\mathbf{v}$ (an eigenvector) for which:

  $$A\mathbf{v} = \lambda\mathbf{v}$$

  The spectrum gives a complete picture of the behavior of the matrix's action in terms of scaling along the directions of its eigenvectors.

- **Spectral Radius**

  The spectral radius of a matrix A, denoted $\rho(A)$, is the largest absolute value of its eigenvalues. If $S(A)$ is the set of eigenvalues of A, then the spectral radius is:

  $$\rho(A) = \max\{|\lambda| : \lambda \in S(A)\}$$

  In other words, it gives the magnitude of the "dominant" eigenvalue, which has the greatest influence on the long-term behavior of the matrix's powers.

- **Spectral Shift**

  A spectral shift refers to modifying the eigenvalues of a matrix by adding a scalar multiple of the identity matrix I to the original matrix A. If A has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then adding cI to A results in a new matrix $A + cI$, whose eigenvalues are shifted by c:

  $$Eigenvalues of A + cI = \{\lambda_1 + c, \lambda_2 + c, \ldots, \lambda_n + c\}$$

  This spectral shift preserves the relative spacing between the eigenvalues but translates them by a fixed amount.

Problems

a) The set of eigenvalues of the matrix $A = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}$ is?

Is necessary find the eigenvalues, solving the characteristic equation:

$$\det(A - \lambda I) = 0$$

Where I is the identity matrix, and λ represents the eigenvalues. The characteristic equation becomes:

$$\det \begin{bmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{bmatrix} = 0$$

The determinant is:

$$(-3-\lambda)(-3-\lambda) - (1)(1) = \lambda^2 + 6\lambda + 9 - 1 = \lambda^2 + 6\lambda + 8 = 0$$

Solve the quadratic equation:

$$\lambda^2 + 6\lambda + 8 = 0$$

Thus, the eigenvalues are:

$$\lambda_1 = -2, \quad \lambda_2 = -4$$

The set of eigenvalues is $\{-2, -4\}$.

b) The spectral radius of the matrix $A = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}$ is?

The absolute values of the eigenvalues are, according to a):

$$|\lambda_1| = 2, \quad |\lambda_2| = 4$$

The spectral radius is the largest of these values:

$$\rho(A) = 4$$

c) Consider the set $\{-2,\ 2,\ 5\}$ of eigenvalues of the real 3×3 matrix $A + 3I$, where I denotes the identity matrix. Find the eigenvalues of A.

The eigenvalues of $A + 3I$ are related to the eigenvalues of A by a shift of $+3$. In other words:

$$\text{Eigenvalues of } A + 3I = \{\lambda_1 + 3, \lambda_2 + 3, \lambda_3 + 3\}$$

Are given that the eigenvalues of $A + 3I$ are $\{-2, 2, 5\}$. Therefore, to find the eigenvalues of A, it is necessary to subtract 3 from each of these values:

$$\text{Eigenvalues of } A = \{-2 - 3, 2 - 3, 5 - 3\} = \{-5, -1, 2\}$$

$$\text{Eigenvalues of } A = \{-5, -1, 2\}$$

Thus, the eigenvalues of A are $\{-5, -1, 2\}$.

Reference: Nicholson, W. K. (2006). Elementary Linear Algebra. ISBN 85-86804-92-4.