Biinvariant Generalized Barycentric Coordinates on Lie Groups

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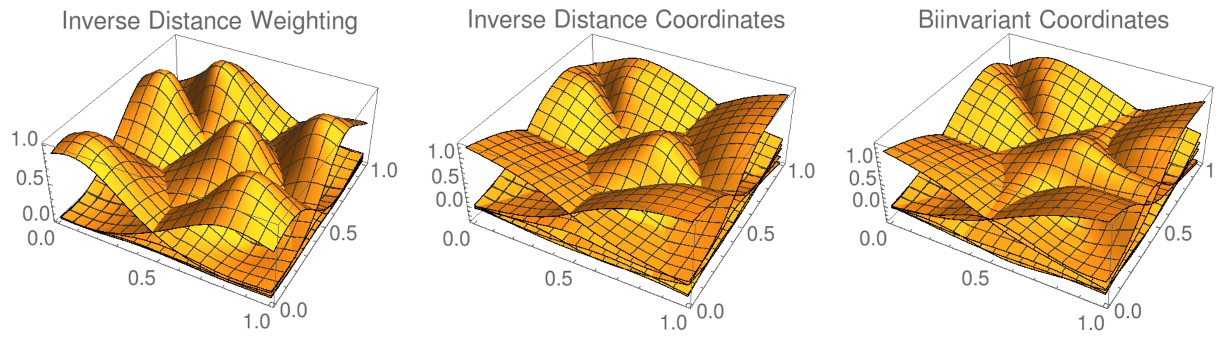

Figure: Basis functions of inverse distance weighting, inverse distance coordinates, and biinvariant coordinates with exponent $\beta = 2$ for an example set of six points in the unit square. ■

Abstract: We construct biinvariant generalized barycentric coordinates for scattered sets of points in any Lie group. The coordinates are invariant under left-action, right-action, and inversion, and satisfy the Lagrange property. The construction does not utilize a metric on the Lie group, unlike inverse distance coordinates. Instead, proximity is determined in a vector space of higher dimensions than the group using the Euclidean norm. The coordinates that we propose are an inverse to the unique, biinvariant weighted average in the Lie group.

Keywords: generalized barycentric coordinates, biinvariant mean, Lie groups, homogeneous spaces

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Introduction

Let G be a d-dimensional Lie group, and $P = \{p_1, p_2, \ldots, p_n\}$ a set of points $p_i \in G$ for $i = 1, \ldots, n$ with $n > d$. Two problems can be posed from the following equations where $x \in G$ and $w \in \mathbb{R}^n$

(1) $\sum_{i=1}^{n} w_i = 1$ (partition of unity)

(2) $\sum_{i=1}^{n} w_i \log\left(x^{-1}.p_i\right) = 0$ (barycentric equation)

Forward problem: Given P and w, find the *weighted average* $x \in G$ that satisfies (2).

Inverse problem: Given P and x, find a *barycentric coordinate* $w \in \mathbb{R}^n$ that satisfies (1) and (2).

Weighted Averages

Given P and w, [2012 Pennec/Arsigny] prove that if the data points in P belong to a sufficiently small normal convex neighborhood $\mathcal{V} \subset G$ of some point then there exists a unique solution $x \in G$ of (2). The weighted average $\mu_P(w) := x$ is referred to as the *biinvariant mean*, because when the points in P are subject to a group transformation by any element $g \in G$, then the weighted average undergoes the same transformation:

$\mu_{g.P}(w) = g.x$ (left-action)

$\mu_{P.g}(w) = x.g$ (right-action)

$\mu_{P^{-1}}(w) = x^{-1}$ (inversion)

where $g.P := \{g.p_1, \ldots, g.p_n\}$, $P.g := \{p_1.g, \ldots, p_n.g\}$, and $P^{-1} := \left\{p_1^{-1}, \ldots, p_n^{-1}\right\}$. The property of biinvariance holds always, regardless of whether the group permits the definition of a biinvariant metric. Finding $x \in G$ is generally a non-linear problem that can be solved using an iterative fixed point algorithm.

Remark: [2012 Pennec/Arsigny] assume that the weights w_i are non-negative. However, in all examples that we have encountered, the weighted average is biinvariant and unique even if weights are allowed to be slightly outside the unit interval [0, 1] given that "*the dispersion of the data is small enough*". ■

Generalized Barycentric Coordinates

The contribution of this article is the derivation of a solution to the inverse problem: Given P, we construct a function $c_P : G \to \mathbb{R}^n$ that yields a generalized barycentric coordinate $c_P(x) = w$ that satisfies (1) and (2) for $x \in G$. We quietly assume that "*the dispersion of the data is small enough*", that $\log(x^{-1}.p_i)$ exists, and that the definition of $c_P : \mathcal{V} \subset G \to \mathbb{R}^n$ is restricted to a sufficiently small neighborhood $\mathcal{V} \subset G$ of x.

Since the weighted average μ_P is always biinvariant, we find it intuitive to impose biinvariance to the generalized barycentric coordinate $c_P : G \to \mathbb{R}^n$. We refer to the function c_P as *biinvariant*, if c_P is invariant under left-action, right-action, and inversion

$$c_{g.P}(g.x) = w \qquad \text{(invariance under left-action)}$$

$$c_{P.g}(x.g) = w \qquad \text{(invariance under right-action)}$$

$$c_{P^{-1}}(x^{-1}) = w \qquad \text{(invariance under inversion)}$$

Additionally, we can design the biinvariant coordinate $c_P(x) = w$ to satisfy the Lagrange property

$$\text{(3)} \qquad x = p_i \quad \Rightarrow \quad w_j = \delta_{i,j} := \begin{cases} 1 & i = j \\ 0 & \text{else} \end{cases} \qquad \text{(Lagrange property)}$$

The Lagrange property is relevant for applications that require interpolation: Given a set of tuples $\{(p_i, q_i) : i = 1, \ldots, n\}$ with $p_i \in G$, and $q_i \in H$, a biinvariant coordinate c_P function that satisfies the Lagrange property, and a weighted average μ_Q in the space H. Then, the concatenation $c_P \circ \mu_Q : G \to H$ is a biinvariant function that interpolates the data points, i.e. $\mu_Q(c_P(p_i)) = q_i$ for all $i = 1, \ldots, n$.

Coordinates are an inverse of the weighted average, because of the identity $\mu_P(c_P(x)) = x$ for all $x \in G$. In contrast to the weighted average, the biinvariant barycentric coordinate $w \in \mathbb{R}^n$ is typically not unique. Apart from the generally non-linear computation of the vectors $v_i := \log(x^{-1}.p_i)$ for $i = 1, \ldots, n$, our construction of c_P uses only concepts from linear algebra.

Related work

We are not aware of any prior publication on generalized barycentric coordinates for Lie groups. This section gives an overview on the literature that has inspired the creation of the biinvariant generalized barycentric coordinates for Lie groups.

Lie Groups and Biinvariant Mean

[2012 Xavier Pennec, Vincent Arsigny] show that on any Lie group the weighted average $\mu_P(w) = x$ exists uniquely "*provided that the dispersion of the data is small enough*", and has the property of biinvariance, and therefore can be referred to as the *biinvariant mean* determined by P and w. The authors derive explicit formulas for the biinvariant mean in the group of scalings and translations ST(d), the Heisenberg group He(d), and the special Euclidean group SE(2). Their results have motivated our search for generalized barycentric coordinates on Lie groups that are biinvariant and satisfy the Lagrange property.

[2017 Ethan Eade] derives explicit formulas for the group action, inverse, exp, and log for the Lie groups SO(3), SE(2), and SE(3) that are relevant in robotics.

Generalized Barycentric Coordinates

[2011 Shayne Waldron] derives *affine coordinates* for scattered sets of points in $\mathbb{R}^d$ that satisfy (2), but do not have the Lagrange property. We will show that for an arbitrary Lie group G, the concept of affine coordinates corresponds to a uniquely determined biinvariant generalized barycentric coordinate function.

[2013 Daniele Panozzo, Ilya Baran, Olga Diamanti, Olga Sorkine-Hornung] define weighted averages on triangular meshes, and an inverse with the Lagrange property that for a set of anchor points P maps any x

on the mesh to affine weights that are consistent with the weighted average. Their approach is motivated as: "*Combining the forward and inverse problems allows us to define a correspondence mapping between two different meshes based on provided corresponding point pairs, enabling texture transfer, compatible remeshing, morphing and more.*"

[1968 Donald Shepard] introduces *inverse distance weighting* for the interpolation of irregularly-spaced data in $\mathbb{R}^d$. Shepard's inverse distance weights do not qualify as generalized barycentric coordinates due to the violation of (2). [2020 Hakenberg] projects Shepard's weights to the closest solution of (2) in the least-square sense, which results in *inverse distance coordinates*. If a biinvariant metric is defined on a Lie group, then the metric induces inverse distance coordinates that are biinvariant and satisfy the Lagrange property. Because not all Lie groups can be equipped with a biinvariant metric, we propose a different construction for c_P in this article that applies universally to any Lie group.

Construction of Coordinates

We derive biinvariant generalized barycentric coordinates for a d-dimensional Lie group G. Given a set of points $P = \{p_1, p_2, \ldots, p_n\}$ with $p_i \in G$ for $i = 1, \ldots, n$, and $n > d$. Denote with $v_i := \log(x^{-1}.p_i)$ the vector in the Lie algebra $\mathfrak{g}$ of G. Any non-zero vector from the nullspace of the matrix $V = [v_1, \ldots, v_n]$ of dimensions $d \times n$ that may be normalized to sum up to 1 produces a solution to (1) and (2).

Denote with $N := \text{nullspace}(V)$ a matrix of dimensions $r \times n$ with row vectors that span the nullspace of V, i.e. that satisfies $V.N^T = 0$. The nullspace is non-trivial because $n > d$ implies $r > 0$.

Remark: The coordinate $w \in \mathbb{R}^n$ consists of n variables w_i. Equation (1) eliminates one degree of freedom. In the special case when the number of points is $n = d + 1$, the barycentric coordinate w is uniquely determined by (2), if matrix V has maximal rank d. ■

Biinvariance

Theorem 11 of [2012 Pennec/Arsigny, p. 21] states that if $x \in G$ is the weighted average of $P = \{p_1, p_2, \ldots, p_n\}$, i.e. a solution to the barycentric equation for a fixed vector $w \in \mathbb{R}^n$ that satisfies (1), then $g.x$ is the weighted average of $g.P$, $x.g$ is the weighted average of $P.g$, and x^{-1} is the weighted average of P^{-1} for all $g \in G$. Their proof argues that the nullspace of V (as a subspace in $\mathbb{R}^n$) is invariant under left-action, right-action, and inversion that are applied simultaneously to x and $p_i \in P$ for $i = 1, \ldots, n$.

Let N^+ denote the pseudoinverse with dimensions $n \times r$ of the matrix N. The matrix $M := N^+.N$ is a linear projection from $\mathbb{R}^n$ to the subspace in $\mathbb{R}^n$ spanned by the row vectors in $N = \text{nullspace}(V)$. A projection means that any eigenvalue of M is equal to either 1 or 0. The projection to the subspace is independent of the choice of vectors that span the subspace, i.e. $M = (A.N)^+.(A.N)$ for any invertible matrix A of dimensions $r \times r$.

We consider a vector $\alpha \in \mathbb{R}^n$ that is subject to the projection by matrix M

(4) $\tilde{w} = \alpha.M$

The vector α represents *target weights* since the projection yields $\tilde{w} \in \mathbb{R}^n$ in the subspace spanned by N closest to α that also satisfies (2). If $\sum_{i=1}^n \tilde{w}_i \neq 0$, we obtain the vector of coordinates $w \in \mathbb{R}^n$ that additionally satisfies (1) using scaling

(5) $w = \frac{1}{\sum_{i=1}^n \tilde{w}_i} \tilde{w}$

The degree of freedom is in the design of *target weights* $\alpha \in \mathbb{R}^n$. If the construction of α is biinvariant, the resulting generalized barycentric coordinate w in (5) is also biinvariant.

Remark: The choice of $\alpha = (1, \ldots, 1)$ for all $x \in G$ results in biinvariant generalized barycentric coordinates for an arbitrary Lie group G that we call *affine coordinates* in the spirit of [2011 Waldron]. These coordinates generally do not satisfy the Lagrange property. ■

Lagrange Property

Remark: If G is equipped with a biinvariant metric $\mathrm{d} : G \times G \to \mathbb{R}$ that measures distance between two points, then the projection of the target weights $\alpha_i = 1 \big/ \mathrm{d}(x, p_i)^\beta$ for $i = 1, \ldots, n$ and $\beta \geq 1$ followed by normalization result in biinvariant generalized barycentric coordinates, which we refer to as *inverse distance coordinates* that satisfy the Lagrange property. This approach is not universal, because a biinvariant metric does not exist on every Lie group, for instance on the special Euclidean group SE(d) for $d \geq 2$. ∎

The challenge is to define target weights $\alpha \in \mathbb{R}^n$ similar to the inverse distance but using terms that are biinvariant, and apply to any Lie group G, and therefore do not rely on the existence of a biinvariant metric. Our solution to the design problem only employs the matrix M, which is biinvariant: We define the target weights as

$$\alpha_i = 1 / \| z_i \|^\beta \qquad \text{for } i = 1, \ldots, n,$$

where the vector $z_i \in \mathbb{R}^n$ is defined as $z_i := (\delta_{i,j} - M_{i,j} : j = 1, \ldots, n)$, i.e. z_i is the i-th row of the matrix $I - M$. $\| z \|$ denotes the Euclidean norm, i.e. the 2-norm of a vector $z \in \mathbb{R}^n$. The exponent $\beta \geq 1$ is chosen typically as $\beta = 1$ for linear-like interpolation, or $\beta = 2$ for smooth interpolation.

We argue that the coordinates $c_P(x) = w$ satisfy (3) as x approaches a point p_i from the input set. As $x \to p_i$, the $n \times n$ projection matrix M converges to have entries $M_{i,j} = M_{j,i} = \delta_{i,j}$ for $j = 1, \ldots, n$. That means, the entry α_i that tends to infinity as $z_i \to 0$, while α_j for $j \neq i$ stays bounded, only manifests itself in the entry $\tilde{w}_i$ in (4). The normalization in (5) results in the convergence of w to the unit vector $e_i := (\delta_{i,j} : j = 1, \ldots, n)$ thereby establishing the Lagrange property.

The construction is well-defined for any $x \in G$ that results in $\alpha < \infty$ and $\sum_{i=1}^n \tilde{w}_i \neq 0$.

Remark: Any Lie group G can be equipped with a left-invariant metric $\mathrm{d}_L : G \times G \to \mathbb{R}$ that defines the distance between two points. Then, the projection of the inverse distance weights $\alpha_i = 1 \big/ \mathrm{d}_L(x, p_i)^\beta$ followed by normalization result in left-invariant generalized barycentric coordinates that are generally not biinvariant, but satisfy the Lagrange property. ∎

Implementation

The implementations below are for the Lie group $\mathbb{R}^d$ in *Mathematica*.

```
Shepard[β_][points_, x_] := Module[{Vt = # - x & /@ points, α},
  α = 1 / (Norm /@ Vt) ^ β;
  Normalize[α, Total]]
```

```
IDC[β_][points_, x_] := Module[{Vt = # - x & /@ points, α, N, M},
  N = NullSpace[Transpose[Vt]]; M = PseudoInverse[N].N;
  α = 1 / (Norm /@ Vt) ^ β;
  Normalize[α.M, Total]]
```

```
BIC[β_][points_, x_] := Module[{Vt = # - x & /@ points, α, N, M},
  N = NullSpace[Transpose[Vt]]; M = PseudoInverse[N].N;
  α = 1 / (Norm /@ (IdentityMatrix[Length[points]] - M)) ^ β;
  Normalize[α.M, Total]]
```

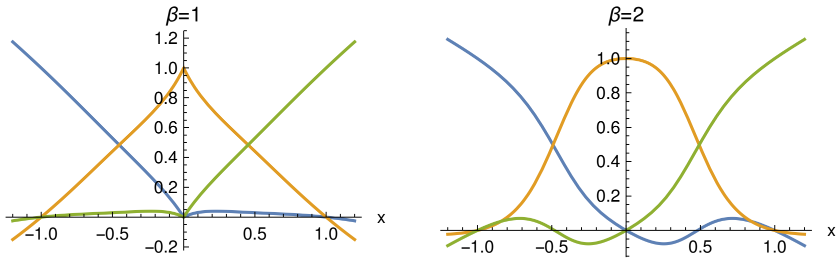
The examples in the next section were generated using the open source, non-linear geometry software library *sophus.* The library implements biinvariant coordinates c_P and weighted averages μ_P for the Lie groups $\mathbb{R}^d$, SO(3), ST(d), He(d), SE(2), $\overline{\mathrm{SE}}$(2), SE(3), and the homogeneous spaces S^d, $\mathrm{Sym}_+(d)$. Inverse distance coordinates are available for $\mathbb{R}^d$, SO(3), and S^d, $\mathrm{Sym}_+(d)$.

Examples

For the 1-dimensional Lie groups $\mathbb{R}^1$ and SO(2), we found that inverse distance coordinates and biinvariant coordinates are identical.

$\mathbb{R}^1$

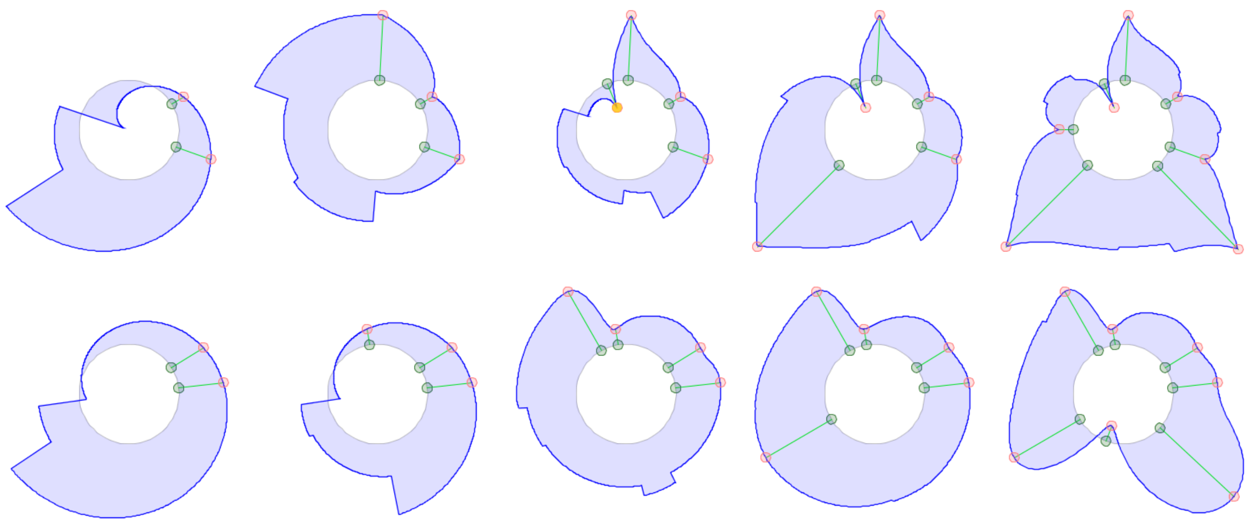
Example: For set of points $P = \{-1, 0, 1\}$ in the group $(\mathbb{R}, +)$, *Mathematica* yields $c_P : \mathbb{R} \to \mathbb{R}^3$ as



$$c_P(x) = \frac{1}{2}\begin{pmatrix} 1 - x + \frac{x^4-1}{1-x^2+2\,|x|+3\,|x|^3} \\ \frac{2-2\,x^4}{1-x^2+2\,|x|+3\,|x|^3} \\ 1 + x + \frac{x^4-1}{1-x^2+2\,|x|+3\,|x|^3} \end{pmatrix} \text{ for } \beta = 1, \text{ and } c_P(x) = \frac{1}{2+18\,x^4}\begin{pmatrix} x(1-x)\left(-1+9\,x^3\right) \\ 2-2\,x^2 \\ x\,(1+x)\left(1+9\,x^3\right) \end{pmatrix} \text{ for } \beta = 2. \blacksquare$$

S^1

Example: The 1-dimensional Lie group of rotations in the plane SO(2) can be identified with the 1-dimensional sphere S^1. Each illustration below shows a set of anchor points $\{p_i\}$ on S^1 with associated real values $q_i \in \mathbb{R}$ indicated in normal direction. $\beta = 1$ in the top row, and $\beta = 2$ in the bottom row.



The graph of the function $c_P \circ \mu_Q : S^1 \to \mathbb{R}$ is plotted as the blue line. Discontinuities are obvious when the points p_i are not distributed well across S^1. ■

The characteristics of discontinuities was also noticed by [2013 Panozzo et al.] for coordinates on surface meshes, who address the issue as: "*We leave it up to the user to ensure that there are enough close-by anchors; our experiments show that this is not difficult.*"

$\mathbb{R}^2$

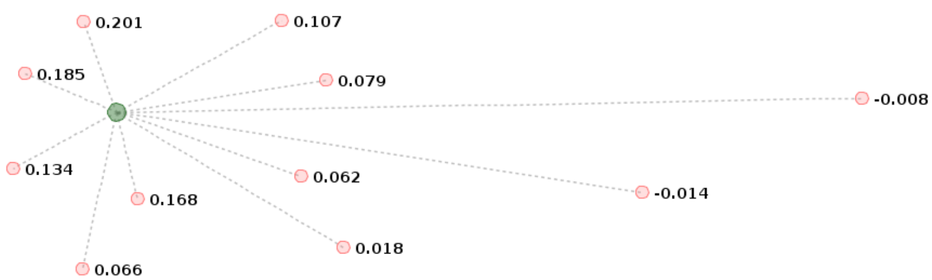
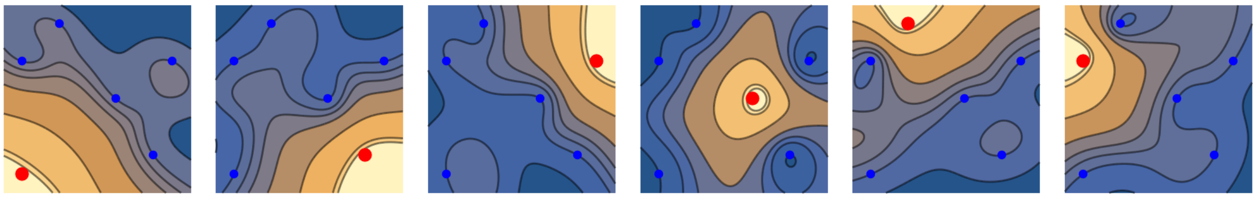
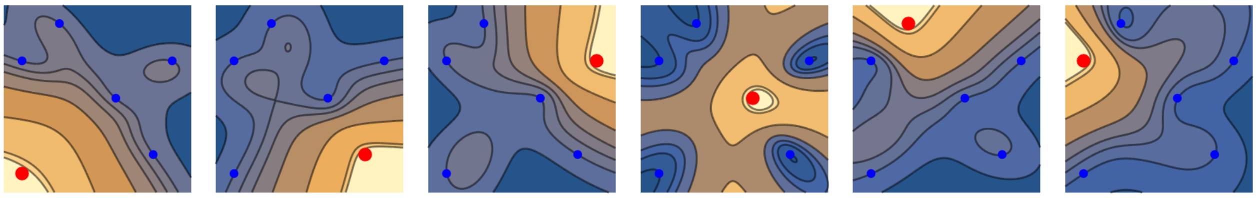


Figure: A set of 11 points $p_i \in \mathbb{R}^2$ in the plane and a generalized barycentric coordinate $w \in \mathbb{R}^{11}$ that yields the weighted average indicated in green. ■

Example: Let $p_1 = (0.1, 0.1)$, $p_2 = (0.8, 0.2)$, $p_3 = (0.9, 0.7)$, $p_4 = (0.6, 0.5)$, $p_5 = (0.3, 0.9)$, $p_6 = (0.1, 0.7)$ with $p_i \in \mathbb{R}^2$. We show approximate contour plots of the basis functions w_i for $i = 1, \ldots, 6$ evaluated over the unit square. Inverse distance coordinates with $\beta = 2$:



Biinvariant coordinates with $\beta = 2$:



At the coordinate $x = (0.3, 0.4)$ for instance, the inverse distance coordinate evaluates to $w_{\mathrm{IDC}} \approx (0.339, 0.097, -0.037, 0.3, 0.059, 0.242)$, the biinvariant coordinate evaluates to $w_{\mathrm{BIC}} \approx (0.352, 0.054, -0.041, 0.367, 0.057, 0.211)$. ■

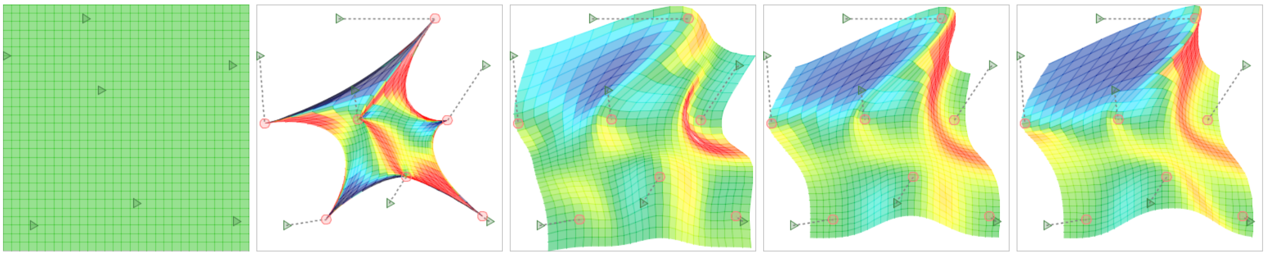


Figure: The left image shows the square domain $D = [0, 1]^2 \subset \mathbb{R}^2$ with a set of $n = 7$ anchor points $\{p_i\}$. Each point p_i has an associated target location $q_i \in \mathbb{R}^2$. The images to the right visualize the image of D using different deformation functions: Shepard's inverse distance weights, moving least squares with Shepard's inverse distance weights, inverse distance coordinates, and biinvariant coordinates, each with exponent $\beta = 2$. In the two latter cases, the deformation is the concatenation $c_P \circ \mu_Q : D \to \mathbb{R}^2$. ■

[2006 Scott Schaefer, Travis McPhail, Joe Warren] and [2016 Olga Sorkine-Hornung, Michael Rabinovich] are references for the *moving least squares* deformation method in $\mathbb{R}^d$.

S^2

The 2-dimensional sphere $S^2 = \mathrm{SO}(3)/\mathrm{SO}(2)$ is not a Lie group, but a homogeneous space. [2005 Scott Schaefer, Ron Goldman] state the functions exp and log on S^d.

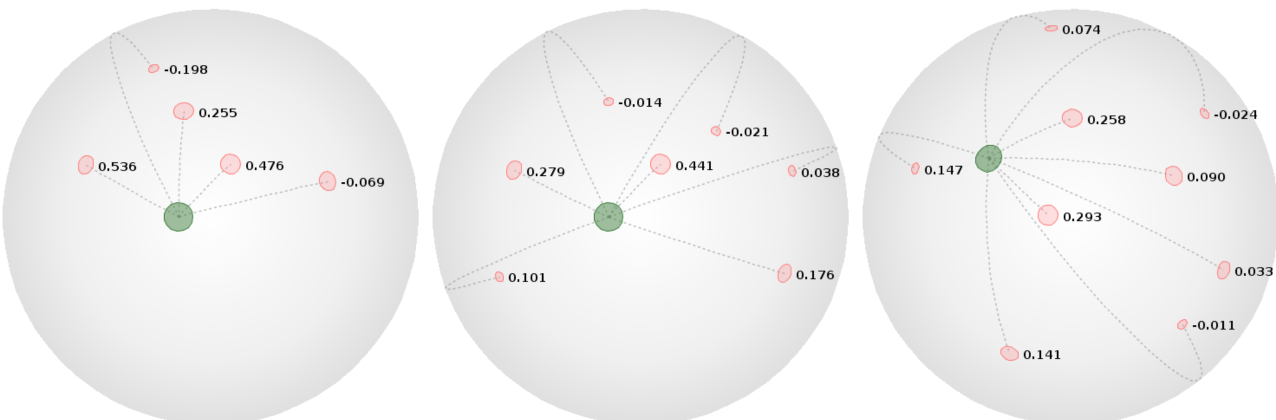
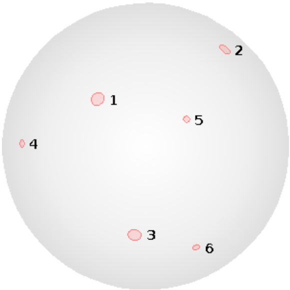


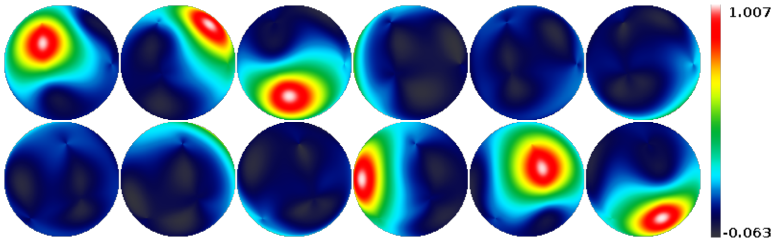
Figure: Three examples of a set of points on S^2 with generalized barycentric coordinate that yields in the weighted average indicated in green. ■

Example: We place 6 anchor points on the 2-dimensional sphere of which p_1, p_2, p_3 are located on the front hemisphere, and p_4, p_5, p_6 are on the back-side:

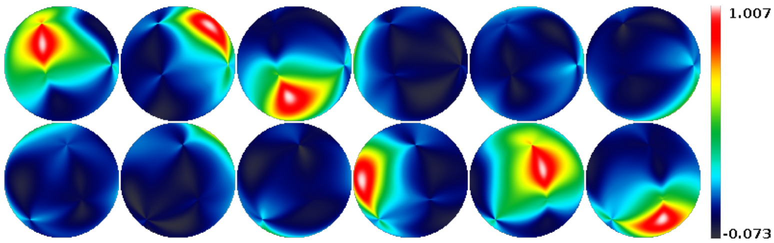


From the points $P = \{p_i\}$ we compute the following generalized barycentric coordinate functions $c_P : S^2 \to \mathbb{R}^6$

inverse distance coordinates:



biinvariant coordinates:



using $\beta = 2$. The 6 orbs on the top row show the respective coordinate of c_P evaluated on the front hemisphere. The orbs on the bottom row show the coordinates of c_P evaluated on the back hemisphere. ∎

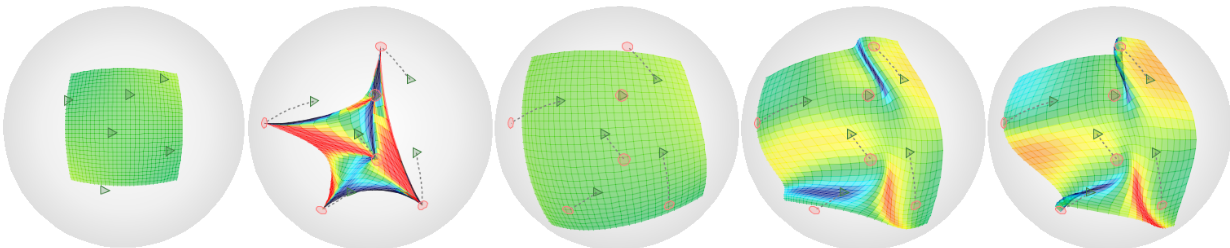


Figure: Deformation of a "square" domain on S^2. Left shows the neutral configuration with anchor points $p_i \in S^2$ for $i = 1, \ldots, 6$. Then: Deformation induced by Shepard's inverse distance weighting, affine coordinates, inverse distance coordinates, and biinvariant coordinates using the mapping μ_Q for target points $q_i \in S^2$. The last two deformation methods are iterpolatory $\mu_Q(c_P(p_i)) = q_i$ for $i = 1, \ldots, n$ because the coordinates satisfy the Lagrange property. ∎

SE(2)

The 3-dimensional Lie group of orientation preserving, rigid transformations of the 2-dimensional plane is the special Euclidean group SE(2). A point in the group is represented by a triple (px, py, θ), where (px, py) $\in \mathbb{R}^2$ represents a location in the plane, and $\theta \in [-\pi, \pi)$ an orientation. We visualize such a triple by an arrowhead. The covering group of SE(2) is denoted $\overline{\text{SE}}(2)$ and accounts for windings in the orientation. In $\overline{\text{SE}}(2)$ the angular component is any real number $\theta \in \mathbb{R}$. A biinvariant metric does not exist on SE(2) or $\overline{\text{SE}}(2)$. The explicit formulas for the group action, inverse, exp and log are stated in [2018 Hakenberg].

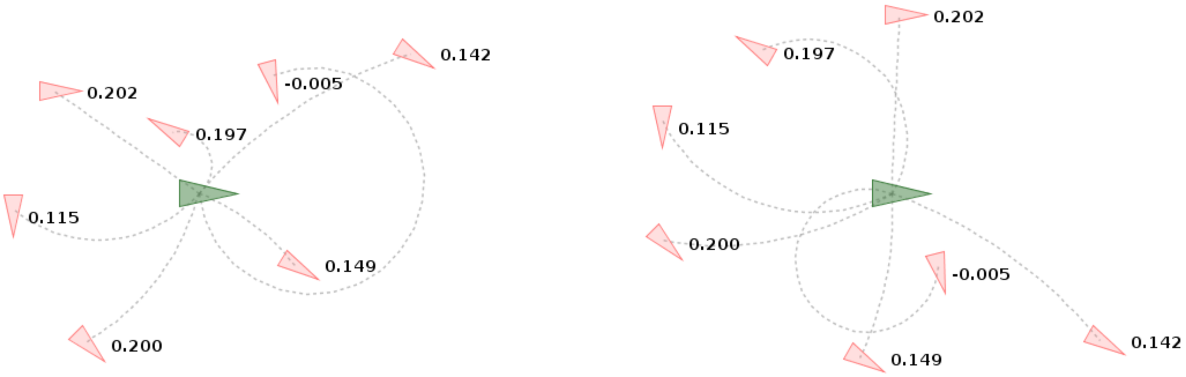


Figure: Points $p_1, \ldots, p_7$ from the Lie group $\overline{\mathrm{SE}}(2)$ are mapped to their average x (in green) according to weights $w_i \in \mathbb{R}$ associated to each point p_i for $i = 1, \ldots, n$. The right figure shows the points $q_i := g.p_i.h$ for $i = 1, \ldots, 7$ that are the result of transformation by left-action and right-action for some $g, h \in \overline{\mathrm{SE}}(2)$. Due to the biinvariance property of the weighted average, the biinvariant mean of the transformed points is $g.x.h$. The biinvariant coordinates obtained in both cases are identical, i.e. $w = c_P(x) = c_Q(g.x.h)$. A dashed line indicates a geodesic $\overline{\mathrm{SE}}(2)$ projected to the xy-plane. ▪

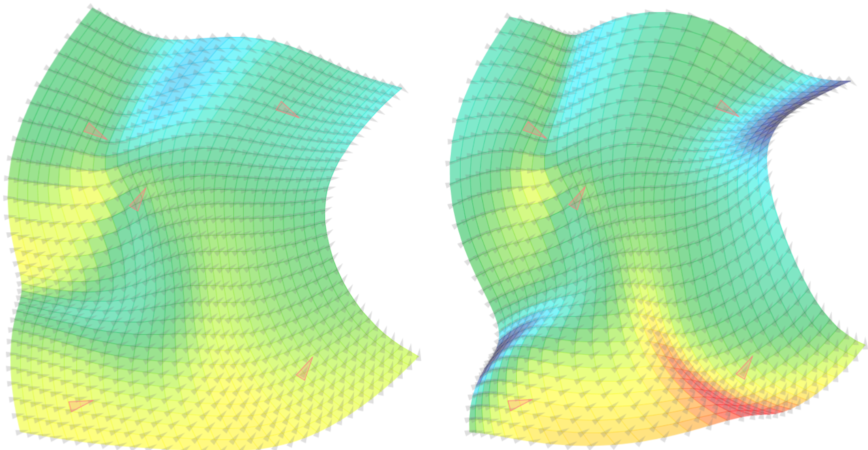


Figure: The graphics compare inverse distance coordinates (left) and biinvariant coordinates (right) with $\beta = 2$ for an input set of $n = 5$ points $p_i = (\mathrm{px}_i, \mathrm{py}_i)$ in $\mathbb{R}^2$, that have associated values $q_i = (\mathrm{px}_i, \mathrm{py}_i, \theta_i)$ in the 3-dimensional Lie group $\overline{\mathrm{SE}}(2)$. The coordinates are evaluated over a rectangular domain $D \subset \mathbb{R}^2$ and mapped into the Lie group $\overline{\mathrm{SE}}(2)$ using the concatenation of $c_P : \mathbb{R}^2 \to \mathbb{R}^5$ and the weighted average μ_Q in $\overline{\mathrm{SE}}(2)$. ▪

$\mathrm{Sym}_+(2)$

[2007 P. Thomas Fletcher, Sarang Joshi] and [2020 Xavier Pennec, Stefan Sommer, Tom Fletcher] state formulas for exp, log, and distance in the homogeneous space of positive symmetric definite matrices $\mathrm{Sym}_+(d)$. The space $\mathrm{Sym}_+(2)$ is 3-dimensional.

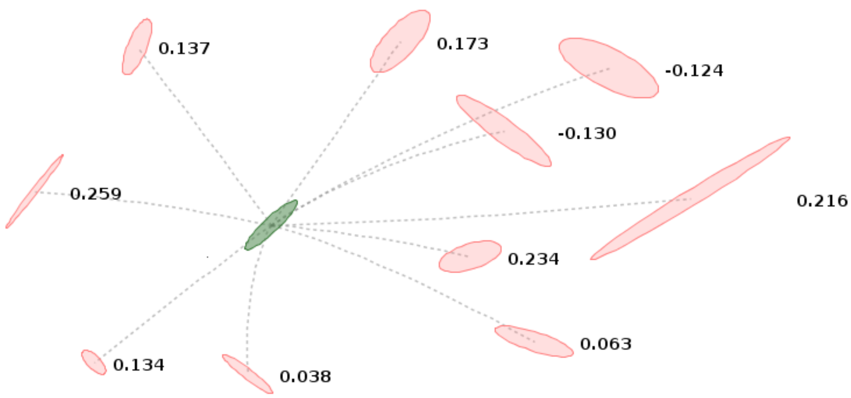


Figure: Visualization of the identity $c_P(\mu_P(x)) = x$ for a set of points $P = \{p_i\}$ and x (in green) in $\mathrm{Sym}_+(2)$. The 2×2 matrices are represented by ellipses as is common for covariance matrices. ▪

Conclusion

We have constructed biinvariant generalized barycentric coordinates that satisfy the Lagrange property on an arbitrary Lie group G. The construction of the function c_P depends on the choice of an exponent β.

Given a set of tuples $\{(p_i, q_i) : i = 1, \ldots, n\}$ with $p_i \in G$, and $q_i \in H$, the coordinates c_P, and a weighted average μ_Q in the space H. Then, the concatenation $c_P \circ \mu_Q : G \to H$ is a biinvariant function that interpolates the data points, i.e. $\mu_Q(c_P(p_i)) = q_i$ for all $i = 1, \ldots, n$, as was illustrated in several examples.

[2012 Pennec/Arsigny] show that the biinvariant mean can always be obtained using an iterative fixed-point algorithm. This approach is in fact necessary in the case of SO(3) and SE(3) for instance. In contrast, for some groups such as He(d), and SE(2), equation (2) reduces to a system of linear equations. The computation of the biinvariant generalized barycentric coordinate $c_P(x) = w$ that we propose in this article requires the computation of the logarithm in $v_i := \log_x(p_i)$ for $i = 1, \ldots, n$, the nullspace $N = \text{nullspace}(V)$, and the pseudoinverse of N regardless of the Lie group.

Our construction of coordinates also applies to the homogeneous spaces S^d, and $\text{Sym}_+(d)$. On a homogeneous space, the barycentric equation is written as

$$\sum_{i=1}^{n} w_i \log_x(p_i) = 0.$$

Future work

The construction of biinvariant generalized barycentric coordinates for Lie groups is not unique but depends on design choices. For instance, the target weights proposed in this article generally do not yield back $c_P(\mu_P(w)) \neq w$ for the special weight vector $w_i = 1/n$ for $i = 1, \ldots, n$.

[2012 Pennec/Arsigny] prove the uniqueness of the solution to the barycentric equation on Lie groups and show that the solution is biinvariant. Weighted averages on homogeneous spaces seem to have eluded investigation so far. Is the solution to the barycentric equation on a homogeneous space always unique and biinvariant?

We plan to investigate applications of biinvariant generalized barycentric coordinates on the Lie groups SE(2) and SE(3) in the field of robotics.

References

1968 Donald Shepard: *A two-dimensional interpolation function for irregularly-spaced data*
2005 Scott Schaefer, Ron Goldman: *Freeform Curves on Spheres of Arbitrary Dimension*
2006 Scott Schaefer, Travis McPhail, Joe Warren: *Image Deformation Using Moving Least Squares*
2007 P. Thomas Fletcher, Sarang Joshi: *Riemannian Geometry for the Statistical Analysis of Diffusion Tensor Data*
2011 Shayne Waldron: *Affine generalised barycentric coordinates*
2012 Xavier Pennec, Vincent Arsigny: *Exponential Barycenters of the Canonical Cartan Connection and Invariant Means on Lie Groups*
2013 Daniele Panozzo, Ilya Baran, Olga Diamanti, Olga Sorkine-Hornung: *Weighted Averages on Surfaces*
2016 Olga Sorkine-Hornung, Michael Rabinovich: *Least-Squares Rigid Motion Using SVD*
2016 Kai Hormann, N. Sukumar: *Generalized Barycentric Coordinates in Computer Graphics and Computational Mechanics*
2017 Ethan Eade: *Lie Groups for 2D and 3D Transformations*
2018 Jan Hakenberg: *Curve Subdivision in SE(2)*
2020 Xavier Pennec, Stefan Sommer, Tom Fletcher: *Riemannian Geometric Statistics in Medical Image Analysis*
2020 Jan Hakenberg: *Inverse Distance Coordinates for Scattered Sets of Points*
2020 https://github.com/datahaki/sophus: *Non-linear geometry library in Java*