Biinvariant Distance Vectors

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Abstract: We construct biinvariant vector valued functions of relative distances using the influence matrix, and the Mahalanobis distance defined by scattered sets of points on Lie groups. The functions are invariant under all group operations. Distance vectors define an ordering of the points in the scattered set with respect to a group element. Applications are classification, inverse distance weighting, and the construction of generalized barycentric coordinates for the purpose of deformation, and domain transfer. ∎

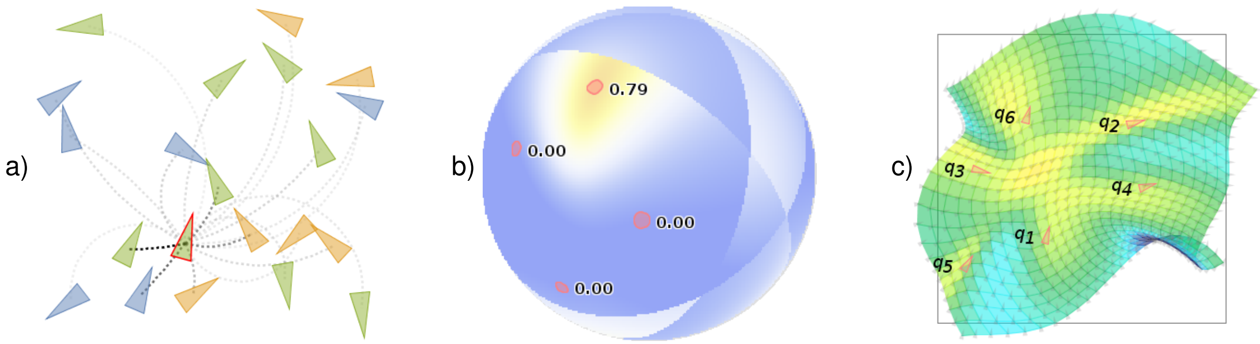

Figure: Applications of biinvariant distance vectors and weightings: a) classification task in the Lie group SE(2), b) kriging function $f : \mathbb{R}P^2 \to \mathbb{R}$ on the real projective space based on four control points with associated values, c) smooth domain transfer $[0, 1]^2 \subset \mathbb{R}^2 \to \overline{\mathrm{SE}}(2)$ based on control points of the form $p_i = (\mathrm{px}_i, \mathrm{py}_i) \in \mathbb{R}^2$ mapped to $q_i = (\mathrm{px}_i, \mathrm{py}_i, \theta_i) \in \overline{\mathrm{SE}}(2)$ for $i = 1, \ldots, 6$. ∎

Keywords: influence matrix, Mahalanobis distance, generalized barycentric coordinates, Lie group, homogeneous space

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Introduction

For a set of points $P = \{p_1, \ldots, p_n\}$ and a point x, we discuss distance vectors $d_P(x) \in \mathbb{R}^n$ where the i-th entry $d_P(x)_i$ corresponds to the relative distance of x to p_i for $i = 1, \ldots, n$. The term *relative* means that $d_P(x)_i$ may depend not only on x and p_i but on all points in P. The distance vector satisfies $d_P(x)_i \geq 0$ for general x, and $d_P(p_i)_i = 0$ in particular, for all $i = 1, \ldots, n$.

The vector $d_P(x) \in \mathbb{R}^n$ defines a preorder on the elements in P as $p_i \preccurlyeq p_j \Leftrightarrow d_P(x)_i \leq d_P(x)_j$. The distance vector can be used for classification: one can simply assign a point x the label associated to the point p_j where $j = \mathrm{argmin}_{i=1,\ldots,n}\, d_P(x)_i$. The distance vectors that we consider in the article are invariant under all symmetries of the space.

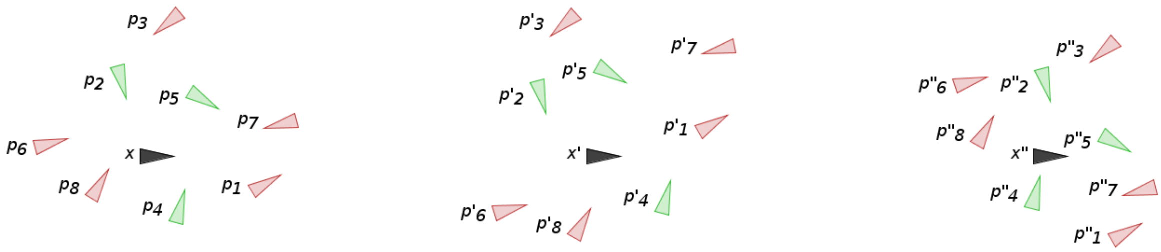


Figure: The illustration shows an example in the 3-dimensional non-linear Lie group SE(2) where the distance vector $d_P(x) \in \mathbb{R}^8$ determines p_2, p_4, p_5 to be the 3 nearest neighbors of x in P. The vector $d_P(x)$, and the induced preorder in particular, remain invariant after applying a group transformation simultaneously to x and p_i for $i = 1, \ldots, 8$. Three equivalent configurations are shown. ∎

Raising each entry of the distance vector to the power of $-\beta$ for some exponent $\beta \geq 0$ followed by normalization of the vector to sum up to 1 defines *inverse distance weighting* $w_P(x)$. The projection of $w_P(x)$ to the solution space of the barycentric equation results in generalized barycentric coordinates $c_P(x)$. The coordinate functions $c_P(x)$ define deformations within the domain, and allow the transfer of the domain into any other space that is equipped with a weighted average.

The article is structured as follows: In the first section, we restrict the discussion to the d-dimensional vector space $\mathbb{R}^d$ and already introduce three types of distance vectors: *metric*, *leverage*, and *harbor*. The generalization to Lie groups later on is achieved simply by modification of the design matrix in the construction. Then, we introduce the *garden* distance vector. We show that the biinvariant influence matrix, and Mahalanobis distance in the definition of the distance vectors result in biinvariant weightings, and biinvariant generalized barycentric coordinates. Examples and applications are illustrated in the spaces $\mathbb{R}^d$, S^2, and SE(2). We conclude by outlining possible future work.

Notation: We use the symbol "." as the dot product according to the convention in *Mathematica*, where a vector, i.e. a tensor of rank 1, cannot be transposed. Instead of the expression $w^T B\, v$ for vectors v, w and a matrix B, we write either of $w.B.v = (B.v).w = B.v.w$. Since the group action of most Lie groups can be formulated as matrix multiplication, "." also denotes the group action as in $g.h$ for $g, h \in G$. $\| v \|$ denotes the Euclidean norm. $\mathbb{1}_n := (1, \ldots, 1)$ is the vector of length n with all entries equal to 1. By "weighting" we refer to weights that sum up to 1.

Related Work

The construction of distance vectors, weightings, and generalized barycentric coordinates that we encounter in this article rely on concepts from linear algebra that also appear in the context of statistical data analysis:

Design matrix of dimensions $n \times d$:	V
Pseudo inverse of V:	$V^+ = (V^T.V)^+.V^T$
Hat matrix, or *influence matrix*:	$H = V.V^+$
Residual maker matrix:	$M = I - H$

where I denotes the $n \times n$ identity matrix, see [2013 Cardinali]. [Wikipedia: Proofs involving the Moore-Penrose inverse] render M as the projection to the left-nullspace of V, since

$$M.V = (I - V.V^+).V = V - V.V^+.V = 0$$

A diagonal element $H_{i,i}$ of the influence matrix H is referred to as *leverage*. Because the projection matrix H is symmetric and idempotent $H = H.H$, we have $H_{i,i} = \sum_{j=1}^n H_{i,j}^2$, which asserts that

$$H_{i,i} \in [0, 1] \qquad \text{for } i = 1, \ldots, n$$

as well as $\sqrt{H_{i,i}} = \| h_i \|$, where h_i denotes the i-th row of matrix H, see [Wikipedia: Leverage (Statistics)]. The matrix equation $H = V.V^+ = V.(V^T.V)^+.V^T$ allows to compute the leverage as

$$H_{i,i} = (V^T.V)^+.v_i.v_i \qquad \text{for } i = 1, \ldots, n$$

where v_i denotes the i-th row of matrix V. The right hand side is an evaluation of the *Mahalanobis distance* squared.

Finally, we define the operator $\eta : \mathbb{R}^n \to \mathbb{R}^n$ that normalizes the entries of a vector $w \in \mathbb{R}^n$ to sum up to 1

$$\eta(w) := \begin{cases} \frac{1}{\sum_{i=1}^n w_i} w & \sum_{i=1}^n w_i \neq 0 \\ \text{undefined} & \text{otherwise} \end{cases}$$

Euclidean Space

For a scattered set of points $P = \{p_1, \ldots, p_n\}$ from the d-dimensional Euclidean space $p_i \in \mathbb{R}^d$ for $i = 1, \ldots, n$ it is a straightforward choice to define a distance vector $d_P : \mathbb{R}^d \to \mathbb{R}^n$ with respect to a location $x \in \mathbb{R}^d$ as

$$d_P(x)_i := \| p_i - x \| \qquad \text{for } i = 1, \ldots, n.$$

[1968 Shepard] has introduced *inverse distance weighting*, i.e. the function $w_P : \mathbb{R}^d \backslash P \to \mathbb{R}^n$ that raises each entry $d_P(x)_i$ to the power of $-\beta$ with $\beta \geq 0$

$\tilde{w}_P(x)_i := d_P(x)_i^{-\beta}$ for $i = 1, \ldots, n$

followed by normalization of the entries to sum up to 1

$w_P(x) := \eta(\tilde{w}_P(x))$

Common choices are $\beta \in \{1, 2\}$. Using the terminology introduced above, we define

Design matrix: $V_P(x) := \begin{pmatrix} p_1 - x \\ \ldots \\ p_n - x \end{pmatrix}$

Influence matrix: $H_P(x) := V_P(x).V_P(x)^+$

Residual maker matrix: $M_P(x) := I - H_P(x)$

[2020 Hakenberg a)] constructs *inverse distance coordinates* $c_P : \mathbb{R}^d \backslash P \to \mathbb{R}^n$ as

$c_P(x) := \eta(M_P(x).w_P(x))$

Because of $M_P(x).V_P(x) = 0$, any linear combination of rows of the symmetric matrix $M_P(x)$ lies in the left-nullspace of $V_P(x)$, and therefore, the vector $c_P(x)$ in particular satisfies

$c_P(x).V_P(x) = 0$ (barycentric equation)

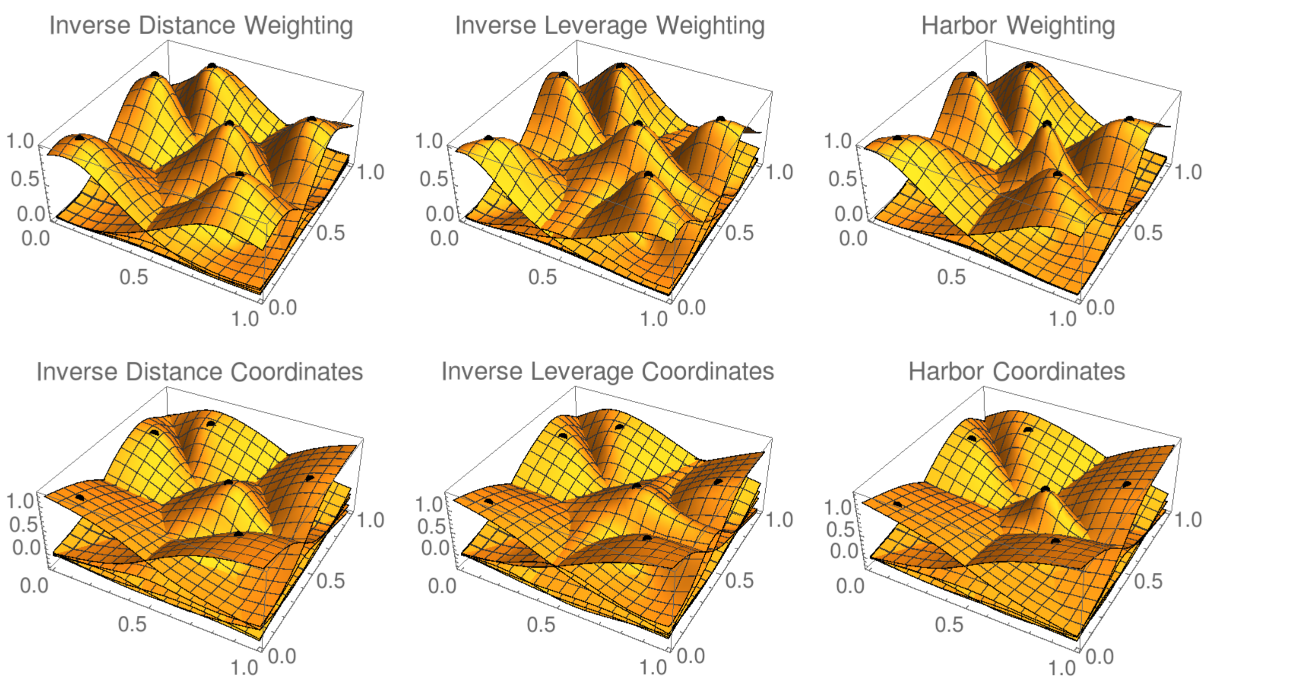
d_P is continuous but not differentiable at the isolated points $x \in P$. On $\mathbb{R}^d \backslash P$, the construction of w_P and c_P is a combination of C^∞ functions. At points where the construction is well defined, i.e. does not involve a division by 0, the functions w_P and c_P are also C^∞.

Let $e_{n,i}$ denote the unit vector of length n with 1 at the i-th entry, and 0 otherwise. Due to continuity and boundedness of w_P and c_P, the limits exist as $\lim_{x \to p_i} w_P(x) = e_{n,i}$ and $\lim_{x \to p_i} c_P(x) = e_{n,i}$ and allow the continuation of the functions at $x \in P$ as $w_P(p_i) := e_{n,i}$ and $c_P(p_i) := e_{n,i}$ for all $i = 1, \ldots, n$.

The construction of $w_P(x)$ and $c_P(x)$ also succeeds for alternative distance vectors:

	definition of distance vector	weighting	coordinates
metric	$d_P^{M}(x)_i := \lVert v_i \rVert = \lVert p_i - x \rVert$	w_P^{ID}	c_P^{ID}
leverage	$d_P^{L}(x)_i := \sqrt{H_P(x)_{i,i}}$	w_P^{IL}	c_P^{IL}
harbor	$d_P^{H}(x)_i := \lVert H_P(p_i) - H_P(x) \rVert_F$	w_P^{H}	c_P^{H}

Example: Let P consist of the six points $p_1 = (0.1, 0.1)$, $p_2 = (0.8, 0.2)$, $p_3 = (0.9, 0.7)$, $p_4 = (0.6, 0.5)$, $p_5 = (0.3, 0.9)$, $p_6 = (0.1, 0.7)$ from the 2-dimensional plane $\mathbb{R}^2$. We plot the weightings w_P^{ID}, w_P^{IL}, w_P^{H} as well as the coordinates c_P^{ID}, c_P^{IL}, c_P^{H} over the unit square $[0, 1]^2 \subset \mathbb{R}^2$ for $\beta = 2$.



Each graphic aggregates the $n = 6$ entries of the vector valued function. ■

The generalized barycentric coordinates $c_P^{\mathrm{IL}}(x)$ that are based on inverse leverags were derived in [2020 Hakenberg b)]. The motivation was to create coordinates without the use of a metric. For $x \in P$, $d_P^{\mathrm{L}}(p_i)_i = 0$ follows from $H_{i,i} = (V^\top.V)^+.v_i.v_i$ since then $v_i = p_i - p_i = 0$ for all $i = 1, \ldots, n$.

The *harbor* distance vector d^{H} is introduced in this article. $d_P^{\mathrm{H}}(x)$ is determined by the Frobenius norm of the difference between the influence matrices $H_P(x)$ and $H_P(p_i)$ for $i = 1, \ldots, n$. Clearly, $d_P^{\mathrm{H}}(p_i)_i = \| H_P(p_i) - H_P(p_i) \|_F = 0$ for all $i = 1, \ldots, n$.

The i-th entry $d_P^{\mathrm{M}}(x)_i$ of the metric distance vector only depends on the two locations p_i and x, and is independent of the remaining points $p_j \in P$ for $j \neq i$. Therefore, one can refer to d_P^{M} as a vector of *absolute* distances. In contrast, the i-th entry $d_P^{\mathrm{L}}(x)_i$ depends on all points in P and x. In that case, we refer to d_P^{L} as a vector of *relative* distances. Analogous, d_P^{H} is a vector of *relative* distances.

Remark: The special choice of $\beta = 0$ results in the constant $w_P(x) = (1/n)\, \mathbb{1}_n$ for all x, and *affine coordinates* $c_P^{\mathrm{AF}}(x) = \eta(M_P(x).\mathbb{1}_n)$ as proposed in [2011 Waldron], see also [2016 Hormann, Sukumar; p.12]. ∎

Lie Groups

A continuous distance function between two points $x, y \in G$ in a Lie group G cannot always be defined in a way that is invariant under all group actions. "*To overcome the lack of existence of bi-invariant Riemannian metrics for general Lie groups*", [2012 Pennec, Arsigny] propose to rely on the canonical Cartan connection. The geodesics of the connection are consistent with group composition and inversion:

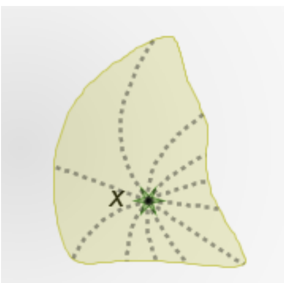
$$\log_x(y) = \mathrm{d}L_x.\log\left(x^{-1}.y\right) = \mathrm{d}R_x.\log\left(y.x^{-1}\right)$$

for all $y \in \mathcal{V}$ in a neighborhood of $x \in \mathcal{V} \subset G$. The function $\log : \mathcal{V} \subset G \to \mathfrak{g}$ without subindex refers to the mapping of group elements to the Lie algebra $\mathfrak{g}$. The function $L_x(y) := x.y$ denotes left-action, and $R_x(y) := y.x$ is right-action.

Let G be a d-dimensional Lie group, and $P = \{p_1, \ldots, p_n\}$ a set of points $p_i \in G$ for $i = 1, \ldots, n$ with $n > d$. [2012 Pennec, Arsigny; p.14] generalize $V_P(x)$ for $x \in G$ as

Design matrix: $$V_P(x) := \begin{pmatrix} \log_x(p_1) \\ \ldots \\ \log_x(p_n) \end{pmatrix}$$

The vectors $v_i := \log_x(p_i)$ for $i = 1, \ldots, n$ are from the same vector space, namely the tangent space $T_x G$. The logarithm $\log_x : \mathcal{V} \subset G \to T_x G$ maps geodesics emanating from x to straight lines in $T_x G$ with $\log_x(x) = 0$. We quietly assume that "*the dispersion of the data is small enough*", that means all points $p_1, \ldots, p_n, x$ are from a sufficiently small neighborhood $\mathcal{V} \subset G$ so that $\log_x(p_i)$ exists for all $i = 1, \ldots, n$. The definition of $V_P(x)$ is consistent with the previous section regarding the Euclidean space, since in $\mathbb{R}^d$ we have $\mathrm{d}L_x = I$, and $\log_x(y) = \log\left(x^{-1}.y\right) = y - x$.

a) 

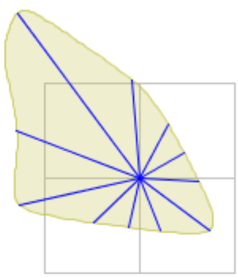
b) 

Figure: a) shows $\mathcal{V} \subset G$ as a neighborhood of x in yellow, and several geodesics emanating from x. b) visualizes the $\log_x$ image of $\mathcal{V}$ and of the geodesics in $T_x G$. ∎

Two problems can be posed from the following equations where $x \in G$ and $w \in \mathbb{R}^n$

(1) $w.\mathbb{1}_n = 1$ (partition of unity)

(2) $w.V_P(x) = 0$ (barycentric equation)

Forward problem: Given P and w, find the *weighted average* $x \in G$ that satisfies (2).

Inverse problem: Given P and x, find a *barycentric coordinate* $w \in \mathbb{R}^n$ that satisfies (1) and (2).

[2012 Pennec, Arsigny] refer to (2) as the characterization of the weighted average as the *exponential barycenter*. The authors show that the weighted average $\mu_P(w) := x \in G$ is unique, and biinvariant regardless whether a biinvariant metric exists on the Lie group G. In Theorem 3, p.21 the authors show that the left-nullspace of $V_P(x)$ is biinvariant, i.e. for any $w \in \mathbb{R}^n$

$$w.V_P(x) = 0 \quad \Leftrightarrow \quad w.V_{g.P}(g.x) = 0 \quad \Leftrightarrow \quad w.V_{P.g}(x.g) = 0 \quad \Leftrightarrow \quad w.V_{P^{-1}}\left(x^{-1}\right) = 0$$

where $g.P := \{g.p_1, \ldots, g.p_n\}$, $P.g := \{p_1.g, \ldots, p_n.g\}$ for $g \in G$, and $P^{-1} := \left\{p_1^{-1}, \ldots, p_n^{-1}\right\}$.

The authors derive explicit formulas for weighted averages in selected Lie groups, for instance SE(2), and He(d). In $\mathbb{R}^d$, the weighted average is simply $\mu_P(w) = \sum_{i=1}^n w_i\, p_i$. Generally, the weighted average may be obtained using an iterative fixed point algorithm. The uniqueness of the weighted average allows to generalize constructions that involve affine linear combinations originally conceived for $\mathbb{R}^d$ to Lie groups: Bézier curves, subdivision, and smoothing filters, see the references in [2018 Hakenberg b); p.10].

The solution to the inverse problem is generally not unique. In the 2-dimensional Euclidean space $\mathbb{R}^2$, numerous constructions for barycentric coordinates exist, see [2016 Hormann, Sukumar; p.15]. However, these functions involve notions of length, area, angle, and convexity, which complicates their generalization to arbitrary Lie groups.

[2016 Pennec] refers to the set of solutions of the inverse problem as the *exponential barycentric subspace*. The article "*shows that barycentric subspaces locally define a submanifold of dimension* $n-1$". The main aim of the author is the generalization of principal component analysis to Riemannian manifolds referred to as *barycentric subspaces analysis*.

[2020 Hakenberg b)] constructs the specific solution $c_P^{\text{IL}}(x)$ to the inverse problem, i.e. generalized barycentric coordinates that are moreover biinvariant on arbitrary Lie groups.

The terms "forward problem" and "inverse problem" were introduced in [2013 Panozzo, Baran, Diamanti, Sorkine-Hornung].

Biinvariant Distance Vectors

A distance vector $d_P : \mathcal{V} \subset G \to \mathbb{R}^n$ is called *biinvariant* if

$$d_P(x) = d_{g.P}(g.x) = d_{P.g}(x.g) = d_{P^{-1}}\left(x^{-1}\right) \qquad \text{for all } g \in G.$$

[2012 Pennec, Arsigny; p.21] show that the left-nullspace of $V_P(x)$ as a subspace in $\mathbb{R}^n$ is biinvariant. That means the projection $M_P(x)$ to the left-nullspace of $V_P(x)$ is also biinvariant. The projection $H_P(x)$ to the orthogonal complement of the left-nullspace of $V_P(x)$ is therefore also biinvariant. That implies that both functions d_P^{L}, and d_P^{H} defined as

$$d_P^{\text{L}}(x)_i := \sqrt{H_P(x)_{i,i}} \qquad \text{for } i = 1, \ldots, n$$

$$d_P^{\text{H}}(x)_i := \| H_P(p_i) - H_P(x) \|_F \qquad \text{for } i = 1, \ldots, n$$

are biinvariant. For *metric* distances $d_P^{\text{M}}(x)_i := \| v_i \| = \| \log_x(p_i) \|$ to be biinvariant, a biinvariant metric has to exist on the Lie group, which is not always the case. For instance, "*there is No Bi-invariant Metric for Rigid Transformations*" SE(d) for $d \geq 2$, see [2012 Pennec, Arsigny; p.16]. The impossible often has a kind of integrity to it which the merely improbable lacks.

[2012 Pennec, Arsigny; p.39] define the *Mahalanobis distance* $m_{P,x} : \mathcal{V} \subset G \to \mathbb{R}$ as

$$m_{P,x}^2(y) := S_P(x)^+.\log_x(y).\log_x(y) \qquad \text{where} \qquad S_P(x) := V_P(x)^\top.V_P(x)$$

The authors argue that $m_{P,x}(y)$ is biinvariant under simultaneous transformation of all parameters P, x, y by an arbitrary group element $g \in G$, and inversion. The 2-covariant tensor $S_P(x)$ is generally positive definite, but may be only positive *semi*definite if the tangent space $T_x G$ is embedded in a vector space of higher dimensions for convenience, or for pathological constellations of points P. Therefore, we define the bilinear form using the pseudo inverse $S_P(x)^+$. Furthermore, we omit the factor n^{-1} in the definition of $S_P(x)$ in order to achieve equivalence with leverages as argued in the previous section

$$d_P^L(x)_i = \sqrt{H_P(x)_{i,i}} = m_{P,x}(p_i) \qquad \text{for } i = 1, \ldots, n.$$

In other words, the bilinear form $S_P(x)^+$ on the tangent space $T_x G$ evaluated with the vectors $v_i = \log_x(p_i)$ for $i = 1, \ldots, n$ yield the same distance notion as the projection matrix $H_P(x)$.

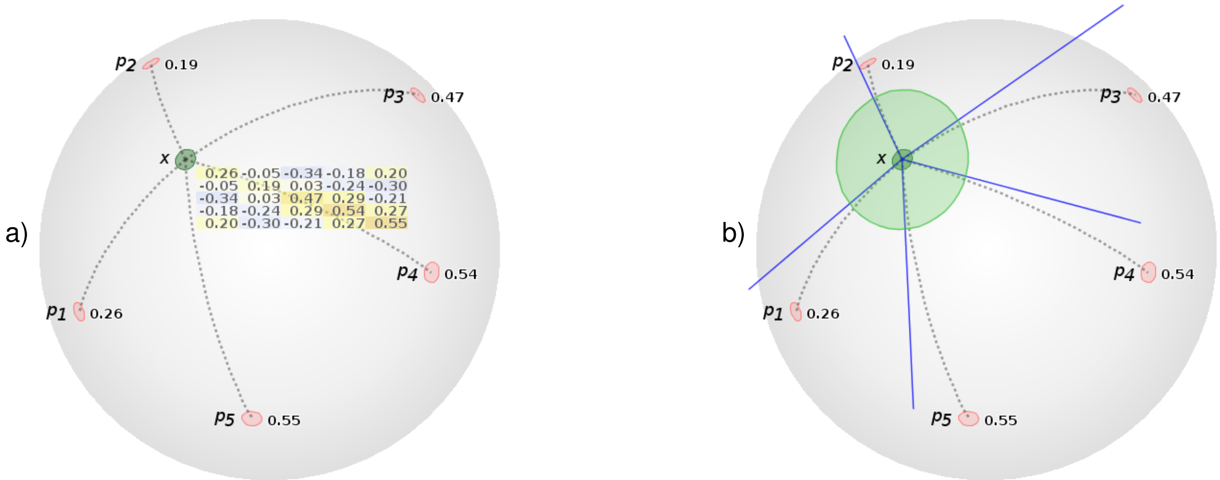


Figure: Two equivalent approaches to compute $d_P^L(x) \in \mathbb{R}^5$ for $P = \{p_1, \ldots, p_5\}$ and x on the 2-dimensional sphere S^2: a) Influence matrix $H_P(x)$ of dimensions 5×5. b) Positive semidefinite form $S_P(x)^+$ indicated as ellipse, and tangent vectors $v_i = \log_x(p_i) \in T_x S^2$ for $i = 1, \ldots, 5$ embedded in $\mathbb{R}^3$ that determine $m_{P,x}(p_i)$. ■

The *garden* distance vector makes use of the bilinear form $S_P(p_i)^+$ on the tangent space $T_{p_i} G$ in the evaluation of the Mahalanobis distance at each $p_i \in P$ as

$$d_P^G(x)_i := m_{P,p_i}(x) = \sqrt{S_P(p_i)^+ . \log_{p_i}(x) . \log_{p_i}(x)} \qquad \text{for } i = 1, \ldots, n.$$

The distance vector $d_P^G(x)$ is biinvariant, because all entries $m_{P,p_i}(x)$ are biinvariant. We yield $d_P^G(p_i)_i = m_{P,p_i}(p_i) = 0$ due to $\log_{p_i}(p_i) = 0$ for all $i = 1, \ldots, n$.

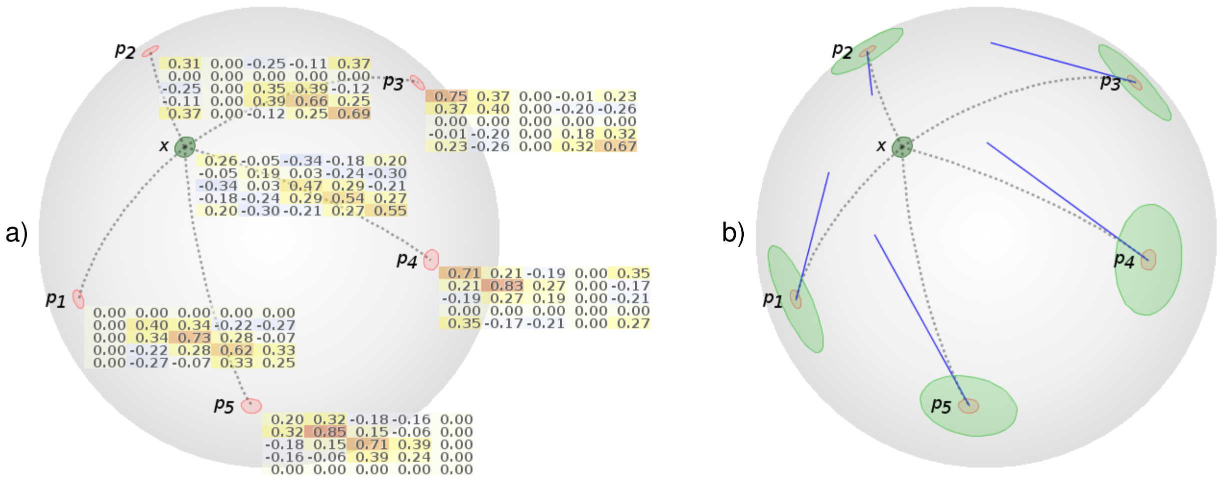


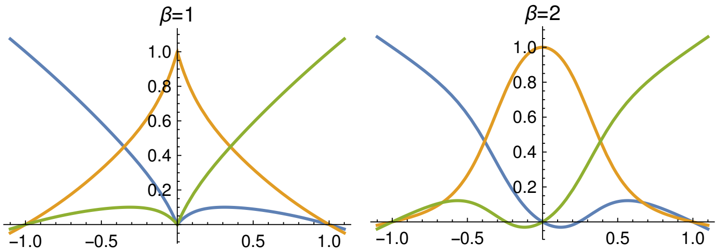
Figure: a) Projection matrices $H_P(x)$ and $H_P(p_i)$ for $i = 1, \ldots, 5$ involved in the computation of *harbor* distances $d_P^H(x) \in \mathbb{R}^5$. b) Mahalanobis bilinear form $S_P(p_i)^+$ and tangent vector $\log_{p_i}(x) \in T_{p_i} S^2$ for $i = 1, \ldots, 5$ that determine *garden* distances $d_P^G(x) \in \mathbb{R}^5$. ■

Remark: We have coined the distance vectors “harbor”, and “garden” respectively, due to lack of awareness of a preexisting concept. Is there any tea on this spaceship? ■

Examples

$\mathbb{R}^d$ is a d-dimensional Lie group with vector addition as group action. The tangent space $T_x \mathbb{R}^d$ at a point $x \in \mathbb{R}^d$ is a d-dimensional vector space and therefore can be identified with $T_x \mathbb{R}^d = \mathbb{R}^d$. The mapping of a point $y \in \mathbb{R}^d$ into the vector space $T_x \mathbb{R}^d$ is simply by translation $\log_x(y) = \log(x^{-1}.y) = y - x$, so that $\log_x(x) = 0$.

Example: For the set $P = \{-1, 0, 1\}$ of three points in $\mathbb{R}^1$ the inverse distance coordinates c_P^{ID} and inverse leverage coordinates c_P^{IL} are identical, see [2020 Hakenberg b); p.5]. *Mathematica* yields the components of the generalized barycentric *harbor* coordinates $c_P^{\text{H}}(x)_i$ for $i \in \{1, 2, 3\}$ as



The exponent $\beta = 1$ results in

$$c_P^{\text{H}}(x) = \frac{1}{2\sqrt{2}\,(x^2-1) - \sqrt{5}\,(2+3x^2)\,|x|} \begin{pmatrix} (-1+x)\left(-\sqrt{2}\,x(x+1)^2 + \sqrt{5}\,(2+3x^2)\,|x|\right) \\ 2\sqrt{2}\,(x^4-1) \\ (-1-x)\left(+\sqrt{2}\,x(x-1)^2 + \sqrt{5}\,(2+3x^2)\,|x|\right) \end{pmatrix}$$

The exponent $\beta = 2$ results in smooth $c_P^{\text{H}} \in C^\infty$ namely

$$c_P^{\text{H}}(x) = \frac{1}{4+12x^2+84x^4} \begin{pmatrix} x(x-1)\,(2+3x((x-13)x^2-2)) \\ 2(1-x^2)(2+3x^4) \\ x(x+1)(2+3x((x+13)x^2+2)) \end{pmatrix} \blacksquare$$

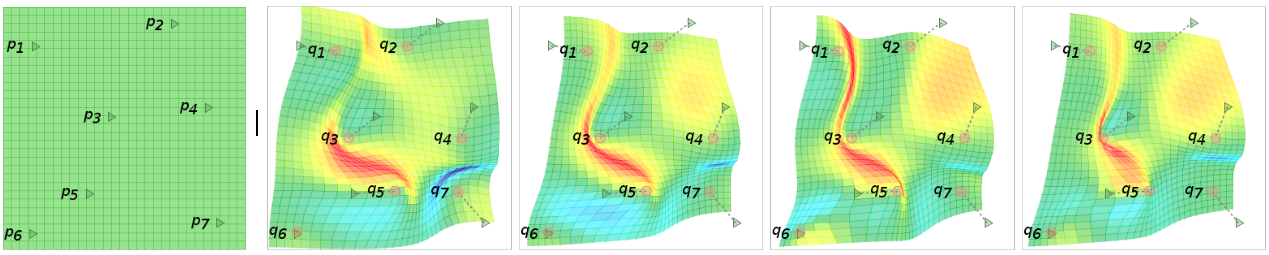


Figure: The left image shows the square domain $\mathcal{V} = [0, 1]^2 \subset \mathbb{R}^2$ and the set P of seven control points. Each point $p_i \in P$ has an associated target location $q_i \in \mathbb{R}^2$ for $i = 1, \ldots, 7$. The images to the right show the deformation of $\mathcal{V}$ subject to different interpolatory methods that have in common that p_i is mapped to q_i for $i = 1, \ldots, 7$: Moving least squares with Shepard's inverse distance weighting w_P^{ID}. Then, deformation as the concatenation $c_P \circ \mu_Q : \mathcal{V} \to \mathbb{R}^2$ with c_P^{ID}, c_P^{IL}, and c_P^{H}. The exponent $\beta = 2$ is used in all instances. ■

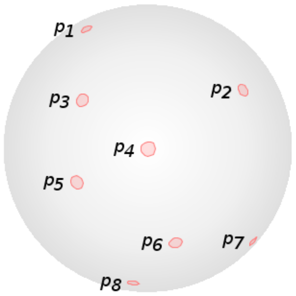
Remark: In $\mathbb{R}^1$, $w^{\text{ID}} = w^{\text{IL}}$. In $\mathbb{R}^d$, $w^{\text{H}} = w^{\text{G}}$. I'd far rather be happy than right any day. ■

The d-dimensional sphere is the homogeneous space $S^d = \text{SO}(d+1)/\text{SO}(d)$. [2016 Pennec; p.8] states the formula for $\log_x : \mathcal{V} \subset S^d \to T_x S^d$.

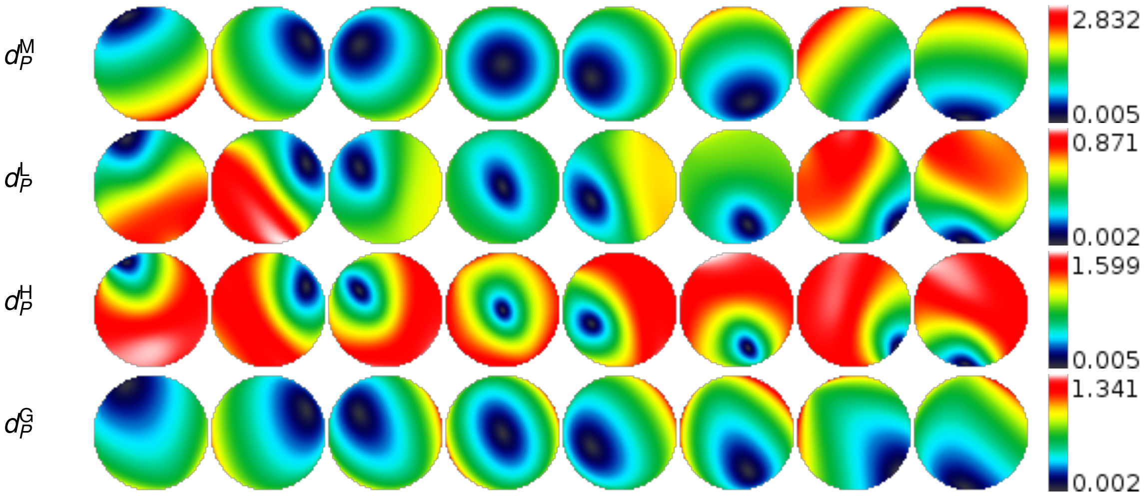
a) b)

Figure: a) subset $\mathcal{V} \subset S^2$ in yellow, and several geodesics emanating from $x \in \mathcal{V}$. b) $\log_x$ image of $\mathcal{V}$ and of the geodesics in $T_x S^2$. ■

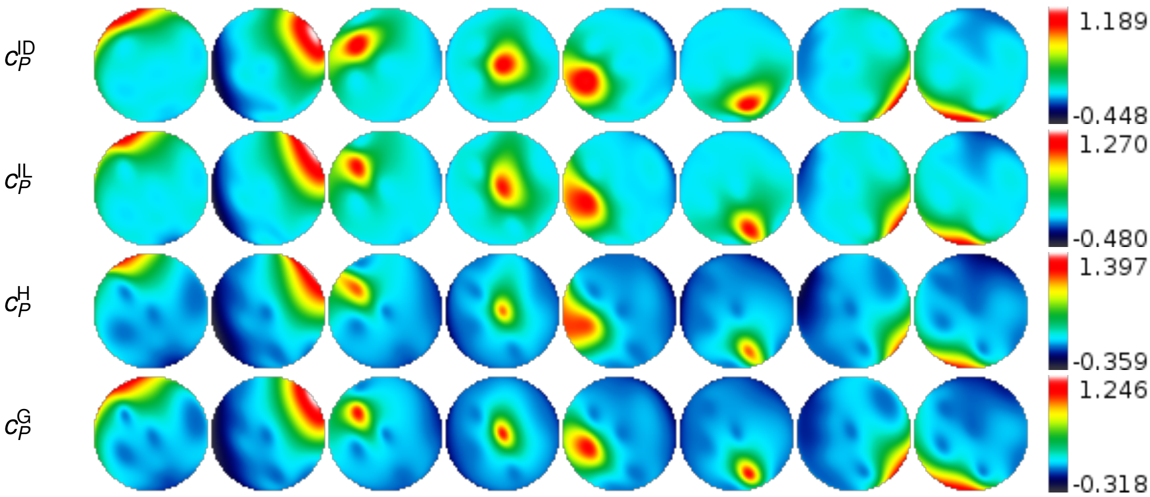
Example: Eight control points $P = \{p_1, \ldots, p_8\}$ are placed on the front hemisphere $\mathcal{V} \subset S^2$ as



The distance vectors $d_P : \mathcal{V} \subset S^2 \to \mathbb{R}^8$ are



The exponent $\beta = 2$ results in the smooth coordinate functions $c_P : \mathcal{V} \subset S^2 \to \mathbb{R}^8$ as



Only the front hemisphere is shown. The back hemisphere contains the antipodes $-p_i$ for $i = 1, \ldots, 8$ where the d_P's are not smooth. Consequently, the c_P's also lack regularity on the back hemisphere. ■

[2001 Buss, Fillmore; p.114] "*prove existence and uniqueness properties of the* [spherical] *weighted averages, and give fast iterative algorithms with linear and quadratic convergence rates.*"

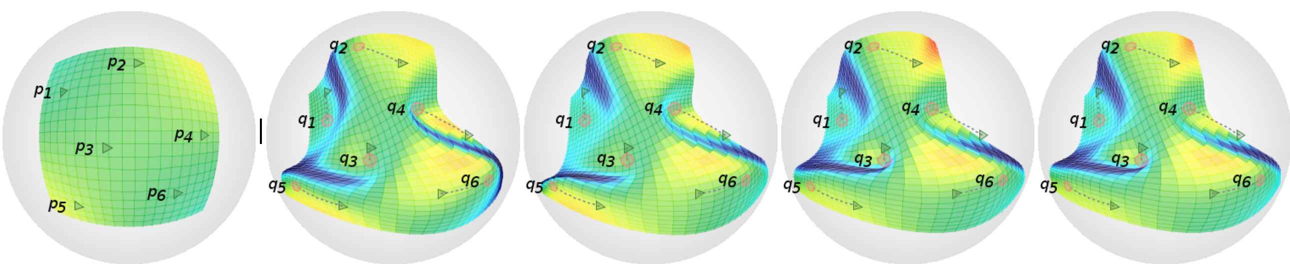


Figure: Deformation of a "square" patch $\mathcal{V} \subset S^2$ as the concatenation $\mu_Q \circ c_P : \mathcal{V} \to S^2$ defined by points $p_i, q_i \in S^2$ for $i = 1, \ldots, 6$. The deformations are produced using c_P^{ID}, c_P^{IL}, c_P^H, and c_P^G respectively. ■

The 3-dimensional special Euclidean group SE(2) is a Lie group, that means $\log_x(y) = dL_x.\log(x^{-1}.y)$ for all $x, y \in$ SE(2). [2017 Eade; p.16] states the formula for log : SE(2) → $\mathfrak{se}$(2). [2018 Hakenberg b); p.3] states the formula for the logarithm of the covering group $\overline{\text{SE}}(2)$. A point $p \in$ SE(2) consists of a position (px, py) $\in \mathbb{R}^2$ and an angular orientation $\theta \in [-\pi, \pi)$. We visualize $p = (\text{px}, \text{py}, \theta)$ as an arrowhead in the plane.

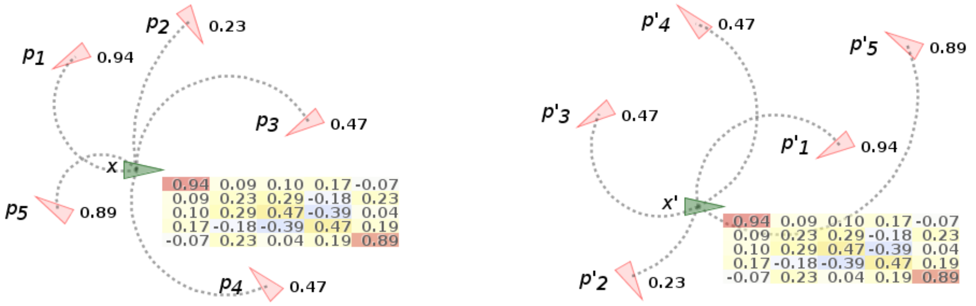


Figure: Points p_i, $x \in$ SE(2) and the transformed points $p'_i = g.p_i.h$, $x' = g.x.h$ for some $g, h \in$ SE(2). The influence matrix is biinvariant, i.e. $H_P(x) = H_{P'}(x')$. Numbers indicated at p_i and p'_i are the i-th leverage identical to the diagonal entry $H_P(x)_{i,i}$ for $i = 1, \ldots, 5$. ■

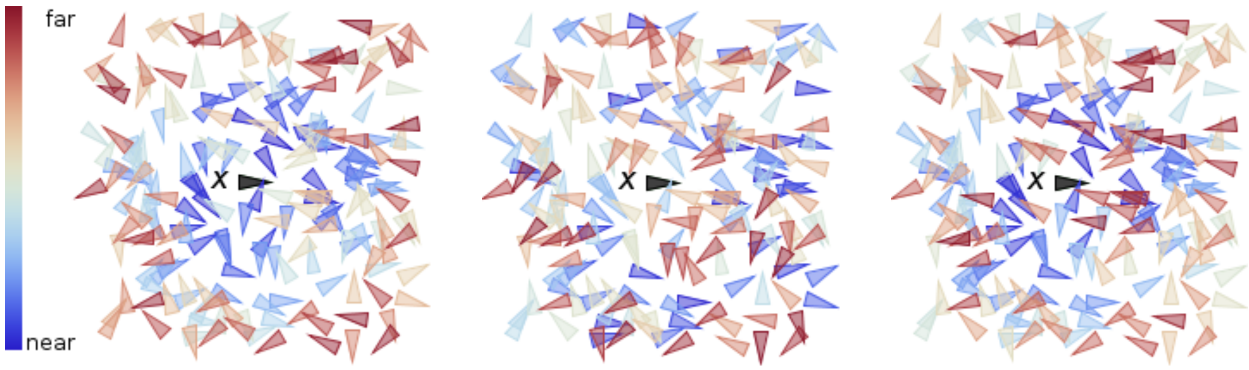


Figure: The points in $P \subset$ SE(2) are color coded according to their position in the ordering based on their relative distance from $x \in$ SE(2). From left to right: $d_P^L(x)$, $d_P^H(x)$, $d_P^G(x)$. ■

Implementation

The *Mathematica* code below computes *metric*, *leverage*, and *harbor* distance vectors, weightings, and generalized barycentric coordinates. For spaces other than $\mathbb{R}^d$, the implementation of $\log_x(y)$ has to be adapted accordingly.

```
η[v_] := Normalize[v, Total]
log[x_][y_] := y - x
H[P_][x_] := With[{V = log[x] /@ P}, V.PseudoInverse[V]]
M[P_][x_] := IdentityMatrix[Length[P]] - H[P][x]
HP[P_] := HP[P] = H[P] /@ P

dM[P_, x_] := Norm /@ log[x] /@ P
dL[P_, x_] := Diagonal[H[P][x]] ^ (1 / 2)
dH[P_, x_] := With[{Hx = H[P][x]}, Norm[Flatten[# - Hx]] & /@ HP[P]]

wID[β_][P_, x_] := η[dM[P, x] ^ -β]
wIL[β_][P_, x_] := η[dL[P, x] ^ -β]
wH[β_][P_, x_] := η[dH[P, x] ^ -β]

cID[β_][P_, x_] := η[M[P][x].wID[β][P, x]]
cIL[β_][P_, x_] := η[M[P][x].wIL[β][P, x]]
cH[β_][P_, x_] := η[M[P][x].wH[β][P, x]]
```

The visualizations of non-linear geometry in this article were computed with the open source software library *sophus*. The library implements weighted averages, distance vectors, weightings, and generalized barycentric coordinates for the Lie groups $\mathbb{R}^d$, SO(3), ST(d), He(d), SE(2), $\overline{\text{SE}}(2)$, SE(3), and the homogeneous spaces S^d, H^d, $\text{Sym}_+(d)$, $\mathbb{RP}^d$. Weblink: https://github.com/datahaki/sophus

Future Work

The preorder induced by the biinvariant distance vector $d_P(x)$ allows to arrange the points $p_i \in P$ in a priority queue according to their relative distance to x. This prioritization may be useful in motion planning applications when the shortest paths between x and $p_i \in P$ should be tested first.

[2007 Press, Teukolsky, Vetterling, Flannery; p.145] give a recipe for *kriging*, also known as *Gauss-Markov estimation*, or *Gaussian process regression*. At the heart of the method is a (symmetric) matrix of distances. We plan to investigate the use of the matrix $D_{i,j} := d_P(p_i)_j$ in kriging on Lie groups and homogeneous spaces.

d_P^H measures the distance between the projection matrices $H_P(x)$ and $H_P(p_i)$ as the Frobenius norm of the matrix difference. The use of other distances are possible. In particular, the geodesic distance between two projection matrices H_1, H_2 in the Grassmannian manifold $\mathrm{Gr}(n, k)$ is

$$\mathbf{d}_{\mathrm{Gr}}^2(H_1, H_2) = -\frac{1}{4}\,\mathrm{tr}\big(\log^2((I - 2\,H_2).(I - 2\,H_1))\big)$$

as derived in [2015 Batzies, Hüper, Machado, Silva Leite; p.91]. However, "*notice that we are assuming that H_1 and H_2 can be joined by a unique geodesic. So, there is an implicit condition on these two matrices, namely that the orthogonal matrix* $(I - 2\,H_1)(I - 2\,H_2)$ *has no negative real eigenvalues*." Furthermore, $H_P(x)$ and $H_P(p_i)$ may not always be of the same rank.

The weighted average $\mu_P(w)$ defined as exponential barycenter exists uniquely and can be computed using an iterative fixed point algorithm. [2012 Pennec, Arsigny] prove this for all Lie groups. [2001 Buss, Fillmore] provide the proof for the homogeneous space S^d. Both results assume non-negative weights $w_i \geq 0$. However, the explicit formulas that [2012 Pennec, Arsigny] derive for the weighted average in $\mathbb{R}^d$, SE(2), and He(d) do not require $w_i \geq 0$ at all. Furthermore, what about weighted averages on general homogeneous spaces?

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