BML 4. Computational methods for Bayesian decision analysis. Intro

DataLab CSIC

Brief description

Introduce key computational methods in Bayesian decision analysis

- Optimisation problems in Bayesian decision analysis
- Regression metamodels
- Augmented probability simulation
- Solutions for games

Resources

Sources:

French and DRI (2000) Ch 7

Ekin et al (2023)

Powell (2019) EJOR

GeNIe https://www.bayesfusion.com/downloads/

Stan https://mc-stan.org/docs/2_32/stan-users-guide-2_32.pdf

Recap: Inference as decision analysis. Standard loss

$$A = \Theta \qquad l(a,\theta) = (\theta - a)^2$$

$$\arg\min_a \int (\theta-a)^2 p(\theta|x)\, d\theta = \arg\min_a \int \theta^2 p(\theta|x)\, d\theta - 2a \int \theta\, p(\theta|x)\, d\theta + a^2 \int p(\theta|x)\, d\theta$$

$$= \arg\min_a \; a^2 - 2a\, E(\theta|x) \implies \hat{a} = E(\theta|x)$$

Tens of ‘stylised’ examples in Chapter 3. Inference as DA… and as MC simulation
French, DRI (2000) chap 6

Computational problems in Bayesian analysis

Computing the posterior

$$p(\theta|x) = \frac{p(x|\theta)\, p(\theta)}{p(x)} = \frac{p(x|\theta)\, p(\theta)}{\int p(x|\theta)\, p(\theta)\, d\theta} \propto p(x|\theta)\, p(\theta)$$

Computing the predictive

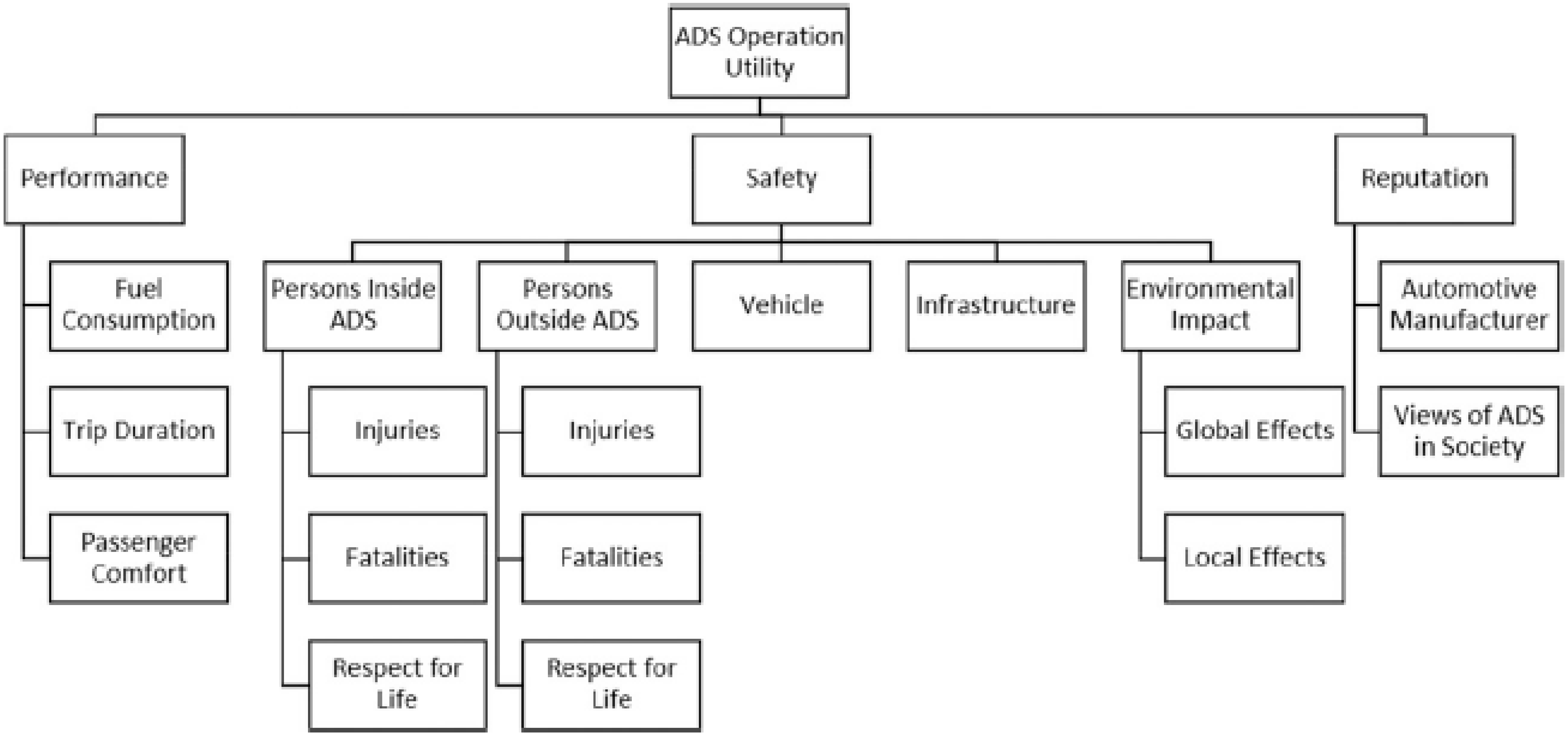
$$p(y|x) = \int p(y|\theta)\, p(\theta|x)\, d\theta$$

Finding the optimal alternative

$$\max_a \int u(a,\theta)\, p(\theta|x)\, d\theta$$

$$\max_a \int u(a,y)\, p(y|x)\, dy$$

Non-standard losses/utilities



Strategies

Problem

Focus on finding alternative with maximum posterior expected utility

$$\underset{a \in A}{\operatorname{argmax}} \ \Psi(a) = \int u(a,\theta) \, f(\theta \mid x) \, d\theta$$

(or minimum posterior expected loss)

$$\underset{a \in A}{\operatorname{argmin}} \ \Psi(a) = \int l(a,\theta) \, f(\theta \mid x) \, d\theta$$

Maximising predictive expected utility is analogous

Many Variants

Is the set of alternatives finite (small or large) or continuous?

Is the problem static or sequential?

Do alternatives influence uncertainties (probs conditional on alternatives)?

Are there several agents?

And more…

Core needs (depending on optimisation algos)

Estimate the objective function

$$\Psi(a) = \int u(a,\theta)\, p(\theta|x)\, d\theta$$

Estimate the gradient of the objective function

$$\nabla_a \Psi(a) = \int \nabla_a u(a,\theta)\, p(\theta|x)\, d\theta$$

Estimate the (sign) of the difference in evaluations

$$\Psi(a_1) - \Psi(a_2) \geq 0$$

Idea 1. Ignore Bayes' formula denominator, sample from prior & approximate by MC

Take into account

$$\int u(a,\theta) f(\theta|x)\, d\theta \propto \int u(a,\theta)\, f(x|\theta)\, f(\theta)\, d\theta$$

Use

SAMPLE $\theta_1, \dots, \theta_n \sim f(\theta)$

APPROXIMATE $\frac{1}{n} \sum_{i=1}^{n} u(a,\theta_i)\, f(x|\theta_i)$

May be a terrible idea…

Idea 2. Sample from posterior and approximate by MCMC

SAMPLE $\theta_1, \dots, \theta_n \sim p(\theta|x)$ (MCMC)

APPROXIMATE $\frac{1}{n} \sum_{i=1}^{n} u(a, \theta_i)$ -

$$\max_a \sum_{i=1}^{n} u(a, \theta_i)$$

Potentially expensive (function and gradient evaluations with large n)

What if dependence on a?

Idea 3. Use a regression metamodel

SAMPLE $\theta_1, \dots, \theta_n \sim p(\theta|x)$

APPROXIMATE $\frac{1}{n} \sum_{i=1}^{n} u(a, \theta_i) = \hat{\Psi}(a)$

APPROXIMATE $\hat{\Psi}(a_1), \dots, \hat{\Psi}(a_m)$

REGRESS $((a_1, \hat{\Psi}(a_1)), \dots, (a_m, \hat{\Psi}(a_m))) \rightarrow \hat{g}(a)$

$\max_a \hat{g}(a)$

Sometimes dependence on *a* of posterior….. (and importance sampling infeasible…)

Keep this in mind when we do Bayesian optimization

Idea 4. Use augmented probability simulation (I)

Expected utility when probabilities depend on alternative

$$\psi(a) = \int u(a,\theta) \; p(\theta \mid x, a) \; d\theta$$

If utility positive and integrable, define augmented probability distribution

$$h(a,\theta) \propto u(a,\theta) \; p(\theta \mid x, a)$$

Mode of marginal of AP is the optimal alternative

$$\int h(a,\theta) \, d\theta \propto \int u(a,\theta) \, p(\theta \mid x, a) \, d\theta = \psi(a)$$

Idea 4. Use augmented probability simulation (II)

Proposed scheme

1. Generate a sample $((\theta^1, a^1), ..., (\theta^m, a^m))$ from density $h(a, \theta)$.
2. Convert it to a sample $(a^1, ..., a^m)$ from the marginal $h(a)$.
3. Find the sample mode.

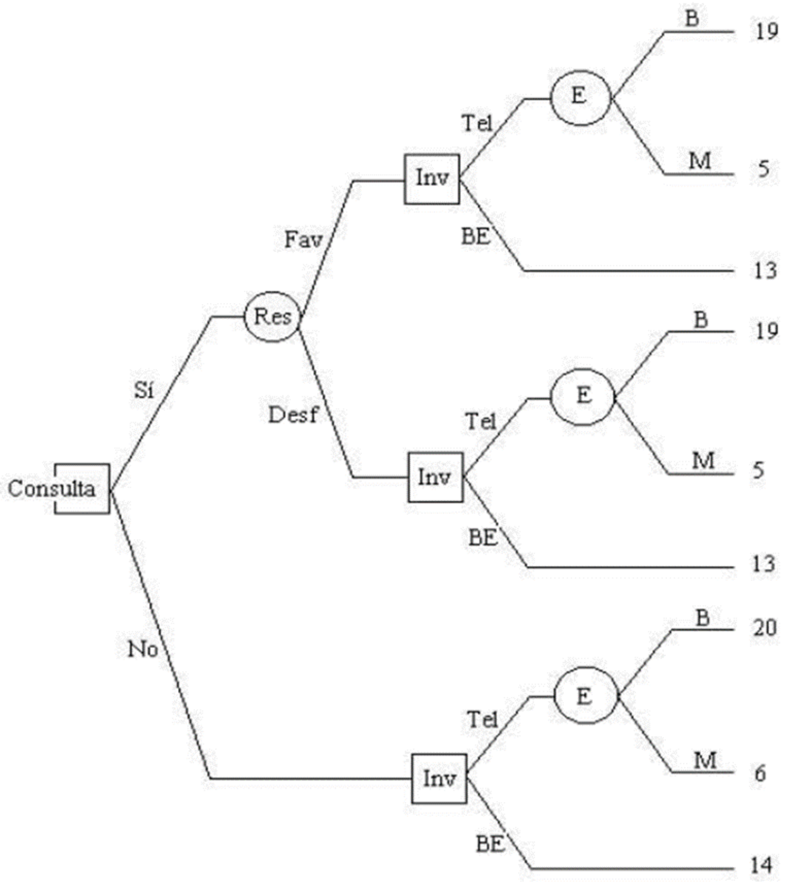
For 1, MCMC technology

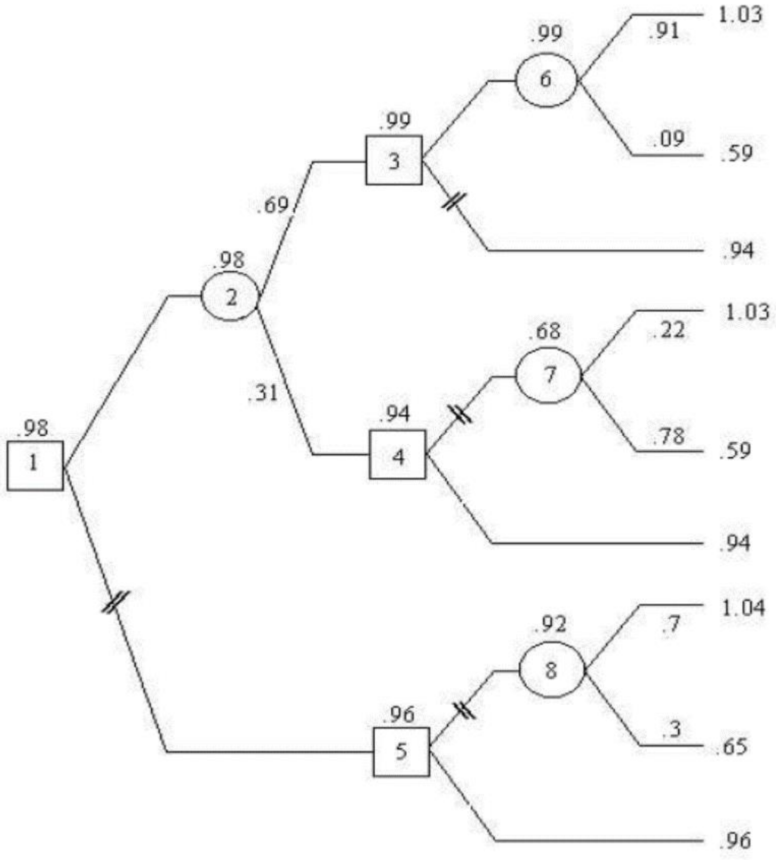
For 2, cluster analysis, density estimation,….

Variants

Sequential problems. I

- Dynamic programming

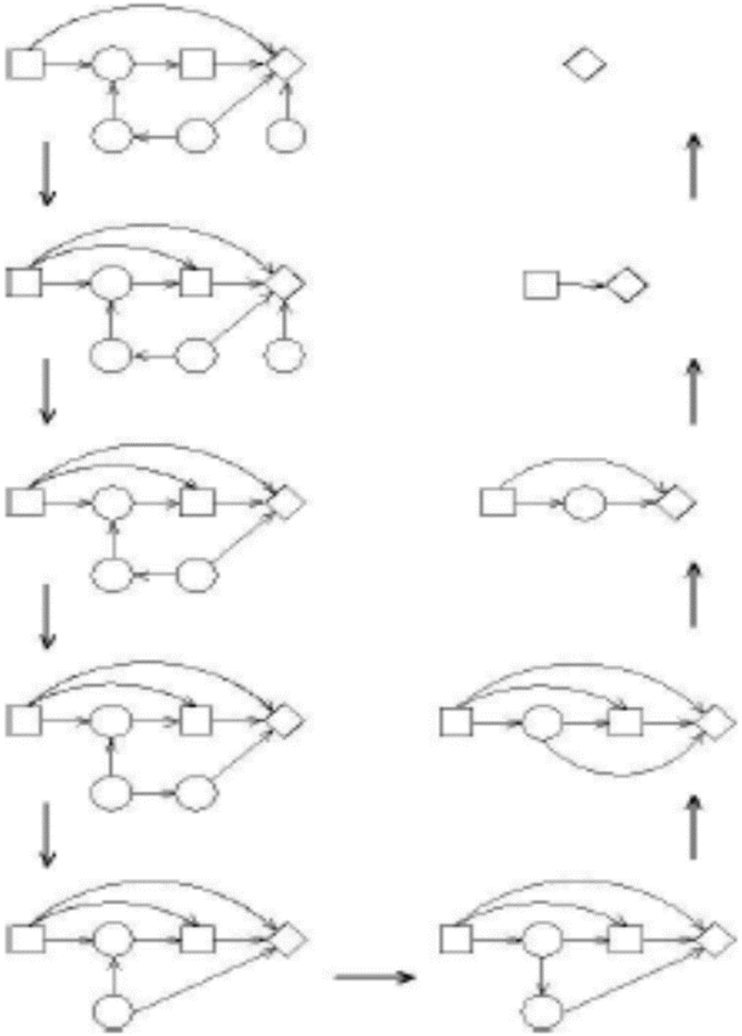




Sequential problems. II

- Influence diagrams

Algos that combine Bayes formula, dynamic programming and expected utility estimation (+ possibly MC and MCMC and APS simulation)

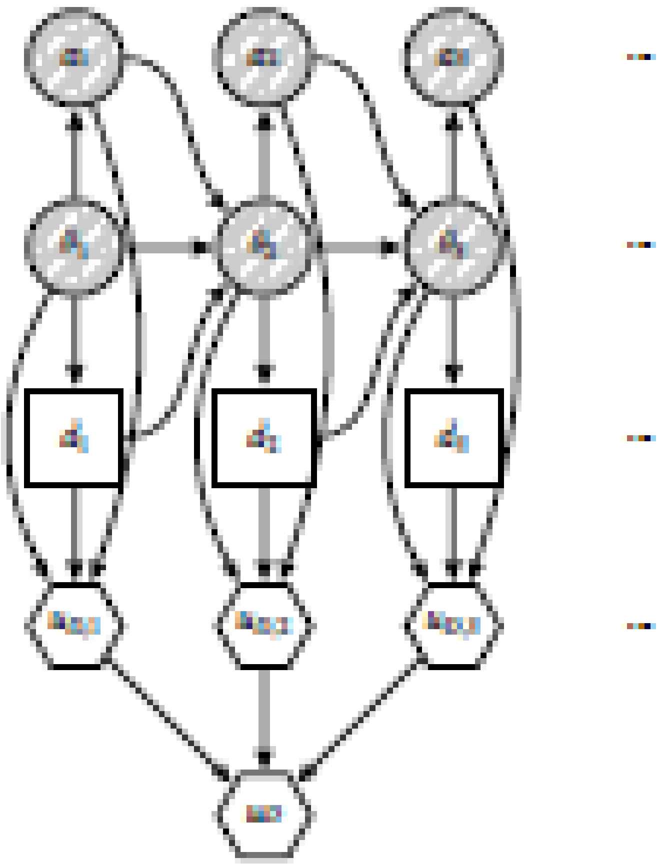


Check GeNIe's manual

Shachter's http://www.dia.fi.upm.es/~jafernan/teaching/dss/references/eval-id-shachter.pdf

Sequential problems. III

Markov decision processes and reinforcement learning. ML

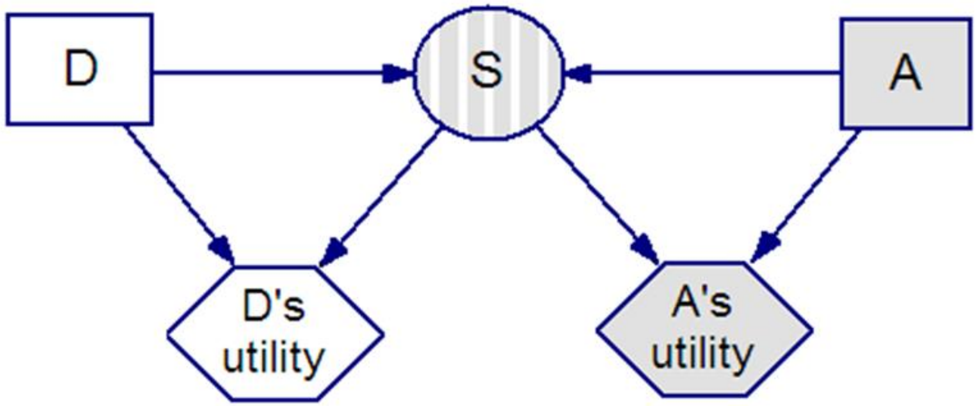


- Long-term discounted expected utility

Check Powell’s paper (2019) and book

IntroML course, chapter 10

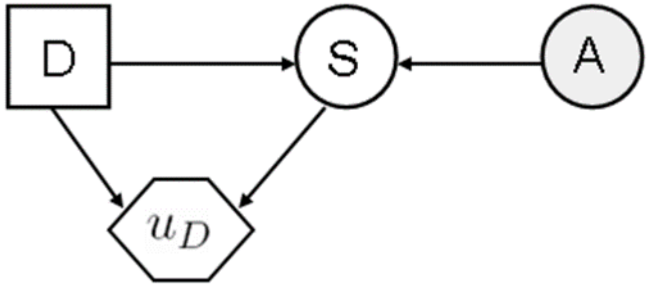
Games I. Simultaneous game



Nash equilibria

$$\psi_D(d^*, a^*) \geq \psi_D(d, a^*) \ \ \forall\, d \in \mathcal{D}$$
$$\psi_A(d^*, a^*) \geq \psi_A(d^*, a) \ \ \forall\, a \in \mathcal{A}$$

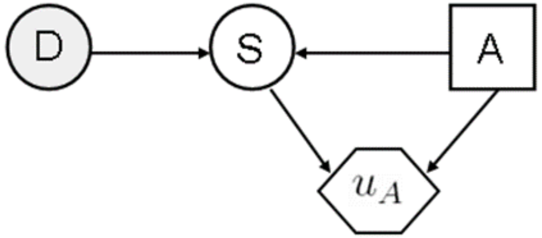
Games II. Simultaneous game from a Bayesian perspective. Defender problem



$$d^* = \text{argmax}_{d \in \mathcal{D}} \sum_{a \in \mathcal{A}} \left[\sum_{s \in \{0,1\}} u_D(d,s)\, p_D(S = s \mid d, a) \right] \pi_D(A = a)$$

This ONE??

Games III. Simultaneous game from a Bayesian perspective. Attacker problem



$$a^* = \operatorname{argmax}_{a \in \mathcal{A}} \sum_{d \in \mathcal{D}} \left[\sum_{s \in \{0,1\}} u_A(a,s)\, p_A(S = s \mid d, a) \right] \pi_A(D = d)$$

$$(u_A, p_A, \pi_A) \sim (U_A, P_A, \Pi_A)$$

$$A \mid D \sim \operatorname{argmax}_{a \in \mathcal{A}} \sum_{d \in \mathcal{D}} \left[\sum_{s \in \{0,1\}} U_A(a,s)\, P_A(S = s \mid d, a) \right] \Pi_A(D = d)$$

APS for this? And for more complex structures? Single stage procedure?

Final comments

Summary

This has been a super brief intro

If curious and/or interested check some classics

Clemen, Reilly (2013) Making hard decisions

Myerson (2013) Game theory

Sutton, Barto (2018) Reinforcement learning: An introduction

And references mentioned earlier