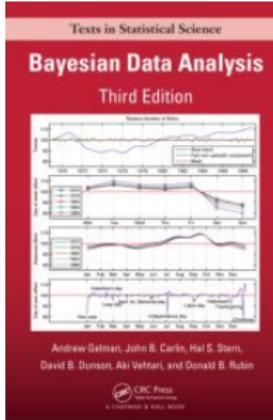


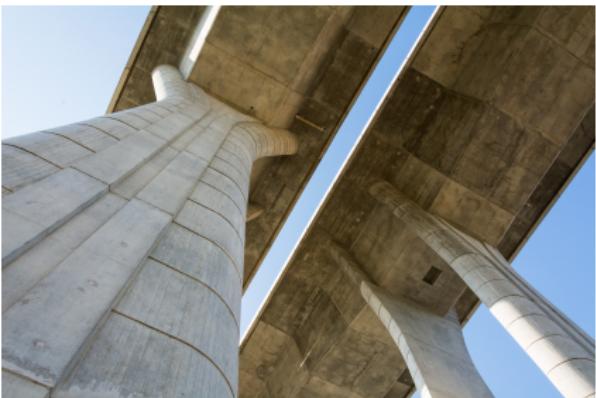
Bayesian data analysis (Aalto fall 2022)

- ▶ Book: Gelman, Carlin, Stern, Dunson, Vehtari & Rubin: Bayesian Data Analysis, Third Edition. (online pdf available)
- ▶ The course website has more detailed information than these slides
https://avehtari.github.io/BDA_course_Aalto/Aalto2022.html
- ▶ Timetable: see the course website
- ▶ TAs: Anna Riha, Elena Shaw, Kunal Ghosh, Andrew Johnson, Noa Kallioinen, David Kohns, Leevi Lindgren, Yann McLatchie, Teemu Sailynoja, Niko Siccha



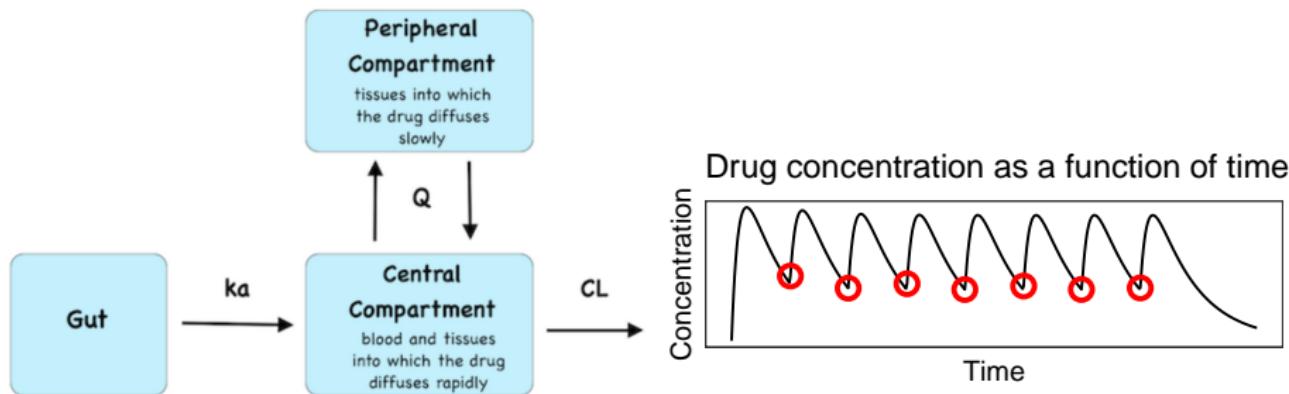
Uncertainty and decision making

- ▶ Predicting concrete quality



Uncertainty and decision making¹

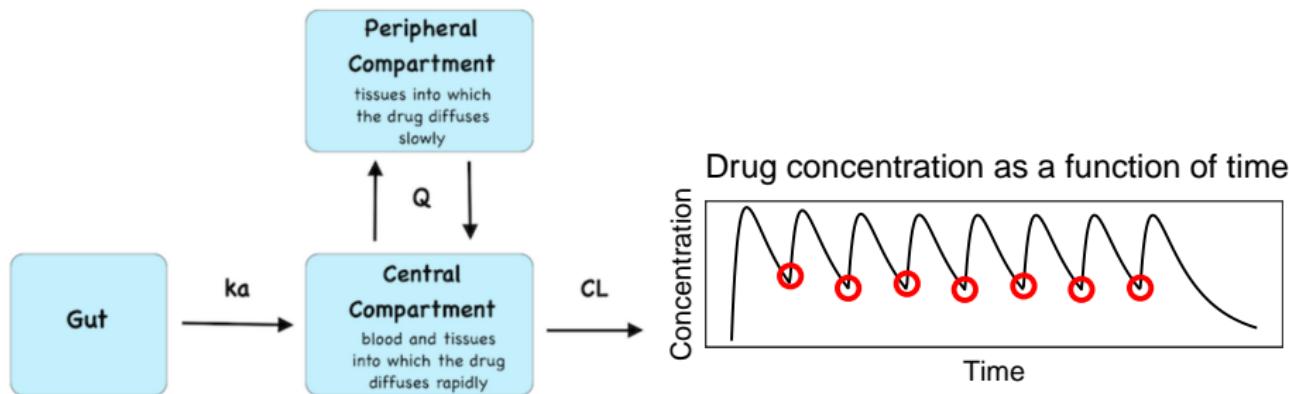
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¹with E. Siivola, Aalto and S. Weber, Novartis Pharma

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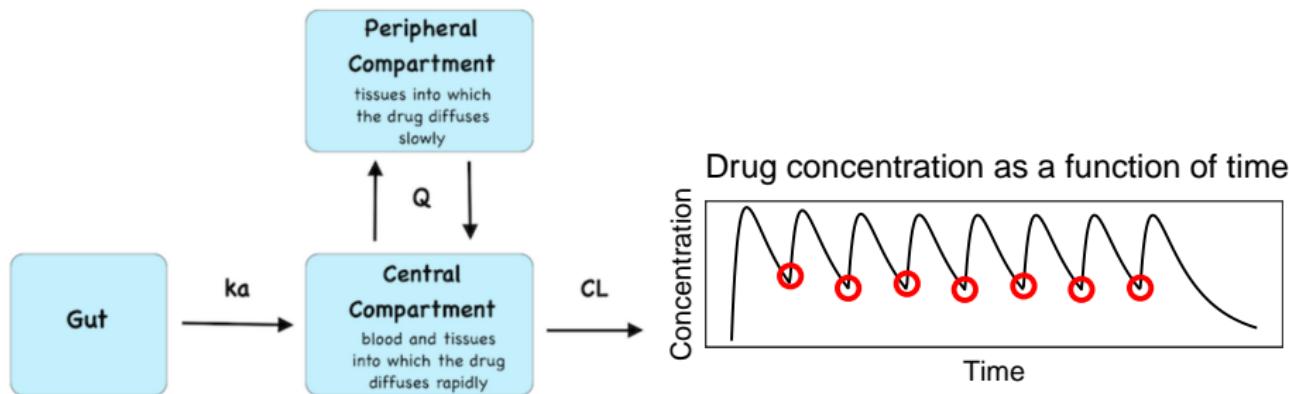


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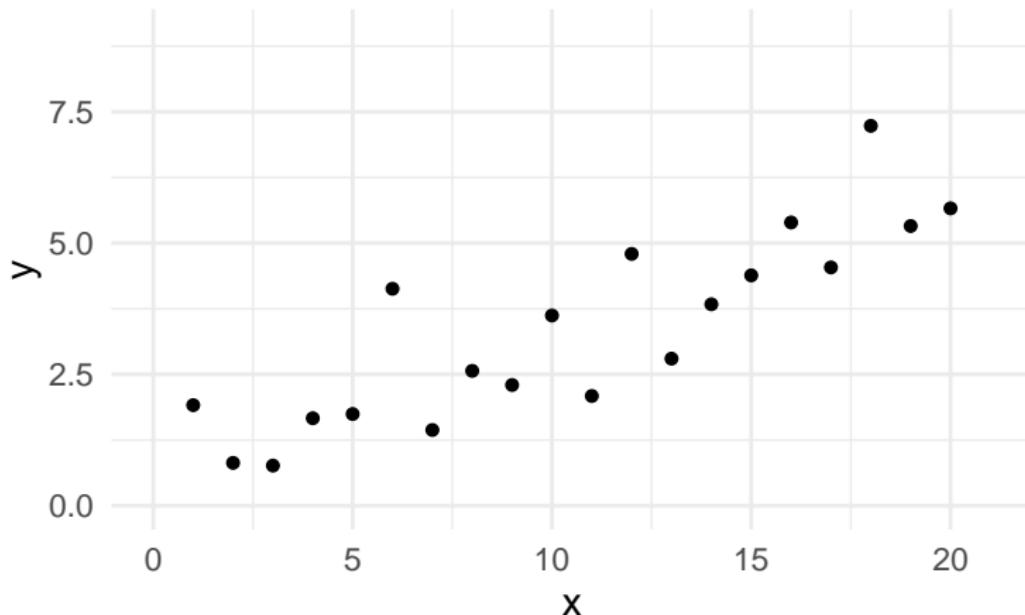


- ▶ Model fitted with 500 adults, extrapolation to children?
- ▶ Maturation effect, 17 observations from children

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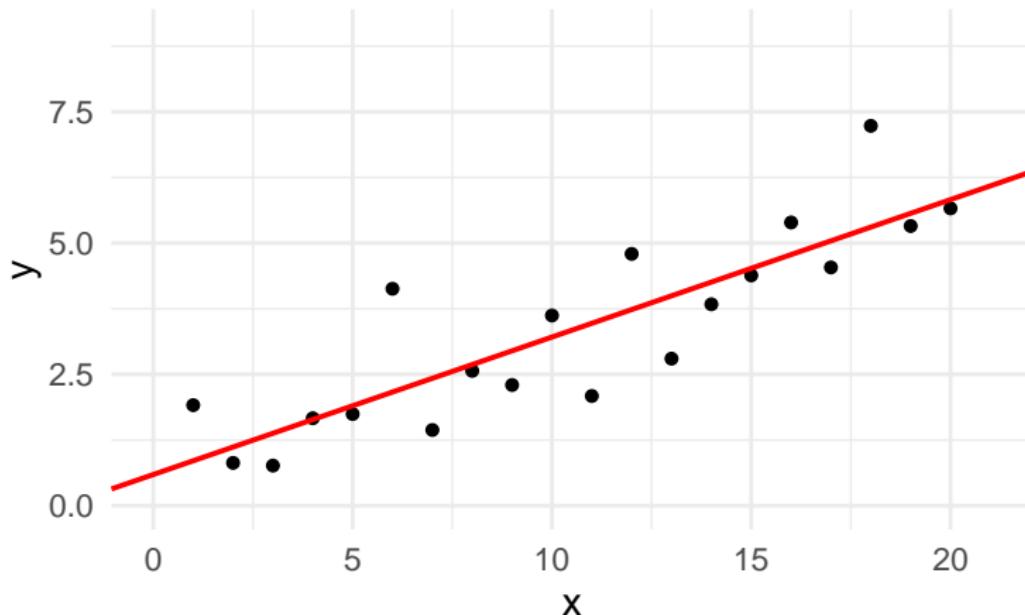
Uncertainty in modeling

Data



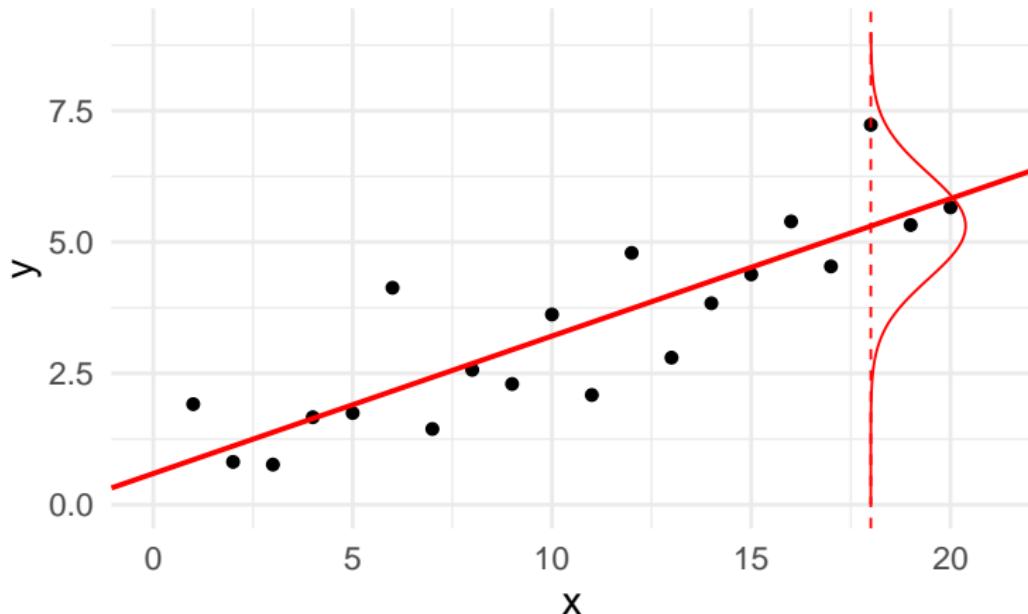
Uncertainty in modeling

Posterior mean



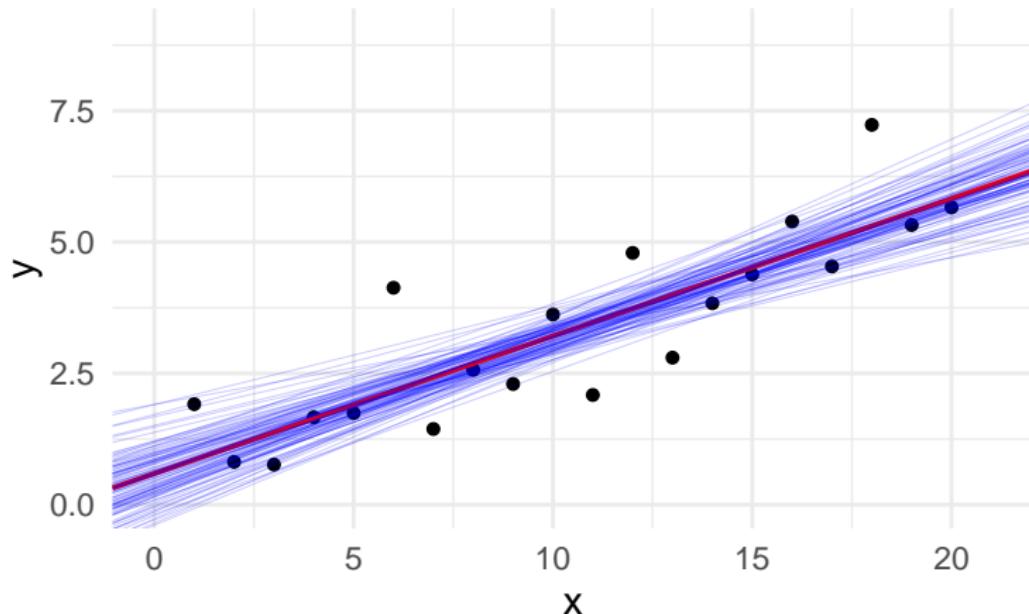
Uncertainty in modeling

Predictive distribution given posterior mean



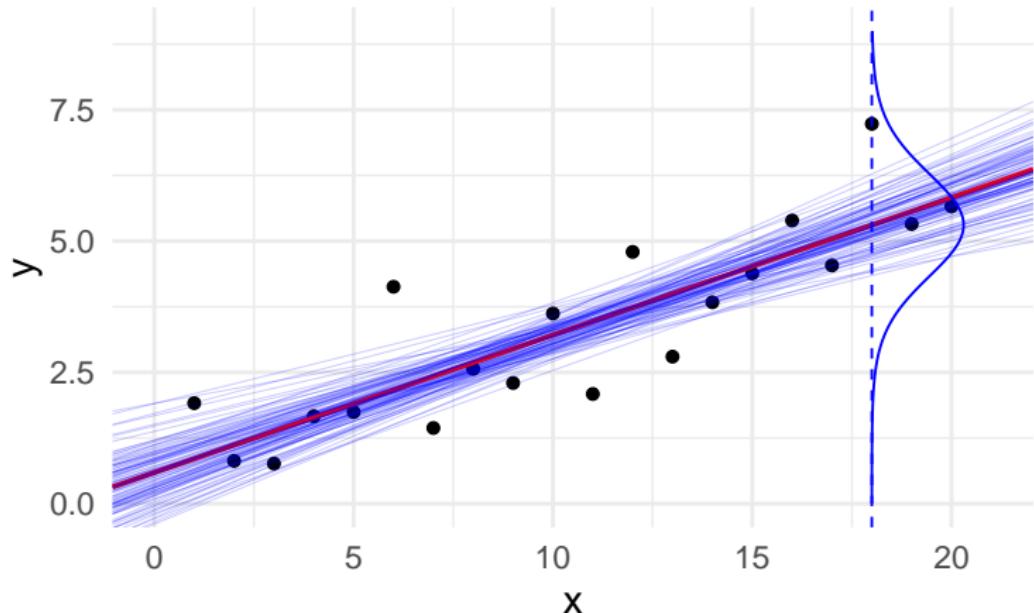
Uncertainty in modeling

Posterior draws



Uncertainty in modeling

Posterior draws and predictive distribution



Bayesian probability theory

expert information

Bayesian probability theory

expert information



mathematical model

+

uncertainty with probabilities

Bayesian probability theory



Bayesian probability theory

expert information



mathematical model

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uncertainty with probabilities

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Bayesian probability theory



$$p(\theta|y) = \frac{p(y|\theta)p(\theta)}{p(y)}$$

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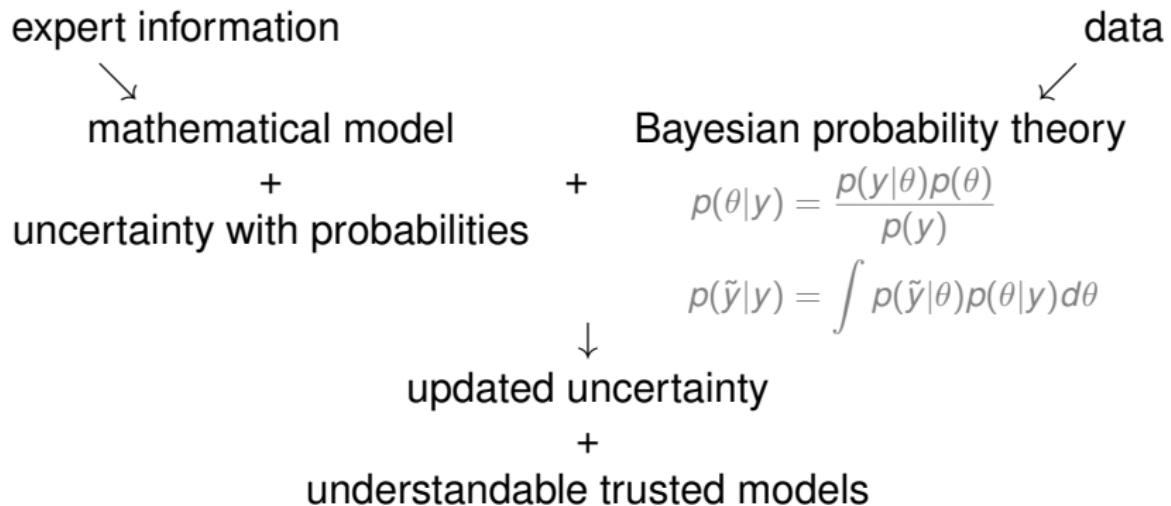
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updated uncertainty

Bayesian probability theory



Bayesian inference with computers

mathematical model to computer

probabilistic programming

computation, automatic inference algorithms

limitations of computers

Yes, but did it work?

computation + inference diagnostics

model diagnostics

limitations of mathematical models

improve and iterate

Probabilistic programming and Stan

Stan is a probabilistic programming framework and ecosystem
40+ developers, 100+ contributors, 100K+ users



mc-stan.org

Bayesian Data Analysis course

- ▶ Probability distributions as model building blocks
 - ▶ need to understand the math part (prereq.)
 - ▶ continuous vs discrete (prereq.)
 - ▶ observation model, likelihood, prior
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 - ▶ We need to be able to compute expectations

$$E_{\theta|y} [g(\theta)] = \int p(\theta|y)g(\theta)d\theta$$

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- ▶ grid, importance sampling, Monte Carlo, Markov chain Monte Carlo
- ▶ Workflow
 - ▶ steps of model building, inference, and diagnostics

Impact on society

Better modelling and quantification of uncertainty

→ better science

→ better informed decision making
in companies, government, and NGOs

Bayesian probability theory

- ▶ Based on Bayesian probability theory
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 - ▶ probabilities are updated based on new information

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- ▶ A nice book about history: Sharon Bertsch McGrayne, *The Theory That Would Not Die*, 2012.

Term Bayesian used first time in mid 20th century

- ▶ Earlier there was just "probability theory"
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- ▶ R. A. Fisher used in 1950 first time term "Bayesian" to emphasize the difference to general term "probability theory"
 - ▶ term became quickly popular, because alternative descriptions were longer

Uncertainty and probabilistic modeling

- ▶ Two types of uncertainty: aleatoric and epistemic
- ▶ Representing uncertainty with probabilities
- ▶ Updating uncertainty

Two types of uncertainty

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 - ▶ we are able to obtain observations which can reduce this uncertainty
 - ▶ two observers may have different epistemic uncertainty

Updating uncertainty

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Model vs. likelihood

- ▶ Bayes rule $p(\theta|y) \propto p(y|\theta)p(\theta)$
- ▶ Model: $p(\mathbf{y}|\theta)$ as a function of \mathbf{y} given fixed θ describes the aleatoric uncertainty
- ▶ Likelihood: $p(y|\theta)$ as a function of θ given fixed y provides information about epistemic uncertainty, but is not a probability distribution

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- ▶ Bayes rule combines the likelihood with prior uncertainty $p(\theta)$ and transforms them to updated posterior uncertainty

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The art of probabilistic modeling

- ▶ The art of probabilistic modeling is to describe in a mathematical form (model and prior distributions) what we already know and what we don't know
- ▶ “Easy” part is to use Bayes rule to update the uncertainties
 - ▶ computational challenges
- ▶ Other parts of the art of probabilistic modeling are, for example,
 - ▶ model checking: is data in conflict with our prior knowledge?
 - ▶ presentation: presenting the model and the results to the application experts

Modeling nature

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 - ▶ air resistance, air pressure, shape and surface structure of the ball
 - ▶ relativity
- ▶ Taking into account the accuracy of the measurements, how accurate model is needed?
 - ▶ often simple models are adequate and useful
 - ▶ *All models are wrong, but some of them are useful*, George P. Box

Reminder: Uncertainty and probabilistic modeling

- ▶ Two types of uncertainty: aleatoric and epistemic
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Questions

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- ▶ Is the quantum uncertainty aleatoric or epistemic?
- ▶ What is your own example with both aleatoric and epistemic uncertainty?

Pre-requisites

- probability
- probability density
- probability mass
- probability density function (pdf)
- probability mass function (pmf)
- probability distribution
- discrete probability distribution
- continuous probability distribution
- cumulative distribution function (cdf)
- likelihood

Ambiguous notation in statistics

$\ln p(y | \theta)$

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In $p(y | \theta)$

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- y and θ can also be mix of continuous and discrete
- due to the sloppiness sometimes likelihood is used to refer $P_{Y,\theta}(Y | \Theta)$, $p_{Y,\theta}(Y | \Theta)$

Chapter 1

Reading instructions

- ▶ 1.1-1.3 important terms
- ▶ 1.4 a useful example
- ▶ 1.5 foundations
- ▶ 1.6 & 1.7 examples (can be skipped, but may be useful to read)
- ▶ 1.8 & 1.9 background material, good to read before doing the exercises
- ▶ 1.10 a point of view for using Bayesian inference