Modeling Growth 1

Daniel Anderson

Week 5

Agenda

- More practice with equation/models
- Thinking flexibly about time
  - Coefficient interpretation by time coding
- A few methods for handling non–linearity

Note

The last bullet is not necessarily specific to growth models

Equation/model practice

The data

Please load the following data

```
library(tidyverse)
d <- read_csv(here::here("data", "longitudinal-sim.csv"))
```

02:00

Model 1

Please translate the following model into `lme4::lmer()` code

$$
\begin{aligned}
\text{g5\_spring}_i &\sim N\left(\alpha_{j[i],k[i]} + \beta_1(\text{g4\_spring}) + \beta_2(\text{g3\_spring}), \sigma^2\right) \\
\alpha_j &\sim N\left(\mu_{\alpha_j}, \sigma^2_{\alpha_j}\right), \text{ for scid j} = 1, \dots, \text{J} \\
\alpha_k &\sim N\left(\mu_{\alpha_k}, \sigma^2_{\alpha_k}\right), \text{ for distid k} = 1, \dots, \text{K}
\end{aligned}
$$

Three equivalent specifications

```r
m1a <- lmer(g5_spring ~ g4_spring + g3_spring +
              (1|scid) + (1|distid),
            data = d)

m1b <- lmer(g5_spring ~ g4_spring + g3_spring + (1|distid/scid),
            data = d)

m1c <- lmer(g5_spring ~ g4_spring + g3_spring +
              (1|distid) + (1|distid:scid),
            data = d)
```

Compute starting group means

```
d <- d %>%
  group_by(scid) %>%
  mutate(sch_mean_start = mean(g3_fall)) %>%
  group_by(distid) %>%
  mutate(dist_mean_start = mean(g3_fall))
```

00:30

Model 2

Don't worry if you run into convergence warnings

$$
\begin{aligned}
\text{g5\_spring}_i &\sim N\left(\alpha_{j[i],k[i]} + \beta_1(\text{g4\_spring}) + \beta_{2j[i]}(\text{g3\_spring}), \sigma^2\right) \\
\begin{pmatrix} \alpha_j \\ \beta_{2j} \end{pmatrix} &\sim N\left(\begin{pmatrix} \gamma_0^\alpha + \gamma_1^\alpha(\text{sch\_mean\_start}) \\ \mu_{\beta_{2j}} \end{pmatrix}, \begin{pmatrix} \sigma^2_{\alpha_j} & \rho_{\alpha_j\beta_{2j}} \\ \rho_{\beta_{2j}\alpha_j} & \sigma^2_{\beta_{2j}} \end{pmatrix}\right), \text{ for scid j} = 1, \dots, \text{J} \\
\alpha_k &\sim N\left(\mu_{\alpha_k}, \sigma^2_{\alpha_k}\right), \text{ for distid k} = 1, \dots, \text{K}
\end{aligned}
$$

```
lmer(g5_spring ~ g4_spring + g3_spring + sch_mean_start +
       (g3_spring|scid) + (1|distid),
     data = d)
```

02:00

Model 3

$$\text{g5\_spring}_i \sim N\left(\alpha_{j[i],k[i]} + \beta_{1j[i],k[i]}(\text{g4\_spring}) + \beta_{2j[i],k[i]}(\text{g3\_spring}), \sigma^2\right)$$

$$\begin{pmatrix} \alpha_j \\ \beta_{1j} \\ \beta_{2j} \end{pmatrix} \sim N\left(\begin{pmatrix} \gamma_0^\alpha + \gamma_1^\alpha(\text{sch\_mean\_start}) \\ \mu_{\beta_{1j}} \\ \mu_{\beta_{2j}} \end{pmatrix}, \begin{pmatrix} \sigma^2_{\alpha_j} & \rho_{\alpha_j\beta_{1j}} & \rho_{\alpha_j\beta_{2j}} \\ \rho_{\beta_{1j}\alpha_j} & \sigma^2_{\beta_{1j}} & \rho_{\beta_{1j}\beta_{2j}} \\ \rho_{\beta_{2j}\alpha_j} & \rho_{\beta_{2j}\beta_{1j}} & \sigma^2_{\beta_{2j}} \end{pmatrix}\right), \text{for scid j} = 1, \ldots$$

$$\begin{pmatrix} \alpha_k \\ \beta_{1k} \\ \beta_{2k} \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_{\alpha_k} \\ \mu_{\beta_{1k}} \\ \mu_{\beta_{2k}} \end{pmatrix}, \begin{pmatrix} \sigma^2_{\alpha_k} & \rho_{\alpha_k\beta_{1k}} & \rho_{\alpha_k\beta_{2k}} \\ \rho_{\beta_{1k}\alpha_k} & \sigma^2_{\beta_{1k}} & \rho_{\beta_{1k}\beta_{2k}} \\ \rho_{\beta_{2k}\alpha_k} & \rho_{\beta_{2k}\beta_{1k}} & \sigma^2_{\beta_{2k}} \end{pmatrix}\right), \text{for distid k} = 1, \ldots, \text{K}$$

```
lmer(g5_spring ~ g4_spring + g3_spring + sch_mean_start +
       (g4_spring + g3_spring|scid) +
       (g4_spring + g3_spring|distid),
     data = d)
```

02:00

Model 4

$$\text{g5\_spring}_i \sim N\left(\alpha_{j[i],k[i]} + \beta_{1j[i],k[i]}(\text{g4\_spring}) + \beta_{2j[i],k[i]}(\text{g3\_spring}), \sigma^2\right)$$

$$\begin{pmatrix} \alpha_j \\ \beta_{1j} \\ \beta_{2j} \end{pmatrix} \sim N\left(\begin{pmatrix} \gamma_0^{\alpha} + \gamma_1^{\alpha}(\text{sch\_mean\_start}) \\ \mu_{\beta_{1j}} \\ \gamma_0^{\beta_2} + \gamma_1^{\beta_2}(\text{sch\_mean\_start}) \end{pmatrix}, \begin{pmatrix} \sigma^2_{\alpha_j} & \rho_{\alpha_j\beta_{1j}} & \rho_{\alpha_j\beta_{2j}} \\ \rho_{\beta_{1j}\alpha_j} & \sigma^2_{\beta_{1j}} & \rho_{\beta_{1j}\beta_{2j}} \\ \rho_{\beta_{2j}\alpha_j} & \rho_{\beta_{2j}\beta_{1j}} & \sigma^2_{\beta_{2j}} \end{pmatrix}\right), \text{ for scid j} = 1,$$

$$\begin{pmatrix} \alpha_k \\ \beta_{1k} \\ \beta_{2k} \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_{\alpha_k} \\ \mu_{\beta_{1k}} \\ \mu_{\beta_{2k}} \end{pmatrix}, \begin{pmatrix} \sigma^2_{\alpha_k} & \rho_{\alpha_k\beta_{1k}} & \rho_{\alpha_k\beta_{2k}} \\ \rho_{\beta_{1k}\alpha_k} & \sigma^2_{\beta_{1k}} & \rho_{\beta_{1k}\beta_{2k}} \\ \rho_{\beta_{2k}\alpha_k} & \rho_{\beta_{2k}\beta_{1k}} & \sigma^2_{\beta_{2k}} \end{pmatrix}\right), \text{ for distid k} = 1, \dots, \text{K}$$

```
lmer(g5_spring ~ g4_spring + g3_spring +
       sch_mean_start + sch_mean_start:g3_spring +
       (g4_spring + g3_spring|scid) +
       (g4_spring + g3_spring|distid),
     data = d)
```

02:00

Model 5

A little bit tricky

$$
\text{g5\_spring}_i \sim N\left(\alpha_{j[i],k[i]} + \beta_{1j[i],k[i]}(\text{g4\_spring}) + \beta_{2j[i],k[i]}(\text{g3\_spring}), \sigma^2\right)
$$

$$
\begin{pmatrix} \alpha_j \\ \beta_{1j} \\ \beta_{2j} \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_{\alpha_j} \\ \mu_{\beta_{1j}} \\ \mu_{\beta_{2j}} \end{pmatrix}, \begin{pmatrix} \sigma^2_{\alpha_j} & 0 & 0 \\ 0 & \sigma^2_{\beta_{1j}} & 0 \\ 0 & 0 & \sigma^2_{\beta_{2j}} \end{pmatrix}\right) \text{, for scid j} = 1, \dots \text{,J}
$$

$$
\begin{pmatrix} \alpha_k \\ \beta_{1k} \\ \beta_{2k} \end{pmatrix} \sim N\left(\begin{pmatrix} \gamma_0^\alpha + \gamma_1^\alpha(\text{dist\_mean\_start}) \\ \mu_{\beta_{1k}} \\ \mu_{\beta_{2k}} \end{pmatrix}, \begin{pmatrix} \sigma^2_{\alpha_k} & 0 & 0 \\ 0 & \sigma^2_{\beta_{1k}} & 0 \\ 0 & 0 & \sigma^2_{\beta_{2k}} \end{pmatrix}\right) \text{, for distid k} = 1, \dots,
$$

```
lmer(g5_spring ~ g4_spring + g3_spring + dist_mean_start +
       (g4_spring + g3_spring||scid) +
       (g4_spring + g3_spring||distid),
     data = d)
```

02:00

Move to long

```
l <- d %>%
  pivot_longer(
    cols = starts_with("g"),
    names_to = "timepoint",
    values_to = "score"
  )
l
```

```
## # A tibble: 202,500 x 7
## # Groups:   distid [100]
##    distid scid  sid   sch_mean_start dist_mean_start timepoint    score
##     <dbl> <chr> <chr>          <dbl>           <dbl> <chr>        <dbl>
##  1      1 1-1   1-1-1       195.1831        189.6743 g3_fall   203.0107
##  2      1 1-1   1-1-1       195.1831        189.6743 g3_winter 202.4761
##  3      1 1-1   1-1-1       195.1831        189.6743 g3_spring 212.2639
##  4      1 1-1   1-1-1       195.1831        189.6743 g4_fall   205.3442
##  5      1 1-1   1-1-1       195.1831        189.6743 g4_winter 214.2586
##  6      1 1-1   1-1-1       195.1831        189.6743 g4_spring 220.2867
##  7      1 1-1   1-1-1       195.1831        189.6743 g5_fall   220.5970
##  8      1 1-1   1-1-1       195.1831        189.6743 g5_winter 220.9811
##  9      1 1-1   1-1-1       195.1831        189.6743 g5_spring 237.0075
## 10      1 1-1   1-1-2       195.1831        189.6743 g3_fall   195.4607
## # … with 202,490 more rows
```

Recode timepoint

First create a data frame that maps the existing values to the new values you want.

```
wave_frame <- tibble(
  timepoint = paste0(
    "g",
    rep(3:5, each = 3),
    rep(c("_fall", "_winter", "_spring"), 3)
  ),
  wave = 0:8
)
wave_frame
```

```
## # A tibble: 9 x 2
##   timepoint  wave
##   <chr>     <int>
## 1 g3_fall       0
## 2 g3_winter     1
## 3 g3_spring     2
## 4 g4_fall       3
## 5 g4_winter     4
## 6 g4_spring     5
## 7 g5_fall       6
```

Join

```r
l <- left_join(l, wave_frame)
```

```
## Joining, by = "timepoint"
```

```r
l
```

```
## # A tibble: 202,500 x 8
## # Groups:   distid [100]
##    distid scid  sid   sch_mean_start dist_mean_start timepoint    score
##     <dbl> <chr> <chr>          <dbl>           <dbl> <chr>        <dbl>
##  1      1 1-1   1-1-1       195.1831        189.6743 g3_fall   203.0107
##  2      1 1-1   1-1-1       195.1831        189.6743 g3_winter 202.4761
##  3      1 1-1   1-1-1       195.1831        189.6743 g3_spring 212.2639
##  4      1 1-1   1-1-1       195.1831        189.6743 g4_fall   205.3442
##  5      1 1-1   1-1-1       195.1831        189.6743 g4_winter 214.2586
##  6      1 1-1   1-1-1       195.1831        189.6743 g4_spring 220.2867
##  7      1 1-1   1-1-1       195.1831        189.6743 g5_fall   220.5970
##  8      1 1-1   1-1-1       195.1831        189.6743 g5_winter 220.9811
##  9      1 1-1   1-1-1       195.1831        189.6743 g5_spring 237.0075
## 10      1 1-1   1-1-2       195.1831        189.6743 g3_fall   195.4607
## # … with 202,490 more rows
```

Model 6

$$\text{score}_i \sim N\left(\alpha_{j[i],k[i],l[i]} + \beta_{1j[i],k[i]}(\text{wave}), \sigma^2\right)$$

$$\begin{pmatrix} \alpha_j \\ \beta_{1j} \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_{\alpha_j} \\ \mu_{\beta_{1j}} \end{pmatrix}, \begin{pmatrix} \sigma^2_{\alpha_j} & \rho_{\alpha_j\beta_{1j}} \\ \rho_{\beta_{1j}\alpha_j} & \sigma^2_{\beta_{1j}} \end{pmatrix}\right), \text{ for sid j} = 1, \dots, \text{J}$$

$$\begin{pmatrix} \alpha_k \\ \beta_{1k} \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_{\alpha_k} \\ \mu_{\beta_{1k}} \end{pmatrix}, \begin{pmatrix} \sigma^2_{\alpha_k} & \rho_{\alpha_k\beta_{1k}} \\ \rho_{\beta_{1k}\alpha_k} & \sigma^2_{\beta_{1k}} \end{pmatrix}\right), \text{ for scid k} = 1, \dots, \text{K}$$

$$\alpha_l \sim N\left(\mu_{\alpha_l}, \sigma^2_{\alpha_l}\right), \text{ for distid l} = 1, \dots, \text{L}$$

```
lmer(score ~ wave +
       (wave|sid) + (wave|scid) + (1|distid),
     data = d)
```

02:00

Model 7

This one takes a while to fit, so don't worry about actually fitting it, just try to write the code.

$$
\begin{aligned}
\text{score}_i &\sim N\left(\alpha_{j[i],k[i],l[i]} + \beta_{1j[i],k[i]}(\text{wave}), \sigma^2\right) \\
\begin{pmatrix} \alpha_j \\ \beta_{1j} \end{pmatrix} &\sim N\left(\begin{pmatrix} \mu_{\alpha_j} \\ \mu_{\beta_{1j}} \end{pmatrix}, \begin{pmatrix} \sigma^2_{\alpha_j} & \rho_{\alpha_j\beta_{1j}} \\ \rho_{\beta_{1j}\alpha_j} & \sigma^2_{\beta_{1j}} \end{pmatrix}\right), \text{ for sid j} = 1, \dots, \text{J} \\
\begin{pmatrix} \alpha_k \\ \beta_{1k} \end{pmatrix} &\sim N\left(\begin{pmatrix} \gamma_0^{\alpha} + \gamma_{1l[i]}^{\alpha}(\text{sch\_mean\_start}) \\ \gamma_0^{\beta_1} + \gamma_1^{\beta_1}(\text{sch\_mean\_start}) \end{pmatrix}, \begin{pmatrix} \sigma^2_{\alpha_k} & \rho_{\alpha_k\beta_{1k}} \\ \rho_{\beta_{1k}\alpha_k} & \sigma^2_{\beta_{1k}} \end{pmatrix}\right), \text{ for scid k} = 1, \dots, \text{K} \\
\begin{pmatrix} \alpha_l \\ \gamma_{1l} \end{pmatrix} &\sim N\left(\begin{pmatrix} \gamma_0^{\alpha} + \gamma_1^{\alpha}(\text{dist\_mean\_start}) + \gamma_2^{\alpha}(\text{dist\_mean\_start} \times \text{wave}) \\ \mu_{\gamma_{1l}} \end{pmatrix}, \begin{pmatrix} \sigma^2_{\alpha_l} & \rho_{\alpha_l\gamma_{1l}} \\ \rho_{\gamma_{1l}\alpha_l} & \sigma^2_{\gamma_{1l}} \end{pmatrix}\right), \text{ for distid l} = 1, \dots
\end{aligned}
$$

```
lmer(score ~ wave + sch_mean_start + dist_mean_start +
       wave:sch_mean_start + wave:dist_mean_start +
       (wave|sid) + (wave|scid) + (sch_mean_start|distid),
     data = l)
```

02:00

Model 8

Last one

$$\text{score}_i \sim N\left(\alpha_{j[i]} + \beta_{1j[i]}(\text{wave}), \sigma^2\right)$$
$$\begin{pmatrix} \alpha_j \\ \beta_{1j} \end{pmatrix} \sim N\left(\begin{pmatrix} \gamma_0^{\alpha} + \gamma_1^{\alpha}(\text{sch\_mean\_start}) + \gamma_2^{\alpha}(\text{dist\_mean\_start}) \\ \gamma_0^{\beta_1} + \gamma_1^{\beta_1}(\text{sch\_mean\_start}) \end{pmatrix}, \begin{pmatrix} \sigma^2_{\alpha_j} & \rho_{\alpha_j\beta_{1j}} \\ \rho_{\beta_{1j}\alpha_j} & \sigma^2_{\beta_{1j}} \end{pmatrix}\right), \text{ for sid j} = 1, \dots, \text{J}$$

```
lmer(score ~ wave * sch_mean_start + dist_mean_start +
       (wave|sid),
     data = l)
```

02:00

Growth modeling

The data

Sample of the Children of the National Longitudinal Study of Youth

Outcome = `piat` = Peabody Individual Achievement Test

Please read in `cnlsy.csv` now

```
library(tidyverse)
d <- read_csv(here::here("data", "cnlsy.csv"))
```

02:00

Look at the data

```
d
```

```
## # A tibble: 267 x 5
##        id  wave agegrp       age  piat
##     <dbl> <dbl>  <dbl>     <dbl> <dbl>
##  1      1     1    6.5  6           18
##  2      1     2    8.5  8.333333    35
##  3      1     3   10.5 10.33333     59
##  4      2     1    6.5  6           18
##  5      2     2    8.5  8.5         25
##  6      2     3   10.5 10.58333     28
##  7      3     1    6.5  6.083333    18
##  8      3     2    8.5  8.416667    23
##  9      3     3   10.5 10.41667     32
## 10      4     1    6.5  6           18
## # … with 257 more rows
```

Fit a basic model

Please try to fit a model that accounts for the within–subjects design in some way and includes a random intercept and slope.

```
library(lme4)
d <- d %>%
  mutate(wave_c = wave - 1)
m_wave <- lmer(piat ~ wave_c + (wave_c|id),
               data = d)
```

02:00

Interpret

```
arm::display(m_wave)
```

```
## lmer(formula = piat ~ wave_c + (wave_c | id), data = d)
##             coef.est coef.se
## (Intercept) 21.16     0.62
## wave_c      10.06     0.59
##
## Error terms:
##  Groups   Name        Std.Dev. Corr
##  id       (Intercept) 3.38
##           wave_c      4.24     0.22
##  Residual             5.20
## ---
## number of obs: 267, groups: id, 89
## AIC = 1830.4, DIC = 1821.5
## deviance = 1819.9
```

But what does a one unit increase in `wave_c` actually mean?

More meaningful

- Note that each `wave` is tied to a specific age group (the approximate age of participants at that age). Can we use this? Try!

```
m_agegrp <- lmer(piat ~ agegrp + (agegrp|id),
                 data = d,
                 control = lmerControl(optimizer = "bobyqa"))
```

01:00

Interpret

What does the intercept mean here? Age group?

```
arm::display(m_agegrp)
```

```
## lmer(formula = piat ~ agegrp + (agegrp | id), data = d, control = lmerCo
##             coef.est coef.se
## (Intercept) -11.54     2.21
## agegrp        5.03     0.30
##
## Error terms:
##  Groups   Name        Std.Dev. Corr
##  id       (Intercept) 13.43
##           agegrp       2.12    -0.97
##  Residual              5.20
## ---
## number of obs: 267, groups: id, 89
## AIC = 1831.8, DIC = 1820.1
## deviance = 1819.9
```

How do we fix the intercept?

Centering

Let's center age group on the first time point

```
d <- d %>%
  mutate(agegrp_c = agegrp - 6.5)
m_agegrp2 <- lmer(piat ~ agegrp_c + (agegrp_c|id),
                  data = d,
                  control = lmerControl(optimizer = "bobyqa"))
```

Interpret

What does the intercept represent now?

```
arm::display(m_agegrp2)
```

```
## lmer(formula = piat ~ agegrp_c + (agegrp_c | id), data = d, control = lm
##             coef.est coef.se
## (Intercept) 21.16     0.62
## agegrp_c     5.03     0.30
##
## Error terms:
##  Groups   Name        Std.Dev. Corr
##  id       (Intercept) 3.38
##           agegrp_c    2.12     0.22
##  Residual             5.20
## ---
## number of obs: 267, groups: id, 89
## AIC = 1831.8, DIC = 1820.1
## deviance = 1819.9
```

Pop Quiz: Without looking, how do you think the fit of the model has changed?

Comparing fit

```
library(performance)
compare_performance(m_agegrp, m_agegrp2) %>%
  print_md()
```

Table: Comparison of Model Performance Indices

Name	Model	AIC	BIC	R2 (cond.)	R2 (marg.)	ICC	RMSE	Sigma
m_agegrp	lmerMod	1831.78	1853.30	0.81	0.48	0.64	4.15	5.20
m_agegrp2	lmerMod	1831.78	1853.30	0.81	0.48	0.64	4.15	5.20

They're identical!

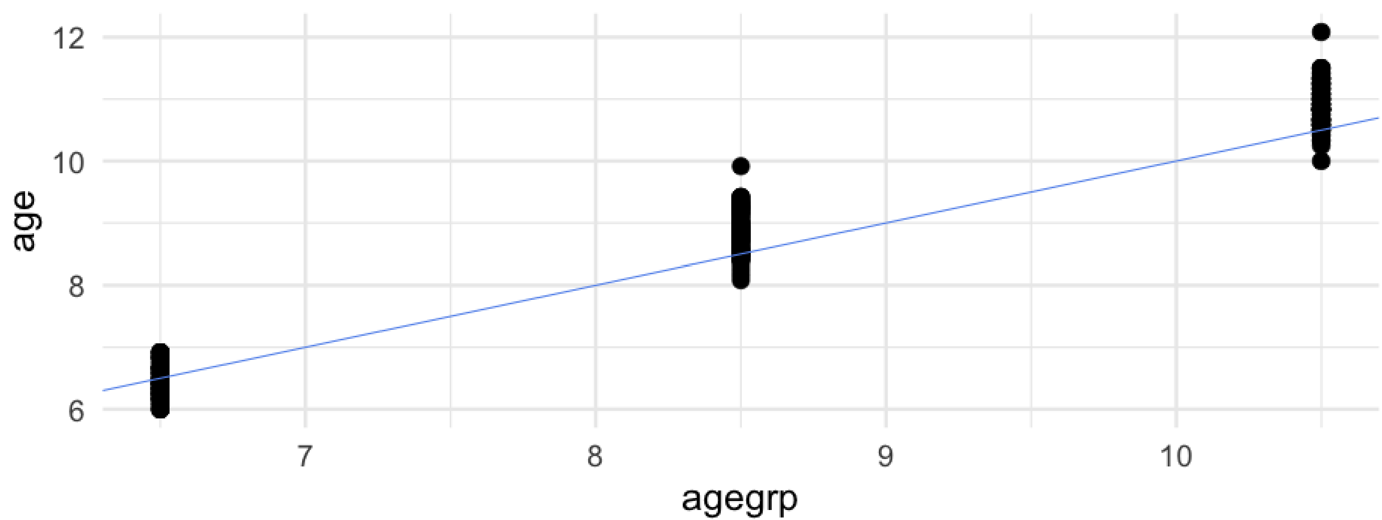
Slightly different

Compare model predictions

Age

- Notice that `agegrp` does not always correspond directly with their *actual* age.

```
ggplot(d, aes(agegrp, age)) +
  geom_point() +
  geom_abline(intercept = 0, slope = 1,
              color = "cornflowerblue")
```



Model assumptions

- When we use the `agegrp` variable, we are assuming that all children are *the exact same age* at each assessment wave.

- Although `agegrp` is more interpretable than `wave`, it doesn't solve all our problems

You try

Fit another model with `age` as the time variable instead. How do the results compare?

02:00

Intercept

How do we want to handle this? Probably need to do something. Look at first time point

```
d %>%
  filter(wave == 1) %>%
  count(age)
```

```
## # A tibble: 12 x 2
##          age     n
##        <dbl> <int>
##   1 6            6
##   2 6.083333     4
##   3 6.166667     9
##   4 6.25         9
##   5 6.333333    10
##   6 6.416667    11
##   7 6.5          7
##   8 6.583333     7
##   9 6.666667     9
##  10 6.75         3
##  11 6.833333     7
##  12 6.916667     7
```

Centering

- I'll choose to subtract 6 from each age
- what will this value represent for students who were 6.91 years old at the first wave?
  - Backwards projection

```
d <- d %>%
  mutate(age6 = age - 6)

m_age <- lmer(piat ~ age6 + (age6|id), data = d)
```

Summary

```
arm::display(m_age)
```

```
## lmer(formula = piat ~ age6 + (age6 | id), data = d)
##             coef.est coef.se
## (Intercept) 18.79     0.61
## age6         4.54     0.26
##
## Error terms:
##  Groups   Name        Std.Dev. Corr
##  id       (Intercept) 2.01
##           age6        1.84     0.17
##  Residual             5.23
## ---
## number of obs: 267, groups: id, 89
## AIC = 1816.1, DIC = 1803.6
## deviance = 1803.9
```

Compare fit

```
compare_performance(m_wave, m_agegrp2, m_age) %>%
  print_md()
```

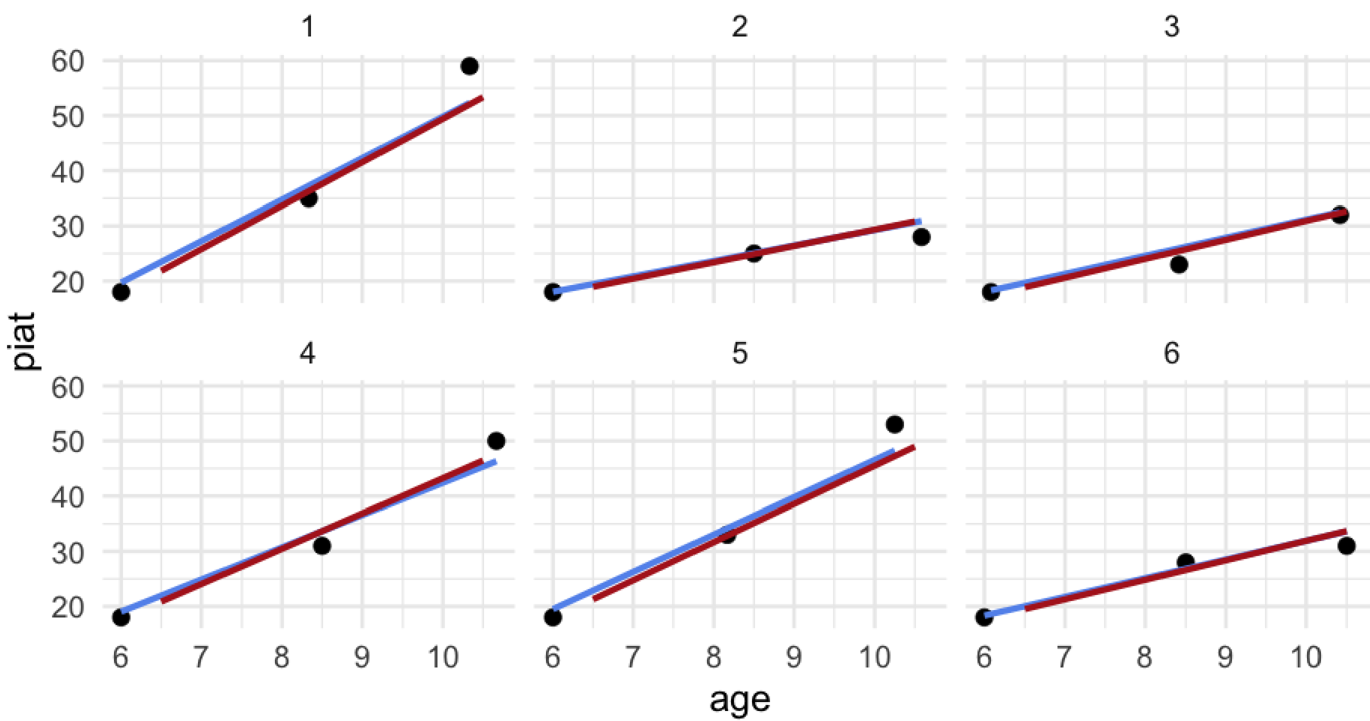
Table: Comparison of Model Performance Indices

Name	Model	AIC	BIC	R2 (cond.)	R2 (marg.)	ICC	RMSE	Sigma
m_wave	lmerMod	1830.39	1851.92	0.81	0.48	0.64	4.15	5.20
m_agegrp2	lmerMod	1831.78	1853.30	0.81	0.48	0.64	4.15	5.20
m_age	lmerMod	1816.14	1837.67	0.81	0.50	0.62	4.34	5.23

Difference in predictions

```
pred_frame <- d %>%
  mutate(pred_agegrp = predict(m_agegrp2),
         pred_age = predict(m_age)) %>%
  filter(id %in% 1:6)
```

```r
ggplot(pred_frame, aes(age, piat)) +
  geom_point() +
  geom_line(aes(x = age6 + 6, y = pred_age),
            color = "cornflowerblue") +
  geom_line(aes(x = agegrp_c + 6.5, y = pred_agegrp),
            color = "firebrick") +
  facet_wrap(~id)
```



Differences

The differences in the model predictions overall appear modest, but it does display better fit to the data, and the assumptions we're making are less stringent.

Changing interpretation

- In this case, our coefficient for age is interepeted in years.

> On average, children gained 4.54 points on the Peabody Individual Achievement Test **per year**.

Challenge

Can you change the model so the coefficient represents monthly growth?

03:00

Solution

Just multiply `age` by 12 to get it coded in months.

```
d <- d %>%
  mutate(age_months = age6 * 12)
d
```

```
## # A tibble: 267 x 9
##        id  wave agegrp       age  piat wave_c agegrp_c       age6 age_mor
##     <dbl> <dbl>  <dbl>     <dbl> <dbl>  <dbl>    <dbl>      <dbl>     <c
##  1      1     1    6.5  6          18      0        0 0              0
##  2      1     2    8.5  8.333333   35      1        2 2.333333      28
##  3      1     3   10.5 10.33333    59      2        4 4.333333      52
##  4      2     1    6.5  6          18      0        0 0              0
##  5      2     2    8.5  8.5        25      1        2 2.5           30
##  6      2     3   10.5 10.58333    28      2        4 4.583333      55
##  7      3     1    6.5  6.083333   18      0        0 0.08333333   1.000
##  8      3     2    8.5  8.416667   23      1        2 2.416667      29
##  9      3     3   10.5 10.41667    32      2        4 4.416667      53
## 10      4     1    6.5  6          18      0        0 0              0
## # … with 257 more rows
```

Refit

```r
m_months <- lmer(piat ~ age_months + (age_months|id), data = d,
                 control = lmerControl(optimizer = "bobyqa"))
arm::display(m_months)
```

```
## lmer(formula = piat ~ age_months + (age_months | id), data = d,
##     control = lmerControl(optimizer = "bobyqa"))
##             coef.est coef.se
## (Intercept) 18.79     0.61
## age_months   0.38     0.02
##
## Error terms:
##  Groups   Name        Std.Dev. Corr
##  id       (Intercept) 2.01
##           age_months  0.15     0.17
##  Residual             5.23
## ---
## number of obs: 267, groups: id, 89
## AIC = 1821.1, DIC = 1798.7
## deviance = 1803.9
```

Which model fits better?

Before we test – what do you suspect?

They are not actually the same

```
compare_performance(m_age, m_months)
```

```
## # Comparison of Model Performance Indices
##
## Name     |   Model |      AIC |      BIC | R2 (cond.) | R2 (marg.) |    I
## -----------------------------------------------------------------------------
## m_age    | lmerMod | 1816.145 | 1837.668 |      0.807 |      0.496 | 0.6
## m_months | lmerMod | 1821.115 | 1842.638 |      0.807 |      0.496 | 0.6
```

But they are essentially

```
pred_frame %>%
  mutate(pred_months = predict(m_months)[1:18]) %>%
  select(id, starts_with("pred"))
```

```
## # A tibble: 18 x 4
##       id pred_agegrp pred_age pred_months
##    <dbl>       <dbl>    <dbl>       <dbl>
##  1     1    21.85047 19.72314    19.72322
##  2     1    37.59817 37.27018    37.27026
##  3     1    53.34587 52.31049    52.31057
##  4     2    18.95405 18.04288    18.04279
##  5     2    24.89376 25.06252    25.06245
##  6     2    30.83348 30.91222    30.91217
##  7     3    18.88021 18.29429    18.29419
##  8     3    25.75976 25.95783    25.95776
##  9     3    32.63931 32.52659    32.52655
## 10     4    20.86038 19.03189    19.03190
## 11     4    33.65203 33.64630    33.64632
## 12     4    46.44368 46.31212    46.31215
## 13     5    21.28110 19.49879    19.49885
## 14     5    35.12409 34.15691    34.15697
## 15     5    48.96708 48.25126    48.25132
## 16     6    19.51079 18.32752    18.32747
## 17     6    26.60084 26.82974    26.82970
## 18     6    33.69089 33.63151    33.63148
```

Another example

With more complications

Wages data

Please read in the `wages.csv` dataset.

```
wages <- read_csv(here::here("data", "wages.csv"))
```

```
## 
## ── Column specification ────────────────────────────────
## cols(
##   id = col_double(),
##   lnw = col_double(),
##   exper = col_double(),
##   ged = col_double(),
##   black = col_double(),
##   hispanic = col_double(),
##   hgc = col_double(),
##   uerate = col_double()
## )
```

02:00

Data

- Mournane, Boudett, and Willett (1999)
- National Longitudinal Survey of Youth
- Studied wages of individuals who dropped out of high school

Variables

- `id`: Participant ID
- `lnw`: Natural log of wages
- `exper`: Experience, in years
- `ged`: Whether or not they completed a GED
- `black`, `hispanic`: Dummy variables for race/ethnicity
- `hgc`: Highest grade completed
- `uerate`: Unemployment rate at the time

Complications

```r
wages %>%
  filter(id %in% c(206, 332))
```

```
## # A tibble: 13 x 8
##        id   lnw exper   ged black hispanic   hgc uerate
##     <dbl> <dbl> <dbl> <dbl> <dbl>    <dbl> <dbl>  <dbl>
##  1    206 2.028 1.874     0     0        0    10   9.2
##  2    206 2.297 2.814     0     0        0    10  11
##  3    206 2.482 4.314     0     0        0    10   6.295
##  4    332 1.63  0.125     0     0        1     8   7.1
##  5    332 1.476 1.625     0     0        1     8   9.6
##  6    332 1.804 2.413     0     0        1     8   7.2
##  7    332 1.439 3.393     0     0        1     8   6.195
##  8    332 1.748 4.47      0     0        1     8   5.595
##  9    332 1.526 5.178     0     0        1     8   4.595
## 10    332 2.044 6.082     0     0        1     8   4.295
## 11    332 2.179 7.043     0     0        1     8   3.395
## 12    332 2.186 8.197     0     0        1     8   4.395
## 13    332 4.035 9.092     0     0        1     8   6.695
```

Complications

- Unbalanced data

```r
wages %>%
  count(id) %>%
  summarize(range = range(n))
```

```
## # A tibble: 2 x 1
##   range
##   <int>
## 1     1
## 2    13
```

- Participants age ranged from 14–17 at first time point
- Unequal spacing between waves

Complications

- Participants dropped out at different times, entered the workforce and different times, and switched jobs at different times
- A decision was made to clock *time* from their first day of work
- The `exper` variable tracks their overall time in the workforce, and time at a given salary

Fitting a model

- The hard part – structuring the data – is already done. We really don't have to do anything special here to account for all these complexities!

```r
m_wage0 <- lmer(lnw ~ exper + (exper|id), data = wages,
                control = lmerControl(optimizer = "bobyqa"))
```

```r
arm::display(m_wage0)
```

```
## lmer(formula = lnw ~ exper + (exper | id), data = wages, control = lmerC
##             coef.est coef.se
## (Intercept) 1.72     0.01
## exper       0.05     0.00
##
## Error terms:
##  Groups   Name        Std.Dev. Corr
##  id       (Intercept) 0.23
##           exper       0.04     -0.30
##  Residual             0.31
## ---
## number of obs: 6402, groups: id, 888
## AIC = 4951.3, DIC = 4903.5
## deviance = 4921.4
```

Every one year of extra experience corresponded to a 0.05 increase in log wages, on average, which varied across participants with a standard deviation of 0.04.

Challenge

Let's fit a more interesting model. Try to fit a model that addresses the following questions:

> Is the relation between experience and log wages the same across coded race/ethnicity categories? Do these relations depend on highest grade completed?

05:00

Centering

Let's center highest grade completed. You could choose whatever value makes the most sense to you. I'll choose Grade 9.

```
wages <- wages %>%
  mutate(hgc_9 = hgc - 9)
```

Is this right?

If not, what is it missing?

```
m_wage1 <- lmer(lnw ~ exper + black + hispanic + hgc_9 +
                  (exper|id),
                data = wages,
                control = lmerControl(optimizer = "bobyqa"))
```

Random effects

In the previous model, I specified `exper` as randomly varying across `id` levels.

Could or should I have set any of the other variables to vary randomly? Why or why not?

Marginal predictions

```r
pred_frame <- expand.grid(
  exper = 0:15,
  black = 0:1,
  hispanic = 0:1,
  hgc_9 = 6:12 - 9,
  id = -999
)

pred_frame <- pred_frame %>%
  mutate(pred = predict(m_wage1,
                        pred_frame,
                        allow.new.levels = TRUE))
```

Race/Ethnicity

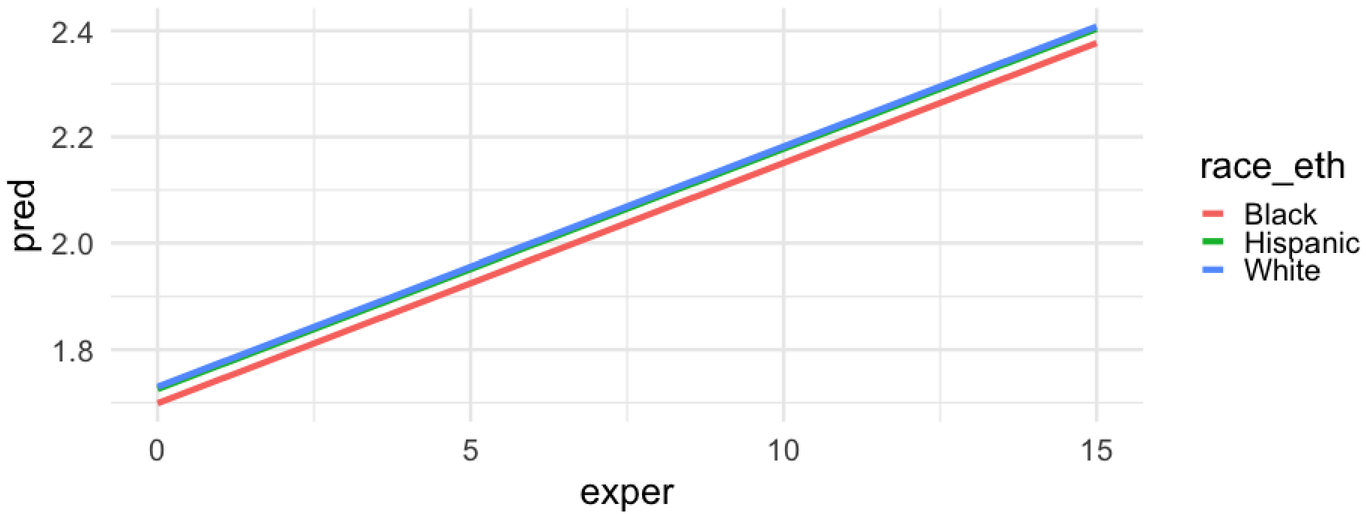
Let's create a new variable that has all the race/ethnicity *labels* instead of the dummy codes.

```
pred_frame <- pred_frame %>%
  mutate(
    race_eth = case_when(
      black == 0 & hispanic == 0 ~ "White",
      black == 1 & hispanic == 0 ~ "Black",
      black == 0 & hispanic == 1 ~ "Hispanic",
      TRUE ~ NA_character_
    )
  )
```

Plots

Look at just `hgc_9 == 0`.

```
pred_frame %>%
  drop_na() %>%
  filter(hgc_9 == 0) %>%
  ggplot(aes(exper, pred)) +
  geom_line(aes(color = race_eth))
```



All hgc

Interactions

If we want to know how the *slope* may or may not depend on these variables, we have to model the interactions.

Just the two–way interactions

```
m_wage2 <- lmer(lnw ~ exper + black + exper:black +
                  exper:hispanic +
                  hgc_9 + exper:hgc_9 +
                  (exper|id),
                data = wages,
                control = lmerControl(optimizer = "bobyqa"))
```

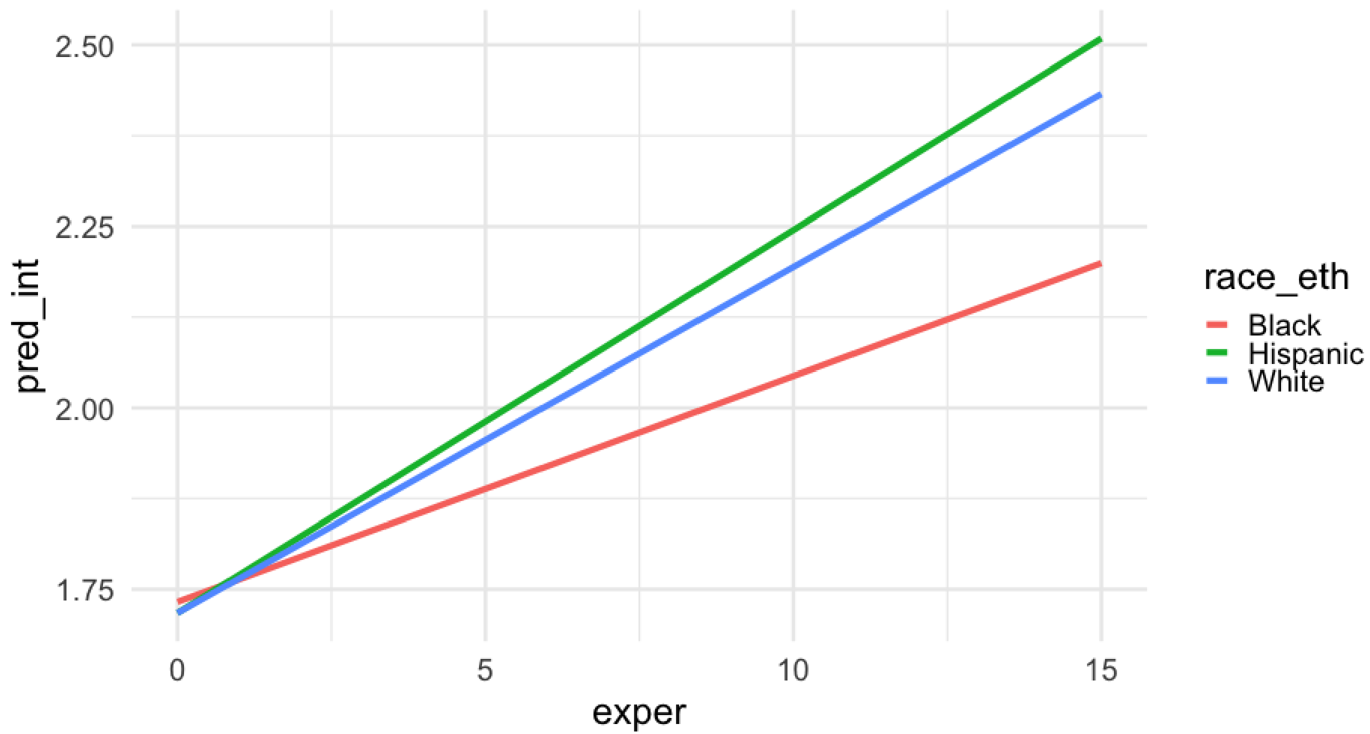
Reproduce the plots

First make new predictions

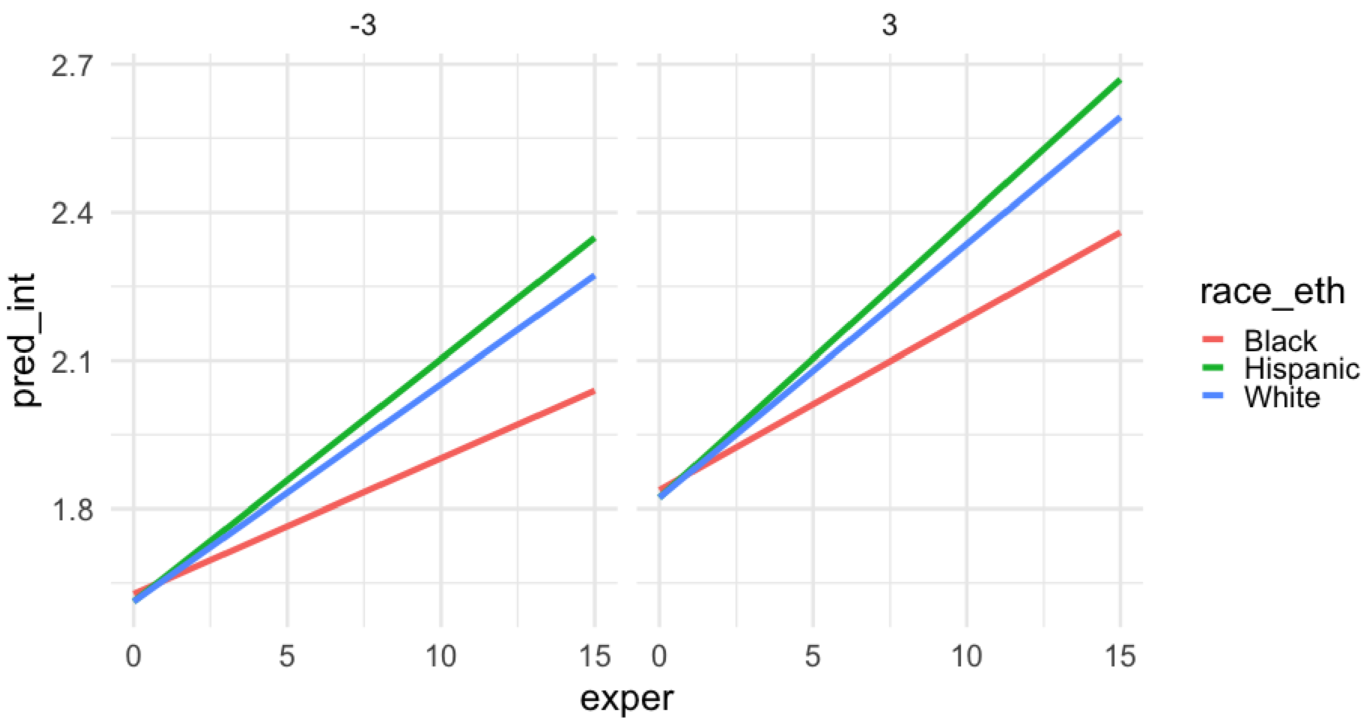
```
pred_frame <- pred_frame %>%
  mutate(pred_int = predict(m_wage2,
                            newdata = pred_frame,
                            allow.new.levels = TRUE))
```

Plot

```
pred_frame %>%
  drop_na() %>%
  filter(hgc_9 == 0) %>%
  ggplot(aes(exper, pred_int)) +
  geom_line(aes(color = race_eth))
```

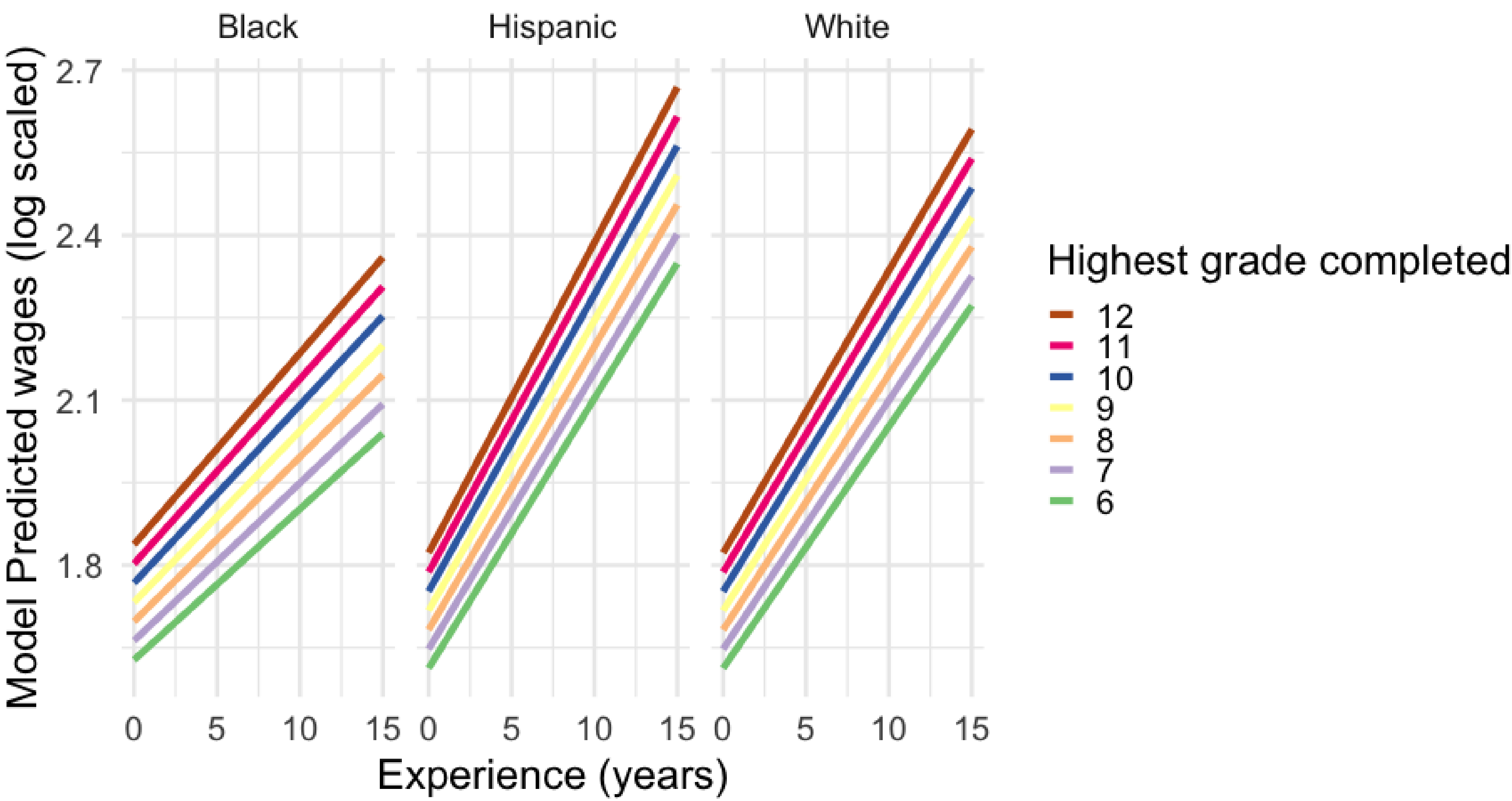


```r
pred_frame %>%
  drop_na() %>%
  filter(hgc_9 == -3 | hgc_9 == 3) %>%
  ggplot(aes(exper, pred_int)) +
  geom_line(aes(color = race_eth)) +
  facet_wrap(~hgc_9)
```



Focus on hgc

```
pred_frame %>%
  drop_na() %>%
  ggplot(aes(exper, pred_int)) +
  geom_line(aes(color = factor(hgc_9))) +
  facet_wrap(~race_eth) +
  scale_color_brewer("Highest grade completed",
                     palette = "Accent",
                     breaks = 3:-3,
                     labels = 12:6) +
  labs(x = "Experience (years)",
       y = "Model Predicted wages (log scaled)")
```



Black
Hispanic
White
Model Predicted wages (log scaled)
2.7
2.4
2.1
1.8
0
5
10
15
Experience (years)
Highest grade completed
12
11
10
9
8
7
6

Coefficient interpretation

Notice I started with the plots

- In *presenting* a model, this is generally what I would do
  - I think this generally helps interpretation
- In practice I generally start by looking at the coefficients

Model summary

```
arm::display(m_wage2)
```

```
## lmer(formula = lnw ~ exper + black + exper:black + exper:hispanic +
##     hgc_9 + exper:hgc_9 + (exper | id), data = wages, control = lmerCont
##                 coef.est coef.se
## (Intercept)      1.72     0.01
## exper            0.05     0.00
## black            0.02     0.02
## hgc_9            0.03     0.01
## exper:black     -0.02     0.01
## exper:hispanic   0.01     0.00
## exper:hgc_9      0.00     0.00
##
## Error terms:
##  Groups   Name        Std.Dev. Corr
##  id       (Intercept) 0.23
##           exper       0.04     -0.31
##  Residual             0.31
## ---
## number of obs: 6402, groups: id, 888
## AIC = 4955.1, DIC = 4811.9
## deviance = 4872.5
```

Handling non–linearity

The data

Simulated data to mimic a common form of non-linearity.

Notice the "true" intercept and slope for each student is actually in the data.

```
sim_d <- read_csv(here::here("data", "curvilinear-sim.csv"))
sim_d
```

```
## # A tibble: 2,760 x 5
##      sid      int    slope date            score
##    <dbl>    <dbl>    <dbl> <date>          <dbl>
##  1     1 31.91237 32.25614 2019-04-26 116.1638
##  2     1 31.91237 32.25614 2019-04-14  50.02688
##  3     1 31.91237 32.25614 2019-05-21 151.8375
##  4     2 22.91502 24.05294 2019-04-25  81.93698
##  5     2 22.91502 24.05294 2019-04-30  93.47374
##  6     2 22.91502 24.05294 2019-05-24 113.8067
##  7     2 22.91502 24.05294 2019-04-27  87.83396
##  8     2 22.91502 24.05294 2019-05-29 112.8697
##  9     2 22.91502 24.05294 2019-04-25  82.40156
## 10     2 22.91502 24.05294 2019-05-27 111.8477
## # … with 2,750 more rows
```

Complexities

Notice these data do have some complexities

Unbalance

```
sim_d %>%
  count(sid) %>%
  summarize(range(n))
```

```
## # A tibble: 2 x 1
##   `range(n)`
##        <int>
## 1          3
## 2          8
```

Varied "starting" points

```
sim_d %>%
  arrange(sid, date) %>%
  group_by(sid) %>%
  slice(1) %>%
  ungroup() %>%
  summarize(range(date))
```

```
## # A tibble: 2 x 1
##   `range(date)`
##   <date>
## 1 2019-04-14
## 2 2019-05-29
```

Overall date range

```
range(sim_d$date)
```

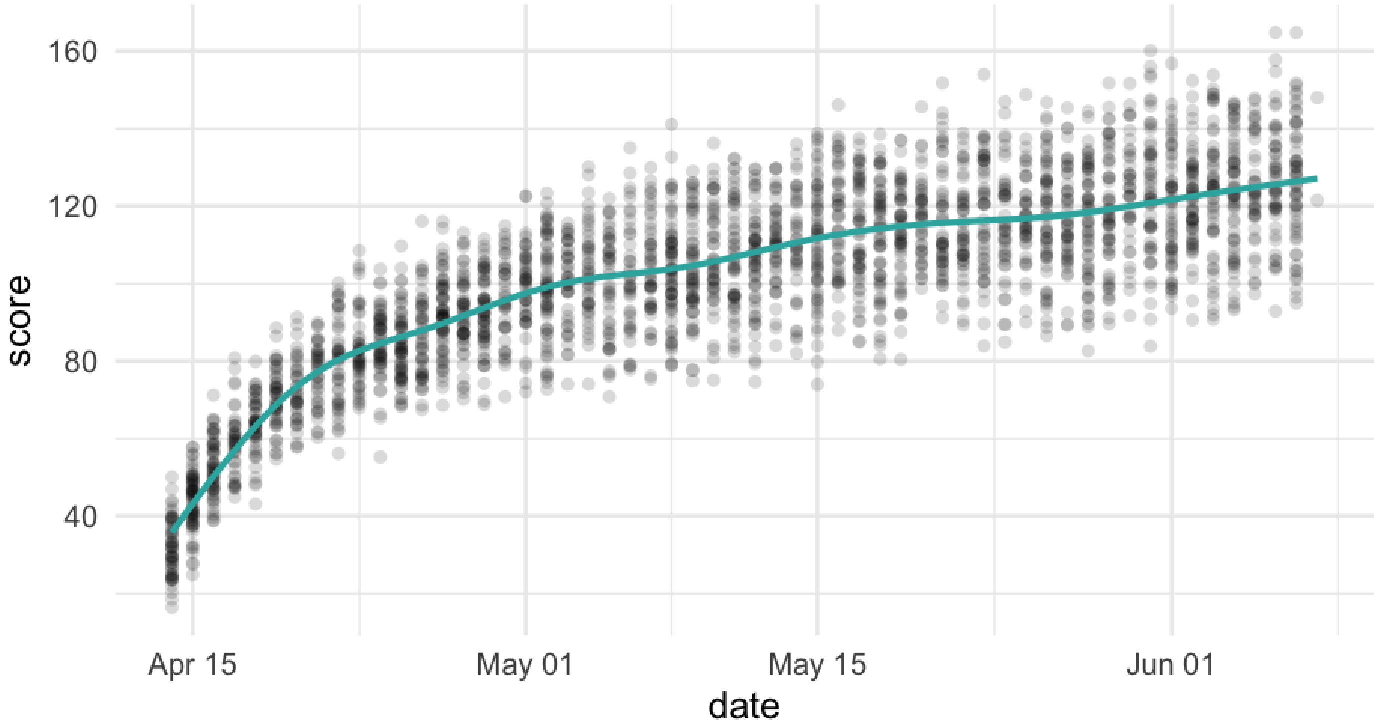
```
## [1] "2019-04-14" "2019-06-08"
```

Plot

Show the overall relation between `date` and `score`. What do you notice?

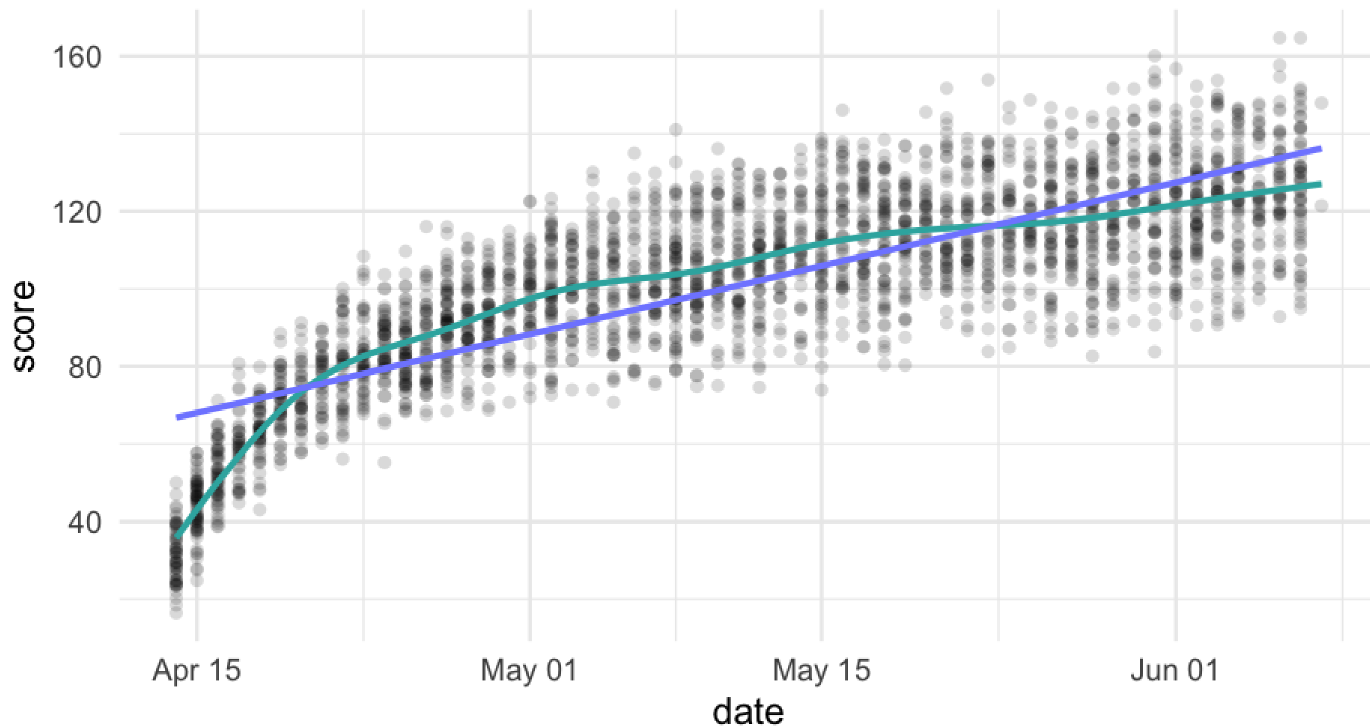
01:00

```r
ggplot(sim_d, aes(date, score)) +
  geom_point(alpha = 0.15, stroke = NA) +
  geom_smooth(se = FALSE, color = "#33B1AE", size = 2)
```



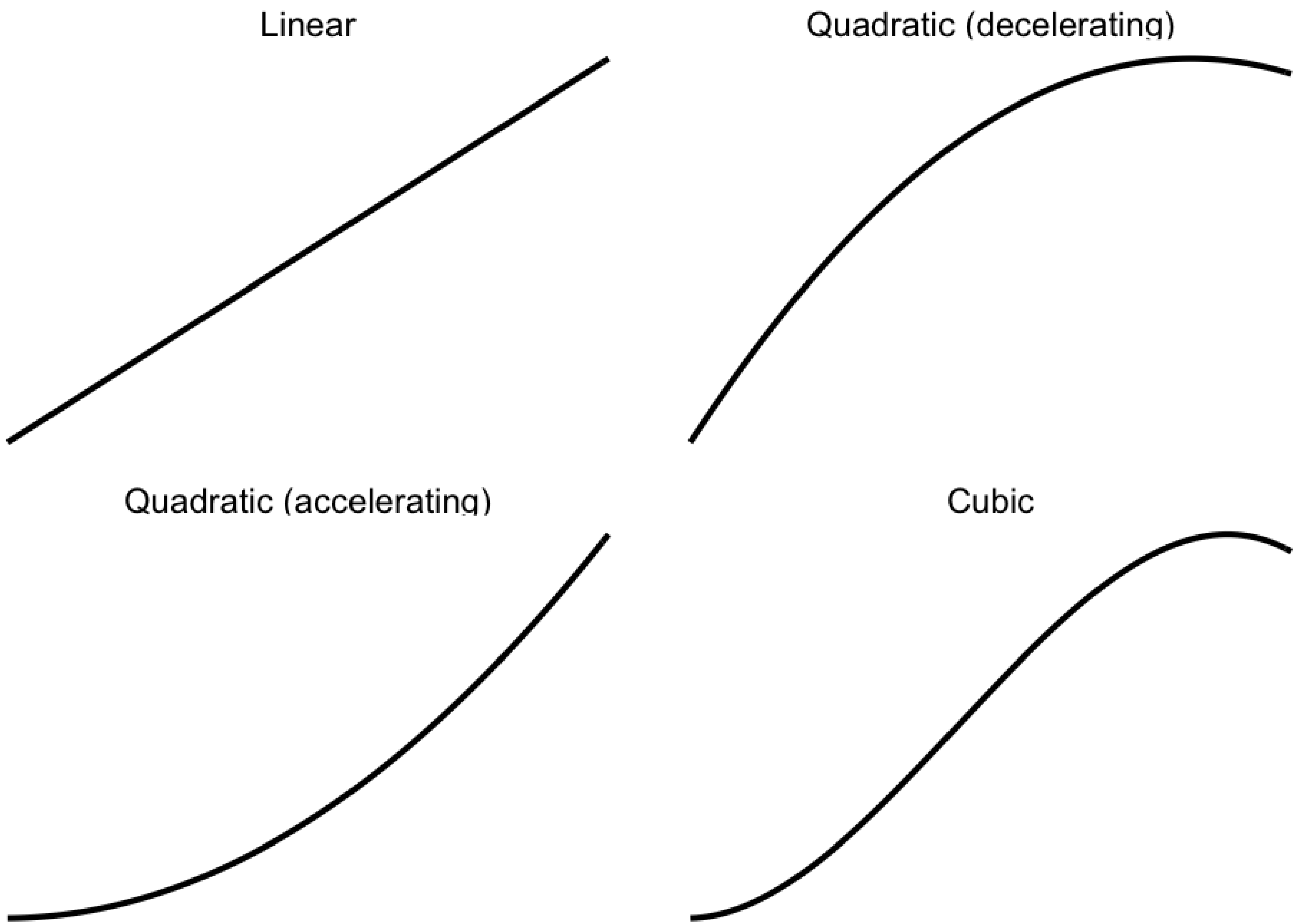
Ideas on how to model this?

```r
ggplot(sim_d, aes(date, score)) +
  geom_point(alpha = 0.15, stroke = NA) +
  geom_smooth(se = FALSE, color = "#33B1AE", size = 2) +
  geom_smooth(se = FALSE, method = "lm", color = "#808AFF", size
```



Linear modeling is not going to work...

Polynomials



Fit a model

Let's try fitting a linear model and a quadratic model and see which fits better.

You try fitting the linear model first, with `date` predicting `score`, and both the intercept and slope varying across students.

01:00

Center date

Let's first center date and put it in interpretable units.

I'll center it on the first time point.

First – what do dates look like when converted to numbers?

```
library(lubridate)
as_date(0)
```

```
## [1] "1970-01-01"
```

```
as_date(1)
```

```
## [1] "1970-01-02"
```

One unit = one day.

Center

```
sim_d <- sim_d %>%
  mutate(
    days_from_start = as.numeric(date) - min(as.numeric(date))
  )
```

Fit linear model

```r
linear <- lmer(score ~ days_from_start + (days_from_start|sid),
               data = sim_d,
               control = lmerControl(optimizer = "Nelder_Mead"))
arm::display(linear)
```

```
## lmer(formula = score ~ days_from_start + (days_from_start | sid),
##     data = sim_d, control = lmerControl(optimizer = "Nelder_Mead"))
##                 coef.est coef.se
## (Intercept)     68.66     0.71
## days_from_start  1.22     0.02
##
## Error terms:
##  Groups   Name            Std.Dev. Corr
##  sid      (Intercept)     13.69
##           days_from_start  0.28    -0.41
##  Residual                  7.91
## ---
## number of obs: 2760, groups: sid, 500
## AIC = 21005, DIC = 20981.9
## deviance = 20987.4
```

Fit quadratic model

```
sim_d <- sim_d %>%
  mutate(days2 = days_from_start^2)

quad <- lmer(score ~ days_from_start + days2 +
               (days_from_start|sid),
             data = sim_d,
             control = lmerControl(optimizer = "Nelder_Mead"))
```

Quadratic summary

```r
arm::display(quad)
```

```
## lmer(formula = score ~ days_from_start + days2 + (days_from_start |
##     sid), data = sim_d, control = lmerControl(optimizer = "Nelder_Mead"))
##                 coef.est coef.se
## (Intercept)     52.09     0.54
## days_from_start  2.98     0.03
## days2           -0.03     0.00
##
## Error terms:
##  Groups   Name            Std.Dev. Corr
##  sid      (Intercept)     10.07
##           days_from_start  0.18    0.31
##  Residual                  4.41
## ---
## number of obs: 2760, groups: sid, 500
## AIC = 18279.8, DIC = 18224.7
## deviance = 18245.2
```

Compare

```
anova(linear, quad)
```

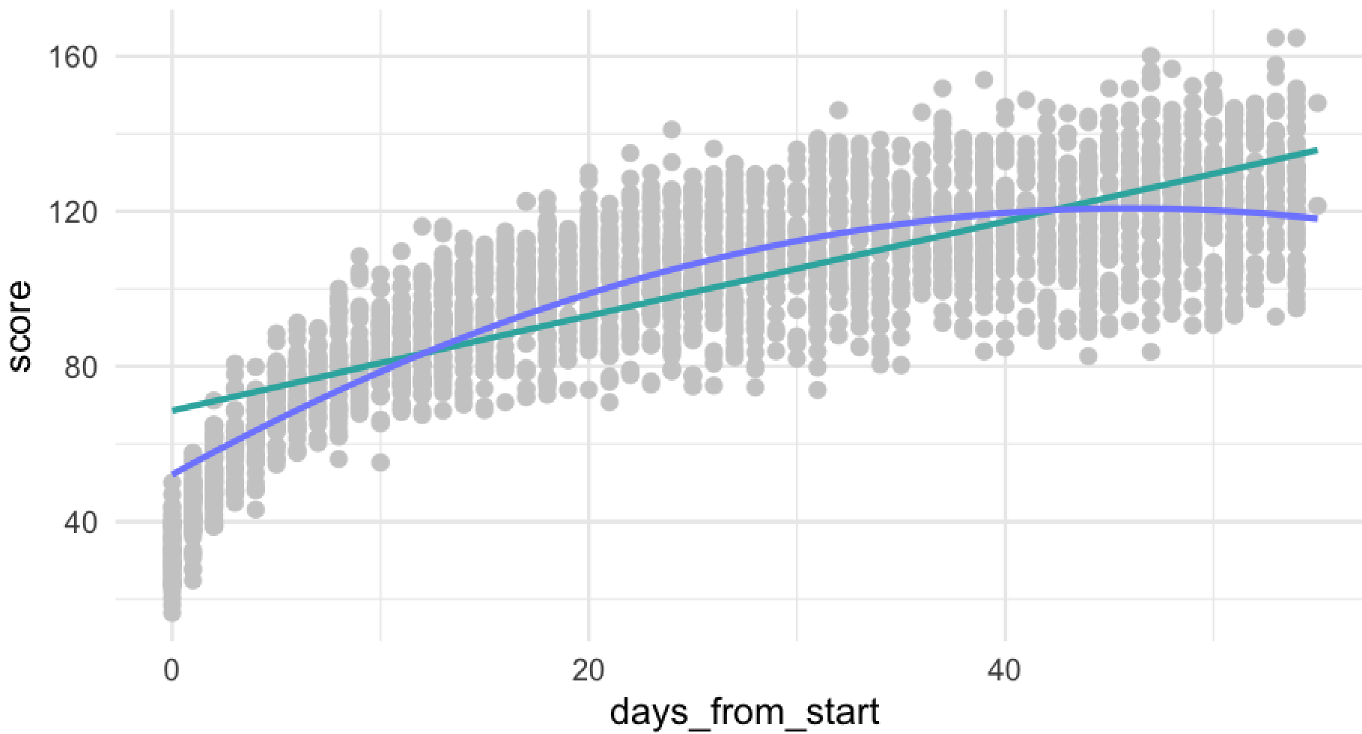
```
## refitting model(s) with ML (instead of REML)

## Data: sim_d
## Models:
## linear: score ~ days_from_start + (days_from_start | sid)
## quad: score ~ days_from_start + days2 + (days_from_start | sid)
##        npar   AIC   BIC   logLik deviance  Chisq Df Pr(>Chisq)
## linear    6 20999 21035 -10493.7    20987
## quad      7 18259 18301  -9122.6    18245 2742.2  1  < 2.2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Plot predictions

```r
pred_frame <- tibble(
    days_from_start = 0:max(sim_d$days_from_start),
    days2 = days_from_start^2,
    sid = -999
  ) %>%
  mutate(pred_linear = predict(linear,
                               newdata = .,
                               allow.new.levels = TRUE),
         pred_quad = predict(quad,
                             newdata = .,
                             allow.new.levels = TRUE))
pred_frame
```

```
## # A tibble: 56 x 5
##    days_from_start days2   sid pred_linear pred_quad
##              <int> <dbl> <dbl>       <dbl>     <dbl>
##  1               0     0  -999    68.65771  52.09304
##  2               1     1  -999    69.87886  55.04428
##  3               2     4  -999    71.10000  57.93071
##  4               3     9  -999    72.32114  60.75233
##  5               4    16  -999    73.54228  63.50914
##  6               5    25  -999    74.76342  66.20114
##  7               6    36  -999    75.98456  68.82834
##  8               7    49  -999    77.20570  71.39072
```

```r
ggplot(pred_frame, aes(days_from_start)) +
  geom_point(aes(y = score), data = sim_d, color = "gray80") +
  geom_line(aes(y = pred_linear), color = "#33B1AE") +
  geom_line(aes(y = pred_quad), color = "#808AFF")
```



This is definitely looking better, but it's too high in the lower tail and maybe a bit too low in the upper

Cubic?

You try first – extend what we just did to model a cubic trend

```
sim_d <- sim_d %>%
  mutate(days3 = days_from_start^3)

cubic <- lmer(score ~ days_from_start + days2 + days3 +
                (days_from_start|sid),
              data = sim_d,
              control = lmerControl(optimizer = "Nelder_Mead"))
```

```
## Warning: Some predictor variables are on very different scales: consider
```

03:00

Cubic summary

```
arm::display(cubic)
```

```
## lmer(formula = score ~ days_from_start + days2 + days3 + (days_from_star
##     sid), data = sim_d, control = lmerControl(optimizer = "Nelder_Mead")
##                 coef.est coef.se
## (Intercept)     43.64     0.49
## days_from_start  4.93     0.04
## days2           -0.12     0.00
## days3            0.00     0.00
##
## Error terms:
##  Groups   Name            Std.Dev. Corr
##  sid      (Intercept)     9.48
##           days_from_start 0.15     0.55
##  Residual                 2.81
## ---
## number of obs: 2760, groups: sid, 500
## AIC = 16311.2, DIC = 16211.2
## deviance = 16253.2
```

Compare

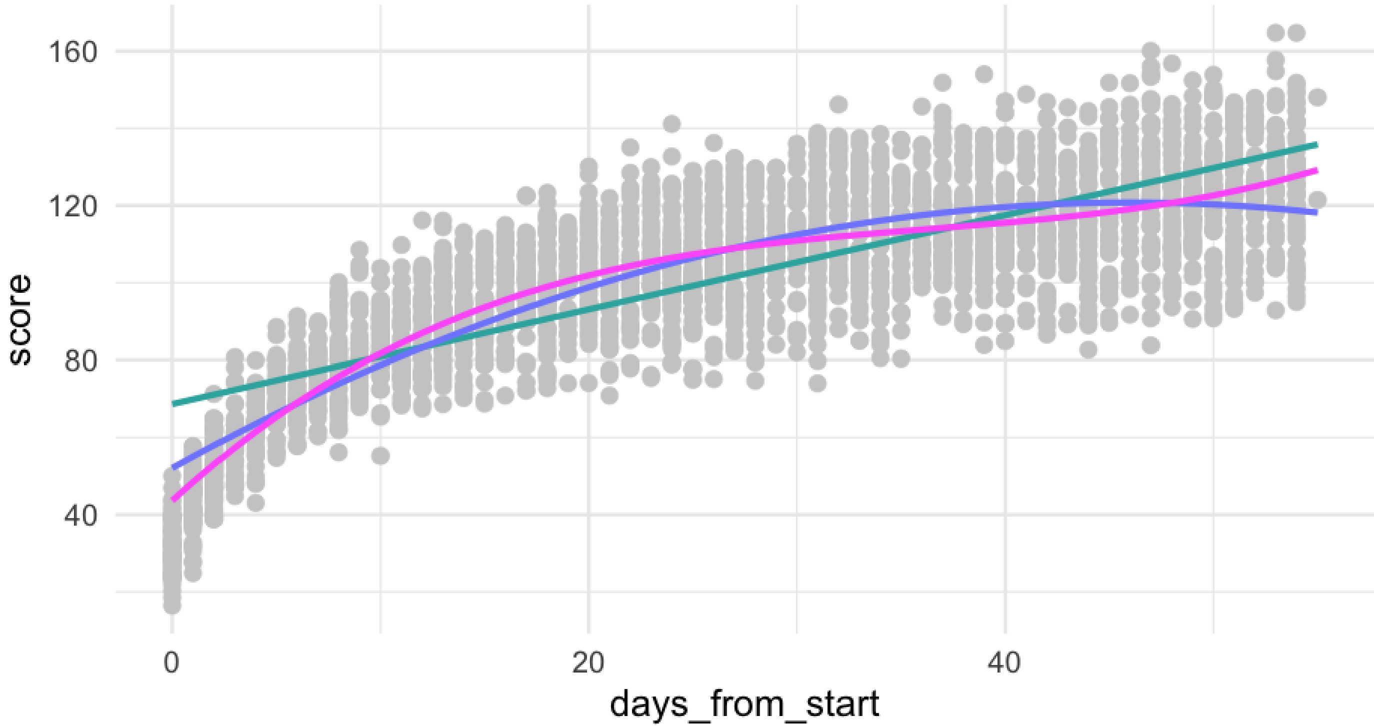
```r
anova(linear, quad, cubic)
```

```
## refitting model(s) with ML (instead of REML)
```

```
## Data: sim_d
## Models:
## linear: score ~ days_from_start + (days_from_start | sid)
## quad: score ~ days_from_start + days2 + (days_from_start | sid)
## cubic: score ~ days_from_start + days2 + days3 + (days_from_start |
## cubic:     sid)
##        npar   AIC   BIC   logLik deviance  Chisq Df Pr(>Chisq)
## linear    6 20999 21035 -10493.7    20987
## quad      7 18259 18301  -9122.6    18245 2742.2  1  < 2.2e-16 ***
## cubic     8 16269 16317  -8126.6    16253 1992.0  1  < 2.2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Predictions

```r
pred_frame <- pred_frame %>%
  mutate(days3 = days_from_start^3)

pred_frame %>%
  mutate(pred_cubic = predict(cubic,
                              newdata = .,
                              allow.new.levels = TRUE)) %>%
  ggplot(aes(days_from_start)) +
  geom_point(aes(y = score), data = sim_d, color = "gray80") +
  geom_line(aes(y = pred_linear), color = "#33B1AE") +
  geom_line(aes(y = pred_quad), color = "#808AFF") +
  geom_line(aes(y = pred_cubic), color = "#ff66fa")
```


160
120
80
40
score
0
20
40
days_from_start

Alternative

Instead of modeling additional parameters, just transform the data

Common transformations

- log
- square root
- inverse $1/x$

Try log

If you're familiar with log growth, the scatterplots we've been looking at probably resemble this trend quite well.

Let's try log transforming our time variable, then fit with it – bonus, we save two estimated parameters.

Note – I will have to use `log(x + 1)` instead of `log(x)` because `log(0)` $= -\infty$ and `log(1)` $= 0$.

```r
sim_d <- sim_d %>%
  mutate(days_log = log(days_from_start + 1))

log_m <- lmer(score ~ days_log + (days_log|sid),
              data = sim_d)

arm::display(log_m)
```

```
## lmer(formula = score ~ days_log + (days_log | sid), data = sim_d)
##             coef.est coef.se
## (Intercept) 26.08     0.32
## days_log    24.43     0.15
##
## Error terms:
##  Groups   Name        Std.Dev. Corr
##  sid      (Intercept) 6.16
##           days_log    3.05     0.20
##  Residual             1.52
## ---
## number of obs: 2760, groups: sid, 500
## AIC = 13703.9, DIC = 13687
## deviance = 13689.5
```

Compare

```r
anova(linear, quad, cubic, log_m)
```

```
## refitting model(s) with ML (instead of REML)
```

```
## Data: sim_d
## Models:
## linear: score ~ days_from_start + (days_from_start | sid)
## log_m: score ~ days_log + (days_log | sid)
## quad: score ~ days_from_start + days2 + (days_from_start | sid)
## cubic: score ~ days_from_start + days2 + days3 + (days_from_start |
## cubic:     sid)
##        npar   AIC   BIC   logLik deviance Chisq Df Pr(>Chisq)
## linear    6 20999 21035 -10493.7    20987
## log_m     6 13702 13737  -6844.7    13690  7298  0
## quad      7 18259 18301  -9122.6    18245     0  1          1
## cubic     8 16269 16317  -8126.6    16253  1992  1     <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

It use the same number of parameters as the linear model, but fits *far* better.

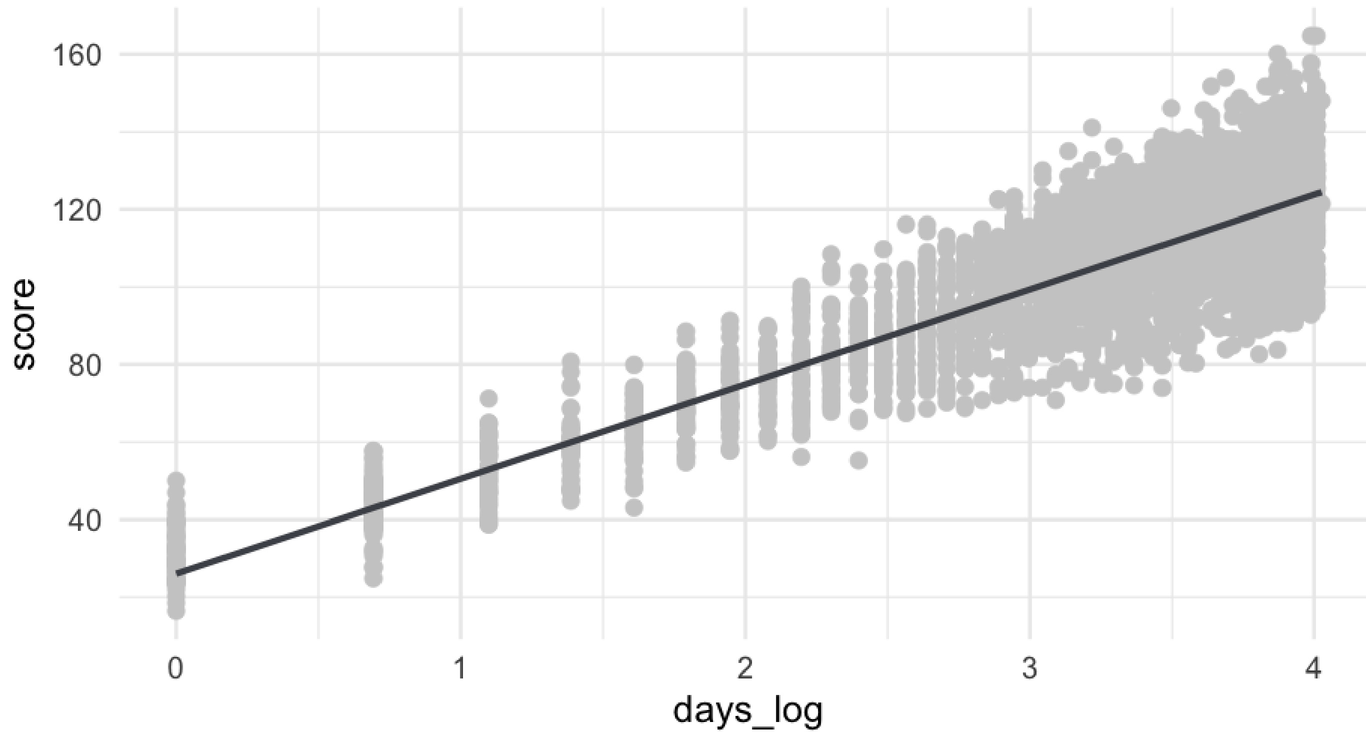
Predictions

```
pred_frame <- pred_frame %>%
  mutate(days_log = log(days_from_start + 1))

pred_frame <- pred_frame %>%
  mutate(pred_log = predict(log_m,
                            newdata = .,
                            allow.new.levels = TRUE))
```

Let's first look at these predictions on the log scale

Predictions on log scale

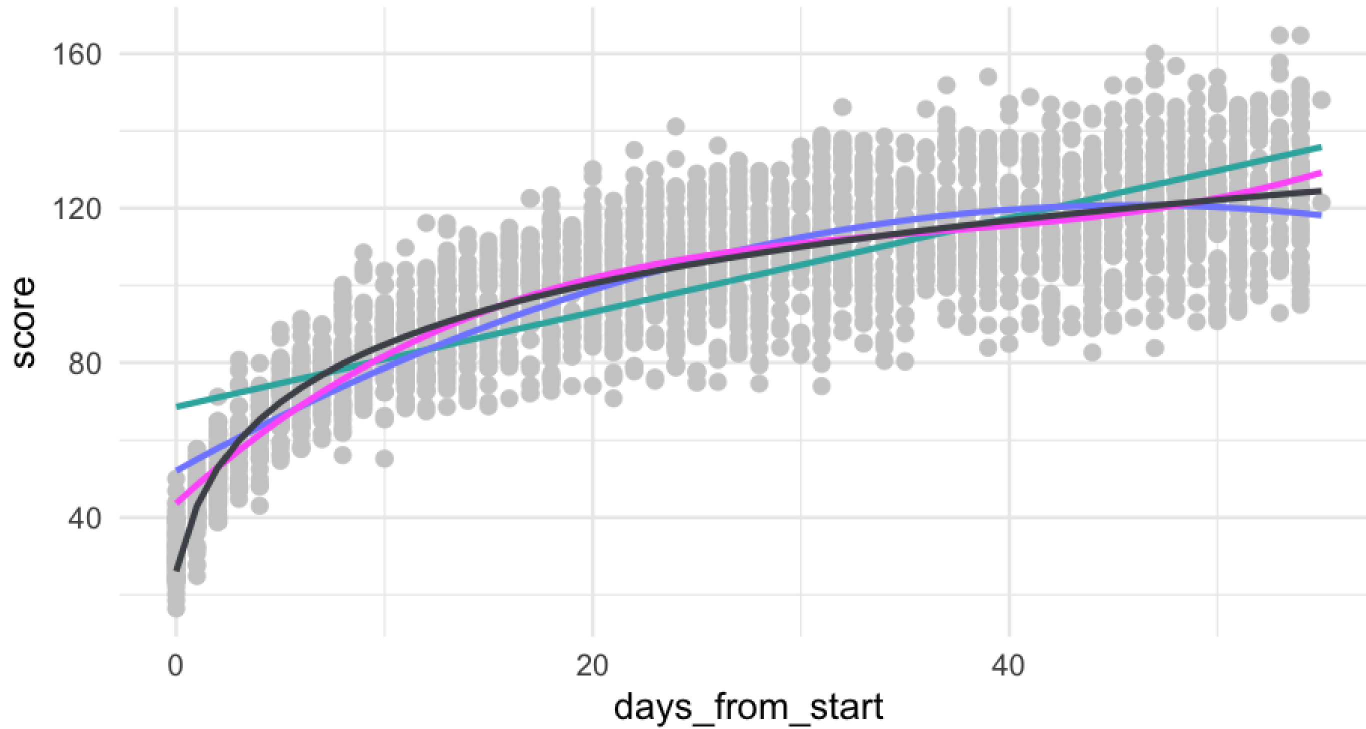
```r
ggplot(pred_frame, aes(days_log)) +
  geom_point(aes(y = score), data = sim_d, color = "gray80") +
  geom_line(aes(y = pred_log), color = "#4D4F57")
```



On raw scale

```r
pred_frame %>%
  mutate(pred_cubic = predict(cubic,
                              newdata = .,
                              allow.new.levels = TRUE)) %>%
  ggplot(aes(days_from_start)) +
  geom_point(aes(y = score), data = sim_d, color = "gray80") +
  geom_line(aes(y = pred_linear), color = "#33B1AE") +
  geom_line(aes(y = pred_quad), color = "#808AFF") +
  geom_line(aes(y = pred_cubic), color = "#ff66fa") +
  geom_line(aes(y = pred_log), color = "#4D4F57")
```

On raw scale



Next time

Bayesian Methods

Now: Homework 2