Clustering nodes in graphs

Why graph clustering is useful?

- Distance matrices are graphs → as useful as any other clustering
- Identification of communities in social networks
- Webpage clustering for better data management of web data

Outline

- k-core decomposition of a graph
- Min s-t cut problem
- Min cut problem
- Spectral graph partitioning

k-core graph decomposition

- Assume an undirected graph $G=(V,E)$
- The core i of G, denoted by G_i, is a **subgraph** of G such that all nodes in G_i have degree at least i

- The core number of a node u is $c(u)$, if u belongs in the $c(u)$ core but not in core $c(u)+1$

Min s-t cut

- Weighted graph $G(V,E)$
- An s-t cut $C = (S,T)$ of a graph $G = (V, E)$ is a cut partition of V into S and T such that $s \in S$ and $t \in T$
- Cost of a cut: $Cost(C) = \Sigma_{e(u,v)\ u \in S,\ v \in T}\ w(e)$
- **Problem:** Given G, s and t find the minimum cost s-t cut

Min-cut problem

- Connected, undirected graph **$G=(V,E)$**
- Assignment of weights to edges: **$w: E \rightarrow R^+$**
- **Cut:** Partition of V into two sets: **V', V–V'**. The set of edges with one end point in **V** and the other in **V'** define the cut
- The removal of the cut disconnects **G**
- **Cost of a cut:** sum of the weights of the edges that have one of their end point in **V'** and the other in **V–V'**

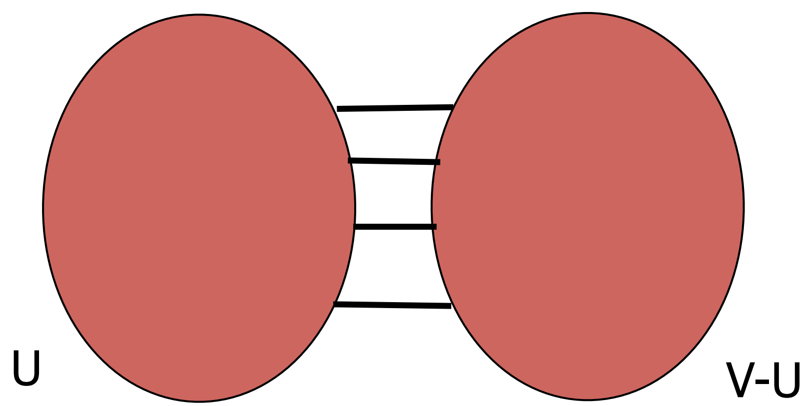
Min cut problem

- Can we solve the min-cut problem using an algorithm for s-t cut?

More on min-cut

- What does it mean that a set of nodes are well or sparsely interconnected?

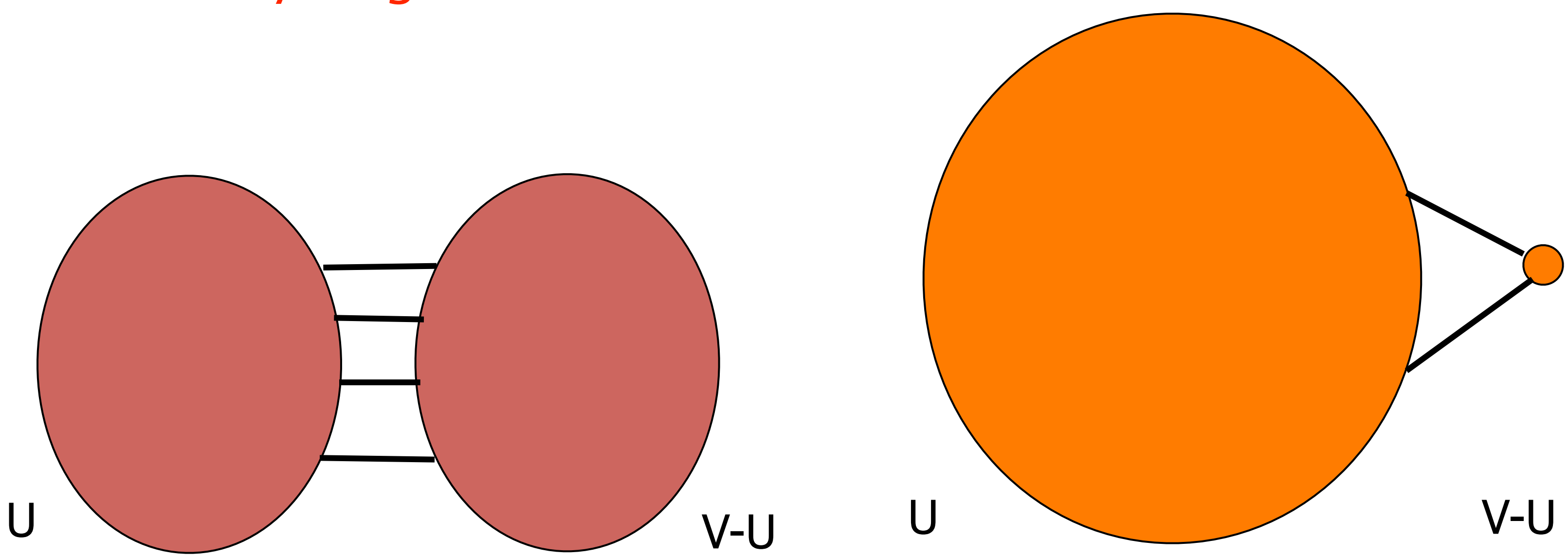
- min-cut: the min number of edges such that when removed cause the graph to become disconnected
  - small min-cut implies sparse connectivity
  - $\min_U E(U, V \setminus U) = \sum_{i \in U} \sum_{j \in V \setminus U} A[i,j]$



Measuring connectivity

- What does it mean that a set of nodes are well interconnected?

- min-cut: the min number of edges such that when removed cause the graph to become disconnected
  - not always a good idea!



Graph expansion

- Normalize the cut by the size of the smallest component
- Cut ratio: $$\alpha = \frac{E(U, V \setminus U)}{\min\{|U|, |V \setminus U|\}}$$
- Graph expansion: $$\alpha(G) = \min_{U} \frac{E(U, V \setminus U)}{\min\{|U|, |V \setminus U|\}}$$
- We will now see how the graph expansion relates to the eigenvalue of the adjacency matrix A

Spectral analysis

- The Laplacian matrix $L = D - A$ where
  - A = the adjacency matrix
  - $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$
    - d_i = degree of node i

- Therefore
  - $L(i,i) = d_i$
  - $L(i,j) = -1$, if there is an edge (i,j)

Laplacian Matrix properties

- The matrix L is symmetric and positive semi-definite
  - all eigenvalues of L are positive

- The matrix L has 0 as an eigenvalue, and corresponding eigenvector $w_1 = (1,1,\ldots,1)$
  - $\lambda_1 = 0$ is the smallest eigenvalue

The second smallest eigenvalue

- The second smallest eigenvalue (also known as Fielder value) λ_2 satisfies

$$\lambda_2 = \min_{\|x\|=1, x \perp w_1} x^T L x$$

- The vector that minimizes λ_2 is called the Fielder vector. It minimizes

$$\lambda_2 = \min_{x \neq 0} \frac{\sum_{(i,j)\in E} (x_i - x_j)^2}{\sum_i x_i^2} \text{ where } \sum_i x_i = 0$$

Spectral ordering

- The values of x minimize

$$\min_{x \neq 0} \frac{\sum_{(i,j)\in E}(x_i - xj)^2}{\sum_i x_i^2} \qquad \sum_i x_i = 0$$

- For weighted matrices

$$\min_{x \neq 0} \frac{\sum_{(i,j)} A[i,j](x_i - xj)^2}{\sum_i x_i^2} \qquad \sum_i x_i = 0$$

- The ordering according to the x_i values will group similar (connected) nodes together

- Physical interpretation: The stable state of springs placed on the edges of the graph

Spectral partition

- Partition the nodes according to the ordering induced by the Fielder vector
- If $u = (u_1, u_2, \ldots, u_n)$ is the Fielder vector, then split nodes according to a value s
  - bisection: s is the median value in u
  - ratio cut: s is the value that minimizes α
  - sign: separate positive and negative values ($s=0$)
  - gap: separate according to the largest gap in the values of u
- This works well (provably for special cases)